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Fourier Transforms in Clifford Analysis

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Abstract

This chapter gives an overview of the theory of hypercomplex Fourier transforms, which are generalized Fourier transforms in the context of Clifford analysis. The emphasis lies on three different approaches that are currently receiving a lot of attention: the eigenfunction approach, the generalized roots of -1 approach, and the characters of the spin group approach. The eigenfunction approach prescribes complex eigenvalues to the L_2 basis consisting of the Clifford–Hermite functions and is therefore strongly connected to the representation theory of the Lie superalgebra $\mathfrak{osp}(1|2)$. The roots of -1 approach consists of replacing all occurrences of the imaginary unit in the classical Fourier transform by roots of -1 belonging to a suitable Clifford algebra. The resulting transforms are often used in engineering. The third approach uses characters to generalize the classical Fourier transform to the setting of the group $Spin(4)$, resp. $Spin(6)$ for application in image processing. For each approach, precise definitions of the transforms under consideration are given, important special cases are highlighted, and a summary of the most important results is given. Also directions for further research are indicated.

Introduction

The classical Fourier transform (FT), defined over $\mathbb{R}^m$, is given by

$$\mathcal{F}(f)(y) := (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} e^{-i\langle x,y\rangle} f(x)dx \tag{1}$$

for functions $f \in L_1(\mathbb{R}^m)$, where $\langle x, y\rangle = \sum_{j=1}^{m} x_j y_j$ is the standard inner product. Without any doubt, it is one of the most important tools of modern mathematics, with a myriad of applications in virtually all branches of engineering and physics. Its mathematical foundation (see, e.g., [53]) is studied in the field of harmonic analysis, i.e. the function theory of the Laplace operator.

In Clifford analysis the Laplace operator is replaced by its square root, namely the Dirac operator. It then becomes a natural problem to investigate generalized Fourier transforms in this framework, both for theoretical reasons and for use in applications. In the recent literature three different approaches to these so-called hypercomplex FTs have been considered. They can be identified as follows:

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- **A:** Eigenfunction approach
- **B:** Generalized roots of -1 approach
- **C:** Characters of spin group approach

The first approach **A** is mainly studied in, e.g., [8–10, 19–23], and aims at constructing new hypercomplex transforms by prescribing eigenvalues to a suitable basis of a Clifford-algebra valued L_2 space. The choice of eigenvalues implies that there is a huge design freedom in this approach. The transforms of this class also reveal a deep connection with quantum mechanics and exhibit a particular underlying algebraic structure, namely that of the Lie superalgebra $\mathfrak{osp}(1|2)$. For a recent review from this point of view, see [18].

The second approach **B** is mainly advocated in [13,14] and boils down to replacing the imaginary unit i in the exponent of the ordinary Fourier transform by a generalized root of -1, belonging to a Clifford algebra (see, e.g., [35, 37] for a detailed study of such roots). It encompasses several of the hypercomplex FTs often used in applications, such as the quaternionic Fourier transform [31], the Sommen–Bülow transform [15, 49], the Clifford Fourier transform (written without hyphen) introduced in [28] and further extended in [1, 36]. Again this approach exhibits a huge design freedom, as the set of roots of -1 is very big and as the roots can in principle be chosen independently for each application. Note that an interesting comparison between approach **A** and **B** is given in the paper [12], especially concerning the different types of convolution products that can be defined in this context.

Finally, a third approach **C** is given in [2–4], and reinterprets the notion of character as a group morphism in order to generalize the ordinary Fourier transform to the setting of the group $Spin(4)$, resp. $Spin(6)$ for direct application in gray scale, resp. color image processing.

In this chapter, an overview will be given of these three approaches. After stating the definition in each case, the emphasis will lie on eigenfunctions and eigenvalues, as well as computation of the inverse transform. For each approach, the examples that have attracted the most attention will be discussed separately. Also an overview of the main results will be given. In a final section, various open problems are indicated.

Readers interested in other aspects of signal processing in Clifford analysis may wish to consult [6] concerning wavelets, and [16] concerning monogenic signals.

Preliminaries on Clifford Algebras and Analysis

The Clifford algebra $\mathcal{C}l_{0,m}$ over $\mathbb{R}^m$ is the algebra generated by e_i, $i = 1, \ldots, m$, under the relations

$$e_i e_j + e_j e_i = 0, \quad i \neq j$$
$$e_i^2 = -1.$$

This algebra has dimension 2^m as a vector space over $\mathbb{R}$. It can be decomposed as $\mathcal{C}l_{0,m} = \oplus_{k=0}^m \mathcal{C}l_{0,m}^k$ with $\mathcal{C}l_{0,m}^k$ the space of k-vectors defined by

$$\mathcal{C}l_{0,m}^k := \operatorname{span}\{e_{i_1} \ldots e_{i_k}, i_1 < \cdots < i_k\}.$$

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In the applied literature, Clifford algebras are usually called geometric algebras. A detailed exposition from this point of view, including geometric interpretations and applications in computer vision, can be found in, e.g., [27].

In the rest of the text, unless stated otherwise, functions f taking values in $\mathcal{C}l_{0,m}$ will be considered. Such functions can be decomposed as

$$f = f_0 + \sum_{i=1}^{m} e_i f_i + \sum_{i<j} e_i e_j f_{ij} + \cdots + e_1 \ldots e_m f_{1\ldots m}$$

with f_0, f_i, $f_{ij}, \ldots, f_{1\ldots m}$ all real- or complex-valued functions on $\mathbb{R}^m$.

The Dirac operator is given by $\partial_{\underline{x}} := \sum_{j=1}^{m} \partial_{x_j} e_j$ and the vector variable by $\underline{x} := \sum_{j=1}^{m} x_j e_j$. The square of the Dirac operator equals, up to a minus sign, the Laplace operator in $\mathbb{R}^m$: $\partial_{\underline{x}}^2 = -\Delta$. For more information regarding Clifford analysis, see, e.g., [7, 25] and [51].

Denote by $\mathcal{P}$ the space of polynomials taking values in $\mathcal{C}l_{0,m}$, i.e.

$$\mathcal{P} := \mathbb{R}[x_1, \ldots, x_m] \otimes \mathcal{C}l_{0,m}.$$

The space of homogeneous polynomials of degree k is then denoted by $\mathcal{P}_k$. The space $\mathcal{M}_k := \ker \partial_{\underline{x}} \cap \mathcal{P}_k$ is called the space of spherical monogenics of degree k.

