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Clifford Analysis: History and Perspective

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Abstract. An introduction is given to basic parts of Clifford analysis including some historical remarks.

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Introduction

W. K. Clifford introduced the algebras named after him in 1878 (see [10]). The importance of these algebras — which he called "geometric algebras" — essentially lies in the fact that they incorporate inside one single structure as well the inner product as the wedge product of vectors. Indeed, by defining a new multiplication rule for vectors in $\mathbb{R}^n$, he obtained — in the Euclidean setting — that for any two vectors x and y in $\mathbb{R}^n$,

$$x \bullet y = \frac{1}{2}(xy + yx),$$

$$x \wedge y = \frac{1}{2}(xy - yx).$$

Clifford algebras were rediscovered at several occasions, in particular by physicists. Let us for instance point out that the γ-matrices introduced by P. Dirac in 1928 (see [15]) in order to linearize the Klein-Gordon equation, are in fact generators for the Clifford algebra $\mathbb{R}_{1,3}$.

Similar observations concerning the linearization of the Laplacian in n-dimensional Euclidean space led — independently — R. Fueter and Gr. Moisil and N. Théodoresco, in the 1930's to their first results in what is nowadays called Clifford analysis.

We have pleasure in thanking the organizers of the fourth CMFT-Conference held at the University of Aveiro for their kind invitation to present some basic results in Clifford analysis to an audience which consisted for the most part of people dealing with classical complex analysis. We do hope that the underlying paper will encourage them to have a closer look at this function theory for the

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Dirac operator in Euclidean space and at its applicability to higher dimensional problems.

The title of our paper may be somewhat misleading: not that much history nor perspective are developed. Instead we have chosen for presenting those items from Clifford analysis we think should be part of any introductory course to the subject. For a state-of-the-art view of the actual research in Clifford analysis and its applications we refer to [5].

Contents

Part I. Clifford algebras in geometry: geometric algebra

The aim of this part is to give some preliminary definitions and properties of real and complex Clifford algebras. For real Clifford algebras, we will restrict ourselves to the Clifford algebra $\mathbb{R}_{0,m+1}$ constructed over the quadratic space $\mathbb{R}^{0,m+1}$ and this since it is essentially this algebra we use further on in our presentation of some basic concepts in Clifford analysis.

1. Real Clifford algebras

1.1. Definitions. Let $m \in \mathbb{N}$ and let $\mathbb{R}^{0,m+1}$ stand for the real vector space $\mathbb{R}^{m+1}$ provided with a non-degenerate symmetric bilinear form $\mathcal{B}$ of signature $(0, m+1)$, i.e. for an appropriate orthonormal basis $e = (e_0, e_1, \ldots, e_m)$ of $\mathbb{R}^{0,m+1}$ w.r.t. $\mathcal{B}$ and any two vectors $\nu, w \in \mathbb{R}^{m+1}$ with $[\nu]_e = x = (x_0, x_1, \ldots, x_m)$ and $[w]_e = y = (y_0, y_1, \ldots, y_m)$, we have that

$$\mathcal{B}(\nu, w) = -\sum_{j=0}^{m} x_j\, y_j.$$

In particular,

$$\mathcal{B}(e_i, e_j) = -\delta_{ij}, \qquad i, j = 0, 1, \ldots, m.$$

For the associated quadratic form $\mathcal{Q}$ we thus have that

$$\mathcal{Q}(\nu) = \mathcal{B}(\nu, \nu) = -\sum_{j=0}^{m} {x_j}^2.$$

In particular

$$\mathcal{Q}(e_j) = -1, \qquad j = 0, 1, \ldots, m.$$

Henceforth we assume the orthonormal basis e for granted and denote an arbitrary element $x \in \mathbb{R}^{0,m+1}$ by $x = \sum_{j=0}^{m} x_j e_j$.

The real Clifford algebra $\mathbb{R}_{0,m+1}$ constructed over $\mathbb{R}^{0,m+1}$ is a real linear associative algebra with identity 1, of dimension 2^{m+1}, containing $\mathbb{R}$ and $\mathbb{R}^{0,m+1}$ as subspaces and in which for each vector $x \in \mathbb{R}^{0,m+1}$,

$$x^2 = \mathcal{Q}(x) = -\sum_{j=0}^{m} {x_j}^2 = -|x|^2$$

where $|x|$ stands for the Euclidean norm of x.

It thus follows that for the elements of the basis e

$$\begin{aligned} {e_i}^2 &= -1, \qquad i = 0, 1, \ldots, m, \\ e_i e_j + e_j e_i &= 0, \qquad i \neq j. \end{aligned}$$

A basis for $\mathbb{R}_{0,m+1}$ then consists of the elements $e_A = e_{i_1} e_{i_2} \cdots e_{i_h}$ where $A = (i_1, i_2, \ldots, i_h) \subset \{0, 1, \ldots, m\}$ is such that $0 \leq i_1 < i_2 < \cdots < i_h \leq m$. For $A = \phi$, $e_\phi = 1$, the identity element of $\mathbb{R}_{0,m+1}$.

In such way, any element $a \in \mathbb{R}_{0,m+1}$ may be written as

$$a = \sum_A a_A e_A, \qquad a_A \in \mathbb{R}.$$

Notice that for any two vectors $x, y \in \mathbb{R}^{0,m+1}$,

$$xy = -x \bullet y + x \wedge y$$

where

$$x \bullet y = \sum_{j=0}^{m} x_j y_j$$

is the standard (Euclidean) inner product and

$$x \wedge y = \sum_{i<j} e_i e_j (x_i y_i - x_j y_j)$$

is the standard outer product in $\mathbb{R}^{m+1}$.

We may also write

$$\begin{aligned} x \bullet y &= -\frac{1}{2}(xy + yx), \\ x \wedge y &= \frac{1}{2}(xy - yx). \end{aligned}$$

Putting for each $k \in \{0, 1, \ldots, m+1\}$,

$$\mathbb{R}_{0,m+1}^{(k)} = \left\{ a \in \mathbb{R}_{0,m+1} : a = \sum_{|A|=k} a_A e_A \right\},$$

we thus have that

$$\mathbb{R}_{0,m+1} = \sum_{k=0}^{m+1} \oplus \mathbb{R}_{0,m+1}^{(k)} = \sum_{k \text{ even}} \oplus \mathbb{R}_{0,m+1}^{(k)} \oplus \sum_{k \text{ odd}} \oplus \mathbb{R}_{0,m+1}^{(k)}.$$

The subspace $\sum_{k \text{ even}} \oplus \mathbb{R}_{0,m+1}^{(k)}$ is denoted by $\mathbb{R}_{0,m+1}^{+}$. It is a subalgebra of $\mathbb{R}_{0,m+1}$, called *even subalgebra*, and is isomorphic to $\mathbb{R}_{0,m}$. Indeed, it is generated by the elements $\varepsilon_j = -e_0 e_j$, $j = 1, \ldots, m$, which satisfy ${\varepsilon_j}^2 = -1$, $j = 1, \ldots, m$, and $\varepsilon_i \varepsilon_j + \varepsilon_j \varepsilon_i = 0$, $i \neq j$, whence the set $\varepsilon = (\varepsilon_j : j = 1, \ldots, m)$ may be regarded as an orthonormal basis for $\mathbb{R}^{0,m}$. For $a \in \mathbb{R}_{0,m+1}$, we may thus write

$$a = \sum_{k=0}^{m+1} [a]_k$$

where $[a]_k$ is the projection of a on $\mathbb{R}_{0,m+1}^{(k)}$.

An arbitrary element of $\mathbb{R}_{0,m+1}^{(k)}$ is called a *k-vector*. In such way, 0-vectors are scalars, 1-vectors are elements of $\mathbb{R}^{0,m+1}$, 2-vectors are elements of the form $\sum_{i<j} a_{ij}\, e_i e_j$, and so on. An $(m+1)$-vector has the form $a_{01\ldots m}\, e_0 e_1 \cdots e_m$.

As we have seen, the product of two vectors $x, y \in \mathbb{R}^{0,m+1}$ splits into a scalar part $-x \bullet y$ and a bivector part $x \wedge y$.

Example 1.1 ($\mathbb{R}_{0,1} \cong \mathbb{C}$)**.** Indeed, $e = \{e_0\}$ with ${e_0}^2 = -1$. Moreover $\dim \mathbb{R}_{0,1} = 2$ where $(1, e_0)$ is a basis for $\mathbb{R}_{0,1}$.

Example 1.2 ($\mathbb{R}_{0,2} \cong \mathbb{H}$)**.** Indeed, $e = (e_0, e_1)$ with ${e_0}^2 = {e_1}^2 = -1$. Moreover $\dim \mathbb{R}_{0,2} = 4$ where $(1, e_0, e_1, e_0 e_1)$ is a basis for $\mathbb{R}_{0,2}$ with $(e_0 e_1)^2 = -1$.

Furthermore $\mathbb{R}_{0,2}^{+} \cong \mathbb{C}$ since $\mathbb{R}_{0,2}^{+}$ is generated by $\varepsilon_1 = -e_0 e_1$.

Example 1.3 ($\mathbb{R}_{0,3}$)**.** An orthonormal basis for $\mathbb{R}^{0,3}$ is given by $e = (e_0, e_1, e_2)$. As we have seen, the (Euclidean) inner product of any two vectors $x, y \in \mathbb{R}^3$ is given by $x \bullet y = -\frac{1}{2}(xy + yx)$. Furthermore, the cross product $x \times y$ may be expressed as

$$x \times y = -x \wedge y\ e_{012}.$$

This implies that the classical vector algebra operations in $\mathbb{R}^3$ may be performed inside $\mathbb{R}_{0,3}$.

We have $\mathbb{R}_{0,3}^{+} \cong \mathbb{R}_{0,2} \cong \mathbb{H}$, its generators $\varepsilon_1, \varepsilon_2$ being given by $\varepsilon_j = -e_0 e_j$, $j = 1, 2$.

1.2. Involutions and norm. Three (anti)-involutions are defined on $\mathbb{R}_{0,m+1}$. They are introduced first on the basic elements e_A and are then extended by linearity to $\mathbb{R}_{0,m+1}$.

(i) The main involution $a \to \hat{a}$

$$\begin{aligned} \hat{e}_A &= (-1)^k e_A \qquad \text{if } |A| = k, \\ \widehat{ab} &= \hat{a}\hat{b}. \end{aligned}$$

In particular $\hat{a} = a$, for $a \in \mathbb{R}^{+}_{0,m+1}$.

(ii) The reversion $a \to a^*$

$$\begin{aligned} e_A^* &= e_{i_h} e_{i_{h-1}} \cdots e_{i_1} \qquad \text{if } e_A = e_{i_1} e_{i_2} \cdots e_{i_h}, \\ (ab)^* &= b^* a^*. \end{aligned}$$

(iii) The conjugation $a \to \alpha(a)$

$$\begin{aligned} \alpha(e_A) &= (\hat{e}_A)^* = (e_A^*)^\wedge, \\ \alpha(ab) &= \alpha(b)\alpha(a). \end{aligned}$$

By means of the anti-involution α, an algebra norm $|a|$ may be defined for each $a \in \mathbb{R}_{0,m+1}$ by putting

$$|a|^2 = 2^{m+1}[a\alpha(a)]_0 = 2^{m+1} \sum_A |a_A|^2.$$

2. Subgroups of $\mathbb{R}_{0,m+1}$

One of the important features of Clifford algebras is their application to groups of quadratic automorphisms.

2.1. The Clifford group $\Gamma(m+1)$. First notice that for each $x \in \mathbb{R}^{0,m+1}$,

$$\begin{aligned} x^2 &= \mathcal{B}(x,x) \\ &= -x\,\hat{x} \\ &= -\hat{x}\,x. \end{aligned}$$

Now let $s \in \mathbb{R}_{0,m+1}$ be such that it is invertible and that for all $x \in \mathbb{R}^{0,m+1}$,

$$s x \hat{s}^{-1} \in \mathbb{R}^{0,m+1}.$$

Defining the associated linear transformation $\chi(s) \colon \mathbb{R}^{0,m+1} \to \mathbb{R}^{0,m+1}$ by

$$\chi(s)(x) = s x \hat{s}^{-1},$$

we have that $(\chi(s)(x))^2 = x^2$ whence $\chi(s) \in O(m+1)$.

We obtain that

$$\Gamma(m+1) = \{ s \in \mathbb{R}_{0,m+1} : \; s x \hat{s}^{-1} \in \mathbb{R}^{0,m+1} \text{ for all } x \in \mathbb{R}^{0,m+1} \}$$

is a group under multiplication in $\mathbb{R}_{0,m+1}$. It is called the *Clifford group* of $\mathbb{R}_{0,m+1}$.

2.2. The groups $\mathrm{Pin}(m+1)$ **and** $\mathrm{Spin}(m+1)$**.** Take $s \in \mathbb{R}^{0,m+1}$ with $s^2 = -1$, i.e. $s \in S^m$, the unit sphere in $\mathbb{R}^{m+1}$. Then $\chi(s)$ is nothing else but the orthogonal reflection w.r.t. the hyperplane $H_s = s^\perp$. Indeed, if we write $x = \lambda s + t$ with $\lambda \in \mathbb{R}$ and $t \perp s$, then a straightforward calculation yields that

$$\chi(s)(x) = -\lambda s + t.$$

Hence

$$\mathrm{Pin}(m+1) = \left\{ \prod_{j=1}^{k} s_j \,:\, k \in \mathbb{N},\ s_j \in S^m \right\}$$

describes $\mathrm{O}(m+1)$, while

$$\mathrm{Spin}(m+1) = \left\{ \prod_{j=1}^{2k} s_j \,:\, k \in \mathbb{N},\ s_j \in S^m \right\}$$

describes $\mathrm{SO}(m+1)$.

$\mathrm{Pin}(m+1)$ and $\mathrm{Spin}(m+1)$ are respectively called the Pin and Spin group for $\mathbb{R}^{0,m+1}$.

Clearly $\mathrm{Pin}(m+1)$ is generated by the set $(e_0, e_1, \ldots, e_m)$ while $\mathrm{Spin}(m+1)$ is generated by the set $(\varepsilon_j : j = 1, \ldots, m)$. Of course

$$\mathrm{Spin}(m+1) \subset \mathrm{Pin}(m+1) \subset \Gamma(m+1).$$

3. Möbius transformations in $\mathbb{R}^{m+1}$

The class of conformal mappings in $\mathbb{R}^2$ is very rich. In fact, one may say that essentially it coincides with the set of holomorphic functions. An important subclass of it consists of the so-called Möbius transformations in $\mathbb{R}^2$, i.e. transformations of the form

$$f(z) = \frac{az+b}{cz+d}, \qquad z \in \mathbb{C},$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$.

They are obtained by composing three types of them, namely translations, dilatations and the inversion. They send circles into circles.

It was first shown by Liouville in 1850 that in $\mathbb{R}^3$ the only conformal mappings are Möbius transformations, i.e. compositions of translations, dilatations and the inversion $x \to x/|x|^2$, or equivalently, reflections in affine hyperplanes and inversions in spheres. The same holds in $\mathbb{R}^{m+1}$ $(m \geq 2)$.

As has been proved at several occasions, Möbius transformations in $\mathbb{R}^{m+1}$ may be fully described by using Clifford algebras. It is essentially through the papers of V. Ahlfors (see e.g. [1]) that this technique has got the interest it deserves.

The Möbius transformations in $\mathbb{R}^{m+1}$ form a group (for composition), denoted by $\mathrm{Mob}(m+1)$. The orientation preserving elements from $\mathrm{Mob}(m+1)$ form a subgroup denoted by $\mathrm{Mob}^+(m+1)$.

We have

Theorem 3.1. *Let* $g \in \mathrm{Mob}(m+1)$. *Then there exists a matrix* $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ *satisfying*

(i) $a, b, c, d, \in \Gamma(m+1) \cup \{0\}$,
(ii) $ab^*, cd^*, c^*a, d^*b \in \mathbb{R}^{0,m+1}$,
(iii) $ad^* - bc^* \in \mathbb{R} \setminus \{0\}$,

and such that for $x \in \mathbb{R}^{m+1}$,

$$g(x) = \frac{ax+b}{cx+d} = (ax+b)(cx+d)^{-1}.$$

Conversely, each such A *determines a* $g \in \mathrm{Mob}(m+1)$.

