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CONSTRUCTION OF THE SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR THE BIHARMONIC OPERATOR IN A RECTANGLE

by N. ARONSZAJN, R. D. BROWN and R. S. BUTCHER

Introduction.

The present paper is the completion and extension of Technical Report 23 (new series) issued in 1970 under the same title and by the same authors.

The subject of the paper is the development of a technique for constructing solutions of the equation

$$\Delta^2 u = F \tag{0.1.1}$$

in the open rectangle $R_{a,b} = \{(x, y) : |x| < a, \ |y| < b\}$, subject to the boundary conditions

$$u = \varphi, \ \frac{\partial u}{\partial n} = \psi \ , \tag{0.1.2}$$

on $\partial R_{a,b}$. This problem, which gives the position of equilibrium for a clamped rectangular plate, has of course been extensively studied in the literature (see, e.g., the references in [10] ; for reasons of space only those references needed in our development are included here in the bibliography). The novelty of the technique presented here is that the construction yields an approximation procedure with both a priori and a posteriori error estimates.

In the case of a clamped plate the solution u represents the (small) deflection of the plate, and it follows from the fact that the strain energy of the plate due to twisting and the strain energy due

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to the bending moments in the direction of the x and y axes are finite [10] that all second order partial derivatives of u are square integrable. It is shown in § 1 that this latter fact implies that u is a Bessel potential of second order in $R_{a,b}$, and that this in turn imposes certain necessary conditions on the assigned functions F, φ, and ψ. (The definitions and facts concerning Bessel potentials which are needed here are all given in § 1.)

If F, φ, and ψ satisfy the above mentioned conditions, then a solution $u \in P^2(R_{a,b})$ does in fact exist and is unique.

Since the first decade of the century (see S. Zaremba [11]) it has been known that the problem of solving (0.1) reduces to the problem of finding the orthogonal projection of a function w in $L^2(R_{a,b})$ onto the closed subspace $\mathcal{H}$ of $L^2(R_{a,b})$ which consists of all square integrable functions which are harmonic in $R_{a,b}$. The problem of finding such a projection would be trivial if one knew an explicitly given complete orthogonal basis for $\mathcal{H}$. Such bases are known, e.g., for the space of those square integrable functions harmonic in a circle, and for the space of those harmonic in an ellipse. No such basis is known, however, in the case of a rectangle.

Nevertheless, one may try to decompose $\mathcal{H}$ into the sum of two closed (not necessarily orthogonal) subspaces, $\mathcal{H} = \mathcal{H}^{(1)} + \mathcal{H}^{(2)}$, such that for each of the spaces $\mathcal{H}^{(i)}$ a complete orthogonal basis is known. Such a decomposition is possible if and only if $\mathcal{H}^{(1)} + \mathcal{H}^{(2)}$ is dense in $\mathcal{H}$ and the minimal angle between $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ is positive. If this is the case, a projection formula from [4] can be used to express the projection P of $L^2(R_{a,b})$ onto $\mathcal{H}$ in terms of the projections $P^{(1)}$ and $P^{(2)}$ onto $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ respectively. If, in addition, $\mathcal{H}^{(1)} \cap \mathcal{H}^{(2)} = (0)$, this projection formula enables one to establish very convenient approximations to the projection P in the uniform operator topology.

If one follows this approach, there are several seemingly promising ways in which one can attempt to decompose $\mathcal{H}$. In [3] the first author tried this approach taking for $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ the harmonic functions which are extendable in the horizontal direction as periodic functions of period $2a$ and in the vertical direction as periodic functions of period $2b$ respectively. Denote by $\mathcal{H}_{\mu\nu}$ the subspace of $\mathcal{H}$ consisting of those functions of parity μ in x and parity ν in y, where $\mu = 0,1$ and $\nu = 0,1$. It was shown in [3] that the minimal angle between $\mathcal{H}^{(1)}_{\mu\nu}$ and $\mathcal{H}^{(2)}_{\mu\nu}$ is positive for $\mu = \nu$, but is zero for $\mu \neq \nu$. Hence this attempt was only partially successful.

In the same paper [3] another possible decomposition of $\mathcal{H}$ was mentioned ; namely ; by taking for $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ those functions in $\mathcal{H}$ which vanish on the vertical and on the horizontal boundaries respectively. At that time, however, the first author mistakenly believed that the results for this decomposition would be similar to those for the previously mentioned case and therefore did not pursue the investigation. It was only a couple of years ago that the authors of the present paper noticed that actually, if $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ are chosen in this last mentioned way, then the minimal angle between $\mathcal{H}^{(1)}_{\mu\nu}$ and $\mathcal{H}^{(2)}_{\mu\nu}$ is positive for all $\mu = 0,1$ and $\nu = 0,1$.

Evaluating the minimal angle between these subspaces comprises the most involved part of the present paper.

The reduction of the original problem to the construction of P is carried out in § 1 (where the conditions on u, F, φ and ψ are also specified).

The projection formulas needed and the corresponding error estimates are given in § 2. The decomposition of $\mathcal{H}$ is given in § 3, where the decomposition theorem (Theorem 3.1) is stated. This theorem not only gives the desired decomposition of $\mathcal{H}$ but also gives bounds for the cosines needed in the above mentioned error estimates. A corollary to the decomposition theorem (Corollary 3.1) gives, for functions in $\mathcal{H}$, representation formulas corresponding to the decomposition in Theorem 3.1.