Next the inner product and the wedge product of two vectors $\underline{x}$ and $\underline{y}$ are defined by

$$\langle x, y \rangle := \sum_{j=1}^{m} x_j y_j$$

$$\underline{x} \wedge \underline{y} := \sum_{j<k} e_j e_k (x_j y_k - x_k y_j).$$

There exist two important different bases for the space $\mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$, where $\mathcal{S}(\mathbb{R}^m)$ denotes the Schwartz space. Define the functions $\psi_{j,k,\ell}$ by

$$\begin{aligned} \psi_{2j,k,\ell} &:= L_j^{\frac{m}{2}+k-1}(|x|^2) M_k^{(\ell)} e^{-|x|^2/2}, \\ \psi_{2j+1,k,\ell} &:= L_j^{\frac{m}{2}+k}(|x|^2) \underline{x} M_k^{(\ell)} e^{-|x|^2/2}, \end{aligned} \tag{2}$$

where $j, k \in \mathbb{N}$, $\{M_k^{(\ell)} \in \mathcal{M}_k : \ell = 1, \ldots, \dim \mathcal{M}_k\}$ is a basis for $\mathcal{M}_k$, and L_j^{α} are the Laguerre polynomials. The set $\{\psi_{j,k,\ell}\}$ forms a basis of $\mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$, see [50]. This basis is called the Clifford–Hermite basis or the spherical basis. Alternatively, define the one-dimensional Hermite functions (see, e.g., [54]) by

$$\begin{aligned} \psi_k(x) &:= \left(x - \tfrac{d}{dx}\right)^k e^{-x^2/2} \\ &= H_k(x) e^{-x^2/2} \end{aligned}$$

for $k \in \mathbb{N}$. Then the set $\{\psi_{j_1, j_2, \ldots, j_m}\}$ with

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$$\psi_{j_1,j_2,\ldots,j_m} = \psi_{j_1}(x_1)\psi_{j_2}(x_2)\ldots\psi_{j_m}(x_m)$$

and $j_1,\ldots,j_m \in \mathbb{N}$ is also a basis of $\mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$, called the tensor product or cartesian basis. Both bases interact nicely with the ordinary FT. One has

$$\mathcal{F}\left(\psi_{j,k,\ell}\right) = (-i)^{j+k}\psi_{j,k,\ell} \tag{3}$$

$$\mathcal{F}\left(\psi_{j_1,j_2,\ldots,j_m}\right) = (-i)^{j_1+\cdots+j_m}\psi_{j_1,j_2,\ldots,j_m}. \tag{4}$$

It is interesting to compare this result with the subsequent Theorems 1 and 4. For more information about these two bases, see [17].

Remark 1. The definition of Clifford algebra immediately generalizes to arbitrary signature. Indeed, the algebra $\mathcal{C}l_{p,q}$ over $\mathbb{R}^{p,q}$ is generated by e_i, $i = 1,\ldots,p+q$, under the relations

$$\begin{aligned} &e_i e_j + e_j e_i = 0, \qquad i \neq j,\\ &e_i^2 = 1, \quad i \leq p\\ &e_i^2 = -1, \quad i > p. \end{aligned}$$

Note, however, that one has to take care with the study of the Dirac operator in arbitrary signature because it no longer remains elliptic.

Hypercomplex Fourier Transforms in Approach A

Definition, Eigenvalues and Eigenfunctions

In this section a general integral kernel of the following form is considered

$$K(x,y) = \Big(A(w,\tilde{z}) + (\underline{x} \wedge \underline{y})B(w,\tilde{z})\Big)\, e^{\frac{i}{2}(\cot\alpha)(|x|^2+|y|^2)} \tag{5}$$

with

$$A(w,\tilde{z}) = \sum_{k=0}^{+\infty} \alpha_k\ (\tilde{z})^{-\lambda} J_{k+\lambda}(\tilde{z}) C_k^{\lambda}(w)$$

$$B(w,\tilde{z}) = \sum_{k=1}^{+\infty} \beta_k\ (\tilde{z})^{-\lambda-1} J_{k+\lambda}(\tilde{z}) C_{k-1}^{\lambda+1}(w)$$

and $\alpha_k, \beta_k \in \mathbb{C}$, $\tilde{z} = (|x||y|)/\sin\alpha$, $w = \langle x, y\rangle/(|x||y|)$, $\lambda = (m-2)/2$ and $\alpha \in [-\pi,\pi]$. Here, J_ν is the Bessel function and C_k^λ the Gegenbauer polynomial. The cases where $\alpha = 0$ or $\alpha = \pm\pi$ will be excluded in the sequel, as they lead to distributions rather than integral transforms.

The integral transform associated with this kernel is defined by

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$$\mathcal{F}_K\,(f)\,(y) = \rho_{\alpha,m} \int_{\mathbb{R}^m} K(x,y)\; f(x)\,dx \tag{6}$$

with

$$\rho_{\alpha,m} = (\pi(1-e^{-2i\alpha}))^{-m/2}$$

and with dx the standard Lebesgue measure on $\mathbb{R}^m$. The precise form of the kernel in formula (5) is inspired by the results obtained in [20]. In particular, it encompasses all previously studied kernels in the eigenfunction approach **A** and yields transforms which are diagonalized by the basis (2) of $\mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$.

Remark 2. In order to ensure good analytic behavior of the transform (6), both $A(w,\tilde{z})$ and $B(w,\tilde{z})$ should satisfy at least polynomial bounds of the following type

$$|A(w,\tilde{z})| \leq c(1+|x|)^j(1+|y|)^j,$$
$$|B(w,\tilde{z})| \leq c(1+|x|)^j(1+|y|)^j,$$

with $j \in \mathbb{N}$ and c a constant.

This is the case for all subsequent examples, and allows for proving that the transform (6) yields a continuous map on $\mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$, using the method of [20].