As is readily seen, the matrix A is defined up to a non-zero factor $\rho \in \mathbb{R}$.

$\Delta(g) = ad^* - bc^*$ is called the *pseudo-determinant* of g.

After normalization we may thus suppose that g corresponds to a matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with $\Delta(g) = \pm 1$. Furthermore we have that $g \in \mathrm{Mob}^+(m+1)$ if $\Delta(g) = +1$. If with $g \in \mathrm{Mob}(m+1)$, the matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is associated, then g^{-1} corresponds to $A' = \left[\begin{smallmatrix} d^* & -b^* \\ -c^* & a^* \end{smallmatrix}\right]$.

Example 3.2 (Möbius transformations)**.** The basic Möbius transformations are represented as follows:

$$\begin{array}{lll} \text{Rotation} & x \to sx\hat{s}^{-1} & \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}, \quad s \in \mathrm{Spin}(m+1) \\[2ex] \text{Translation} & x \to x+b & \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \quad b \in \mathbb{R}^{0,m+1} \\[2ex] \text{Dilation} & x \to \lambda x & \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}, \quad \lambda \in \mathbb{R}^+ \\[2ex] \text{Inversion} & x \to \dfrac{x}{|x|^2} & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \end{array}$$

Example 3.3 (The Cayley transform)**.** Call $\mathbb{R}^{m+1}_+ = \{x = (x_0, x_1, \ldots, x_m) : x_m > 0\}$ and let $\overset{\circ}{B}(1)$ be the open unit ball in $\mathbb{R}^{m+1}$.

The mapping $C\colon \mathbb{R}^{m+1}_+ \to \overset{\circ}{B}(1)$ such that

$$C(x) = \frac{x - e_m}{-e_m x + 1}$$

is called the *Cayley transform.*

Its inverse C^{-1} is given by

$$C^{-1}(y) = \frac{y + e_m}{e_m y + 1}.$$

Example 3.4 (Mob(S^m))**.** By Mob(S^m) we denote the set of those Möbius transformations in $\mathbb{R}^{m+1}$ leaving the unit sphere S^m invariant.

We have

Theorem 3.5.

(i) *Let* $g \in \mathrm{Mob}(m+1)$ *with associated matrix* $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. *Then* g *leaves* S^m *invariant if and only if*

(i.1) $a, b \in \Gamma(m+1) \cup \{0\}$ *with* $|a| \neq |b|$,

(i.2) $c = \hat{b}$ *and* $d = \hat{a}$ *or* $c = -\hat{b}$ *and* $d = -\hat{a}$.

Moreover g *leaves* $\overset{\circ}{B}(1)$ *invariant if* $|b| < |a|$.

(ii) Mob(S^m) *is generated by orthogonal transformations and Möbius transformations of the type*

$$g_\alpha(x) = \frac{x - \alpha}{\alpha x + 1}$$

where $\alpha \in \mathbb{R}^{0,m+1} \cup \{\infty\}$ *with* $|\alpha| \neq 1$.
For $\alpha = \infty$,

$$g_\infty(x) = \frac{x}{|x|^2},$$

i.e. g_∞ *is the inversion.*
Moreover g_α *leaves* $\overset{\circ}{B}(1)$ *invariant if* $|\alpha| < 1$.

4. Complex Clifford algebras

One way of defining the complex Clifford algebra $\mathbb{C}_{m+1}$ is by means of

$$\mathbb{C}_{m+1} = \mathbb{R}_{0,m+1} \otimes_{\mathbb{R}} \mathbb{C}.$$

It follows that $\mathbb{C}_{m+1}$ is a linear associative algebra over $\mathbb{C}$, having dimension 2^{m+1} and basis $(e_A \otimes 1 : \ A \subset \{0, 1, \ldots, m\})$.

We still write $e_A = e_A \otimes 1$.

An element $a \in \mathbb{C}_{m+1}$ may thus be represented as

$$a = \sum_A a_A e_A, \quad a_A \in \mathbb{C}.$$

The (anti)-involutions introduced on $\mathbb{R}_{0,m+1}$ may be extended to $\mathbb{C}_{m+1}$. They are denoted the same way. The anti-involution on $\mathbb{C}_{m+1}$ which is obtained by taking the tensor product of the conjugation α on $\mathbb{R}_{0,m+1}$ and the classical complex conjugation on $\mathbb{C}$ is called the *bar-map* and denoted by $a \to \bar{a}$. We so have

$$a = \sum_A a_A e_A \to \bar{a} = \sum_A \bar{a}_A \alpha(e_A).$$

It follows that for $a \in \mathbb{C}_{m+1}$, its norm $|a|$ may be defined by

$$|a|^2 = 2^{m+1}[a\bar{a}]_0 = 2^{m+1} \sum_A |a_A|^2.$$

5. Remarks

Let $p, q \in \mathbb{N}$ with $p + q = n$ and let $\mathbb{R}^{p,q}$ stand for the real vector space $\mathbb{R}^n$ provided with a non-degenerate symmetric bilinear form $\mathcal{B}$ of signature (p, q). Then $\mathbb{R}_{p,q}$ denotes the universal real Clifford algebra constructed over $\mathbb{R}^{p,q}$. The case $(p, q) = (1, 3)$ — i.e. Minkowski space — is of course of particular importance in physics.

For the structure of real Clifford algebras, their classification and their subgroups $\Gamma(p, q), \mathrm{Pin}(p, q)$ and $\mathrm{Spin}(p, q)$, we refer e.g. to [34].

A complete description of the Möbius transformations on $\mathbb{R}^{p,q}$ and their relation to $\mathrm{Pin}(p + 1, q + 1)$, together with a characterization of matrices in the $\mathrm{Pin}(p+1, q+1)$ group was given by J. Fillmore and A. Springer in [16]. A more straightforward approach was obtained by J. Cnops in [11].

For an excellent introductory monograph on real Clifford algebras and their applicability in geometry and physics, we refer to [28].

For the classification of complex Clifford algebras, in particular the realization of $\mathbb{C}_{2k}$, $k \in \mathbb{N}$, as the full matrix algebra $\mathbb{C}(2^k)$ through the so-called basic spinor representation γ, we refer to [14].

Part II. Clifford algebras in analysis: Clifford analysis

1. Dirac and Weyl operators

The aim of this section is to introduce first order differential operators in $\mathbb{R}^{m+1}$ which linearize Δ_x, the Laplacian in $\mathbb{R}^{m+1}$.

We shall identify $\mathbb{R}^{m+1}$ in two ways with a subspace of $\mathbb{R}_{0,m+1}$.

Let $x = (x_0, \underline{x}) = (x_0, x_1, ..., x_m) \in \mathbb{R}^{m+1}$. Then

$$x \to x = \sum_{j=0}^{m} x_j e_j \in \mathbb{R}^{0,m+1},$$

i.e. x is a 1-vector in $\mathbb{R}_{0,m+1}$.

$$x \to x = x_0 + \sum_{j=1}^{m} x_j \varepsilon_j = x_0 + \underline{x}$$

where $\varepsilon_j = \bar{e}_0 e_j$, $j = 1, ..., m$, i.e. x is a paravector in $\mathbb{R}_{0,m} \cong \mathbb{R}^{+}_{0,m+1}$. We thus have in $\mathbb{R}_{0,m+1}$ that

$$x = x_0 + \sum_{j=1}^{m} \varepsilon_j x_j = \bar{e}_0 (\sum_{j=0}^{m} x_j e_j).$$

As we already pointed out, the classical example where the second identification is made is of course when $m = 1$, i.e. when $x \in \mathbb{R} \bigoplus \mathbb{R}^{0,1}$ is written as $x = x_0 + \varepsilon_1 x_1 \in \mathbb{R}^{+}_{0,2}$ or, putting $\varepsilon_1 = i$, as $x = x_0 + i x_1 \in \mathbb{C}$.

Let us also recall that if

$$x = \sum_{j=0}^{m} x_j e_j, \qquad \text{then} \qquad x^2 = -|x|^2$$

and

$$x = x_0 + \sum_{j=1}^{m} \varepsilon_j x_j, \qquad \text{then} \qquad x\bar{x} = \bar{x}x = |x|^2.$$

In a formal way we now introduce the following first order linear differential operators ∂_x and D_x called, respectively, the Dirac and Weyl (or Cauchy-Riemann) operators in $\mathbb{R}^{m+1}$.

$$\text{The Dirac operator} \quad \partial_x = \sum_{j=0}^{m} e_j \partial_{xj} = e_0 \partial_{x_0} + \partial_{\underline{x}}$$

$$\text{where } \partial_{\underline{x}} = \sum_{i=1}^{m} e_i \partial_{xi}.$$

$$\text{The Cauchy-Riemann operator} \quad D_x = \partial_{x_0} + \sum_{i=1}^{m} \varepsilon_i \partial_{xi} = \overline{e}_0 \partial_x = \partial_{x_0} + \partial_{\underline{x}}$$

$$\text{where } \partial_{\underline{x}} = \sum_{i=1}^{m} \varepsilon_i \partial_{xi}$$

In both cases $\partial_{\underline{x}}$ may thus be considered as the Dirac operator in $\mathbb{R}^m$.

They act on $\mathcal{C}_1$-functions defined in an open subset $\Omega \subset \mathbb{R}^{m+1}$ and having values in $\mathbb{R}_{0,m+1}$ or $\mathbb{C}_{m+1}$.

If $f(x) = \sum_A f_A(x) e_A$, where the f_A's are $\mathbb{R}$-or $\mathbb{C}$-valued, then the action may be from the left, i.e. for instance in the case of ∂_x:

$$\partial_x f = \sum_{j,A} e_j e_A \frac{\partial f_A}{\partial x_j},$$

or from the right, i.e.

$$f\partial_x = \sum_{A,j} e_A e_j \frac{\partial f_A}{\partial x_j}.$$

Formally we have

$$\begin{aligned} {\partial_x}^2 &= -\Delta_x, \\ D_x \bar{D}_x &= \bar{D}_x D_x = \Delta_x, \end{aligned}$$

Δ_x being the Laplacian in $\mathbb{R}^{m+1}$. Here

$$\bar{D}_x = \partial_{x_0} - \sum_{j=1}^{m} \varepsilon_j \partial_{xj} = \partial_x \overline{e}_0.$$

In the case $m = 1$, the paravector formalism thus leads to the classical Cauchy-Riemann operator when putting $\varepsilon_1 = i$:

$$D_x = \partial_{x_0} + i\partial_{x_1},$$

and its conjugate

$$\overline{D}_x = \partial_{x_0} - i\partial_{x_1}.$$

Remark 1.1. From a purely algebraic point of view, identifying $x = (x_0, \underline{x}) \in \mathbb{R}^{m+1}$ with either $x = \sum_{j=0}^{m} x_j e_j \in \mathbb{R}^{0,m+1}$ or $x = x_0 + \sum_{j=1}^{m} \varepsilon_j x_j \in \mathbb{R}^{+}_{0,m+1}$, of course makes a lot of difference. Indeed, as we saw in Part I, Section 2.2 the set $(e_j : j = 0, ..., m)$ generates the Pin-group $\mathrm{Pin}(m+1)$ of $\mathbb{R}_{0,m+1}$, while the set $(\varepsilon_j : j = l, ..., m)$ generates its Spin-group $\mathrm{Spin}(m+1)$.

From the geometrical point of view, we could say that when using the paravector formalism, we have chosen a real axis — namely the x_0-axis — by multiplying the vector $x \in \mathbb{R}^{m+1}$ on the left by $\overline{e}_0$, the square of which is -1. Of course, we could as well have chosen e.g. as real axis the x_m-axis and this by multiplying the vector x (on the left) by $\overline{e}_m$. We then would have that in $\mathbb{R}_{0,m+1}$.

$$x = (x_0, x_1, ..., x_m) \to x = \sum_{j=0}^{m-1} \eta_j x_j + x_m$$

where $\eta_j = \overline{e}_m e_j$, $j = 0, ..., m-1$.

Remark 1.2. A general form of the Dirac and Weyl operators in $\mathbb{R}^{m+1}$ is given by

$$\begin{aligned} \partial_x &= \sum_{j=0}^{m} \rho(e_j) \partial_{xj}, \\ D_x &= \partial_{x_0} + \sum_{j=1}^{m} \rho(\varepsilon_j) \partial_{xj}, \end{aligned}$$

where (ρ, V) is a basic representation space of the group $\mathrm{Pin}(m+1)$, respectively, $\mathrm{Spin}(m+1)$ or a direct sum of such representations. The functions f on which these operators are acting are then belonging to $\mathcal{C}_1(\Omega; V)$. For a detailed description of this more general context we refer the reader to [14].

2. Monogenic functions: definitions, operator equalities and basic integral formulae

2.1. Definitions. Let $\Omega \subset \mathbb{R}^{m+1}$ be open and let f be a $\mathcal{C}_1$-function in Ω which is $\mathbb{C}_{m+1}$-valued. Then we say that f is *left monogenic* in Ω if in Ω

$$\partial_x f = 0 \qquad \text{or} \qquad D_x f = 0.$$

Analogously, f is called *right monogenic* in Ω if in Ω

$$f\partial_x = 0 \qquad \text{or} \qquad f D_x = 0.$$

In what follows, "f monogenic in Ω" will stand for "f is left monogenic in Ω". The set of monogenic functions in Ω is denoted by $M(\Omega)$. It is a right $\mathbb{C}_{m+1}$-module, i.e. if $f, g \in M(\Omega)$ and $a, b \in \mathbb{C}_{m+1}$, than $fa + gb \in M(\Omega)$.

Notice that at some occasions, we will restrict ourselves to $\mathbb{R}_{0,m+1}$ valued functions. In that case the elements a en b should be taken in $\mathbb{R}_{0,m+1}$. Saying that f is monogenic in Ω thus means that its components $(f_A)_{A\subset\{0,\ldots,m\}}$ satisfy 2^{m+1} homogeneous first order linear partial differential equations. It may be proved that the first order system thus obstained is strongly elliptic, whence $M(\Omega) \subset \mathcal{A}(\Omega)$, the right $\mathbb{C}_{m+1}$-module of real-analytic $\mathbb{C}_{m+1}$-valued functions in Ω. In particular, as

$$\Delta_x = -\partial_x^2 = \overline{D}_x D_x,$$

$f \in M(\Omega)$ implies that $f \in \mathrm{Harm}(\Omega)$, the set of harmonic $\mathbb{C}_{m+1}$-valued functions in Ω.

Example 2.1. Let f be $\mathbb{R}^{0,m+1}$-valued in Ω, i.e.

$$f(x) = \sum_{j=0}^{m} e_j f_j(x).$$

Then claiming that $\partial_x f = 0$ in Ω (or f is monogenic w.r.t. the Dirac operator ∂_x) is equivalent to saying that its components $f_j, j = 0, ..., m$, satisfy the system

$$\text{(2.1)} \qquad \begin{aligned} \sum_{j=0}^{m} \frac{\partial f_j}{\partial x_J} &= 0 \\ \frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} &= 0, \qquad i < j. \end{aligned}$$

The system (2.1) is nothing else but the classical Riesz system. As is well known, in the case Ω simply connected, it thus means that there ought to exist a function u, $\mathbb{R}$-valued and harmonic in Ω such that

$$f_j = \frac{\partial u}{\partial_{x_j}}, \qquad j = 0, \dots, m.$$

The set $(f_0, f_1, \dots, f_m)$ is called a set of conjugate harmonic functions in Ω (see [49]).

Remark 2.2. In what follows basic properties will be given for monogenic functions in the case where the Dirac operator ∂_x is considered. A lot of the formulas thus established remain valid in the case of the Cauchy-Riemann operator D_x by formally replacing e_0 by 1 and e_j by ε_j, $j = 1, \dots, m$. However, notice that in some important situations — in particular when the multiplication operator $x\colon f \to xf$ is involved — this substitution cannot be done. For more details we refer to [14].