The proof of Theorem 3.1 is given in § 4. This proof requires two estimates, which are stated in Lemma 4.1. One estimate is proved in § 5 ; the other, in § 6. In § 7 all the preceding results are combined to obtain (in some detail) an approximation to the solution u of (0.1), together with a priori error bounds. In § 8 a posteriori error bounds are derived, and methods are discussed by which they can be used to improve the procedures of § 7.

The a priori estimates are as usual very pessimistic – they cannot take into account the multiple cancellations in the approximating expressions. It may happen, e.g., that to approximate the solution to within one figure accuracy according to the a priori estimates would require the solution of a linear system of one thousand equations for one thousand unknowns. Therefore from a practical point of view the procedure described in § 8 using only a posteriori error estimates would be much preferable-especially since the a posteriori error estimate is quite precise.

We make one additional comment : there are some other problems in partial differential equations where finding the solution reduces to finding the orthogonal projection onto a suitable subspace V of a functional Hilbert space. Suppose that no complete orthogonal basis is known for V, but that V can be decomposed into a direct (non-orthogonal) sum, $V = V^{(1)} + V^{(2)}$, where an explicit complete orthogonal basis is known for each of the closed subspaces $V^{(1)}$ and $V^{(2)}$. If, in addition, the (necessarily positive) minimal angle between $V^{(1)}$ and $V^{(2)}$ can be evaluated, then one can apply *mutatis mutandis* all the approximation procedures of § 8 involving a posteriori error estimates. The techniques used in § 7 to derive the a priori error estimates, on the other hand, even though theoretically applicable to this general situation, may lead to even less practical evaluations than those of the present paper.

In an Appendix at the end of the paper we give results of numerical computations where we used the a posteriori evaluations of the errors.

1. Statement and reduction of the problem.

As noted in the introduction, certain facts concerning Bessel potentials are needed in order to state precisely the conditions which the functions F, φ, and ψ in equations (0.1) must satisfy. We state here only those definitions and results needed for the problem under discussion ; for a complete development of the theory of Bessel potentials see [8], [2], [1].

The space $P^2(R_{a,b})$ of Bessel potentials of second order on $R_{a,b}$ is the perfect functional completion of the space of restrictions of functions in $C_0^\infty(R^2)$ to $R_{a,b}$ with respect to the norm

$$\|u\|_2^2 = \int_{-a}^{a}\int_{-b}^{b}\left[|u(x,y)|^2 + 2\left|\frac{\partial u}{\partial x}\right|^2 + 2\left|\frac{\partial u}{\partial y}\right|^2 + \left|\frac{\partial^2 u}{\partial x^2}\right|^2 + 2\left|\frac{\partial^2 u}{\partial x\,\partial y}\right|^2 + \left|\frac{\partial^2 u}{\partial y^2}\right|^2\right] dy\,dx \,. \tag{1.1}$$

The elements of $P^2(R_{a,b})$ are precisely those functions $u \in L^2(R_{a,b})$ such that

1) u is continuous on $R_{a,b}$,

2) $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ exist pointwise a.e. in $R_{a,b}$ and are equal almost everywhere to functions absolutely continuous on almost all lines parallel to the coordinate axes,

3) $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial x\, \partial y}$, and $\frac{\partial^2 u}{\partial y^2}$ belong to $L^2(R_{a,b})$.

Thus, as noted in the introduction, it is natural for applications to require that the solution u of (0.1) belong to $P^2(R_{a,b})$. Note also that formula (1.1) is valid for all $u \in P^2(R_{a,b})$.

Let $u \in P^2(R_{a,b})$. Then Δu belongs to $L^2(R_{a,b})$ and if we take the Laplacian of Δu in the sense of distributions we obtain that $\Delta^2 u$ belongs to the space

$$H^{-2}(R_{a,b}) = \{F : F = \Delta f \text{ for some } f \in L^2(R_{a,b})\} .$$

$H^{-2}(R_{a,b})$ includes not only every function in $L^1(R_{a,b})$ and in $L^2(R_{a,b})$, considered as the density of a measure, but also every finite Borel measure on $R_{a,b}$. Moreover, if $F \in H^{-2}(R_{a,b})$ then one may obtain an $f \in L^2(R_{a,b})$ such that $\Delta f = F$ by taking the convolution, defined in an appropriate way, of F with the function $(2\pi)^{-1} \log(x^2 + y^2)^{-1/2}$. We therefore assume :

Condition 1. – The function F in (0.1.1) belongs to $H^{-2}(R_{a,b})$.

We next consider the conditions on φ and ψ in (0.1.2). If $u \in P^2(R_{a,b})$, then boundary values for $u, \frac{\partial u}{\partial x}$, and $\frac{\partial u}{\partial y}$ may be defined using Bessel potentials defined on open intervals of $\mathbf{R}^1$.

Accordingly, let $I = (\alpha, \beta)$ be a bounded open interval in $\mathbf{R}^1$. The space $P^{1/2}(I)$ of Bessel potentials of order 1/2 on I is the perfect functional completion of the space of restrictions of functions in $C_0^\infty(\mathbf{R}^1)$ to I with respect to the norm

$$\|f\|_{1/2}^2 = \int_\alpha^\beta |f(x)|^2\, dx + \frac{1}{2\pi} \int_\alpha^\beta \int_\alpha^\beta \frac{|f(x) - f(y)|^2}{|x - y|^2}\, dx\, dy\, . \tag{1.2}$$

Every function $f \in P^{1/2}(I)$ is defined at every point of I except on a subset of I of 1-capacity zero [8] (in particular, therefore, f is defined a.e. on I), belongs to $L^2(I)$, and has norm given by (1.2). Moreover, every function $f \in L^2(I)$ such that $\|f\|_{1/2} < \infty$ is equal a.e. to a function $\tilde{f} \in P^{1/2}(I)$ ($\tilde{f}$ is a "correction" of f [2]).