It is now possible to compute the action of the transform (6) on the basis (2) of $\mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$. First the radial behavior of the integral transform is determined:

Proposition 1. *Let $M_k \in \mathcal{M}_k$ be a spherical monogenic of degree k. Let $f(x) = f_0(|x|)$ be a real-valued radial function in $\mathcal{S}(\mathbb{R}^m)$. Further, put $\underline{\xi} = \underline{x}/|x|$ and $\underline{\eta} = \underline{y}/|y|$. Then one has, putting $\beta_0 = 0$,*

$$\begin{aligned}\mathcal{F}_K\,(f(r)M_k)\,(y) &= c_m \left(\frac{\lambda}{\lambda+k}\alpha_k - \sin\alpha\, \frac{k}{2(k+\lambda)}\beta_k \right) e^{\frac{i}{2}(\cot\alpha)|y|^2} M_k(\eta) \\ &\quad \times \int_0^{+\infty} r^{m+k-1} f_0(r)\;(\tilde{z})^{-\lambda} J_{k+\lambda}(\tilde{z})\; e^{\frac{i}{2}(\cot\alpha)r^2} dr\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}_K\,(f(r)\underline{x}M_k)\,(y) &= c_m \left(\frac{\lambda}{\lambda+k+1}\alpha_{k+1} + \sin\alpha \frac{k+1+2\lambda}{2(k+1+\lambda)}\beta_{k+1} \right) e^{\frac{i}{2}(\cot\alpha)|y|^2} \underline{\eta}\; M_k(\eta) \\ &\quad \times \int_0^{+\infty} r^{m+k} f_0(r)\;(\tilde{z})^{-\lambda} J_{k+1+\lambda}(\tilde{z})\; e^{\frac{i}{2}(\cot\alpha)r^2} dr\end{aligned}$$

with $\tilde{z} = (r|y|)/\sin\alpha$, $\lambda = (m-2)/2$ and

$$c_m = \frac{2}{\Gamma\left(\frac{m}{2}\right)\; (1-e^{-2i\alpha})^{m/2}}.$$

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Proof. The proof goes along similar lines as the proof of Theorem 6.4 in [23]. □

The eigenvalues and eigenfunctions of $\mathcal{F}_K$ are then given in the following theorem.

Theorem 1. *One has, putting $\beta_0 = 0$,*

$$\mathcal{F}_K(\psi_{2j,k,\ell}) = \frac{2^{-\lambda}}{\Gamma(\lambda+1)} \left(\frac{\lambda}{\lambda+k}\alpha_k - \sin\alpha \, \frac{k}{2(\lambda+k)}\beta_k \right) i^k e^{-i\alpha(k+2j)} \, \psi_{2j,k,\ell}$$

and

$$\mathcal{F}_K(\psi_{2j+1,k,\ell}) = \frac{2^{-\lambda}}{\Gamma(\lambda+1)} \left(\frac{\lambda}{\lambda+k+1}\alpha_{k+1} + \sin\alpha \, \frac{k+1+2\lambda}{2(\lambda+k+1)}\beta_{k+1} \right) i^{k+1} e^{-i\alpha(k+2j+1)} \psi_{2j+1,k,\ell}.$$

Proof. This follows from the explicit expression (2) of the basis and the identity (see [33, p. 847, formula 7.421, number 4 with $\alpha = 1$]):

$$\int_0^{+\infty} x^{\nu+1} e^{-\beta x^2} L_n^{\nu}(x^2) J_{\nu}(xy) dx = 2^{-\nu-1}\beta^{-\nu-n-1}(\beta-1)^n y^{\nu} e^{-\frac{y^2}{4\beta}} L_n^{\nu}\left(\frac{y^2}{4\beta(1-\beta)}\right).$$

□

Theorem 1 is very important; it allows for designing a hypercomplex Fourier transform $\mathcal{F}_K$ by prescribing the eigenvalues on the basis $\{\psi_{j,k,\ell}\}$ via

$$\mathcal{F}_K(\psi_{2j,k,\ell}) = \lambda_k e^{-i\alpha 2j} \psi_{2j,k,\ell}$$
$$\mathcal{F}_K(\psi_{2j+1,k,\ell}) = \mu_k e^{-i\alpha(2j+1)} \psi_{2j+1,k,\ell}$$

for any set of numbers $\lambda_k, \mu_k \in \mathbb{C}$. Indeed, it suffices to solve the system of equations

$$\lambda_k = \frac{2^{-\lambda}}{\Gamma(\lambda+1)} \left(\frac{\lambda}{\lambda+k}\alpha_k - \sin\alpha \, \frac{k}{2(\lambda+k)}\beta_k \right) i^k e^{-i\alpha k}$$
$$\mu_k = \frac{2^{-\lambda}}{\Gamma(\lambda+1)} \left(\frac{\lambda}{\lambda+k+1}\alpha_{k+1} + \sin\alpha \, \frac{k+1+2\lambda}{2(\lambda+k+1)}\beta_{k+1} \right) i^{k+1} e^{-i\alpha k}$$

to determine the integral kernel $K(x, y)$ in terms of the coefficients α_k and β_k.

It is possible to obtain the inverse of the general transform $\mathcal{F}_K$ on the basis $\{\psi_{j,k,\ell}\}$.

Theorem 2. *The inverse of $\mathcal{F}_K$ on the basis $\{\psi_{j,k,\ell}\}$ is given by*

$$\mathcal{F}_K^{-1}(f)(y) = \rho_{-\alpha,m} \int_{\mathbb{R}^m} \widetilde{K(x,y)} \, f(x) \, dx,$$

with

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$$\widetilde{K(x,y)} = \Big(C(w,\tilde{z}) + (\underline{x} \wedge \underline{y}) D(w,\tilde{z})\Big) e^{-\frac{i}{2}(\cot\alpha)(|x|^2+|y|^2)}$$

where

$$C(w,\tilde{z}) = \sum_{k=0}^{+\infty} \frac{1}{N_k^\lambda}(\alpha_k + \beta_k \ \sin\alpha)\ (\tilde{z})^{-\lambda} J_{k+\lambda}(\tilde{z}) C_k^\lambda(w)$$

$$D(w,\tilde{z}) = -\sum_{k=1}^{+\infty} \frac{1}{N_k^\lambda}\beta_k \ (\tilde{z})^{-\lambda-1} J_{k+\lambda}(\tilde{z}) C_{k-1}^{\lambda+1}(w),$$

and

$$N_k^\lambda = \frac{1}{2^{2\lambda}(\Gamma(\lambda+1))^2}\left(\frac{\lambda}{\lambda+k}\alpha_k - \sin\alpha \frac{k}{2(\lambda+k)}\beta_k\right)\left(\frac{\lambda}{\lambda+k}\alpha_k + \sin\alpha\frac{k+2\lambda}{2(\lambda+k)}\beta_k\right).$$

This result was obtained in [12], by first computing the radial behavior of $\mathcal{F}_K^{-1}$ as in Proposition 1. Subsequently, one can compute the eigenvalues on the basis $\{\psi_{j,k,\ell}\}$ and check that they are inverse to the ones obtained in Theorem 1.