2.2. Operator equalities. We systematically deal with the Dirac operator. Passing to polar coordinates $x = r\xi$, $r = |x|$, $\xi \in S^m$, we have:

$$\partial_x = \xi \left(\partial_r + \frac{1}{r}\Gamma_\xi \right).$$

Here $\Gamma_\xi = \Gamma_x$ acts on S^m and is called the *spherical Dirac operator*, with

$$\Gamma_x = \bar{x} \wedge \partial_x = -\sum_{i<j} e_i e_j (x_i \partial_{x_j} - x_j \partial_{x_i}).$$

Furthermore

$$\Delta_x = {\partial_r}^2 + \frac{m}{r}\partial_r + \frac{1}{r^2}\Delta_\xi,$$

Δ_ξ being the Laplace-Beltrami operator on S^m, and

$$\begin{aligned} \Delta_\xi &= ((m-1)\mathbf{1} - \Gamma_\xi)\Gamma_\xi, \\ \xi\Gamma_\xi\xi &= \Gamma_\xi - m\mathbf{1}. \end{aligned}$$

Put

$$\mathbf{E} = \sum_{j=0}^{m} x_j \partial_{x_j} = r\partial_r$$

($\mathbf{E}$ is called the *Euler operator*) Then

$$\begin{aligned} \bar{x}\partial_x &= \mathbf{E} + \Gamma_x, \\ \Delta_x x &= 2\partial_x + x\Delta_x, \\ \partial_x x &= -(m+1)\mathbf{1} - \mathbf{E} + \Gamma_x, \\ x\partial_x + \partial_x x &= -2\mathbf{E} - (m+1)\mathbf{1}, \\ \Gamma_x &= (m+1)\mathbf{1} + \mathbf{E} + \partial_x x. \end{aligned}$$

Using some of the above relations, we may prove

Theorem 2.3. *Suppose f is monogenic in Ω. Then*

(i) $(\mathbf{E}+\Gamma_x)f=0$,
(ii) $\partial_x(xf)=-(m+1)f-2\mathbf{E}f$,
(iii) $\Delta_x(xf)=0$, *i.e.* xf *is harmonic in* Ω.

2.3. Basic integral formulae. Let $\Omega\subset\mathbb{R}^{m+1}$ be open, let C be a compact orientable $(m+1)$-dimensional manifold with boundary ∂C and define the oriented $\mathbb{C}_{m+1}$-valued surface element $d\sigma$ on ∂C by

$$d\sigma_x=\sum_{j=0}^{m}(-1)^j e_j d\hat{x}_j,$$

where

$$d\hat{x}_j=dx_0\wedge\cdots\wedge[dx_j]\wedge\cdots\wedge dx_m,\qquad j=0,1,\ldots,m.$$

Then if at $x\in\partial C$, $n(x)$ stands for the outwardly pointing unit normal,

$$d\sigma_x=n(x)dS(x),$$

$dS(x)$ being the elementary surface measure.

Stokes' and Cauchy's Theorems. Suppose that $f,g\in\mathcal{C}_1(\Omega)$. Then

Theorem 2.4 (Stokes)**.** *For each $C\subset\Omega$,*

$$\int_{\partial C}f(x)\,d\sigma_x g(x)=\int_C[(f\partial_x)g+f(\partial_x g)]\,dx.$$

Theorem 2.5 (Cauchy)**.** *If f is right monogenic in Ω and g is left monogenic in Ω, then for each $C\subset\Omega$,*

$$\int_{\partial C}fd\sigma=0.$$

Corollary 2.6. *If g is left monogenic (resp. f right monogenic) in Ω, then for each $C\subset\Omega$,*

$$\int_{\partial C}d\sigma g=0\qquad(\textit{resp.}\ \int_{\partial C}f\,d\sigma=0).$$

Remark 2.7. Multiplication in $\mathbb{C}_{m+1}$ being non-commutative, it is important to notice the order of succession of the factors in the integrands.

Remark 2.8. As $d\sigma_x=n(x)dS(x)$, we may thus replace $d\sigma_x$ by this expression in the left hand side of the above relations. However, keep in mind that $n(x)$ should be put at the right place since $n(x)$ is $\mathbb{R}^{m+1}$-valued, namely

$$n(x)=\sum_{j=0}^{m}e_jn_j(x).$$

We for instance write

$$\int_{\partial C} f(x)n(x)g(x)\, dS(x) = \int_{\partial C} f(x)\, d\sigma(x) g\sigma(x).$$

Remark 2.9. It is worth noticing that Cauchy's Theorem implies the following well known formulae from vector analysis. Let u be an $\mathbb{R}$-valued harmonic function in Ω and put

$$g(x) = \partial_x u(x) = \sum_{j=0}^{m} e_j \frac{\partial u}{\partial x_j}.$$

Then g is monogenic in Ω, whence for any $C \subset \Omega$,

$$0 = \int_{\partial C} d\sigma_x g(x) = \int_{\partial C} n(x)g(x)\, dS(x).$$

But, as $n(x)$ and $g(x)$ are both vector-valued,

$$n(x)g(x) = -n(x) \bullet g(x) + n(x) \wedge g(x)$$

whence

$$\int_{\partial C} n(x) \bullet g(x)\, dS(x) = 0$$

and

$$\int_{\partial C} n(x) \wedge g(x)\, dS(x) = 0$$

Remark 2.10. A particularly important example occurs in the following case: take $f = 1$ and $C = \overset{\circ}{B}(1)$, the unit ball in $\mathbb{R}^{m+1}$, i.e. $\partial C = S^m$, the unit sphere in $\mathbb{R}^{m+1}$. Then at each point $\omega \in S^m$, $n(\omega) = \omega$ whence

$$\int_{S^m} \omega\, dS(\omega) = 0.$$

Cauchy's integral representation theorem. Let us first point out that the fundamental solution of the Dirac operator ∂_x is given by

$$E(x) = \frac{1}{A_{m+1}} \frac{\bar{x}}{|x|^{m+1}}.$$

Here A_{m+1} is the area of the unit sphere S^m in $\mathbb{R}^{m+1}$, i.e.

$$A_{m+1} = \frac{2\pi^{(m+1)/2}}{\Gamma(\frac{m+1}{2})}.$$

We have that

(i) E is $\mathbb{R}^{m+1}$-valued and belongs to $L_1^{loc}(\mathbb{R}^{m+1})$

(ii) E is left and right monogenic in $\mathbb{R}_0^{m+1} = \mathbb{R}^{m+1} \backslash 0$ and $\lim_{x\to\infty} E(x) = 0$.

(iii) $\partial_x E = E\partial_x = \delta(x)$, $\delta(x)$ being the classical δ-function in $\mathbb{R}^{m+1}$, i.e. for each $\varphi \in \mathcal{D}(\mathbb{R}^{m+1}; \mathbb{C}_{m+1})$,

$$\langle \delta(x), \varphi(x) \rangle = \varphi(0)$$

Theorem 2.11 (Borel-Pompeiu formula)**.** *Let $f \in \mathcal{C}_1(\Omega)$. Then for each $C \subset \Omega$,*

$$\int_{\partial C} E(y-x)\, d\sigma_y f(y) - \int_C E(y-x)\partial_y f(y)\, dy = \begin{cases} f(x) & \text{if } x \in \overset{\circ}{C} \\ 0 & \text{if } x \in \Omega \backslash C \end{cases}$$

Corollary 2.12 (Cauchy's integral representation theorem)**.** *Let f be left monogenic in Ω, i.e. $\partial_x f = 0$ in Ω. Then for each $C \subset \Omega$ and $x \in \overset{\circ}{C}$*

$$f(x) = \int_{\partial C} E(y-x)\, d\sigma_y f(y).$$

Remark 2.13. The Borel-Pompeiu formula measures in some sense the difference between a $\mathcal{C}_1$-function and a monogenic function in Ω.

Remark 2.14. The complexification $\mathcal{D}$ of the classical Dirac operator where $\mathcal{D} = \partial_x + i\partial_y$, ∂_x and ∂_y being Dirac operators in $\mathbb{R}^m$, i.e. $\partial_x = \sum_{j=1}^m e_j \partial_{x_j}$, $\partial_y = \sum_{j=1}^m e_j \partial_{y_j}$, gives rise to connections between complex analysis and Clifford analysis.

For $\mathbb{C}$-valued $\mathcal{C}_1$-functions f in $\bar{G} \subset \mathbb{R}^{2m}$, G being open and bounded, this operator allows a Borel-Pompeiu type formula which may adapted to the case $G = \mathbb{R}^{2m}$ provided $f(\infty) = \lim_{|(x,y)| \to \infty} f(x,y)$ exists.

As has been shown in [2], the latter may be applied to inverse scattering problems.

The Borel-Pompeiu type formula for the operator $\mathcal{D}$ in bounded domains $G \subset \mathbb{R}^{2m}$ is closely related to the Bochner-Martinelli type formula established in [44]. The latter contains as a special case the classical Bochner-Martinelli formula for holomorphic functions.

Remark 2.15. The study of null solutions of $\mathcal{D}$ which are moreover holomorphic — so-called complex monogenic functions — was initiated in the beginning of the 1980's (see e.g. [39]). Associated Cauchy integral type formulae contain as well integral formulae of elliptic type as of hyperbolic type (see e.g. [6])

Remark 2.16. Null solutions to a generalized complex Dirac type operator of the form $\mathcal{D} = \sum_{j=0}^m \Phi(e_j)\frac{\partial}{\partial z_j}$, Φ being a representation of $\mathrm{Spin}(m+1)$, were studied in [7]. Applications to the inverse Penrose transform are given.

Remark 2.17. Recently, a function theory has been set up for so-called hyperholomorphic Cauchy-Riemann operators (see [38]). It is an attempt to embrace both several complex variable theory and Clifford analysis.

2.4. Historical notes. As is well known, there are several approaches possible to classical complex analysis, one of them being the Riemann approach based on studying null-solutions $f = u + iv$ of the Cauchy-Riemann operator

$$\partial = \frac{\partial}{\partial_x} + i\frac{\partial}{\partial_y}$$

in $\mathbb{C}$, thus leading to the famous Cauchy-Riemann equations combining $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$:

$$\begin{aligned} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} &= 0, \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} &= 0, \end{aligned}$$

and to the fact that, as $\bar{\partial}\partial = \Delta$, the real and imaginary parts of f are harmonic. As was already pointed out by Gr. Moisil and N. Théodoresco in their 1931-paper (see [32]), these two characteristics of holomorphic functions are easily transferable to higher dimensions when considering Clifford algebra valued functions. The example we gave in Section 2.1 is a nice illustration of this. It was M. Riesz (see [37]) who found out that the system

$$\begin{aligned} \operatorname{div} \vec{f} &= 0 \\ \operatorname{curl} \vec{f} &= 0, \end{aligned}$$

$\vec{f} = (f_1, \ldots f_m)$, could be written in compact form using formally the operator $\nabla = \sum_{j=1}^m e_j \partial_{x_j}$ in $\mathbb{R}^m$ (which is in fact the Dirac operator $\partial_{\underline{x}}$ in $\mathbb{R}^m$). Gr. Moisil and N. Théodoresco obtained the operator ∇ in quite another way. Their starting point was the idea already worked out by D. Pompeiu in 1912 (see [33]) to measure in some sense the difference between a $\mathbb{C}$-valued $\mathcal{C}_1$-function and a holomorphic one. D. Pompeiu was thus led to the notion of *areolar derivative.* Let f be a $\mathcal{C}_1$-function in Ω and let for $z_0 \in \Omega$,

$$Df(z_0) = \lim_{B \downarrow z_0} \frac{\int_{\partial B} f(z)\, dz}{2i \iint_B dxdy} \tag{2.2}$$

where the domain $B \subset \Omega$ is shrinking to z_0. Putting $f(z) = u + iv$, $P(x, y) = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$ and $Q(x, y) = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$, then using Stokes' Theorem, we have that, if $z_0 = x_0 + iy_0$,

$$Df(z_0) = \frac{1}{2}(P(x_0, y_0) + iQ(x_0, y_0)).$$

Notice that by (2.2), $D = \frac{1}{2}\partial$ is obtained. So holomorphy of $f \in \Omega$ is equivalent to $Df(z_0) = 0$ at each $z_0 \in \Omega$. Putting weaker conditions on f, the Romanian school developed systematically since the 1930's the study of the areolar derivative D (see e.g. [30] for a historical survey) In higher dimensional case, using Clifford algebras N. Théodoresco [51] introduced the operator ∇ as follows. Let $\Omega \subset \mathbb{R}^m$ be open and let $f \in \mathcal{C}_1(\Omega; \mathbb{R}_{0,m})$. Then

$$\nabla f(x) = \lim_{B \downarrow x} \frac{\int_{\partial B} n(y) f(y)\, dS(y)}{\int_B dl_m}$$

where the domain $B \subset \Omega$ is shrinking to x. Here $n(y) = \sum_{j=1}^m e_j n_j(y)$ is the outward pointing unit normal to ∂B, the boundary of B. $dS(y)$ and dl_m are, respectively, the Lebesgue measures on ∂B and in $\mathbb{R}^m$. Of course the multiplication $n(y)f(y)$ is done in $\mathbb{R}_{0,m}$. By means of Stokes' Theorem, one easily finds that

$$\nabla = \sum_{j=1}^m e_j \partial_{x_j}.$$

Also from the 1930's on, R. Fueter started developing quaternionic analysis. He introduced the operator $D = \partial_{x_0} + i\partial_{x_1} + j\partial_{x_2} + k\partial_{x_3}$, (i, j, k) being the classical orthonormal basis for $\mathbb{R}^3 \subset \mathbf{H}$ with $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j$. For $\Omega \subset \mathbb{R}^4$ open, $f \in \mathcal{C}_1(\Omega; \mathbf{H})$ is called (left) regular if $Df = 0$ in Ω, thus giving rise to a generalized Cauchy-Riemann system. However, as was explicitly pointed out by R. Fueter, the main concern for introducing the operator D was to ensure the validity of Cauchy's Theorem for $\mathbf{H}$-valued functions (see [18]) R. Fueter and his school also studied Clifford algebra valued regular functions, but then by formally introducing the operator D as we did in this paragraph and also did when starting to develop Clifford analysis from the late 1960's on. For us, the main reason for looking at this operator was that it linearizes the Laplacian and that for functions satisfying $Df = 0$ in $\Omega \subset \mathbb{R}^m$, Cauchy's Theorem was valid, namely $\int_{\partial C} g d\sigma f = 0$ if $gD = 0$ and $Df = 0$ in Ω, $C \subset \Omega$ be compact and orientable. The same formal approach was followed by V. Iftimie (see [22]). Nevertheless, within the same time interval, D. Hestenes re-introduced Théodoresco's ∇-operator and stressed the important role played by Cauchy's Theorem in classical complex function theory (see [21]). In the 1990's, J. Cnops (see [11]) introduced pre-Dirac operators P on semi-Riemannian orientable manifolds M as follows. Let Ω be a domain in the m-dimensional manifold M. A real linear operator $P \colon \mathcal{C}_1(\Omega) \to \mathcal{C}_0(\Omega)$ such that for any $f, g \in \mathcal{C}_1(\Omega)$, Stokes' Theorem holds, is called a pre-Dirac operator on M in Ω, i.e. for any m-dimensional bounded cycle C in Ω having boundary ∂C with the inherited orientation,

$$\int_{\partial C} \overline{f(y)}\, dM_{m-1} g(y) = \int_C \overline{Pf(x)}\, dM_m(x) g(x) + (-1)^m \int_C \overline{f(x)}\, dM_m(x) Pg(x).$$

By imposing a supplementary condition on P, namely that for any $\mathbb{R}$-valued function g and each x in Ω, Pg is a vector tangent to M at x, P is then called a Dirac operator on M and is denoted by D_M. In Euclidean space, it turns out that $D_M = \sum_{j=1}^m e_j \partial_{x_j}$. For further references and historical notes on quaternionic and Clifford analysis, we refer to [13], [14], [19], [20], [50]. Obviously, we do not claim any form of completeness in giving these historical remarks.

3. Generating monogenic functions

In classical complex analysis, the set $\mathcal{M}(\Omega)$ of meromorphic functions in a simply connected open subset Ω of $\mathbb{C}$ is a linear associative algebra. Moreover, composition preserves holomorphy. As monogenic functions do not enjoy of these powerful properties, the question arises of how to construct elementary monogenic functions such as polynomials, an exponential function, etc. The aim of this section is to show that there are a lot of techniques available to generate monogenic functions and some of them will be described in more or less detail.