The space $P^{3/2}(I)$ of Bessel potentials of order 3/2 on I is the perfect functional completion of the space of restrictions of functions in $C_0^\infty(R^1)$ to I with respect to the norm

$$\|f\|_{3/2}^2 = \|f\|_{1/2}^2 + \|f'\|_{1/2}^2 \tag{1.3}$$

and consists of those functions $f \in L^2(I)$ which are continuous and have a derivative f' which is in $P^{1/2}(I)$. Equation (1.3) holds for every $f \in P^{3/2}(I)$.

Next, let $u \in P^2(R_{a,b})$. Then u can be extended to a function $\hat{u} \in P^2(R_{2a,2b})$. The boundary $\partial R_{a,b}$ of $R_{a,b}$, minus the corners, consists of four open segments

$$\begin{aligned} I_1 &= \{(a, y) : |y| < b\}, \quad I_2 = \{(x, b) : |x| < a\}, \\ I_3 &= \{(-a, y) : |y| < b\}, \quad I_4 = \{(x, -b) : |x| < a\}, \end{aligned} \tag{1.4}$$

and the restrictions of $\hat{u}$ to each of these intervals is a Bessel potential of order 3/2, while the restrictions of $\frac{\partial \hat{u}}{\partial x}$ and $\frac{\partial \hat{u}}{\partial y}$ are Bessel potentials of order 1/2. In this way we obtain boundary values for $u, \frac{\partial u}{\partial x}$, and $\frac{\partial u}{\partial y}$ on each I_j, $j = 1, \ldots, 4$. Moreover, these boundary values are independent of the particular extension $\hat{u}$ used to define them.

It is clear therefore that in order that there exist a function $u \in P^2(R_{a,b})$ such that (0.1.2) holds it is necessary that, on each of the segments I_j which make up the boundary of $R_{a,b}$, φ must be in $P^{3/2}$ and ψ must be in $P^{1/2}$. These conditions are not by themselves sufficient to insure the existence of such a u. However, necessary *and* sufficient conditions follow from results concerning manifolds with singularities of polyhedral type, of which $\partial R_{a,b}$ is one of the simplest examples([1]), and are as follows :

([1]) These results will be included in [9].

Condition 2. – The functions φ, ψ in (0.1.2) are such that :

a) φ is continuous on $\partial R_{a,b}$,

b) The restrictions of φ, ψ to any of the four segments (1.4) making up the boundary of $R_{a,b}$ belong to $P^{3/2}$, $P^{1/2}$ respectively.

c) The following integrals are all finite :

$$\int_{-a}^{a}\int_{-b}^{b} \frac{|\psi(a, y) - \frac{\partial\varphi}{\partial x}(x, b)|^2}{|x - y|^2} dy\, dx, \quad \int_{-a}^{a}\int_{-b}^{b} \frac{|\psi(x, b) - \frac{\partial\varphi}{\partial y}(a, y)|^2}{|x - y|^2} dy\, dx,$$

$$\int_{-a}^{a}\int_{-b}^{b} \frac{|\psi(-a, y) + \frac{\partial\varphi}{\partial x}(x, b)|^2}{|x - y|^2} dy\, dx, \quad \int_{-a}^{a}\int_{-b}^{b} \frac{|\psi(x, b) - \frac{\partial\varphi}{\partial y}(-a, y)|^2}{|x - y|^2} dy\, dx,$$

$$\int_{-a}^{a}\int_{-b}^{b} \frac{|\psi(-a, y) + \frac{\partial\varphi}{\partial x}(x, -b)|^2}{|x - y|^2} dy\, dx, \quad \int_{-a}^{a}\int_{-b}^{b} \frac{|\psi(x, -b) + \frac{\partial\varphi}{\partial y}(-a, y)|^2}{|x - y|^2} dy\, dx,$$

$$\int_{-a}^{a}\int_{-b}^{b} \frac{|\psi(a, y) - \frac{\partial\varphi}{\partial x}(x, -b)|^2}{|x - y|^2} dy\, dx, \quad \int_{-a}^{a}\int_{-b}^{b} \frac{|\psi(x, -b) + \frac{\partial\varphi}{\partial y}(a, y)|^2}{|x - y|^2} dy\, dx.$$

Moreover, if condition 2 holds, then a function $\nu \in P^2(R_{a,b})$ can be constructed such that $\nu = \varphi$ and $\frac{\partial \nu}{\partial n} = \psi$ on $\partial R_{a,b}$ (see [9] ; for specific problems an ad hoc construction of ν may be simpler, however).