Important Special Cases

In the special case where $\alpha = \pi/2$, the kernel (5) takes the form

$$K(x,y) = A(w,z) + (\underline{x} \wedge \underline{y})\ B(w,z) \tag{7}$$

with

$$A(w,z) = \sum_{k=0}^{+\infty} \alpha_k \ z^{-\lambda} J_{k+\lambda}(z) C_k^\lambda(w)$$

$$B(w,z) = \sum_{k=1}^{+\infty} \beta_k \ z^{-\lambda-1} J_{k+\lambda}(z) C_{k-1}^{\lambda+1}(w)$$

and $z = |x||y|$, $w = \langle x, y\rangle/(|x||y|)$, $\lambda = (m-2)/2$. The corresponding integral transform is given by

$$\mathcal{F}_K(f)(y) = \rho_{\frac{\pi}{2},m} \int_{\mathbb{R}^m} K(x,y)\ f(x)\ dx$$

where $\rho_{\frac{\pi}{2},m} = (2\pi)^{-m/2}$.

Note that the kernel of the classical Fourier transform (1), which can equally be expressed as the operator exponential $e^{\frac{i\pi m}{4}} e^{\frac{i\pi}{4}(\Delta - |x|^2)}$, takes the form (7) with

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$$\alpha_k = 2^{\lambda}\Gamma(\lambda)(k+\lambda)(-i)^k$$
$$\beta_k = 0.$$

Also the Clifford–Fourier transform (see [8–10, 23]), a generalization of the classical Fourier transform in the framework of Clifford analysis, takes this form. It is defined by the following operator exponential

$$\mathcal{F}_{\pm} := e^{\frac{i\pi m}{4}} e^{\frac{i\pi}{4}(\Delta - |x|^2 \mp 2\Gamma)},$$

with

$$\Gamma := -\sum_{j<k} e_j e_k (x_j \partial_{x_k} - x_k \partial_{x_j}).$$

In case of the Clifford–Fourier transform $\mathcal{F}_{-}$, the coefficients α_k and β_k in the kernel (7) take the form:

$$\alpha_k = 2^{\lambda-1}\Gamma(\lambda+1)(i^{2\lambda+2} + (-1)^k) - 2^{\lambda-1}\Gamma(\lambda)\,(k+\lambda)(i^{2\lambda+2} - (-1)^k)$$
$$\beta_k = -2^{\lambda}\Gamma(\lambda+1)(i^{2\lambda+2} + (-1)^k),$$

as was obtained in [23]. For the transform $\mathcal{F}_{+}$, similar expressions hold. Moreover, in [19], an entire class of kernels of the form (7), for particular values of the coefficients α_k and β_k, was determined. They yield new integral transforms that have the same calculus properties (i.e., interaction with the Dirac operator) as the original Clifford–Fourier transform, but with different spectrum.

Also for general α, concrete examples have been studied. The fractional Fourier transform (see [43]) is a generalization of the classical Fourier transform. It is usually defined using the operator expression

$$\mathcal{F}_{\alpha} = e^{\frac{i\alpha m}{2}} e^{\frac{i\alpha}{2}(\Delta - |x|^2)}, \qquad \alpha \in [-\pi, \pi].$$

Recently, a fractional version of the Clifford–Fourier transform was introduced (see [20]). It is defined by the following exponential operator

$$\mathcal{F}_{\alpha,\beta} = e^{\frac{i\alpha m}{2}} e^{i\beta\Gamma} e^{\frac{i\alpha}{2}(\Delta - |x|^2)}, \qquad \alpha, \beta \in [-\pi, \pi].$$

The integral kernel of this transform takes the form (5) with

$$\alpha_k = 2^{\lambda-1}\Gamma(\lambda)(k+\lambda)i^{-k}(e^{i\beta(k+2\lambda)} + e^{-i\beta k}) - 2^{\lambda-1}\Gamma(\lambda+1)i^{-k}(e^{i\beta(k+2\lambda)} - e^{-i\beta k})$$
$$\beta_k = \frac{2^{\lambda}\Gamma(\lambda+1)}{\sin\alpha} i^{-k}(e^{i\beta(k+2\lambda)} - e^{-i\beta k}).$$

There are many other examples of transforms belonging to approach **A**, such as the class of hypercomplex radially deformed FTs (see [21, 22]), which is constructed using a deformation of

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the classical Dirac operator. The kernels of the related integral transforms have series expansions similar to formula (7).

Results

The most difficult problem related to approach **A** is obtaining closed formulas for series expansions as in formula (5) or (7). This can be illustrated nicely with the Clifford–Fourier transform. Until recently, its integral kernel was only known explicitly in the case $m = 2$ (see [9]); for higher even dimensions, a complicated iterative procedure for constructing the kernel was given in [11], which could only be used practically in low dimensions. A breakthrough was obtained in [23]. In this paper it is found that for m even the kernel can be expressed as follows in terms of a finite sum of Bessel functions:

$$K_+(x,y) = \left(\frac{\pi}{2}\right)^{1/2} \Big(A(s,t) + B(s,t) + (\underline{x} \wedge \underline{y})\, C(s,t)\Big)$$

with

$$\begin{aligned} A(s,t) &= \sum_{\ell=0}^{\lfloor \frac{m}{4}-\frac{3}{4} \rfloor} s^{m/2-2-2\ell}\, \frac{1}{2^\ell \ell!} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}-2\ell-1\right)}\, \tilde{J}_{(m-2\ell-3)/2}(t) \\ B(s,t) &= \sum_{\ell=0}^{\lfloor \frac{m}{4}-\frac{1}{2} \rfloor} s^{m/2-1-2\ell}\, \frac{1}{2^\ell \ell!} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}-2\ell\right)}\, \tilde{J}_{(m-2\ell-3)/2}(t) \\ C(s,t) &= -\sum_{\ell=0}^{\lfloor \frac{m}{4}-\frac{1}{2} \rfloor} s^{m/2-1-2\ell}\, \frac{1}{2^\ell \ell!} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}-2\ell\right)}\, \tilde{J}_{(m-2\ell-1)/2}(t). \end{aligned} \tag{8}$$