3.1. Cauchy-Kowalewska extension. One way of constructing monogenic functions is by extending real-analytic functions in some open connected domain $\underline{\Omega}$ in $\mathbb{R}^m$. In fact, the problem is the following. "Given a real-analytic function f in $\underline{\Omega} \subset \mathbb{R}^m$, does there exist a monogenic function f^* in some open neighbourhood Ω of $\underline{\Omega}$ in $\mathbb{R}^{m+1}$ such that $f^*|_{\underline{\Omega}} = f$?"

In this setting, a real axis should thus be chosen and we take the x_0-axis for it, which means that the variable in $\mathbb{R}^m$ is $\underline{x} = (x_1, \ldots, x_m)$.

As to Ω, it will be an open connected and x_0-normal neighbourhood Ω of $\underline{\Omega}$ in $\mathbb{R}^{m+1}$. This means that for each $x \in \Omega$, the line segment $\{x + te_0 : t \in \mathbb{R}\} \cap \Omega$ is connected and contains exactly one point in $\underline{\Omega}$.

Finally, as to f^*, it should thus satisfy the conditions

(i) $\partial_x f^* = 0$ in Ω,
(ii) $f^*(x_0, \underline{x})|_{x_0=0} = f^*(0, \underline{x}) = f(\underline{x})$,

i.e. $f^*|_{\underline{\Omega}} = f$.

From (i) it follows that

$$\partial_{x_0} f^* = -\overline{e}_0 \partial_{\underline{x}} f^*,$$

$\partial_{\underline{x}}$ being as usual the Dirac operator in $\mathbb{R}^m$.

Combined with (ii), we thus obtain:

$$f^*(x_0, \underline{x}) = \left(e^{-x_0 \overline{e}_0 \partial_{\underline{x}}}\right) f(\underline{x}) = \sum_{k=0}^{\infty} \frac{(-x_0)^k}{k!} (\overline{e}_0 \partial_{\underline{x}})^k f(\underline{x}). \tag{3.1}$$

So the existence of f^* is guaranteed. The same can be proved about the existence of Ω.

The monogenic function f^* in Ω thus obtained is called the *Cauchy-Kowalewska* extension of f. By construction, it is unique.

When instead of the Dirac operator ∂_x, the Cauchy-Riemann operator D_x is used for defining monogenicity, it again suffices to replace e_0 by 1 and e_j by ε_j, $j = 1, \ldots, m$, in the above formula (3.1) in order to obtain the corresponding CK-extension of f.

Example 3.1 (Homogeneous monogenic polynomials)**.** Call $\underline{\mathcal{P}}(k)$ the set of homogeneous polynomials of degree k in $\mathbb{R}^m$, $k \in \mathbb{N}$ fixed. Then $P \in \underline{\mathcal{P}}(k)$ may be expressed as

$$P(\underline{x}) = \sum_{|\underline{\alpha}|=k} \underline{x}^{\underline{\alpha}}\, a_{\underline{\alpha}}$$

where the multi index $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, $a_{\underline{\alpha}} \in \mathbb{C}_{m+1}$ and $\underline{x}^{\underline{\alpha}} = x_1^{\alpha_1} \ldots x_m^{\alpha_m}$. Obviously, P is real-analytic in $\mathbb{R}^m$ and, in view of (3.1), its CK-extension P^* in $\mathbb{R}^{m+1}$ is given by

$$P^*(x) = \sum_{j=0}^{\infty} \frac{(-x_0)^j}{j!} (\overline{e}_0 \partial_{\underline{x}})^j P(\underline{x}).$$

As for $j > k$, $(\overline{e}_0 \partial_{\underline{x}})^j P(\underline{x}) \equiv 0$, we thus have

$$P^*(x) = \sum_{j=0}^{k} \frac{(-x_0)^j}{j!} (\overline{e}_0 \partial_{\underline{x}})^j P(\underline{x}),$$

i.e. P^* is a homogeneous monogenic polynomial of degree k in $\mathbb{R}^{m+1}$.

Conversely, if P_k is a homogeneous monogenic polynomial of degree k in $\mathbb{R}^{m+1}$, its restriction $P_k(0, \underline{x})$ to $\mathbb{R}^m$ is a homogeneous polynomial of degree k and clearly $(P_k(0, \underline{x}))^* = P_k(x)$.

Calling $M^+(k)$ the set of homogeneous monogenic polynomials in $\mathbb{R}^{m+1}$, the CK-extension thus establishes an isomorphism between the right $\mathbb{C}_{m+1}-$modules $\underline{\mathcal{P}}(k)$ and $M^+(k)$.

In particular,

$$M^+(0) = \mathbb{C}_{m+1}.$$

For $k = 1$, consider the basic elements x_j, $j = 1, \ldots, m$, of $\underline{\mathcal{P}}(1)$. Then

$$(x_j)^* = z_j = x_j - x_0\, \overline{e}_0\, e_j.$$

For $k > 1$, it may be shown that for $|\underline{\alpha}| = k$, $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$,

$$(\underline{x}^{\underline{\alpha}})^* = \underline{\alpha}! V_{\underline{\alpha}}(x)$$

where $V_{\underline{\alpha}}(x)$ can be expressed as follows.

Associate with $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, $|\underline{\alpha}| = k$, the sequence $(l_1, \ldots, l_k) \in \{1, \ldots, m\}^k$ such that j is appearing α_j times in $(l_1, \ldots, l_k)$, $j = 1, \ldots, m$. Then

$$V_{\underline{\alpha}}(x) = V_{l_1 \ldots l_k}(x) = \frac{1}{k!} \sum_{\pi(l_1, \ldots, l_k)} z_{l_1} z_{l_2} \ldots z_{l_k}$$

the sum running over all distinguishable permutations of all of $(l_1, \ldots, l_k)$.

We shall return to the set $M^+(k)$ in Section 4.1.

Example 3.2 (Radial Hermite polynomials)**.** Consider the Gauss-distribution in $\mathbb{R}^m$:

$$G_0(\underline{x}) = e^{-|\underline{x}|^2/2} = e^{\underline{x}^2/2}.$$

Obviously, G_0 is real-analytic in $\mathbb{R}^m$ and in view of (3.1), its CK-extension w.r.t. the Cauchy-Riemann operator D_x in $\mathbb{R}^{m+1}$ is given by

$$G_0(x_0, \underline{x}) = \exp(-x_0 \partial_{\underline{x}}) G_0(\underline{x})$$

Putting

$$G_0(x_0, \underline{x}) = e^{\underline{x}^2/2} \sum_{n=0}^{\infty} \frac{x_0^n}{n!} H_n(\underline{x}),$$

it may be proved that the functions $H_n(\underline{x})$ are polynomials in $\underline{x}$ having real coefficients and satisfying the recurrence formula

$$H_{n+1}(\underline{x}) = (\underline{x} - \partial_{\underline{x}}) H_n(\underline{x})$$

The polynomials $H_n(\underline{x})$ are called *radial Hermite polynomials.* They are also determined by the Rodrigues' Formula

$$H_n(\underline{x}) = (-1)^n \, e^{|\underline{x}|^2/2} \, \partial_{\underline{x}}^n (e^{-|\underline{x}|^2/2}).$$

The first radial Hermite polynomials are thus given by

$$\begin{aligned} H_0(\underline{x}) &= 1, \\ H_1(\underline{x}) &= \underline{x}, \\ H_2(\underline{x}) &= \underline{x}^2 + m, \\ H_3(\underline{x}) &= \underline{x}^3 + (m+2)\underline{x}. \end{aligned}$$

Remark 3.3. For further examples of monogenic functions constructed by means of the CK-extension method and more generally for generating classes of special functions in $\mathbb{R}^{m+1}$, we refer to [14, Ch. III].

Remark 3.4. For applications of the radial Hermite polynomials to continuous wavelet transforms in higher dimension and the construction of specific basic wavelet functions, we refer to [3].

3.2. Differentiation. If f is monogenic in Ω and $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m) \in \mathbb{N}^{m+1}$, then, as $[\partial^\alpha, \partial_x] = 0$, where $\partial^\alpha = \partial_{x_0}^{\alpha_0} \ldots \partial_{x_m}^{\alpha_m}$, we of course have that $\partial^\alpha f \in M(\Omega)$.

However, since from $\partial_x f = 0$ it follows that

$$\partial_{x_0} f = -\bar{e}_0 \partial_{\underline{x}} f,$$

it clearly suffices to consider differential operators of the form $\partial^{\underline{\alpha}}$, where $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$.

Obviously, for each differential operator

$$P\left(\frac{\partial}{\partial \underline{x}}\right) = \sum_{\underline{\alpha}} a_{\underline{\alpha}} \partial^{\underline{\alpha}},$$

$a_{\underline{\alpha}} \in \mathbb{C}$, we have that f monogenic in Ω implies $P(\frac{\partial}{\partial \underline{x}})f$ monogenic in Ω.

Example 3.5. Consider the fundamental solution

$$E(x) = \frac{1}{A_{m+1}} \frac{\overline{x}}{|x|^{m+1}}$$

of ∂_x and take $\underline{\alpha} \in \mathbb{N}^m$ with $|\underline{\alpha}| = k$.

Then we put

$$W_{\underline{\alpha}}(x) = (-1)^{|\underline{\alpha}|}\, \partial^{\underline{\alpha}}\, E(x).$$

By the definition itself, we thus have that $W_{\underline{\alpha}}$ is a homogeneous monogenic function of degree $-(m+k)$ in $\mathbb{R}_0^{m+1} = \mathbb{R}^{m+1} \backslash \{0\}$. As $E(x)$ is vector valued, the same holds for each $W_{\underline{\alpha}}(x)$.

We return to the functions $W_{\underline{\alpha}}(x)$ in Section 4.1.

3.3. Monogenicity, harmonicity, and Möbius transforms. Let g be a Möbius transform in $\mathbb{R}^{m+1}$. Then, as we saw in Part I, Section 3, g may be fully described by a (2×2)-matrix over $\Gamma(m+1) \cup \{0\}$, say

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the elements of which are submitted to some algebraic conditions.

The action of g on $\mathbb{R}^{m+1}$ is given by

$$g(x) = \frac{ax+b}{cx+d} = (ax+b)(cx+d)^{-1}.$$

Given a monogenic function f or harmonic function h in $\Omega \subset \mathbb{R}^{m+1}$ open, new monogenic or harmonic functions, respectively denoted by $\gamma_g f$ or $\eta_g h$, can be constructed from it by composing it with a Möbius transformation g. We have

Theorem 3.6. *Let g be a Möbius transformation in $\mathbb{R}^{m+1}$ and let $\Omega \subset \mathbb{R}^{m+1}$ be open.*

(i) *If h is harmonic in Ω, then*

$$\eta_g h(y) = \frac{1}{|yc - a|^{m-1}} h(g^{-1}y)$$

is harmonic in $g\Omega$.

(ii) *If f is monogenic in Ω, then*

$$\gamma_g f(y) = \frac{-yc + a}{|yc + a|^{m+1}} f(g^{-1}y)$$

is monogenic in $g\Omega$.

Example 3.7 (Inversion w.r.t. the unit sphere S^m)**.** As we have seen, the inversion w.r.t. S^m mapping $x \to \frac{x}{|x|^2}$ may be described by the matrix

$$g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \text{with} \qquad g^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

In the case of harmonic functions, we denote $\eta_g = \mathcal{K}$ (the Kelvin transform) while in the case of monogenic functions, we put $I^* = -\gamma_g$ (the inversion).

For h harmonic in Ω, we thus have that

$$\mathcal{K}h(y) = \frac{1}{|y|^{m-1}}\, h\left(\frac{y}{|y|^2}\right)$$

is harmonic in $g\Omega$, while for f monogenic in Ω,

$$I^* f(y) = \frac{\overline{y}}{|y|^{m+1}}\, f\left(\frac{y}{|y|^2}\right)$$

is monogenic in $g\Omega$.

Clearly $\mathcal{K}^2 = \mathbf{1}$ and $I^{*2} = -\mathbf{1}$.

In particular, if S_k is a homogeneous harmonic polynomial of degree k, $k \in \mathbb{N}$ (i.e. a solid harmonic of degree k), then

$$\mathcal{K}S_k(y) = \frac{1}{|y|^{m-1}}\, S_k\left(\frac{y}{|y|^2}\right)$$

is a homogeneous harmonic function of degree $-(m+k-1)$ in $\mathbb{R}^{m+1}\backslash\{0\}$.

Analogously, if P_k is a homogeneous monogenic polynomial of degree k, $k \in \mathbb{N}$, (i.e. an inner spherical monogenic of degree k) then

$$I^* P_k(y) = \frac{\overline{y}}{|y|^{m+1}}\, P_k\left(\frac{y}{|y|^2}\right)$$

is a homogeneous monogenic function of degree $-(m+k)$ in $\mathbb{R}^{m+1} \setminus \{0\}$.

Furthermore, if h is harmonic (resp. f monogenic) in the unit ball $\overset{\circ}{B}(1)$ of $\mathbb{R}^{m+1}$, then $\mathcal{K}h$ is harmonic (resp. $I^* f$ is monogenic) in $\mathbb{R}^{m+1} \setminus \overline{B}(1)$.

Example 3.8 (Rotations)**.** Let us recall that a rotation in $\mathbb{R}^{m+1}$ admits the vectorial representation $\chi(s)\colon x \to \chi(s)(x) = sx\overline{s}$ where $s \in \mathrm{Spin}(m+1)$. In terms of a Möbius transformation, it may thus be described by the matrix

$$g = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \qquad \text{with} \qquad g^{-1} = \begin{bmatrix} \overline{s} & 0 \\ 0 & \overline{s} \end{bmatrix}.$$

We put $\eta_g = \mathcal{R}(s)$ and $\gamma_g = L(s)$.

For h harmonic,

$$\mathcal{R}(s)(y) = h(\overline{s}ys)$$

is harmonic, while for f monogenic,

$$L(s)f(y) = sf(\overline{s}ys)$$

is monogenic.

As is well known $[\Delta_x, \mathcal{R}(s)] = 0$ or Δ_x is invariant for rotations.

As to ∂_x, we have $[\partial_x, L(s)] = 0$, whence ∂_x yields an example of a so-called first order Spin$(m+1)$-invariant differential operator.

Example 3.9 (Cayley transform)**.** Let $\overset{\circ}{B}(1)$ and $\mathbb{R}^{m+1}_+$ with $\mathbb{R}^{m+1}_+ = \{x \in \mathbb{R}^{m+1} : x_m > 0\}$ be, respectively, the unit ball and the upper half space in $\mathbb{R}^{m+1}$. The Möbius transform C mapping $\mathbb{R}^{m+1}_+$ to $\overset{\circ}{B}(1)$ — the so-called *Cayley transform* — may be described by the matrix

$$g = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -e_m \\ -e_m & 1 \end{bmatrix} \qquad \text{with} \qquad g^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & e_m \\ e_m & 1 \end{bmatrix}.$$

So, if h is harmonic in $\mathbb{R}^{m+1}_+$, then

$$\eta_g h(y) = \frac{1}{|ye_m + 1|^{m-1}}\, h\left(\frac{y + e_m}{e_m y + 1}\right)$$

is harmonic in $\overset{\circ}{B}(1)$, while if f is monogenic in $\mathbb{R}^{m+1}_+$, then

$$\gamma_g f(y) = \frac{ye_m + 1}{|-ye_m + 1|^{m+1}}\, f\left(\frac{y + e_m}{e_m y + 1}\right)$$

is monogenic in $\overset{\circ}{B}(1)$.

Remark 3.10. For a systematic treatment of invariance of operators under Möbius transformations and its applications, we refer to [11] and [40], where the latter also deals with the case of complex Dirac operators.

Remark 3.11. In [41], it is shown how the representation of Möbius transformations by (2×2)-matrices over $\Gamma(m+1) \cup \{0\}$ may be used to deal with cross ratio and Schwarzian derivative in higher dimensional setting.

Remark 3.12. For the role played by Möbius transformations in defining automorphic forms in the upper half space $\mathbb{R}^{m+1}_+$, we refer to [26].

3.4. The Cauchy-Riemann system in the plane and monogenic functions. It is a remarkable fact that the Cauchy-Riemann system in the plane generates monogenic functions. This was first observed by R. Fueter in [17] in the setting of quaternionic analysis.

Notice that in this subsection, monogenicity in $\mathbb{R}^{m+1}$ is w.r.t. the Cauchy-Riemann operator D_x.