Remark 1.1. – If a) and b) hold, then c) in condition 2 is equivalent to the following condition, which may be easier to verify :

c′) For $\varepsilon > 0$ sufficiently small, the following integrals are all finite :

$$\int_0^{\varepsilon} \frac{|\psi(a, b - t) - \frac{\partial\varphi}{\partial x}(a - t, b)|^2}{t} dt, \quad \int_0^{\varepsilon} \frac{|\psi(a - t, b) - \frac{\partial\varphi}{\partial y}(a, b - t)|^2}{t} dt,$$

$$\int_0^{\varepsilon} \frac{|\psi(-a, b - t) + \frac{\partial\varphi}{\partial x}(a - t, b)|^2}{t} dt, \quad \int_0^{\varepsilon} \frac{|\psi(a - t, b) - \frac{\partial\varphi}{\partial y}(-a, b - t)|^2}{t} dt,$$

$$\int_0^\varepsilon \frac{|\psi(-a,b-t)+\frac{\partial\varphi}{\partial x}(a-t,-b)|^2}{t}\,dt,\ \int_0^\varepsilon \frac{|\psi(a-t,-b)+\frac{\partial\varphi}{\partial y}(-a,b-t)|^2}{t}\,dt,$$

$$\int_0^\varepsilon \frac{|\psi(a,b-t)-\frac{\partial\varphi}{\partial x}(a-t,-b)|^2}{t}\,dt,\ \int_0^\varepsilon \frac{|\psi(a-t,-b)+\frac{\partial\varphi}{\partial y}(a,b-t)|^2}{t}\,dt.$$

We assume conditions 1 and 2 and seek a solution $u \in \mathrm{P}^2(\mathrm{R}_{a,b})$ of (0.1). For the construction of u we shall make use of the following well known facts. Let $\mathcal{H}$ be the closed subspace of $\mathrm{L}^2(\mathrm{R}_{a,b})$ consisting of those functions in $\mathrm{L}^2(\mathrm{R}_{a,b})$ which are harmonic in $\mathrm{R}_{a,b}$, P be the orthogonal projection of $\mathrm{L}^2(\mathrm{R}_{a,b})$ onto $\mathcal{H}$, and $\mathcal{H}^\perp$ be the orthogonal complement of $\mathcal{H}$ in $\mathrm{L}^2(\mathrm{R}_{a,b})$. Let $\mathcal{P}_0$ be the space consisting of those functions $\nu \in \mathrm{P}^2(\mathrm{R}_{a,b})$ such that $\nu = 0$ on $\partial\mathrm{R}_{a,b}$. For every $\nu \in \mathcal{P}_0$, $\Delta\nu \in \mathrm{L}^2(\mathrm{R}_{a,b})$, and this defines a one to one mapping of $\mathcal{P}_0$ onto $\mathrm{L}^2(\mathrm{R}_{a,b})$. The inverse mapping G of $\mathrm{L}^2(\mathrm{R}_{a,b})$ onto $\mathcal{P}_0$ can be given explicitly by means of the Green's function corresponding to the Laplace operator in $\mathrm{R}_{a,b}$. Formulas for this Green's function are well known ; it can be expressed either by an infinite series, or in closed form by means of elliptic functions. The function $\mathrm{G}f \in \mathcal{P}_0$ can be explicitly constructed whenever $f \in \mathrm{L}^2(\mathrm{R}_{a,b})$.

The map G becomes an isometry when $\mathcal{P}_0$ is provided with the norm

$$\|\nu\|^2 = \int_{-a}^{a}\int_{-b}^{b} |\Delta\nu|^2\,dy\,dx.$$

This norm is equivalent to the norm induced on $\mathcal{P}_0$ by the norm (1.1) on $\mathrm{P}^2(\mathrm{R}_{a,b})$. Thus $\mathrm{G}(\mathcal{H}^\perp) = \mathrm{G}(\mathcal{H})^\perp$ and the orthogonal decomposition $\mathrm{L}^2(\mathrm{R}_{a,b}) = \mathcal{H} \pm \mathcal{H}^\perp$ induces an orthogonal decomposition $\mathcal{P}_0 = \mathrm{G}(\mathcal{H}) \pm \mathrm{G}(\mathcal{H}^\perp)$. It is well known [11] that

$$\mathrm{G}(\mathcal{H}^\perp) = \left\{\nu \in \mathrm{P}^2(\mathrm{R}_{a,b}) : \nu = \frac{\partial\nu}{\partial n} = 0 \text{ on } \partial\mathrm{R}_{a,b}\right\}. \tag{1.5}$$

In order to construct the solution u of (0.1) we proceed as follows : Since the system (0.1) is linear over $\mathbf{R}^1$ we can without loss of generality consider from now on only real valued functions. By condition 1, there exists $f \in \mathrm{L}^2(\mathrm{R}_{a,b})$ such that $\mathrm{F} = \Delta f$, and, by condition 2, we can construct $v \in P^2(R_{a,b})$ such that $v = \varphi$ and $\frac{\partial v}{\partial n} = \psi$ on $\partial R_{a,b}$. Since f and Δv belong to $L^2(R_{a,b})$ we can define

$$g = (I - P)(f - \Delta v) .$$

Then $\Delta g = F - \Delta^2 v$, and from (1.5)

$$Gg = \frac{\partial}{\partial n} Gg = 0 \text{ on } \partial R_{a,b} .$$

Define

$$u = Gg + v = G(I - P)(f - \Delta v) + v . \tag{1.6}$$

Then $\Delta^2 u = \Delta g + \Delta^2 v = F$, while, on $\partial R_{a,b}$, $u = Gg + v = \varphi$ and $\frac{\partial u}{\partial n} = \frac{\partial}{\partial n} Gg + \frac{\partial v}{\partial n} = \psi$. Thus u is the desired solution, and it is clear from (1.6) that we can construct u explicitly provided we can construct the projection Pw of an arbitrary element $w \in L^2(R_{a,b})$ onto $\mathcal{H}$. We shall return to the construction of u in § 7 after more information about the projection P has been obtained.