Here, the notations $s = \langle \underline{x}, \underline{y} \rangle$, $t = |\underline{x} \wedge \underline{y}| = \sqrt{|\underline{x}|^2|\underline{y}|^2 - \langle \underline{x}, \underline{y}\rangle^2}$ and $\tilde{J}_\alpha(t) = t^{-\alpha} J_\alpha(t)$ are used. Moreover, it is shown that $K_+(x,y) = K_-(x,-y)$ holds for m even. For m odd, the question of determining the kernel explicitly was reduced to the case of $m = 3$. There, a more or less complicated integral expression of the kernel was obtained (see [23, Lemma 4.5]). A relatively simple expression as in formula (8) is not known in this case. Also for other hypercomplex Fourier transforms in approach **A**, expressions similar to (8) have been obtained, see [19, 20].

Next, it is worthwhile to note that all the examples discussed in this section are generated by an underlying Lie superalgebra, namely $\mathfrak{osp}(1|2)$. More details on this aspect of the theory can be found in the review [18]. For more information on representation theory and in particular Lie superalgebras in Clifford analysis, see [52].

Another interesting problem is to determine a suitable convolution product for a hypercomplex FT with kernel as in (5). The classical FT and its convolution product serve as a guide to achieve this. Recall that the classical convolution product for two functions f and g is defined by

$$f * g(x) := \int_{\mathbb{R}^m} f(y)\tau_y g(x) dy, \qquad \tau_y g(x) := g(x-y)$$

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and interacts nicely with the FT:

$$\mathcal{F}(f * g) = (2\pi)^{m/2}\mathcal{F}(f)\mathcal{F}(g).$$

A first way to generalize the convolution product is obtained by introducing a generalization of the (geometric) translation operator τ_y, a strategy which is, e.g., also used for the Dunkl transform (see [45,55]). This can be easily illustrated for the ordinary FT. First, the FT of the translation over z of a function f is computed:

$$\begin{aligned}\mathcal{F}(\tau_z f)(y) &= (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} e^{-i\langle x,y\rangle} f(x-z)dx \\ &= e^{-i\langle z,y\rangle}(2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} e^{-i\langle x,y\rangle} f(x)dx \\ &= e^{-i\langle z,y\rangle}\mathcal{F}(f)(y).\end{aligned}$$

This means that, formally, ordinary translation is recovered via

$$\tau_z f(u) = \mathcal{F}^{-1}\left(e^{-i\langle z,y\rangle}\mathcal{F}(f)(y)\right). \tag{9}$$

Here, the inverse FT acts on the y variable and yields the u variable.

A generalized translation operator related to the integral transform $\mathcal{F}_K$ defined in section "Definition, Eigenvalues and Eigenfunctions" can therefore be defined, following (9), as

Definition 1. Let $f \in \mathcal{S}(\mathbb{R}^m)\otimes\mathcal{C}l_{0,m}$. For $y \in \mathbb{R}^m$ the generalized translation operator $f \longmapsto \tau_y^K f$ is defined by

$$\mathcal{F}_K(\tau_y^K f)(x) = K(y,x)\,\mathcal{F}_K(f)(x), \qquad x \in \mathbb{R}^m.$$

It can be expressed, by the inverse of $\mathcal{F}_K$ (see Theorem 2), as an integral operator

$$\tau_y^K f(x) = \rho_{-\alpha,m}\int_{\mathbb{R}^m} \widetilde{K(\xi,x)}\,K(y,\xi)\,\mathcal{F}_K(f)(\xi)\,d\xi.$$

When the generalized translation operator acts on a radial function, the result can be computed explicitly. This was first done in the context of the Clifford–Fourier transform in [23] and subsequently generalized in [12] to:

Theorem 3. *Let $f \in \mathcal{S}(\mathbb{R}^m)$ be a real-valued radial function on $\mathbb{R}^m$, i.e. $f(x) = f_0(|x|)$ with $f_0 : \mathbb{R}_+ \longmapsto \mathbb{R}$, then*

$$\tau_y^K f(x) = \frac{2\alpha_0}{\Gamma(\lambda+1)}(1-e^{2i\alpha})^{-m/2}e^{-\frac{i}{2}(\cot\alpha)(|x|^2-|y|^2)}H_\lambda\left[\mathcal{F}_\alpha(f)\right]\left(\frac{|x-y|}{\sin\alpha}\right)$$

with $\lambda = (m-2)/2$, $\mathcal{F}_\alpha$ the fractional version of the classical Fourier transform given by the integral transform

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$$\mathcal{F}_\alpha(f)(y) = \rho_{\alpha,m} \int_{\mathbb{R}^m} e^{-\frac{i\langle x,y\rangle}{\sin\alpha}} e^{\frac{i}{2}\cot\alpha(|x|^2+|y|^2)} f(x)dx$$

and H_λ the Hankel transform defined by

$$H_\lambda f(s) := \int_0^\infty f(r) \frac{J_\lambda(rs)}{(rs)^\lambda} r^{2\lambda+1} dr.$$

In the special case when $\alpha = \pi/2$, it follows from the previous result that the generalized translation operator τ_y^K coincides, up to a constant, with geometric translation if f is a radial function:

$$\tau_y^K f(x) = \frac{2^{-\lambda}\alpha_0}{\Gamma(\lambda+1)} f(|x-y|). \tag{10}$$

Using this generalized translation, two types of convolution for functions with values in the Clifford algebra are obtained:

Definition 2. For $f, g \in \mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$, the generalized convolution $f *_L g$, resp. $f *_R g$, is defined for $x \in \mathbb{R}^m$ by

$$(f *_L g)(x) := \int_{\mathbb{R}^m} [\tau_y^K f(x)]\, g(y)\, dy$$

resp.

$$(f *_R g)(x) := \int_{\mathbb{R}^m} f(y)[\tau_y^K g(x)] dy,$$

with $\tau_y^K f$ and $\tau_y^K g$ as in Definition 1.