Fueter's Theorem. Assume f to be holomorphic in some open subset $\Omega \subset \mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and put $f(z) = u + iv$ where as usual $u = \operatorname{Re} f$, $v = \operatorname{Im} f$.

Consider first the case of $\mathbb{H}$-valued functions in $\mathbb{R}^4$, $\mathbb{H}$ being the algebra of real quaternions.

In a classical way, a vector $\mathbf{x} = (x_0, \underline{x}) \in \mathbb{R}^4$ is identified with $\mathbf{x} = x_0 + \underline{x} = x_0 + ix_1 + jx_2 + kx_3$, where $\underline{x} = ix_1 + jx_2 + kx_3 \in \mathbb{R}^3$. The associated Cauchy-Fueter operator $D_{\mathbf{x}}$ is given by

$$D_{\mathbf{x}} = \partial_{x_0} + i\partial_{x_1} + j\partial_{x_2} + k\partial_{x_3}.$$

Passing to polar coordinates in $\mathbb{R}^3$, i.e. putting $\underline{x} = r\underline{\omega}$ where $r = |\underline{x}|$ and $\underline{\omega} = \underline{x}/|\underline{x}|$ with $\underline{\omega} \in S^2$, we may write $\mathbf{x} = x_0 + \underline{\omega}r$. Notice that $\underline{\omega}^2 = -1$. We thus have that for $\underline{\omega}$ fixed, $x_0 + r\underline{\omega}$ behaves like the complex variable $z = x + iy$ when making the identification $x \to x_0$, $y \to z$ and $i \to \underline{\omega}$. Hence for any $\underline{\omega}$ fixed, we may associate with f the $\mathbb{H}$-valued function

$$f(x_0 + \underline{\omega}r) = u(x_0, r) + \underline{\omega}v(x_0, r).$$

We have

Theorem 3.13 (Fueter)**.** *Let f be holomorphic in some open subset $\Omega \subset \mathbb{C}^+$ and let $f(x_0 + \underline{\omega}r)$ be the associated $\mathbb{H}$-valued function. Then $\Delta_{\mathbf{x}}f$ is as well left as right monogenic, i.e. $D_{\mathbf{x}}\Delta_{\mathbf{x}}f = (\Delta_{\mathbf{x}}f)D_{\mathbf{x}} = 0$.*

In the case of the Cauchy-Riemann operator $D_{\mathbf{x}}$ in $\mathbb{R}^{m+1}$, this result has been extended successively by M. Sce [42] (m odd) and T. Qian [35] (m even).

Let us comment about their result.

Considering the Clifford algebra $\mathbb{R}_{0,m+1}$, $\mathbf{x} = (x_0, \underline{x}) \in \mathbb{R}^{m+1}$ is identified with the paravector $\mathbf{x} = x_0 + \underline{x} = x_0 + \sum_{j=1}^{m} \varepsilon_j x_j \in \mathbb{R}_{0,m} \cong \mathbb{R}^+_{0,m+1}$. One may then put $\mathbf{x} = x_0 + r\underline{\omega}$ where $r = |\underline{x}|$ and $\underline{\omega} = \underline{x}/|\underline{x}|$ with $\underline{\omega}^2 = -1$.

For a function $f = u + iv$ holomorphic in $\Omega \subset \mathbb{C}^+$, the corresponding paravector-valued function $f(x_0 + \underline{x}) = u(x_0, r) + \underline{\omega}v(x_0, r)$ may then be defined. It was thus obtained that $\Delta_{\mathbf{x}}^{(m-1)/2}$ is monogenic in $\widetilde{\Omega} = \{x_0 + \underline{x} : (x_0, r) \in \Omega\}$, i.e.

$$D_{\mathbf{x}}\Delta_{\mathbf{x}}^{(m-1)/2} f(x_0 + \underline{x}) = 0 \qquad \text{in } \widetilde{\Omega}.$$

Notice here that in the case m even, $\Delta_{\mathbf{x}}^{(m-1)/2}$ is defined through Fourier multiplier theory (see [35]).

This version of Fueter's Theorem provides us with so-called axial monogenic functions (see [14]), i.e. monogenic functions of the form

$$A(x_0, r) + \underline{\omega}B(x_0, r) = \Delta_x^{(m-1)/2} f(x_0 + \underline{x}),$$

A and B being $\mathbb{R}$-valued and satisfying the Vekua-type system

$$\begin{aligned} \partial_{x_0}A - \partial_r B &= \frac{m-1}{2}B, \\ \partial_{x_0}B + \partial_r A &= 0. \end{aligned}$$

More generally, one may consider axial monogenic functions of the type

$$(A(x_0,r)+\underline{\omega}B(x_0,r))P_k(\underline{x})$$

where $P_k\in M^+(k)$, $k\in\mathbb{N}$, i.e. $P_k(\underline{x})=r^kP_k(\underline{\omega})$ is a homogeneous polynomial of degree k satisfying $\partial_{\underline{x}}P_k=0$ in $\mathbb{R}^m$ (P_k is monogenic in $\mathbb{R}^m$).

Recently, it was proved that (see [24] and [46])

Theorem 3.14. *Let f be holomorphic in an open subset $\Omega\subset\mathbb{C}^+$ and let P_k be a homogeneous monogenic polynomial of degree k in $\mathbb{R}^m$. Then the function*

$$\Delta_{\mathbf{x}}^{k+(m-1)/2}f(x_0+\underline{x})P_k(\underline{x})$$

is monogenic in $\widetilde{\Omega}$.

Remark 3.15. For $k=0$, the generalized version of the classical Fueter Theorem is of course obtained.

Remark 3.16. The result given plays a crucial role in the study of Fourier multipliers and singular integrals on the unit sphere (see [36]).

Remark 3.17. The classical Fueter Theorem for the quaternionic case has also been discussed in [13] and [50].

Monogenic plane waves. Once again, we consider the configuration $\mathbb{R}^{m+1}=\mathbb{R}\oplus\mathbb{R}^m\subset\mathbb{R}_{0,m}\cong\mathbb{R}^+_{0,m+1}$ and the associated Cauchy-Riemann operator $D_{\mathbf{x}}=\partial_{x_0}+\sum_{j=1}^m\varepsilon_j\partial_{x_j}$.

Let g be a plane wave, i.e. g is a $\mathbb{C}$-valued real-analytic function of the scalar product $\langle\underline{x},\underline{t}\rangle$, $\underline{x},\underline{t}\in\mathbb{R}^m$ (see [23]).

Its CK-extension $G(x_0,\underline{x},\underline{t})$($\underline{t}\in\mathbb{R}^m$ fixed), monogenic in an appropriate open subset $\Omega\subset\mathbb{R}^{m+1}$, is then given by (see (3.1))

$$G(x_0,\underline{x},\underline{t})=\left(e^{-x_0\partial_{\underline{x}}}\right)g(\langle\underline{x},\underline{t}\rangle)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\,x_0^k\,\partial_{\underline{x}}^kg(\langle\underline{x},\underline{t}\rangle).$$

Straightforward calculations yield:

$$G(x_0,\underline{x},\underline{t})=g_1(\langle\underline{x},\underline{t}\rangle,x_0|\underline{t}|)-\frac{\underline{t}}{|\underline{t}|}\,g_2(\langle\underline{x},\underline{t},\rangle,x_0|\underline{t}|).$$

Here g_1 and g_2 are $\mathbb{C}$-valued functions in the two real variables $x=\langle\underline{x},\underline{t}\rangle$ and $y=x_0|\underline{t}|$. They satisfy the Cauchy-Riemann system in the plane, i.e.

$$\begin{aligned}\frac{\partial g_1}{\partial x}-\frac{\partial g_2}{\partial y}&=0\\ \frac{\partial g_2}{\partial x}+\frac{\partial g_1}{\partial y}&=0\end{aligned}\qquad(3.2)$$

Conversely, if g_1 and g_2 are $\mathbb{C}$-valued $\mathcal{C}_1$-functions in some open subset $\widetilde{\Omega} \subset \mathbb{C}$ satisfying the Cauchy-Riemann system (3.2), then for $\underline{t} \in \mathbb{R}^m$ fixed, the function

$$G(x_0, \underline{x}, \underline{t}) = g_1(\langle \underline{x}, \underline{t}\rangle, x_0|\underline{t}|) - \frac{\underline{t}}{|\underline{t}|}\, g_2(\langle \underline{x}, \underline{t}\rangle, x_0|\underline{t}|)$$

is clearly monogenic in

$$\Omega_{\underline{t}} = \{(x_0, \underline{x}) : (\langle \underline{x}, \underline{t}\rangle, x_0|\underline{t}|) \in \widetilde{\Omega}\}.$$

Monogenic functions thus obtained are called *monogenic plane waves.*

Example 3.18. Let $k \in \mathbb{N}$ be fixed and put

$$g_k(\langle \underline{x}, \underline{t}\rangle) = (\langle \underline{x}, \underline{t}\rangle)^k.$$

Then its associated CK-extension is given by

$$G_k(x_0, \underline{x}, \underline{t}) = (\langle \underline{x}, \underline{t}\rangle - \underline{t}x_0)^k.$$

It is a paravector-valued function. Its components g_1 and g_2 being real-valued, it thus follows that $f = g_1 + ig_2$ is holomorphic.

Example 3.19. Consider

$$g(\langle \underline{x}, \underline{t}\rangle) = e^{i\langle \underline{x}, \underline{t}\rangle}.$$

Its associated CK-extension $\mathcal{E}(x_0, \underline{x}, \underline{t})$ defines the exponential function (see [43])

$$\mathcal{E}(x_0, \underline{x}, \underline{t}) = (e^{-x_0\partial_{\underline{x}}})(e^{i\langle \underline{x}, \underline{t}\rangle}) = e^{i\langle \underline{x}, \underline{t}\rangle}\left(\mathrm{ch}(x_0|\underline{t}|) - i\,\frac{\underline{t}}{|\underline{t}|}\,\mathrm{sh}(x_0|\underline{t}|)\right).$$

Its components g_1 and g_2 are $\mathbb{C}$-valued.

τ-monogenic functions. Consider $\mathbb{C}^{m+1} = \mathbb{R}^{m+1} \oplus i\mathbb{R}^{m+1}$.

Then if $\mathbb{R}^{m+1}$ is again identified with the set of paravectors in $\mathbb{R}_{0,m} \cong \mathbb{R}^+_{0,m+1}$, $\mathbb{C}^{m+1} \subset \mathbb{C}^+_{m+1} \cong \mathbb{R}^+_{0,m+1} \otimes_{\mathbb{R}} \mathbb{C}$. Now take $\tau \in \mathbb{C}^{m+1}$ such that $\tau\alpha(\tau) = 0$, i.e. if $\tau = u + iv$, $\alpha(\tau) = \overline{u} + i\overline{v}$, then $u \perp v$ and $|u| = |v|$ in $\mathbb{R}^{m+1}$.

Furthermore, put for $x \in \mathbb{R}^{m+1}$

$$\langle \tau, x\rangle = \langle u, x\rangle_{\mathbb{R}^{m+1}} + i\langle v, x\rangle_{\mathbb{R}^{m+1}}.$$

Suppose φ is a holomorphic function in $\Omega \subset \mathbb{C}$ open and associate with it the $\mathbb{C}^{m+1}$-valued function

$$\Phi_\tau(x) = \overline{\tau}\varphi(\langle u, x\rangle_{\mathbb{R}^{m+1}} + i\langle v, x\rangle_{\mathbb{R}^{m+1}}).$$

A straightforward calculation leads to

Theorem 3.20. *The function Φ_τ is monogenic in the associated open subset $\widetilde{\Omega} = \{x \in R^{m+1} : (\langle u, x\rangle, \langle v, x\rangle) \in \Omega\}$.*

A monogenic function thus constructed is called *τ-monogenic.*

Notice that if we write $\Phi_\tau(x) = \Phi_{\tau,1}(x) + i\Phi_{\tau,2}(x)$, we have that $\Phi_{\tau,1}$ and $\Phi_{\tau,2}$ are monogenic paravector-valued functions.

3.5. Series of meromorphic functions. Let $\Omega \subset \mathbb{R}^{m+1}$ be open and let $(f_k)_{k\in\mathbb{N}}$ be a sequence of meromorphic functions in Ω (for the definition of meromorphy see Section 4).

If all f_k are monogenic in Ω and the sequence $(f_k)_{k\in\mathbb{N}}$ converges uniformly on each compact subset $K \subset \Omega$, then if f denotes the limit of this sequence, f is monogenic in Ω. In particular, if $\sum_{k=0}^{\infty} f_k$ is a series of monogenic functions in Ω which converges either normally or uniformly on each colpact subset K of Ω, then its sum f is also monogenic in Ω. The results thus stated are in fact nothing else but Weierstrass' Theorem for the case of monogenic functions. This theorem follows immediately from the fact that monogenic functions are real-analytic.

If instead a series of meromorphic functions in Ω is considered then, just as in complex analysis, one should impose the series to be either compactly convergent or normally convergent in Ω, meaning, respectively, that for each $K \subset \Omega$ compact, there exists $N(K) \in \mathbb{N}$ such that

(i) (compactly convergent) For each $k \geq N(K)$, the set of poles S_k of f_k satisfies $S_k \cap K = \emptyset$ and $\sum_{k=0}^{\infty} f_k|_K$ converges uniformly on K.

or

(ii) (normally convergent) For each $k \geq N(K)$, the set of poles S_k of f_k satisfies $S_k \cap K = \emptyset$ and $\sum_{k=0}^{\infty} \sup_{x\in K} |f_k|_K(x)| < +\infty$.

The sum of a compactly or normally convergent series of meromorphic functions in Ω is again meromorphic in Ω and for its set of poles S we have that $S \subset \bigcup_{k\in\mathbb{N}} S_k$.

Example 3.21 (The cotangent function $COT^{(m)}$)**.** Let $(w_1, w_2, \ldots w_m)$ be a set of m linearly independent paravectors in $\mathbb{R}^{m+1}$ and let $W_m = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_m$ be the associated m-dimensional lattice.

Furthermore, call

$$q_0(x) = \frac{\overline{x}}{|x|^{m+1}} = A_{m+1}E(x).$$

Then the series

$$q_0(x) + \sum_{w\in W'_m} (q_0(x+w) - q_0(w))$$

where $W'_m = W_m \setminus \{0\}$, converges normally in $\mathbb{R}^{m+1} \setminus W_m$.

The meromorphic function thus defined in $\mathbb{R}^{m+1}$ — and hence forth denoted by $\mathrm{COT}^{(m)}(x)$ — generalizes the classical normalized cotangent function

$$\pi \operatorname{cotg} \pi z = \frac{1}{z} + \sum_{k\in\mathbb{Z}'} \left(\frac{1}{z+k} - \frac{1}{k} \right).$$

It may be proved that $\mathrm{COT}^{(m)}$ is odd, has poles of minimal order in the lattice points of W_m and zeros in the points in between, namely in the set

$$\left(\mathbb{Z}+\frac{1}{2}\right)w_1+\cdots+\left(\mathbb{Z}+\frac{1}{2}\right)w_m.$$

It is an example of an m-periodic meromorphic function.

Remark 3.22. Besides the function $\mathrm{COT}^{(m)}$, S. Kraußhar succeeded in introducing by means of generalized Eisenstein series, whole classes of p-periodic meromorphic functions in $\mathbb{R}^{m+1}$, $1 \leq p \leq m+1$, thus generalizing the classical classes of trigonometric and elliptic functions (see [25]).

Remark 3.23. Recently, G. Laville and I. Ramadanoff introduced the class of so-called holomorphic cliffordian functions, i.e. functions which are nullsolutions in an open subset of $\mathbb{R}^{2m+1}$ of the operator $D_x\Delta_x^m$. In this setting they also defined the notion of elliptic functions (see [27]).

3.6. Integral transforms. Let Ω be a domain in $\mathbb{R}^{m+1}$ with boundary $\partial\Omega=\Sigma$. In this subsection, monogenicity is w.r.t. the Dirac operator ∂_x.