2. Review of some projection formulas.

Let $\mathcal{L}$ be a Hilbert space and $\mathcal{H}_0$ be a closed subspace of $\mathcal{L}$. We recall some projection formulas which express the orthogonal projection P_0 of $\mathcal{L}$ onto $\mathcal{H}_0$ in terms of projections onto subspaces of $\mathcal{H}_0$. For proofs and further details see [4].

One simple and well known formula arises in case $\mathcal{H}_0$ is separable. For then there exists a complete orthonormal basis $\{u_n\}$ in $\mathcal{H}_0$ and

$$P_0 = \sum_{n=1}^{\infty} P_n , \tag{2.1}$$

where P_n is the orthogonal projection of $\mathcal{L}$ onto the subspace spanned by u_n : $P_n h = (h , u_n) u_n$ for $h \in \mathcal{L}$. In order that (2.1) be practical for computations, however, the orthonormal basis $\{u_n\}$ must be known explicitly and one must be able to evaluate the error committed by truncating the series in (2.1) to finitely many terms.

A similar but more complicated formula holds in case $\mathcal{H}_0$ is the closed direct sum of two subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$:

$$\mathcal{H}_0 = \mathcal{H}_1 \oplus \mathcal{H}_2 .$$

In this case P_0 is expressed in terms of the orthogonal projections P_1 and P_2 of $\mathcal{L}$ onto $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively by means of the following series :

$$P_0 = \sum_{n=1}^{\infty} [P_1(P_2P_1)^{n-1} + P_2(P_1P_2)^{n-1} - (P_2P_1)^n - (P_1P_2)^n] . \quad (2.2)$$

If the minimal angle θ between $\mathcal{H}_1$ and $\mathcal{H}_2$ is positive, then

$$P_0 - \sum_{n=1}^{m} [P_1(P_2P_1)^{n-1} + P_2(P_1P_2)^{n-1} - (P_2P_1)^n - (P_1P_2)^n] = $$
$$= [(P_0 - P_1)(P_0 - P_2)]^m + (P_2P_1)^m ,$$

and

$$\| [(P_0 - P_1)(P_0 - P_2)]^m \| \leqslant \cos^{2m-1}\theta , \ \| (P_2P_1)^m \| \leqslant \cos^{2m-1}\theta .$$

(The norm $\| \ \|$ here means the bound of the operator.) Thus we have the error estimate

$$\| P_0 - \sum_{n=1}^{m} [P_1(P_2P_1)^{n-1} + P_2(P_1P_2)^{n-1} - (P_2P_1)^n - (P_1P_2)^n] \| \leqslant$$
$$\leqslant 2\cos^{2m-1}\theta \quad (2.3)$$

In view of (2.3), (2.2) is practical for computations provided one knows P_1 and P_2 (or sufficiently good estimates for P_1 and P_2) and $\cos\theta$ (or a sufficiently good upper bound for $\cos\theta$).

Another useful inequality is obtained if (2.2) is rewritten in the form

$$P_0 = \sum_{n=0}^{\infty} (I - P_2)(P_1P_2)^n P_1 + \sum_{n=0}^{\infty} (I - P_1)(P_2P_1)^n P_2 .$$

Since for $h \in \mathcal{L}$,

$$\| (P_1P_2)^n P_1 h \| \leqslant (\cos\theta)^{2n} \| P_1 h \| , \ \| (P_2P_1)^n P_2 h \| \leqslant (\cos\theta)^{2n} \| P_2 h \| ,$$

therefore

$$\|P_0 h\| \leqslant \sum_{n=0}^{\infty} (\cos^2\theta)^n \,[\|P_1 h\| + \|P_2 h\|] =$$
$$= (1 - \cos^2\theta)^{-1}[\|P_1 h\| + \|P_2 h\|] .$$

Hence for every $h \in \mathfrak{L}$,

$$\max[\|P_1 h\| , \|P_2 h\|] \leqslant \|P_0 h\| \leqslant (1 - \cos^2\theta)^{-1}[\|P_1 h\| + \|P_2 h\|] . \tag{2.4}$$

In the remainder of this paper $\mathfrak{L}$ will be $L^2(R_{a,b})$ and the subspaces considered will be subspaces of $\mathcal{H}$.

3. The decomposition of $\mathcal{H}$.

Let $\mathcal{H}$ be the subspace of $L^2(R_{a,b})$ defined in § 1, and for $\mu = 0,1$ and $\nu = 0,1$ define

$$\mathcal{H}_{\mu\nu} = \{h : h \in \mathcal{H} \text{ and } h(x , y) \text{ is of parity } \mu \text{ in } x \text{ and } \nu \text{ in } y\}. \tag{3.1}$$

For any $h \in \mathcal{H}$ we have the decomposition $h(z) = \sum_{\mu=0}^{1} \sum_{\nu=0}^{1} h_{\mu\nu}(z)$, where $h_{\mu\nu}(z) = \frac{1}{4} \sum_{l=0}^{1} \sum_{k=0}^{1} (-1)^{\mu k + \nu l}\, h((-1)^k x + i(-1)^l y)$. Thus one sees easily that

$$\mathcal{H} = \sum_{\mu=0}^{1} \sum_{\nu=0}^{1}{}^{\perp} \mathcal{H}_{\mu\nu} , \tag{3.2}$$

where $\perp$ indicates the sum is orthogonal.