The importance of the generalized translation and the related convolution product lies in obtaining inversion theorems for hypercomplex FTs that go beyond the Schwartz space. For the Clifford–Fourier transform this was achieved in Theorem 8.4 in [23], by making use of formula (10) applied to a scaled gaussian.

Hypercomplex Fourier Transforms in Approach B

Definition, Eigenvalues and Eigenfunctions

The so-called geometric Fourier transform (GFT) can be defined in the following way.

Definition 3. Denote by $\mathcal{I}_m$ the set $\{i \in \mathcal{C}l_{0,m} | i^2 = -1\}$ of geometric square roots of minus one. Let $F_1 := \{i_1, \ldots, i_\mu\}$, $F_2 := \{i_{\mu+1}, \ldots, i_m\}$ be two ordered finite sets of such square roots, $i_k \in \mathcal{I}_m, \forall k = 1, \ldots, m$. The GFT $\mathcal{F}_{F_1,F_2}$ of a function $f : \mathbb{R}^m \to \mathcal{C}l_{0,m}$ takes the form:

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$$\mathcal{F}_{F_1,F_2}(f)(u) := (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} \left(\prod_{k=1}^{\mu} e^{-i_k x_k u_k}\right) f(x) \left(\prod_{k=\mu+1}^{m} e^{-i_k x_k u_k}\right) dx.$$

This definition yields a subset of the set of all general GFTs as introduced in [13], where also non-linear functions in the exponentials are allowed. The restriction to linear factors as in Definition 3 and following [12] allows for obtaining the eigenvalues and eigenfunctions of the GFT. They are given in the following theorem.

Theorem 4. *The basis* $\{\psi_{j_1,j_2,\ldots,j_m}\}$ *of* $\mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$ *diagonalizes the GFT. One has*

$$\mathcal{F}_{F_1,F_2}(\psi_{j_1,j_2,\ldots,j_m}) = \left(\prod_{k=1}^{\mu}(-i_k)^{j_k}\right) \psi_{j_1,j_2,\ldots,j_m} \left(\prod_{k=\mu+1}^{m}(-i_k)^{j_k}\right).$$

Proof. By direct computation it follows that

$$\begin{aligned}\mathcal{F}_{F_1,F_2}(\psi_{j_1,j_2,\ldots,j_m}) &= (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} \left(\prod_{k=1}^{\mu} e^{-i_k x_k u_k}\right) \psi_{j_1,j_2,\ldots,j_m}(x) \left(\prod_{k=\mu+1}^{m} e^{-i_k x_k u_k}\right) dx \\ &= \left(\prod_{k=1}^{\mu}(2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-i_k x_k u_k} \psi_{j_k}(x_k) dx_k\right) \left(\prod_{k=\mu+1}^{m}(2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \psi_{j_k}(x_k) e^{-i_k x_k u_k} dx_k\right) \\ &= \left(\prod_{k=1}^{\mu}(-i_k)^{j_k} \psi_{j_k}(u_k)\right) \left(\prod_{k=\mu+1}^{m} \psi_{j_k}(u_k)(-i_k)^{j_k}\right) \\ &= \left(\prod_{k=1}^{\mu}(-i_k)^{j_k}\right) \psi_{j_1,j_2,\ldots,j_m} \left(\prod_{k=\mu+1}^{m}(-i_k)^{j_k}\right).\end{aligned}$$

Here, use was made of

$$(2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \psi_{j_k}(x_k) e^{-i_k x_k u_k} dx_k = (-i_k)^{j_k} \psi_{j_k}(u_k),$$

which is a special case of formula (4). □

As a consequence of this theorem, the inverse transform of any GFT is a GFT itself, given by

$$\mathcal{F}_{F_1,F_2}^{-1} = \mathcal{F}_{\{-i_\mu,\ldots,-i_1\},\{-i_m,\ldots,-i_{\mu+1}\}}.$$

Remark 3. Note that all results in this section also carry over to the more general setup of functions $f : \mathbb{R}^m \to \mathcal{C}l_{p,q}$, compare [13]. However, this is not the case for the transforms of approach **A**, due to the fact that the Dirac operator is no longer elliptic in arbitrary signature.

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Important Special Cases

Several examples of GFTs have received considerable attention in the literature, before the general definition was stated in [13].

Historically, the first example of a GFT was given by Sommen in [49] and studied further by Bülow and Sommer in, e.g., [15]. In this case, one considers the Clifford algebra $\mathcal{C}l_{0,m}$. The transform takes the form

$$\mathcal{F}(f)(y) := \int_{\mathbb{R}^m} f(x)\, e^{e_1 x_1 y_1} \ldots e^{e_m x_m y_m} dx.$$

Derivation properties of this transform can, e.g., be found in [7].

Another example is the Clifford Fourier transform (written without hyphen) introduced by Ebling and Scheuermann in [28] and extended by Bahri and Hitzer [1,36]. This transform is defined for $\mathcal{C}l_{m,0}$ with $m = 2 \mod 4$ or $m = 3 \mod 4$ by

$$\mathcal{F}(f)(y) := \int_{\mathbb{R}^m} f(x)\, e^{e_{1\ldots m}\langle x,y\rangle} dx.$$

with $e_{1\ldots m} = e_1 e_2 \ldots e_m$ the pseudoscalar in $\mathcal{C}l_{m,0}$. Note the different signature of the Clifford algebra involved. The most important example is probably the quaternionic FT (qFT). First remark that the quaternion algebra $\mathbb{H}$ is isomorphic with the Clifford algebra $\mathcal{C}l_{0,2}$ under the identification $\mathbf{i} = e_1$, $\mathbf{j} = e_2$ and $\mathbf{k} = e_1 e_2$. Let $\mu, \nu \in \mathbb{H}$ be quaternions with $\mu^2 = \nu^2 = -1$. Then, following [38], the two-sided qFT is defined as

$$\mathcal{F}^{\mu,\nu}(f)(y_1, y_2) := (2\pi)^{-1} \int_{\mathbb{R}^2} e^{-\mu x_1 y_1} f(x_1, x_2) e^{-\nu x_2 y_2} dx_1 dx_2$$

for functions $f \in L_1(\mathbb{R}^2; \mathbb{H})$. The first definition of this two-sided transform, with $\mu = \mathbf{j}$ and $\nu = \mathbf{k}$, was introduced in the Ph.D. thesis [29]. In earlier work, devoted to nuclear magnetic resonance (NMR) imaging, a one sided version was given by Ernst et al. [32] and by Delsuc [26]. The applicability of the qFT to color image processing was first demonstrated in [47] using a discrete version. At that point, the switch was made to two general orthogonal axes μ and ν instead of $\mathbf{j}$ and $\mathbf{k}$. Indeed, for color image processing there is a preferred axis of the gray-line in color space, so the transform kernel axes are generally aligned to or perpendicular to this axis. At the same time [48], a change was again made to one-sided transforms, mostly driven by the complexity of the resulting operational formula when using the two-sided qFT definition. Finally, the orthogonality condition on μ and ν was relaxed in [38]. For a recent review on the use of the qFT in image processing, the reader may consult [31]. An overview of all variants of the qFT can be found in [30]. More information on quaternionic analysis from the mathematical point of view can be found in [34].