The Cauchy transform $\mathcal{C}_\Sigma$. For f belonging to a suitable class of functions defined a.e. on Σ, its *Cauchy transform* $\mathcal{C}_\Sigma f$ is given by

$$\mathcal{C}_\Sigma f(x)=\int_\Sigma E(x-y)d\sigma_y f(y)=\int_\Sigma E(x-y)n(y)f(y)dS(y).$$

It defines a monogenic function in $\mathbb{R}^{m+1}\setminus\Sigma$.
This is the case e.g. if $f\in L_p(\Sigma)$, $1<p<+\infty$, (see [19]).

The Cauchy transform of distributions in $\mathbb{R}^{m+1}$. As $E\in L_1^{loc}(\mathbb{R}^{m+1})$, for each distribution $\mathcal{T}\in\mathcal{E}'(\mathbb{R}^{m+1})$, $E*\mathcal{T}$ is well defined and $\partial_x(E*\mathcal{T})=\mathcal{T}$. This implies that $\partial_x(E*\mathcal{T})=0$ in $\mathbb{R}^{m+1}\setminus[\mathcal{T}]$, $[\mathcal{T}]$ being the suppport of $\mathcal{T}$. Hence $E*\mathcal{T}$ determines a monogenic function in $\mathbb{R}^{m+1}\setminus[\mathcal{T}]$. It is called the *Cauchy transform* of $\mathcal{T}$.

Example 3.24 (Borel measures)**.** Let μ be a $\mathbb{C}_{m+1}$-valued regular Borel measure in $\mathbb{R}^{m+1}$ with compact support $[\mu]$. Then its Cauchy transform $E*\mu$ defines an $L_1^{loc}(\mathbb{R}^{m+1})$-function which is monogenic in $\mathbb{R}^{m+1}\setminus[\mu]$. Moreover, for a.e. $x\in\mathbb{R}^{m+1}$,

$$E*\mu(x)=\int_{\mathbb{R}^{m+1}}E(x-y)\,d\mu(y)$$

with $\lim_{x\to\infty}E*\mu(x)=0$.

Example 3.25 (The Théodoresco transform)**.** Let $G\subset\mathbb{R}^{m+1}$ be a bounded domain with piecewise smooth boundary $\partial G=\Sigma$, let $f\in L_p(\overline{G})$, $1\leq p\leq+\infty$ and consider $\tilde{f}=f\chi_G$, χ_G being the characteristic function of G. Then its Cauchy transform $E*\tilde{f}\in L_p^{loc}(\mathbb{R}^{m+1})$ and it is monogenic in $\mathbb{R}^{m+1}\setminus\overline{G}$ with

$$E*\tilde{f}(x)=\int_{\mathbb{R}^{m+1}}E(x-y)\tilde{f}(y)\,dy=\int_G E(x-y)f(y)\,dy.$$

It is common to call

$$T_G f = E * \tilde{f}$$

the *Théodoresco transform* of f.

Its properties have been intensively studied (see e.g. [20]). Notice e.g. that for suitable f

$$\partial_x T_G f = f,$$

which implies that T_G is a right inverse for ∂_x. Hence the inhomogeneous Dirac equation

$$\partial_x f = g$$

admits the solution $f = T_G g$.

4. Spherical monogenics and series expansions

In classical harmonic analysis on the unit circle S^1, the sequence $(e^{ik\theta})_{k\in\mathbb{Z}}$ forms a generating set for $L_2(S^1)$. For each $k \in \mathbb{Z}$, $e^{ik\theta}$ is the restriction of z^k to S^1, where the sequence $(z^k)_{k\in\mathbb{Z}}$ is a generating set for the Laurent series of each function f holomorphic in an open annular neighbourhood of S^1. Furthermore, z^k, $k \in \mathbb{N}$, is a holomorphic homogeneous polynomial of degree k, while z^{-k}, $k \in \mathbb{N}_0$, is a holomorphic homogeneous function of order $-k$ in $\mathbb{C} \setminus \{0\}$. It thus seems natural to look for analogues of the functions z^k, $k \in \mathbb{Z}$, in Clifford analysis. We investigate this problem in the case of the Dirac operator ∂_x.

4.1. Spherical monogenics: definitions. In what follows all functions we are considering are $\mathbb{C}_{m+1}$-valued.

Definition 4.1. *A homogeneous monogenic polynomial P_k of degree k, $k \in \mathbb{N}$, in $\mathbb{R}^{m+1}$ will be called an (inner) spherical monogenic of order k. The set of all such polynomials is denoted by $M^+(k)$.*

In Section 3.1 we have seen that the polynomials $V_{\underline{\alpha}}(x)$, $|\underline{\alpha}| = k$, are homogeneous of degree k and monogenic, whence $V_{\underline{\alpha}}(x) \in M^+(k)$.

Let us recall that for any multi index $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ with $|\underline{\alpha}| = k$,

$$(\underline{x}^{\underline{\alpha}})^* = \underline{\alpha}! V_{\underline{\alpha}}(x).$$

Definition 4.2. *For each $k \in \mathbb{N}$, we put $M^-(k) = I^* M^+(k)$, I^* being the inversion introduced in Section 3.3.*

By definition, for $Q_k \in M^-(k)$ there exists $P_k \in M^+(k)$ such that

$$Q_k(x) = \frac{\overline{x}}{|x|^{m+1}}\, P_k\left(\frac{x}{|x|^2}\right).$$

We have seen that Q_k is a homogeneous monogenic function of order $-(m+k)$. Each element of $M^-(k)$ is called an (outer) spherical monogenic of order k. Notice that, as $I^{*2} = -\mathbf{1}$, $I^* M^-(k) = M^+(k)$. Notice also that

$$I^* 1 = \frac{\overline{x}}{|x|^{m+1}} = A_{m+1} E(x)$$

and that for each multi index $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, the functions $W_{\underline{\alpha}}(x) = (-1)^{|\underline{\alpha}|} \partial^{\underline{\alpha}} E(x)$ all belong to $M^-(k)$ for $|\underline{\alpha}| = k$.

Obviously, $M^+(k)$ and $M^-(k)$ are right $\mathbb{C}_{m+1}$-modules.

We have

Theorem 4.3. *Let $k \in \mathbb{N}$. Then*

(i) *$(V_{\underline{\alpha}}(x) : |\underline{\alpha}| = k)$ is a basis for $M^+(k)$.*
(ii) *$(W_{\underline{\alpha}}(x) : |\underline{\alpha}| = k)$ is a basis for $M^-(k)$.*

As $M^+(k) = CK(\underline{\mathcal{P}}(k))$, $\underline{\mathcal{P}}(k)$ being the right $\mathbb{C}_{m+1}$-module of homogeneous polynomials of degree k in $\underline{x} = (x_1, \ldots, x_m)$, we thus obtain that

$$\dim M^+(k) = \dim M^-(k) = K(m; k)$$

with

$$K(m, k) = \frac{(k+m-1)!}{k!(m-1)!}$$

4.2. Inner products on $\mathcal{P}(k)$. Let for $k \in \mathbb{N}$ fixed, $\mathcal{P}(k)$ be the set of all homogeneous $\mathbb{C}_{m+1}$-valued polynomials in $\mathbb{R}^{m+1}$. Then $\mathcal{P}(k)$ contains the important submodules $H(k)$ and $M^+(k)$ consisting of, respectively, all harmonic and monogenic homogeneous polynomials of degree k with

$$M^+(k) \subset H(k) \subset \mathcal{P}(k).$$

As is well known, for $k \geq 2$,

$$\mathcal{P}(k) = H(k) \oplus r^2 \mathcal{P}(k-2)$$

leading to

$$\mathcal{P}(k) = \sum_{j=0}^{[k/2]} \oplus\, r^{2j} H(k-2j).$$

We now wish to refine this important decomposition and derive a splitting of (solid) spherical harmonics into (inner) spherical monogenics. To this end we introduce two inner products on $\mathcal{P}(k)$.

The inner product $\langle \cdot, \cdot \rangle_k$. For $P(x) = \sum_{|\alpha|=k} x^\alpha a_\alpha$ and $Q(x) = \sum_{|\beta|=k} x^\beta b_\beta$ belonging to $\mathcal{P}(k)$, we put

$$\langle P, Q \rangle_k = \left[\sum_{|\alpha|=k} \alpha!\, \overline{a}_\alpha b_\alpha \right]_0 = \left[\overline{P\left(\frac{\partial}{\partial x}\right)}\, Q \right]_0$$

where

$$P\left(\frac{\partial}{\partial x}\right) = \sum_{|\alpha|=k} \partial^\alpha \, a_\alpha,$$

i.e. $P(\partial/\partial x)$ is the differential operator obtained by replacing x_i by ∂_{x_i} in $P(x)$.

For $P \in \mathcal{P}(k-1)$ and $Q \in \mathcal{P}(k)$, we so have that

$$\langle xP, Q\rangle_k = \langle P, (-\partial_x)Q\rangle_{k-1}$$

whence the multiplication operator $x\colon \mathcal{P}(k-1) \to \mathcal{P}(k)$ with $P(x) \to xP(x)$ has $(-\partial_x)$ as its adjoint. It thus follows that

$$\mathcal{P}(k) = M^+(k) \oplus x\mathcal{P}(k-1),$$

the sum being orthogonal.

We have

Theorem 4.4 (Fischer decomposition)**.** *Let* $k \in \mathbb{N}$*. Then*

(i) $\mathcal{P}(k) = \sum_{s=0}^k \, x^s M^+(k-1)$,
(ii) $H(k) = M^+(k) \oplus xM^+(k-1)$.

Remark 4.5. Let $S_k \in H(k)$. Then due to the Fischer decomposition (i) of Theorem 4.4,

$$S_k = P_k + x \, p_{k-1}$$

where $P_k \in M^+(k)$ and $p_{k-1} \in \mathcal{P}(k-1)$.

Part (ii) of Theorem 4.4 tells us that p_{k-1} in fact belongs to $M^+(k-1)$. Indeed, it may even be proved that

$$p_{k-1} = \frac{-1}{m+1+2k} \, \partial_x S_k.$$

Remark 4.6. As follows directly from Theorem 4.4 (ii), $p_{k-1} \in M^+(k-1)$ implies that $xp_{k-1} \in H(k)$. It may be proved that if for $p_{k-1} \in \mathcal{P}(k-1)$, $xp_{k-1} \in H(k)$, then $p_{k-1} \in M^+(k-1)$.

Remark 4.7. For a Fischer type decomposition of homogeneous polynomials in the case of a q-deformed Dirac equation, we refer to [45].

The inner product on $L_2(S^m)$. Let, as usual, S^m be the unit sphere in $\mathbb{R}^{m+1}$. For functions $f, g \in L_2(S^m)$, we put

$$(f,g) = \int_{S^m} \overline{f(\omega)} g(\omega) \, dS(\omega).$$

Considering $L_2(S^m)$ as a right $\mathbb{C}_{m+1}$-module, we have that for$f, g \in L_2(S^m)$

$$(f,g) = \overline{(g,f)}$$

and for $a, b \in \mathbb{C}_{m+1}$

$$(fa, gb) = \overline{a}(f,g,)b.$$

Furthermore, $(\cdot, \cdot)$ induces a norm on $L_2(S^m)$ by putting

$$\|f\|^2 = [(f, f)]_0 = \int_{S^m} [\overline{f(\omega)} f(\omega)]_0 \, dS(\omega) = \int_{S^m} |f(\omega)|^2 \, dS(\omega).$$

Of course, this norm is also derived from the $\mathbb{C}$-valued inner product

$$(f, g)_0 = [(f, g)]_0.$$

Definition 4.8. *Let $k \in \mathbb{N}$. Then $\mathcal{M}^+(k)$ and $\mathcal{M}^-(k)$ are the spaces consisting of the restrictions to S^m of, respectively, the elements belonging to $M^+(k)$ and $M^-(k)$. Putting $\mathcal{M}(k) = \mathcal{M}^+(k) + \mathcal{M}^-(k)$, the elements of $\mathcal{M}(k)$ are called spherical monogenics of degree k. Arbitrary elements of $\mathcal{M}^+(k)$ and $\mathcal{M}^-(k)$ are denoted by P_k and Q_k.*

In a classical way, $\mathcal{H}(k)$ stands for the space of spherical harmonics of degree k, i.e. the space of the restrictions to S^m of the elements belonging to $H(k)$.

Obviously, $\mathcal{M}^+(k)$, $\mathcal{M}^-(k)$ and $\mathcal{H}(k)$ are all subspaces of $L_2(S^m)$.

From the relation

$$H(k) = M^+(k) \oplus x M^+(k-1)$$

it immediately follows by restriction to S^m that

$$\mathcal{H}(k) = \mathcal{M}^+(k) \oplus \omega \mathcal{M}^+(k-1).$$

But, taking into account that for each $P_k \in M^+(k)$,

$$Q_k(x) = I^* P_k(x) = \frac{\overline{x}}{|x|^{m+1}} \, P_k\left(\frac{x}{|x|^2}\right),$$

we find that for each $Q_k \in \mathcal{M}^-(k)$, $Q_k(\omega) = \overline{\omega} P_k(\omega)$.

Hence $\omega \mathcal{M}^+(k-1) = \mathcal{M}^-(k-1)$ and so for $k \in N$ fixed,

$$\mathcal{H}(k) = \mathcal{M}^+(k) \oplus \mathcal{M}^-(k-1).$$

Here it is understood that $\mathcal{M}^-(-1) = \{0\}$.

The following theorem tells us much more about the interrelationship of the spaces considered. To this end, let us first recall that the elements $S_k \in \mathcal{H}(k)$ are the (only) eigenfunctions of the Laplace-Beltrami operator Δ_ω — considered as an essentially self adjoint operator on $L_2(S^m)$ — with

$$\Delta_\omega S_k(\omega) = (-k)(k + m - 1) S_k(\omega), \quad k \in \mathbb{N}.$$

But we have more, namely

Theorem 4.9. *The operator Γ_ω is essentially self-adjoint on $L_2(S^m)$. Moreover for each $k \in \mathbb{N}$,*

(i) $\Gamma_\omega P_k(\omega) = (-k) P_k(\omega)$, $P_k \in \mathcal{M}^+(k)$,
(ii) $\Gamma_\omega Q_{k-1}(\omega) = (k + m - 1) Q_{k-1}(\omega)$, $Q_{k-1} \in \mathcal{M}^-(k-1)$,
(iii) $\mathcal{H}(k) = \mathcal{M}^+(k) \oplus_\perp \mathcal{M}^-(k-1)$.

Remark 4.10. As is well known, each $f \in L_2(S^m)$ admits a decomposition in spherical harmonics:

$$f(\omega) = \sum_{k=0}^{\infty} \mathbb{S}_k f(\omega)$$

where $\mathbb{S}_k f \in \mathcal{H}(k)$, $k \in \mathbb{N}$.

Denoting the decomposition of $\mathbb{S}_k f$ in spherical monogenics by

$$\mathbb{S}_k f(\omega) = \mathbb{P}_k f(\omega) + \mathbb{Q}_{k-1} f(\omega)$$

where $\mathbb{P}_k f \in \mathcal{M}^+(k)$, $\mathbb{Q}_{k-1} f \in \mathcal{M}^-(k-1)$, we thus obtain for $f \in L_2(S^m)$ the decomposition into spherical monogenics:

$$f(\omega) = \sum_{k=0}^{\infty} \mathbb{P}_k f(\omega) + \sum_{k=0}^{\infty} \mathbb{Q}_k f(\omega).$$

Remark 4.11. As for $k \neq l$, $\mathcal{H}(k) \perp \mathcal{H}(l)$, we thus also have that $\mathcal{M}^+(k) \perp \mathcal{M}^+(l)$ and $\mathcal{M}^-(k) \perp \mathcal{M}^-(l)$.

4.3. Series expansions. From the preceding paragraphs, it becomes clear that the role of the sequences $(e^{ik\theta})_{k\in\mathbb{N}}$ and $(e^{-ik\theta})_{k\in\mathbb{N}_0}$ in classical Fourier analysis on the unit circle, is taken over by the sequences of spherical monogenics $\mathcal{M}^+(k)$ and $\mathcal{M}^-(k-1)$, $k \in \mathbb{N}$, where by definition we put $\mathcal{M}^-(-1) = \{0\}$. One may thus expect that the local behaviour of a monogenic function will be governed by the sequences $M^+(k)$ and $M^-(k-1)$, $k \in \mathbb{N}$, of respectively, inner and outer spherical monogenics.