Next define the closed subspaces

$$\mathcal{H}_{\mu\nu}^{(1)} = \{h \in \mathcal{H}_{\mu\nu} : \text{as } x \to \pm a,\ h(x , y) \to 0 \text{ uniformly in } y \text{ on compact subsets of } (-b , b)\} , \tag{3.3}$$

$$\mathcal{H}_{\mu\nu}^{(2)} = \{h \in \mathcal{H}_{\mu\nu} : \text{as } y \to \pm b,\ h(x , y) \to 0 \text{ uniformly in } x \text{ on compact subsets of } (-a , a)\} . \tag{3.4}$$

The following theorem, proved in § 4, completes the decomposition of $\mathcal{H}$:

THEOREM 3.1. – *Let $\mathcal{H}_{\mu\nu}$, $\mathcal{H}^{(1)}_{\mu\nu}$, $\mathcal{H}^{(2)}_{\mu\nu}$ be defined by (3.1), (3.3), (3.4) respectively, and let $\theta_{\mu\nu}$ be the minimal angle between $\mathcal{H}^{(1)}_{\mu\nu}$ and $\mathcal{H}^{(2)}_{\mu\nu}$. Then for $\mu = 0,1$ and $\nu = 0,1$*

$$\mathcal{H}_{\mu\nu} = \mathcal{H}^{(1)}_{\mu\nu} \oplus \mathcal{H}^{(2)}_{\mu\nu} \tag{3.5}$$

and

$$\cos\theta_{\mu\nu} \leqslant (\mathrm{I}_\mu \mathrm{I}_\nu)^{1/2}, \tag{3.6}$$

where $0 \leqslant \mathrm{I}_0 \leqslant .80$ and $0 \leqslant \mathrm{I}_1 \leqslant 2/\pi$.

Any function $h_{\mu\nu}(x, y) \in \mathrm{L}^2(\mathrm{R}_{a,b})$ which is of parity μ in x and parity ν in y has a Fourier series expansion of the form

$$h_{\mu\nu}(x, y) \sim \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \gamma^{mn}_{\mu\nu}\, u_{\mu,m}(x)\, v_{\nu,n}(y),$$

where the $\gamma^{mn}_{\mu\nu}$'s are constants and for m, $n = 1, 2, \ldots$

$$\begin{cases} u_{0,m}(x) = \cos\left(\dfrac{2m-1}{2}\dfrac{\pi x}{a}\right), & v_{0,n}(y) = \cos\left(\dfrac{2n-1}{2}\dfrac{\pi y}{b}\right), \\ u_{1,m}(x) = \sin\dfrac{m\pi x}{a}, & v_{1,n}(y) = \sin\dfrac{n\pi y}{b}. \end{cases} \tag{3.7}$$

If in addition $h_{\mu\nu} \in \mathcal{H}_{\mu\nu}$, then this expansion takes a special form illustrating the decomposition $\mathcal{H}_{\mu\nu} = \mathcal{H}^{(1)}_{\mu\nu} \oplus \mathcal{H}^{(2)}_{\mu\nu}$:

COROLLARY 3.1. – *A function $h_{\mu\nu} \in \mathrm{L}^2(\mathrm{R}_{a,b})$ belongs to $\mathcal{H}_{\mu\nu}$ if and only if $h_{\mu\nu} = h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu}$, where $h^{(1)}_{\mu\nu} \in \mathcal{H}^{(1)}_{\mu\nu}$ and $h^{(2)}_{\mu\nu} \in \mathcal{H}^{(2)}_{\mu\nu}$ have Fourier expansions of the form*

$$h^{(1)}_{\mu\nu}(x, y) \sim \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+n} \frac{\alpha_\mu(m)\,\beta_\nu(n)}{\alpha_\mu(m)^2 + \beta_\nu(n)^2}\, a^{(m)}_{\mu\nu}\, u_{\mu,m}(x) v_{\nu,n}(y), \tag{3.8.1.$\mu\nu$}$$

$$h^{(2)}_{\mu\nu}(x, y) \sim \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+n} \frac{\alpha_\mu(m)\,\beta_\nu(n)}{\alpha_\mu(m)^2 + \beta_\nu(n)^2}\, b^{(n)}_{\mu\nu}\, u_{\mu,m}(x)\, v_{\nu,n}(y), \tag{3.8.2.$\mu\nu$}$$

with $a^{(m)}_{\mu\nu}$, $b^{(n)}_{\mu\nu}$ constants ; $u_{\mu,m}(x)$, $v_{\nu,n}(y)$ given by (3.7) ; and

$$\alpha_0(m) = \frac{2m-1}{2a}, \quad \alpha_1(m) = \frac{m}{a}, \quad \beta_0(n) = \frac{2n-1}{2b}, \quad \beta_1(n) = \frac{n}{b}.$$