Results

Recently, there has been an increased interest in applying GFTs in various aspects of signal processing where higher dimensional or vector signals are used, such as color image processing [31], flow visualization [28], and even spoken word recognition [5]. The main idea behind these

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applications is the representation of a signal (say, a color image) as a pure quaternion or as an element of a suitable Clifford algebra. This representation is subsequently analysed using a GFT, which takes into account the multi-dimensional and multi-component nature of the signal under consideration. This stands in stark contrast to a component based classical analysis, sometimes also called marginal analysis. Successful further developments of the hypercomplex approach include the design of a color edge filter [46], construction of FFT methods to compute qFTs [44], etc.

The main issue that hinders further development of applications (and in particular of filter design not based on ad hoc assumptions or ideas) is the lack of a suitable convolution theorem. Indeed, in [14] it was shown that GFTs, as given in Definition 3, do not interact nicely with the classical convolution product, but instead lead to very complicated expressions in the Fourier domain. This means that, up to now, no filter design was possible in the Fourier domain, and hence that no fast implementations as multiplication operators have been obtained so far.

A first step towards solving this problem was undertaken in [12]. There, a new convolution product for GFTs was defined, based on the observation that in the classical case the following interaction between convolution and Fourier transform holds:

$$\mathcal{F}(f * g) = (2\pi)^{m/2} \mathcal{F}(f)\, \mathcal{F}(g)\,.$$

This was first exploited in a different context by Mustard [42] to define a new convolution product for the fractional Fourier transform. Based on his idea, a generalized convolution can then be defined for any GFT as follows:

Definition 4. For any GFT $\mathcal{F}_{F_1,F_2}$ the convolution $*_{F_1,F_2}$ is given by

$$(f *_{F_1,F_2} g)(x) := (2\pi)^{\frac{m}{2}} \mathcal{F}^{-1}_{F_1,F_2}(\mathcal{F}_{F_1,F_2}(f)\mathcal{F}_{F_1,F_2}(g))(x).$$

An important problem is to express the convolution $*_{F_1,F_2}$ by means of the standard convolution

$$(f * g)(x) = \int_{\mathbb{R}^m} f(y) g(x-y)\, dy.$$

Using the following notation

Definition 5. For functions $f, g : \mathbb{R}^m \to \mathcal{C}l_{0,m}$ and multi-indices $\vec{\phi}, \vec{\gamma} \in \{0,1\}^m$, put

$$f^{\vec{\phi}}(x) := f((-1)^{\phi_1} x_1, \ldots, (-1)^{\phi_m} x_m),$$
$$g^{\vec{\gamma}}(x) := g((-1)^{\gamma_1} x_1, \ldots, (-1)^{\gamma_m} x_m).$$

the result can be written as follows (see [12]):

Theorem 5. *Let* $J = \{0,1\}^{4\times m}$ *with* $j_{1,k} + j_{2,k} + j_{3,k} \in \{0,2\}$ *and* $j_{4,k} = 0$ *for all* $k = 1,\ldots,m$ *be a set of multi-indices. Any generalized convolution* $*_{F_1,F_2}$ *from Definition 4 can be expressed as a sum of classical convolutions using Definition 5 by*

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$$(f *_{F_1,F_2} g)(x) = \frac{1}{4^m} \sum_{\vec{j}\in J} \sum_{\vec{\phi},\vec{\gamma}\in\{0,1\}^m} c_{\vec{j},\vec{\phi},\vec{\gamma}} \left(\prod_{k=\mu}^{1} (i_k)^{j_{1,k}} \prod_{k=1}^{\mu} (-i_k)^{j_{2,k}} f^{\vec{\phi}} \prod_{k=\mu+1}^{m} (-i_k)^{j_{2,k}} \right)$$
$$* \left(\prod_{k=1}^{\mu} (-i_k)^{j_{3,k}} g^{\vec{\gamma}} \prod_{k=\mu+1}^{m} (-i_k)^{j_{3,k}} \prod_{k=m}^{\mu+1} (i_k)^{j_{1,k}} \right)(x)$$

with the sign $c_{\vec{j},\vec{\phi},\vec{\gamma}}$ *given by*

$$c_{\vec{j},\vec{\phi},\vec{\gamma}} = \prod_{k=1}^{m} (-1)^{(j_{(2\phi_k+\gamma_k+1,k)}+1)(\delta_{(j_{1,k}+j_{2,k}+j_{3,k})}-1)},$$

where

$$\delta_{(\ell)} := \begin{cases} 1, & \text{if } \ell = 0, \\ 0, & \text{if } \ell \neq 0. \end{cases}$$

Hypercomplex Fourier Transforms in Approach C

Definition

The general definition used in this approach was established in [2]. The transform acts on functions on $\mathbb{R}^2$, taking values in the vector part of $\mathcal{C}l_{n,0}$ when n is even, and in the vector part of $\mathcal{C}l_{n+1,0}$ when n is odd. Here, only the case n even will be treated. Note also the different signature of Clifford algebra compared to section "Hypercomplex Fourier Transforms in Approach A". Let ϕ be a group morphism (or spin character, in the terminology of [2]) from the abelian group $\mathbb{R}^2$ to $Spin(n)$. Then the transform is defined as

$$\begin{aligned} \hat{f}(\phi) &:= \int_{\mathbb{R}^2} f(x,y) \perp \phi(-x,-y)\,dxdy \\ &= \int_{\mathbb{R}^2} \phi(x,y) f(x,y) \phi(-x,-y)\,dxdy \end{aligned} \tag{11}$$

where $\perp$ denotes the natural action of $Spin(n)$ on n-dimensional vectors. The result of the transform is again a $\mathcal{C}l_{n,0}$ vector-valued function.