We have

Theorem 4.12 (Taylor expansion)**.** *Let f be monogenic in $\overset{\circ}{B}(R)$. Then there exists a sequence $(\mathbb{P}_k f)_{k\in\mathbb{N}}$ of inner spherical monogenics such that in $\overset{\circ}{B}(R)$*

$$f(x) = \sum_{k=0}^{\infty} \mathbb{P}_k f(x),$$

where for each $k \in \mathbb{N}$

$$\mathbb{P}_k f(x) = \frac{1}{A_{m+1}} \frac{|x|^k}{r^k} \int_{S^m} C^+_{m+1,k}(\xi,\omega) f(r\omega)\, dS(\omega),$$

$0 < r < R$ being arbitrarily chosen.

Moreover, this series converges normally on each $\overline{B}((1-\varepsilon)R)$.

Theorem 4.13 (Laurent expansion)**.** *Let f be monogenic in the open annular domain $G = \overset{\circ}{B}(R_2) \setminus \overline{B}(R_1)$. Then there exist sequences $(\mathbb{P}_k f)_{k\in\mathbb{N}}$ and $(\mathbb{Q}_k f)_{k\in\mathbb{N}}$ of inner and outer spherical monogenics respectively such that in G*

$$f(x) = \sum_{k=0}^{\infty} \mathbb{P}_k f(x) + \sum_{k=0}^{\infty} \mathbb{Q}_k f(x),$$

the convergence being normal on each closed annular subdomain $\overline{B}(r_2') \setminus \overset{\circ}{B}(r_1')$ *of* G.

Moreover

$$\begin{aligned} f_1(x) &= \sum_{k=0}^{\infty} \mathbb{P}_k f(x) \in M(\overset{\circ}{B}(R_2)), \\ f_2(x) &= \sum_{k=0}^{\infty} \mathbb{Q}_k f(x) \in M(\mathbb{R}^{m+1} \setminus \overline{B}(R_1)) \end{aligned}$$

with $\lim_{x\to\infty} f_2(x) = 0$.

Furthermore

$$\begin{aligned} \mathbb{P}_k f(x) &= \frac{1}{A_{m+1}} \frac{|x|^k}{r^k} \int_{S^m} C^+_{m+1,k}(\xi,\omega) f(r\omega)\, dS(\omega), \\ \mathbb{Q}_k f(x) &= \frac{1}{A_{m+1}} \frac{r^{m+k}}{|x|^{m+k}} \int_{S^m} C^-_{m+1,k}(\xi,\omega) f(r\omega)\, dS(\omega), \end{aligned}$$

$0 < R_1 < r < R_2$ *being taken arbitrarily.*

Remark 4.14. For the functions appearing in the integral representations of the projections $\mathbb{P}_k f$ and $\mathbb{Q}_k f$ on $M^+(k)$ and $M^-(k)$, we have

$$C^-_{m+1,k}(\omega,\xi)\overline{\xi}\omega = C^+_{m+1,k}(\xi,\omega)$$

where

$$\begin{aligned} C^-_{m+1,k}(\omega,\xi) &= \frac{1}{m-1} \Big[(k+1) C^{(m-1)/2}_{k+1}(t) \\ &\qquad + (1-m) C^{(m+1)/2}_{k}(t)((\xi_0 \underline{\omega} - \omega_0 \underline{\xi}) e_0 + \underline{\omega} \wedge \underline{\xi}) \Big]. \end{aligned}$$

Here C^ν_k stands for the Gegenbauer polynomial of degree k associated to ν and $t = \langle \xi, \omega \rangle = \xi \bullet \omega$, the Euclidean inner product between $\xi, \omega \in S^m$.

Remark 4.15. As may be expected, the series expansions are obtained by applying Cauchy's integral formula and considering appropriate expansions of the Cauchy kernel function $E(y-x)$. To this end, we wish to point out that for $|x| < R < |y|$,

$$E(y-x) = \sum_{k=0}^{\infty} \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x) W_{\underline{\alpha}}(y) = \sum_{k=o}^{\infty} \sum_{|\underline{\alpha}|=k} W_{\underline{\alpha}}(y) V_{\underline{\alpha}}(x).$$

Remark 4.16. It is clear that if f is monogenic in an open annular domain G about $a \in \mathbb{R}^{m+1}$, say $G = \overset{\circ}{B}(a, R_2) \setminus (a, R_1)$, then f admits in G a Laurent series

$$f(x) = \sum_{k=0}^{\infty} \mathbb{P}^{(a)}_k f(x) + \sum_{k=0}^{\infty} \mathbb{Q}_k f(x).$$

If $a \in \mathbb{R}^{m+1}$ is an isolated singular point of the monogenic function f, i.e. there exists $R > 0$ such that f is monogenic in $\overset{\circ}{B}(a, R) \setminus \{a\}$, then a is said to be a *pole of order* p for f io in the Laurent series of f about a, $\mathbb{Q}_k^{(a)} f(x) \equiv 0$ for all $k \geq p$.

Finally a function f is called *meromorphic* in an open subset $\Omega \subset \mathbb{R}^{m+1}$ if there exists a subset S of Ω such that

(i) S has no accumulation point in Ω,
(ii) f is monogenic in $\Omega \setminus S$,
(iii) f has a pole at each point of S.

It is clear that 0 is a pole of order $k+1$ for all $Q_k \in M^-(k)$, in particular for $W_{\underline{\alpha}}(x)$, $|\underline{\alpha}| = k$.

Remark 4.17. Taylor's expansion immediately yields

Theorem 4.18 (Liouville)**.** *Let f be monogenic in $\mathbb{R}^{m+1}$ such that for some $C > 0$, $|f(x)| \leq C$. Then f is a constant function.*

Remark 4.19. The local behaviour (about the origin) of a monogenic function may thus be described by the space of formal Laurent series

$$\sum_{k=0}^{\infty} \oplus M^+(k) \oplus \sum_{k=0}^{\infty} \oplus M^-(k).$$

Algebraically, the first part, $\sum_{k=0}^{\infty} \oplus M^+(k)$ contains all possible monogenic polynomials. The second part, $\sum_{k=0}^{\infty} \oplus M^-(k)$ defines a special class of rational functions, namely with poles at 0 and vanishing at ∞. Combining both parts gives rise to the class of rational functions with poles at 0.

These function classes are of particular importance in establishing Runge approximation theorems for functions monogenic on compact subsets of $\mathbb{R}^{m+1}$ (see [4]).

Remark 4.20. The space $M^+(k)$ of spherical monogenics of degree k is an irreducible Spin$(m+1)$-module under the natural action of Spin$(m+1)$. The Taylor series $\sum_{k=0}^{\infty} \oplus M^+(k)$ thus yields the decompopsition of solutions of the Dirac equation into elements of irreducible Spin$(m+1)$-modules. It gives rise to the study of classes of conformally invariant first order operators, among which the Rarita-Schwinger operator associated to higher Spin representations (see [7], [8], and [9]) For an approach to Clifford analysis as a study of invariant operators, we refer to [47].

Remark 4.21. A straightforward approach to Spin-groups and spherical functions can be found in [52]. The same article also describes how, using the notion of symmetric product, a Weierstrass' power series approach to monogenic function theory may be obtained.

4.4. More about formal Laurent series. The formal series

$$\sum_{k=0}^{\infty} \oplus M^+(k) + \sum_{k=0}^{\infty} \oplus M^-(k)$$

can be put into a nice scheme.

Let us first recall that the Fischer decomposition for $\mathcal{P}(k)$, $k \in \mathbb{N}$, fixed, reads:

$$\mathcal{P}(k) = \sum_{S=0}^{k} \oplus\, x^S M^+(k-1)$$

where

$$H(k) = M^+(k) \oplus xM^+(k-1).$$

Now putting $\mathcal{Q}(k) = I^*\mathcal{P}(k),\ k \in \mathbb{N}$ fixed, we claim that

$$\mathcal{Q}(k) = \sum_{s=0}^{k} \oplus\, x^{-s} M^-(k-s) \tag{4.1}$$

where

$$I^*H(k) = M^-(k) \oplus x^{-1}M^-(k-1) \tag{4.2}$$

Indeed, as to (4.1) it suffices to observe that

$$\mathcal{P}(k) = M^+(k) + x\mathcal{P}(k-1)$$

and that for $P \in \mathcal{P}(k-1)$,

$$I^*(xP)(x) = -x^{-1}I^*P(x),$$

whence

$$I^*\mathcal{P}(k) = M^-(k) \oplus x^{-1}\mathcal{Q}(k-1).$$

Iteration on k yields the desired result.

As for (4.2), notice that if $P_{k-1} \in M^+(k-1)$, then $I^*P_{k-1} = Q_{k-1} \in M^-(k-1)$ by definition.

For $k \geq s$, a direct calculation results into

$$I^*(x^s M^+(k-s)) = x^{-s}M^-(k-s).$$

It should be noticed that none of the elements belonging to $x^{-s}M^-(k-s)$, $k \geq s$, is harmonic (besides the constant zero function) and this in contrast with the property that if $P_k \in M^+(k)$, then xP_k is harmonic and belongs to $H(k+1)$.

In order to preserve harmonicity, one should of course consider $\mathcal{K}H(k)$ instead of $I^*H(k)$. Notice that, as straightforward calculations show,

$$\mathcal{K}x = -I^*$$

and so, using the fact that $\mathcal{K}^2 = \mathbf{1}$,

$$x = -\mathcal{K}I^*. \tag{4.3}$$

Observe that by means of relation (4.3), an elegant proof may be given of the property that f monogenic in Ω implies xf harmonic in Ω.

Finally notice that

$$\mathcal{K}H(k) = xM^-(k) \oplus M^-(k-1).$$

The foregoing considerations lead to the following scheme

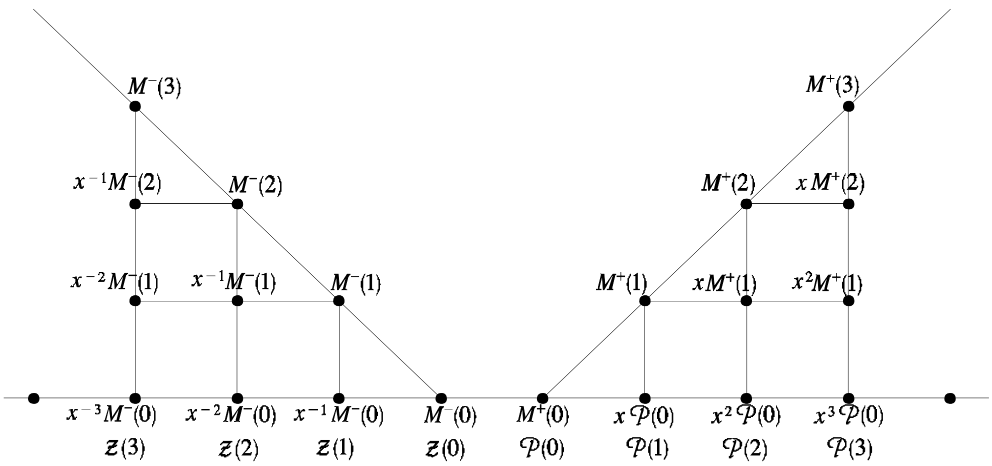

FIGURE 1

Let us have a closer look at the right hand block. As for each $k \in \mathbb{N}$ fixed,

$$\begin{aligned} \mathcal{P}(k) &= \text{range}\left(x|_{\mathcal{P}(k-1)}\right) \oplus ker\partial_x, \\ \mathcal{P}(k-1) &= \text{range}\left(\partial_x|_{\mathcal{P}(k)}\right) \end{aligned}$$

and

$$\partial_x(xM^+(k-1)) = M^+(k-1),$$

the corresponding horizontal lines describe stepwise the injectivity of the multiplication operators x (from the left to the right), respectively, the surjectivity of ∂_x (from the right to the left).

Again for $k \in \mathbb{N}$ fixed, the corresponding vertical line gives the Fischer decomposition of $\mathcal{P}(k)$ in terms of spherical monogenics. Moreover, in each vertical line, the highest two knots yield the components of the decomposition of the space $H(k)$ into $M^+(k) \oplus xM^+(k-1)$.

In the left hand block, for $k \in \mathbb{N}$ fixed, each vertical line represents the decomposition of $\mathcal{Q}(k)$ and the two highest knots in it yield the decomposition of $I^*H(k)$ into $M^-(k) \oplus x^{-1}M^-(k-1)$.

Part III. Clifford algebras in harmonic analysis

Classical H^p-theory on the unit disc D in $\mathbb{C}$ has been the subject of intensive research ever since its origin in 1915 when, following considerations made by G. H. Hardy, F. Riesz introduced the subspace $H^p(D)$ of $\mathcal{O}(D)$ — the space of holomorphic functions in D — consisting of those $F \in \mathcal{O}(D)$ satisfying

$$\sup_{0<r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{i\theta})|^p \, d\theta < +\infty, \qquad 1 < p < +\infty.$$

In an analogous way, the space $H^p(\mathbb{C}^+)$,$1 < p < +\infty$, consists of those $F \in \mathcal{O}(\mathbb{C}^+)$ — $\mathbb{C}^+$ being the upper half space in $\mathbb{C}$ — such that

$$\sup_{y>0} \int_{-\infty}^{+\infty} |F(x+iy)|^p \, dy < +\infty.$$

It turns out that $H^p(\Omega)$-theory ($\Omega = \mathbb{C}^+$ or D) is strongly related to $L_p(\partial\Omega)$-boundary value theory and in particular to singular integral operator theory in $L_p(\partial\Omega)$, more precisely to the study of the Hilbert transform on $L_p(\partial\Omega)$.

In the beginning of the 1980's (see [12]), the boundedness was proved of the Hilbert transform H_Σ on $L_2(\Sigma), \Sigma$ being a Lipschitz curve in $\mathbb{C}$ and this was the starting point of considering analogous higher dimensional problems within the setting of Clifford analysis (see e.g. [19], [29], [31]).

In this section we present some basic results obtained in this context, thus connecting Clifford analysis and classical harmonic analysis.

For the classical real variable approach to H^p-spaces we refer to [48].

1. Hardy spaces: general situation

Unless stated otherwise, all functions considered in this section are $\mathbb{C}_{m+1}$-valued.

Let $\Sigma \subset \mathbb{R}^{m+1}$ be the graph of a Lipschitz function $g\colon \mathbb{R}^m \to \mathbb{R}$ and call $\Omega^\pm$ the domains in $\mathbb{R}^{m+1}$ which lie above, respectively, below Σ. Let furthermore $n(y)$ stand for the outward unit normal on Σ at $y \in \Sigma$ ($n(y)$ is a.e. defined on Σ) and $dS(y)$ for the elementary surface element on Σ. For convenience we put $\Omega^+ = \Omega$.

Let $1 < p < +\infty$. Then we call

$$H^p(\Omega^\pm) = \{F \in M(\Omega^\pm) : \sup_{\delta>0} \int_\Sigma |F(y \pm \delta)|^p \, dS(y) < +\infty\}$$

the *Hardy space* of (left) monogenic functions in $\Omega^\pm$, monogenicity being w.r.t. the Cauchy-Riemann operator D_x.

For $f \in L_p(\Sigma)$ and $x \in \mathbb{R}^{m+1} \setminus \Sigma$,

$$\mathcal{C}_\Sigma f(x) = \int_\Sigma E(x-y) n(y) f(y) \, dS(y)$$

the *Cauchy transform* of f.

For $f \in L_p(\Sigma)$ and a.e. $x \in \Sigma$,

$$\begin{aligned} H_\Sigma f(x) &= 2\,\text{P.v.}\int_\Sigma E(x-y)n(y)f(y)\,dS(y) \\ &= 2\lim_{\varepsilon\to 0+}\int_{\{y\in\Sigma:|x-y|>\varepsilon\}} E(x-y)n(y)f(y)\,dS(y) \end{aligned}$$

the *Hilbert transform* of f.

We have

Theorem 1.1. *Let $f \in L_p(\Sigma)$,$1 < p < +\infty$. Then*

(i) $\mathcal{C}_\Sigma f \in H^p(\Omega^\pm)$.