The expansions $(3.8.1.\mu\nu)$ *and* $(3.8.2.\mu\nu)$ *can also be written in the form*

$$h_{\mu\nu}^{(1)} \sim \sum_{m=1}^{\infty} a_{\mu\nu}^{(m)} \mathrm{U}_m^{\mu\nu}, \tag{3.9.1.$\mu\nu$}$$

$$h_{\mu\nu}^{(2)} \sim \sum_{n=1}^{\infty} b_{\mu\nu}^{(n)} \mathrm{V}_n^{\mu\nu}, \tag{3.9.2.$\mu\nu$}$$

where

$$\mathrm{U}_m^{00}(x, y) = \frac{\pi}{2}\frac{b}{a}\frac{(-1)^{m-1}(2m-1)}{2\cosh\left(\frac{2m-1}{2}\frac{\pi b}{a}\right)}\cos\left(\frac{2m-1}{2}\frac{\pi x}{a}\right)\cosh\left(\frac{2m-1}{2}\frac{\pi y}{a}\right),$$

$$\mathrm{U}_m^{01}(x, y) = \frac{\pi}{2}\frac{b}{a}\frac{(-1)^{m-1}(2m-1)}{2\sinh\left(\frac{2m-1}{2}\frac{\pi b}{a}\right)}\cos\left(\frac{2m-1}{2}\frac{\pi x}{a}\right)\sinh\left(\frac{2m-1}{2}\frac{\pi y}{a}\right),$$

$$\mathrm{U}_m^{10}(x, y) = \frac{\pi}{2}\frac{b}{a}\frac{(-1)^{m-1}m}{\cosh\frac{m\pi b}{a}}\sin\frac{m\pi x}{a}\cosh\frac{m\pi y}{a},$$

$$\mathrm{U}_m^{11}(x, y) = \frac{\pi}{2}\frac{b}{a}\frac{(-1)^{m-1}m}{\sinh\frac{m\pi b}{a}}\sin\frac{m\pi x}{a}\sinh\frac{m\pi y}{a},$$

$$\mathrm{V}_n^{00}(x, y) = \frac{\pi}{2}\frac{a}{b}\frac{(-1)^{n-1}(2n-1)}{2\cosh\left(\frac{2n-1}{2}\frac{\pi a}{b}\right)}\cosh\left(\frac{2n-1}{2}\frac{\pi x}{b}\right)\cos\left(\frac{2n-1}{2}\frac{\pi y}{b}\right),$$

$$\mathrm{V}_n^{01}(x, y) = \frac{\pi}{2}\frac{a}{b}\frac{(-1)^{n-1}n}{\cosh\frac{n\pi a}{b}}\cosh\frac{n\pi x}{b}\sin\frac{n\pi y}{b},$$

$$\mathrm{V}_n^{10}(x, y) = \frac{\pi}{2}\frac{a}{b}\frac{(-1)^{n-1}(2n-1)}{2\sinh\left(\frac{2n-1}{2}\frac{\pi a}{b}\right)}\sinh\left(\frac{2n-1}{2}\frac{\pi x}{b}\right)\cdot\cos\left(\frac{2n-1}{2}\frac{\pi y}{b}\right),$$

$$\mathrm{V}_n^{11}(x, y) = \frac{\pi}{2}\frac{a}{b}\frac{(-1)^{n-1}n}{\sinh\frac{n\pi a}{b}}\sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b}.$$

It is well known that the sequences

$$\{U_m^{\mu\nu}\}_{m=1,2,\ldots} \text{ and } \{V_n^{\mu\nu}\}_{n=1,2,\ldots}$$

form complete orthogonal bases for the spaces $\mathcal{H}_{\mu\nu}^{(1)}$ and $\mathcal{H}_{\mu\nu}^{(2)}$ respectively. Thus, a function $h_{\mu\nu}^{(j)} \in L^2(R_{a,b})$ belongs to $\mathcal{H}_{\mu\nu}^{(j)}$ if and only if it has an expansion of the form (3.9.j.$\mu\nu$). Using the Fourier expansions

$$\sinh \alpha x \sim \frac{2}{\pi} (\sinh \alpha\pi) \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{\alpha^2 + n^2} \sin nx \;, \tag{3.10}$$

$$\cosh \alpha x \sim \frac{1}{\pi} (\cosh \alpha\pi) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1)}{\alpha^2 + \left(\frac{2n-1}{2}\right)^2} \cos\left(\frac{2n-1}{2}\, x\right), \tag{3.11}$$

which converge uniformly in x on compact subsets of $(-\pi, \pi)$, one sees easily that for such functions $h_{\mu\nu}^{(j)}$ expansion (3.9.j.$\mu\nu$) holds if and only if (3.8.j.$\mu\nu$) does. Thus, Corollary 3.1 follows immediately from Theorem 3.1.

It remains to prove Theorem 3.1.

4. The proof of Theorem 3.1.

The proof of Theorem 3.1 breaks down into several steps. We shall prove that for $\mu, \nu = 0,1$:

1) $\mathcal{H}_{\mu\nu}^{(1)} \cap \mathcal{H}_{\mu\nu}^{(2)} = (0)$; i.e., $\mathcal{H}_{\mu\nu}^{(1)} + \mathcal{H}_{\mu\nu}^{(2)}$ is a direct sum.

2) $\mathcal{H}_{\mu\nu}^{(1)} \dot{+} \mathcal{H}_{\mu\nu}^{(2)}$ is dense in $\mathcal{H}_{\mu\nu}$.