Important Special Cases

In [2], particular attention is paid to the case where the value space has dimension four. First one computes all group morphisms from $\mathbb{R}^2$ to $Spin(4)$ as follows. Define the set $\mathbb{S}^2_{3,0}$ of unit bivectors as the set of bivectors B in $\mathcal{C}l^2_{3,0}$ with $B^2 = -1$. Then the group morphisms from $\mathbb{R}^2$ to $Spin(4)$ are the morphisms $\tilde{\phi}_{u,v,B,w,z,C}$ that send (x, y) to

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$$e^{\frac{1}{4}[x(u+w)+y(v+z)][B+C+I(B-C)]}e^{\frac{1}{4}[x(u-w)+y(v-z)][B-C+I(B+C)]}$$

with u, v, w, z real, B, C two elements of $\mathbb{S}^2_{3,0}$ and $I = e_1e_2e_3e_4$ the pseudoscalar. Putting

$$D = \frac{1}{2}(B + C + I(B - C)),$$

this allows for simplifying the definition of the generalized Fourier transform (11) to

Definition 6. Let $f : \mathbb{R}^2 \to \mathcal{C}l^{(1)}_{4,0}$ be a vector-valued function. Its Clifford Fourier transform is then defined as follows:

$$\tilde{\mathcal{F}}(f)(u, v, w, z, D) = \int_{\mathbb{R}^2} e^{\frac{1}{2}[(x(u+w)+y(v+z))D]}e^{\frac{1}{2}[(x(u-w)+y(v-z))ID]}f(x, y)$$
$$e^{-\frac{1}{2}[(x(u+w)+y(v+z))D]}e^{-\frac{1}{2}[(x(u-w)+y(v-z))ID]}dxdy.$$

In the specific case of $w = z = 0$, one obtains

$$\mathcal{F}(f)(u, v, D) = \int_{\mathbb{R}^2} e^{\frac{1}{2}(xu+yv)D}e^{\frac{1}{2}(xu+yv)ID}f(x, y)e^{-\frac{1}{2}(xu+yv)D}e^{-\frac{1}{2}(xu+yv)ID}dxdy.$$

Note that D is a bivector in $\mathcal{C}l_{4,0}$ with $D^2 = -1$. The resulting transforms are invertible, and simple formulas can be found in [2].

Results

In [2], the generalized FT from Definition 6 is applied to color images, allowing to perform low pass, high pass, and directional filtering in the Fourier domain. For these applications, it is important to make suitable choices for the bivector D in the definition of the transform.

Discrete counterparts of the continuous transforms discussed in this section were obtained in [3].

In order to take the geometry of the color image to be analyzed even better into account, a generalization of the transform in Definition 6 was proposed in [4]. The essential change is to make the bivector D depend locally on the geometry of the image (interpreted as a Riemann surface), thus better incorporating all of its features. The resulting transform is of course more complicated to handle, but nevertheless remains invertible and shows nice properties.

Conclusion and Future Directions

In the present chapter, an overview has been given of the three main approaches to the study of hypercomplex Fourier transforms. In this final section, some important open problems and directions for future research are indicated.

The most important problem in approach **A** is to find explicit closed formulas for the integral kernels that are only available as a series expansion. For the Clifford–Fourier transform, the kernel

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is, e.g., not known explicitly in the case of odd dimension (see [23]). The kernel of the radially deformed hypercomplex FT (see [22]) depends on a deformation parameter $c > -1$. It is only computed explicitly for $c = 0$ (which corresponds with the ordinary FT). Especially in order to obtain sharp bounds on the kernel functions, it is crucial to have such explicit formulas available.

However, it seems that finding these closed formulas is a hard problem, as there is no transparent method available for that purpose. Rather, for each kernel a combination of ad hoc methods is necessary. One possibility would be to try and find a method that allows to compute bounds on kernel functions without needing the closed formula (in analogy with the work done for the Dunkl transform in [24]). An additional difficulty is that there seems to be a huge difference between the even and odd dimensional cases. This is due to the fact that one often exploits a recursion of size 2 on the dimensions, combined with an explicit, manageable computation in dimension 2. The odd dimensional case turns out to be much harder, because one cannot go from dimension $m = 1$ (where the kernel is usually explicitly known) to dimension $m = 3$.

An important issue regarding approach **B** is the so-called implementation problem. As explained in section "Results", the transforms in this approach are used, e.g., in color image processing and the main issue that hinders further development of applications, and in particular of filter design not based on ad hoc assumptions or ideas, is the lack of a suitable convolution theorem. A first step towards solving this problem was undertaken in [12], by designing a new convolution product, which can still be expressed in terms of classical convolution (see Theorem 5). The main challenge now is twofold. First, as guiding example one should consider the quaternionic color edge filter [46], which generalizes the classical Prewitt edge detector and detects sharp changes of color in images. Its definition is based on a representation of the color RGB space in terms of quaternions. It is still implemented using the classical convolution product, thus leading to a slow algorithm as the qFT cannot be applied in a meaningful way. It is expected that application of Theorem 5 will lead to a faster and much more elegant implementation. Second, one should use the result to construct a general theoretical framework for the filtering of color images, based on the qFT and the related convolution product.

Concerning approach **C**, a deeper mathematical study of the transforms it contains is required. A first step would be the computation of eigenfunctions and eigenvalues, as well as the interaction with Dirac type differential operators. Subsequently, also here one could investigate the possibility of defining suitable convolution products.

Finally, not all hypercomplex transforms studied so far are covered by the three approaches mentioned in this chapter. Most notably, the transform obtained by considering the monogenic extension of the usual exponential kernel is not included. For a detailed study, we refer the reader to the papers [39–41]. Also the so-called cylindrical Fourier transform (see [10]), defined by

$$\mathcal{F}(f)(y) := \int_{\mathbb{R}^m} e^{\underline{x} \wedge \underline{y}} f(x) dx,$$

is not included in the restricted version of approach **B** presented here. It does however belong to the entire class as described in [13]. It remains a difficult open problem to obtain its eigenvalues and eigenfunctions in closed form.

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