(ii) *$\mathcal{C}_\Sigma f$ has non-tangential limits $(\mathcal{C}_\Sigma f)^\pm$ at almost all $x^* \in \Sigma$, i.e.*

$$\lim_{\substack{x \overset{\text{non-tang}}{\longrightarrow} x^* \\ x\in\Omega^\pm}} \mathcal{C}_\Sigma f(x) = (\mathcal{C}_\Sigma f)^\pm(x^*)$$

(iii) *Putting*

$$\begin{aligned} \mathbb{P}^+_\Sigma f(x^*) &= (\mathcal{C}_\Sigma f)^+(x^*), \\ \mathbb{P}^-_\Sigma f(x^*) &= -(\mathcal{C}_\Sigma f)^-(x^*), \end{aligned}$$

then $\mathbb{P}^\pm_\Sigma$ are bounded projections in $L_p(\Sigma)$.

(iv) *(Plemelj-Sokhotzki Formulae) For a.e. $x^* \in \Sigma$,*

$$\begin{aligned} \mathbb{P}^+_\Sigma f(x^*) &= \frac{1}{2}(f(x^*) + H_\Sigma f(x^*)), \\ \mathbb{P}^-_\Sigma f(x^*) &= \frac{1}{2}(f(x^*) - H_\Sigma f(x^*)) \end{aligned}$$

whence

$$\begin{aligned} \mathbf{1} &= \mathbb{P}^+_\Sigma + \mathbb{P}^-_\Sigma, \\ H_\Sigma &= \mathbb{P}^+_\Sigma - \mathbb{P}^-_\Sigma. \end{aligned}$$

In particular H_Σ is a bounded linear operator on $L_p(\Sigma)$ and, putting $L^\pm_p(\Sigma) = \mathbb{P}^\pm_\Sigma L_p(\Sigma)$,

$$L_p(\Sigma) = L^+_p(\Sigma) \oplus L^-_p(\Sigma).$$

Notice also that, as for any $u, v \in \mathbb{R}^{m+1}$, $[\overline{u}v]_0 = u \bullet v$, we have that if for any $\mathbb{R}$-valued $f \in L_p(\Sigma)$, we define for a.e. $x \in \Sigma$ the integral transform $H_{\Sigma_0} f$ by

$$H_{\Sigma_o} f(x) = [H_\Sigma f(x)]_0 = \frac{2}{A_{m+1}}\,\text{P.v.}\int_\Sigma \frac{(x-y)\bullet n(y)}{|x-y|^{m+1}} f(y)\,dS(y),$$

then $H_{\Sigma_0} f$ is also $\mathbb{R}$-valued.

This gives rise to the so-called *singular double-layer potential operators* H_{Σ_0} and $\mathbb{P}^+_{\Sigma_0}$ on Σ, where $\mathbb{P}^+_{\Sigma_0} = \frac{1}{2}(\mathbf{1} + H_{\Sigma_0})$. Both operators H_{Σ_0} and $\mathbb{P}^+_{\Sigma_0}$ are bounded linear operators on $L_p(\Sigma)$.

For details concerning the proofs of the results mentioned, we refer e.g. to [29] and [31].

Remark 1.2 (Riemann-Hilbert Problems)**.** Let Σ and Ω^+ be as before. Furthermore, let a, b, and h be given in $L_p(\Sigma)$. Then the following boundary value problem is a straightforward generalization of the classical RH-problem in $\mathbb{C}$: "Find w monogenic in $\mathbb{R}^{m+1} \setminus \Sigma$ such that on Σ, $aw^+ - bw^- = h$."

Here, for a.e. $y \in \Sigma$,

$$w^{\pm}(y) = \lim_{\substack{x \to y \\ x \in \Omega^{\pm}}} w(x).$$

Using the Plemelj-Sokhotzki formulae, the boundary value condition may be reformulated as: "Find $u \in L_p(\Sigma)$ such that on Σ, $a\mathbb{P}^+u + b\mathbb{P}^-u = h$", or, putting $c = (a+b)/2$ and $d = (a-b)/2$, as: "Find $u \in L_p(\Sigma)$ such that on Σ, $cu + dH_\Sigma u = h$."

This problem and related ones were intensively studied in the last years (see e.g. [2] and its references).

2. The case $\mathbb{R}^m$

Put for $\mathbf{x} = (x_0, x) \in \mathbb{R}^{m+1} = \mathbb{R} \times \mathbb{R}^m$,

$$\mathbf{x} = x_0 + x = x_0 + \sum_{j=1}^{m} \varepsilon_j x_j,$$

and associate with it the Cauchy-Riemann operator

$$D_{\mathbf{x}} = \partial_{x_0} + \sum_{j=1}^{m} \varepsilon_j \partial_{x_j},$$

with fundamental solution

$$E(\mathbf{x}) = \frac{1}{A_{m+1}} \frac{\overline{\mathbf{x}}}{|\mathbf{x}|^{m+1}} = \frac{1}{A_{m+1}} \frac{x_0 + \sum_{j=1}^{m} x_j \overline{\varepsilon_j}}{\left({x_0}^2 + \sum_{j=1}^{m} {x_j}^2\right)^{(m+1)/2}}.$$

Moreover, put $\mathbb{R}^{m+1}_{\pm} = \{(x_0, x) : x_0 \lesssim 0\}$.

Define for $f \in L_p(\mathbb{R}^m)$,$1 < p < +\infty$, its Cauchy transform $\mathcal{C}f$ by

$$\mathcal{C}f(\mathbf{x}) = \int_{\mathbb{R}^m} E(\mathbf{x} - y) f(y)\, dS(y), \qquad \mathbf{x} \in \mathbb{R}^{m+1} \setminus \mathbb{R}^m,$$

and for a.e. $x \in \mathbb{R}^m$, its Hilbert transform $\mathcal{H}f$ by

$$\begin{aligned} \mathcal{H}f(x) &= \lim_{\varepsilon \to 0^+} \frac{2}{A_{m+1}} \int_{\{y \in \mathbb{R}^m : |x-y| > \varepsilon\}} \frac{\overline{x-y}}{|x-y|^{m+1}} f(y)\, dS(y) \\ &= \sum_{j=1}^{m} \overline{\varepsilon_j} R_j f(x), \end{aligned}$$

R_j being the j-the Riesz transform in $\mathbb{R}^m$, $j = 1, \ldots, m$.

The Cauchy transform $\mathcal{C}f$ is monogenic in $\mathbb{R}^{m+1} \setminus \mathbb{R}^m$ (w.r.t. D_x) and it belongs to $H^p(\mathbb{R}^{m+1}_\pm)$. Furthermore the Hilbert transform $\mathcal{H}$ is bounded on $L_p(\mathbb{R}^m)$, $1 < p < +\infty$.

Writing $E(\mathbf{x})$ as

$$E(\mathbf{x}) = \frac{1}{A_{m+1}} \frac{\overline{\mathbf{x}}}{|\mathbf{x}|^{m+1}} = \frac{1}{2} P_{x_0}(x) + \frac{1}{2} Q_{x_0}(x)$$

where

$$P_{x_0}(x) = \frac{1}{A_{m+1}} \frac{2x_0}{\left({x_0}^2 + \sum_{j=1}^m {x_j}^2 \right)^{(m+1)/2}}$$

is the Poisson kernel for $\mathbb{R}^{m+1}_+$ and

$$Q_{x_0}(x) = \frac{2}{A_{m+1}} \sum_{j=1}^m \frac{x_j \overline{\varepsilon_j}}{\left({x_0}^2 + \sum_{j=1}^m {x_j}^2 \right)^{(m+1)/2}}$$

the conjugate Poisson kernel, we thus have that e.g. for $\mathbf{x} = (x_0, x) \in \mathbb{R}^{m+1}_+$,

$$\mathcal{C}f(x_0, x) = \frac{1}{2}((P_{x_0} * f)(x) + (Q_{x_0} * f)(x)).$$

Consequently

$$\mathbb{P}^+ f(x) = BV^+ \mathcal{C}f(x) = \lim_{x_0 \to 0^+} \mathcal{C}f(x_0, x) = \frac{1}{2}(f(x) + \mathcal{H}f(x))$$

and, analogously,

$$\mathbb{P}^- f(x) = -BV^- \mathcal{C}f(x) = -\lim_{x_0 \to 0^+} \mathcal{C}f(-x_0, x) = \frac{1}{2}(f(x) - \mathcal{H}f(x)).$$

Remark 2.1. For an $\mathbb{R}$-valued $f \in L_p(\mathbb{R}^m)$, we have that $\mathcal{C}f(x_0, x)$ is $\mathbb{R} \oplus \mathbb{R}^m$-valued.

Putting

$$\begin{aligned} u_0 &= \frac{1}{2} P_{x_0} * f, \\ u_j &= \frac{1}{2} Q^{(j)}_{x_0} * f, \qquad j = 1, \ldots, m \end{aligned}$$

where

$$Q^{(j)}_{x_0}(x) = \frac{1}{A_{m+1}} \frac{2x_j}{({x_0}^2 + \sum_{i=1}^m {x_i}^2)^{(m+1)/2}},$$

we so have that in $\mathbb{R}^{m+1}_+$,

$$\mathcal{C}f(\mathbf{x}) = u_0(\mathbf{x}) + \sum_{j=1}^m \bar{\varepsilon}_j u_j(\mathbf{x}).$$

It thus follows from $D_{\mathbf{x}}\mathcal{C}f = 0$ in $\mathbb{R}^{m+1}_+$ that $(u_0, u_1, \ldots, u_m)$ is a set of conjugate harmonic functions in the sense of Stein-Weiss (see [49]).

Remark 2.2. Defining the principal value kernel $\mathrm{Pv}\,(x/|x|^{m+1})$ in $\mathbb{R}^m$ by

$$\mathrm{Pv}\left(\frac{x}{|x|^{m+1}}\right) = \frac{2}{A_{m+1}}\frac{\overline{x}}{|x|^{m+1}},$$

we thus have that for $f \in L_p(\mathbb{R}^m)$, $1 < p < +\infty$,

$$\mathcal{H}f = \mathrm{Pv}\left(\frac{x}{|x|^{m+1}}\right) * f.$$

In the sense of distributions we so obtain that

$$\mathbb{P}^+ f = \frac{1}{2}\left(\delta + \mathrm{Pv}\left(\frac{x}{|x|^{m+1}}\right)\right) * f = \delta^+ * f.$$

Here δ^+, with

$$\delta^+ = \frac{1}{2}\left(\delta + \mathrm{Pv}\left(\frac{x}{|x|^{m+1}}\right)\right),$$

generalizes the Heisenberg delta-function to $\mathbb{R}^m$.

Using the fact that

$$\mathcal{F}^-\left(\frac{2x_j}{A_{m+1}|x|^{m+1}}\right) = \frac{1}{i}\frac{y_j}{|y|}, \qquad j = 1, \ldots, m,$$

putting (see also [29])

$$\chi_\pm(y) = \frac{1}{2}\left(1 \pm i\frac{y}{|y|}\right)$$

and noticing that

$${\chi_\pm}^2 = \chi_\pm, \qquad \chi_+ + \chi_- = 1,$$

we can now formulate the following characterisation of $\mathbb{P}^+ L_p(\mathbb{R}^m) = L_p^+(\mathbb{R}^m)$.

Theorem 2.3. *For $f \in L_p(\mathbb{R}^m)$ are equivalent:*

(i) $BV^+\mathcal{C}f = f$,
(ii) $\mathcal{H}f = f$,
(iii) $\mathcal{F}^- f = \chi_+ \mathcal{F}^- f$,
(iv) $f = \delta^+ * f$.

This theorem should be compared with its classical version in complex analysis.

3. The case S^m

Put for $\mathbf{x} = (x_0, x) \in \mathbb{R}^{m+1}$, $\mathbf{x} = \sum_{j=0}^m e_j x_j$ and associate with it the Dirac operator $\partial_{\mathbf{x}} = \sum_{j=0}^m e_j \partial_{x_j}$, with fundamental solution

$$E(\mathbf{x}) = \frac{1}{A_{m+1}} \frac{\overline{\mathbf{x}}}{|\mathbf{x}|^{m+1}} = \frac{1}{A_{m+1}} \frac{\sum_{j=0}^m x_j \overline{e_j}}{|\mathbf{x}|^{m+1}}.$$

Passing to polar coordinates we have $\mathbf{x} = r\xi$ where $r = |\mathbf{x}|$ and $\xi \in S^m$.
Define for $f \in L_p(S^m)$, $1 < p < +\infty$, its Cauchy transform $\mathcal{C}f$ by

$$\mathcal{C}f(\mathbf{x}) = \frac{1}{A_{m+1}} \int \frac{\overline{\omega - \mathbf{x}}}{|\omega - \mathbf{x}|^{m+1}} \omega f(\omega)\, dS(\omega), \qquad \mathbf{x} \in \mathbb{R}^{m+1} \setminus S^m.$$

and for a.e. $\xi \in S^m$, its Hilbert transform $\mathcal{H}f$ by

$$\begin{aligned} Hf(\xi) &= 2 \lim_{\varepsilon \to 0^+} \frac{1}{A_{m+1}} \int_{\{\omega \in S^m : |\xi - \omega| > \varepsilon\}} \frac{\overline{\omega - \xi}}{|\xi - \omega|^{m+1}} \omega f(\omega)\, dS(\omega) \\ &= 2 \lim_{\varepsilon \to 0^+} \frac{1}{A_{m+1}} \int_{\{\omega \in S^m : |\xi - \omega| > \varepsilon\}} \frac{1 + \xi\omega}{|1 + \xi\omega|^{m+1}} f(\omega)\, dS(\omega). \end{aligned}$$

Notice that

$$\mathcal{C}f(\mathbf{x}) = f^+(\mathbf{x}) + f^-(\mathbf{x})$$

where

$$\begin{aligned} f^+(\mathbf{x}) &= Y(1 - |\mathbf{x}|)\mathcal{C}f(\mathbf{x}) \in M(\mathring{B}(1)), \\ f^-(\mathbf{x}) &= Y(|\mathbf{x}| - 1)\mathcal{C}f(\mathbf{x}) \in M(\mathbb{R}^{m+1} \setminus \bar{B}(1)), \end{aligned}$$

Y being the Heaviside function.
So we have

$$\begin{aligned} \mathbb{P}^+ f(\xi) &= BV^+ \mathcal{C}f(\xi) = \lim_{\substack{r \to 1 \\ <}} f^+(r\xi) = \frac{1}{2}(f(\xi)) + \mathcal{H}f(\xi)) \\ \mathbb{P}^- f(\xi) &= -BV^- \mathcal{C}f(\xi) = -\lim_{\substack{r \to 1 \\ <}} f^-(r\xi) = \frac{1}{2}(f(\xi)) - \mathcal{H}f(\xi)). \end{aligned}$$

Remark 3.1. We have seen in Part II, Section 4.3 that in the case $p = 2$, each $f \in L_2(S^m)$ admits a decomposition in spherical monogenics:

$$f(\xi) = \sum_{k=0}^{\infty} \mathbb{P}_k f(\xi) + \sum_{k=0}^{\infty} \mathbb{Q}_k f(\xi).$$

We then have that

$$\begin{aligned} \mathbb{P}^+ f(\xi) &= \sum_{k=0}^{\infty} \mathbb{P}_k f(\xi), \\ \mathbb{P}^- f(\xi) &= \sum_{k=0}^{\infty} \mathbb{Q}_k f(\xi). \end{aligned}$$

Putting $H^2(S^m) = \mathbb{P}^+ L_2(S^m)$, a closed subspace of $L_2(S^m)$ is obtained, isomorphic to the Hardy space $H^2(\overset{\circ}{B}(1))$ consisting of those $F \in M(\overset{\circ}{B}(1))$ such that

$$\sup_{0<r<1} \int_{S^m} |F(r\omega)|^2 \, dS(\omega) < +\infty.$$

The space $H^2(\overset{\circ}{B}(1))$ is a Hilbert space with reproducing kernel the Szegő-kernel

$$S_+(\omega, \mathbf{x}) = \frac{1}{A_{m+1}} \frac{1 - \omega\overline{\mathbf{x}}}{|1 - \omega\overline{\mathbf{x}}|^{m+1}}, \qquad (\omega, \mathbf{x}) \in S^m \times \overset{\circ}{B}(1).$$

We have

Theorem 3.2. *For $f \in L_2(S^m)$ are equivalent:*

(i) $f \in H^2(S^m)$,
(ii) $\mathcal{H}f = f$,
(iii) $f = \sum_{k=0}^{\infty} \mathbb{P}_k f$.

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