3) For every $h_{\mu\nu}^{(1)} \in \mathcal{H}_{\mu\nu}^{(1)}$ and $h_{\mu\nu}^{(2)} \in \mathcal{H}_{\mu\nu}^{(2)}$,

$$\| h_{\mu\nu}^{(1)} + h_{\mu\nu}^{(2)} \|^2 \geq (1 - (I_\mu I_\nu)^{1/2}) (\| h_{\mu\nu}^{(1)} \|^2 + \| h_{\mu\nu}^{(2)} \|^2) \;, \tag{4.1}$$

where $0 < I_0 \leq .8, \quad 0 < I_1 \leq \frac{2}{\pi}$.

It follows from (4.1) (see [7], [3], [5]) that the direct sum $\mathcal{H}_{\mu\nu}^{(1)} + \mathcal{H}_{\mu\nu}^{(2)}$ is closed in $\mathcal{H}$ and that $\cos\theta_{\mu\nu} \leq (I_\mu I_\nu)^{1/2}$, where $\theta_{\mu\nu}$ is the minimal angle between $\mathcal{H}_{\mu\nu}^{(1)}$ and $\mathcal{H}_{\mu\nu}^{(2)}$. Theorem 3.1 is thus immediate.

In order to prove 1) we note that if $h \in \mathcal{H}_{\mu\nu}^{(1)} \cap \mathcal{H}_{\mu\nu}^{(2)}$, then h can be extended by reflection to a function $\hat{h}$ which is harmonic in $R_{3a,3b}$, except possibly for isolated singularities at $(-a, -b)$, $(-a, b)$, $(a, -b)$, and (a, b). Since $h \in L^2(R_{a,b})$, therefore $\hat{h} \in L^2(R_{3a,3b})$ and these singularities, if they exist, can therefore only be logarithmic singularities. But for $|x| < a$ and $|y| < b$,

$$\hat{h}(\pm a, y) = \hat{h}(x, \pm b) = 0 .$$

Thus, for example,

$$\lim_{y \uparrow b} h(a, y) = 0 \neq \infty ,$$

so (a, b) is not a logarithmic singularity ; similarly for $(-a, b)$, $(a, -b)$, and $(-a, -b)$. It follows that $\hat{h}$ is harmonic in all of $R_{3a,3b}$. Hence h is harmonic in $R_{a,b}$, continuous in $\overline{R}_{a,b} = R_{a,b} \cup \partial R_{a,b}$, and equal to zero on $\partial R_{a,b}$; i.e., $h \equiv 0$.

In order to prove 2), we first note that, if $h_{\mu\nu} \in \mathcal{H}_{\mu\nu} \cap C(\overline{R}_{a,b})$, then the Green's function $G(x, y, x', y')$ for Laplace's equation in $R_{a,b}$ can be used to write $h_{\mu\nu}$ in the form

$$h_{\mu\nu} = h_{\mu\nu}^{(1)} + h_{\mu\nu}^{(2)} ,$$

where, with $\dfrac{\partial G}{\partial n}$ the exterior normal of G,

$$h_{\mu\nu}^{(1)}(x, y) = \int_{-b}^{b} \left[\frac{\partial G}{\partial n}(x, y, x', y')\, h_{\mu\nu}(x', y')\right] \Bigg|_{x' = -}^{x' = a} dy' \in \mathcal{H}_{\mu\nu}^{(1)} ,$$

$$h_{\mu\nu}^{(2)}(x, y) = \int_{-a}^{a} \left[\frac{\partial G}{\partial n}(x, y, x', y')\, h_{\mu\nu}(x', y')\right] \Bigg|_{y' = b}^{y' = -b} dx' \in \mathcal{H}_{\mu\nu}^{(2)} ,$$

Thus, $\mathcal{H}_{\mu\nu} \cap C(\overline{R}_{a,b}) \subset \mathcal{H}_{\mu\nu}^{(1)} + \mathcal{H}_{\mu\nu}^{(2)}$.

For arbitrary $h_{\mu\nu} \in \mathcal{H}_{\mu\nu}$, define $h_\alpha(z) = h_{\mu\nu}(\alpha z)$, $0 < \alpha < 1$. Then $h_\alpha \in \mathcal{H}_{\mu\nu} \cap C(\overline{R}_{a,b}) \subset \mathcal{H}_{\mu\nu}^{(1)} + \mathcal{H}_{\mu\nu}^{(2)}$, while (see e.g. the proof of Theorem 15 in [3]) $h_\alpha \to h_{\mu\nu}$ in $L^2(R_{a,b})$ as $\alpha \to 1$. Thus $\mathcal{H}_{\mu\nu}^{(1)} + \mathcal{H}_{\mu\nu}^{(2)}$ is dense in $\mathcal{H}_{\mu\nu}$.

The proof of 3) is somewhat more involved and makes use of the following lemma, proved in § 5 and § 6 :

LEMMA 4.1. – *Let* $F(x) = x(1 + x^2)^{-2}$, $G(x) = x^2(1 + x^2)^{-2}$. *Then*

$$\mathrm{I}_0 = \sup_{\alpha > 0} \left[\sum_{n=1}^{\infty} \mathrm{F}\left(\frac{2n-1}{\alpha}\right)\right]\left[\sum_{n=1}^{\infty} \mathrm{G}\left(\frac{2n-1}{\alpha}\right)\right]^{-1} \leqslant .8\,, \tag{4.2}$$

$$\mathrm{I}_1 = \sup_{\alpha > 0} \left[\sum_{n=1}^{\infty} \mathrm{F}\left(\frac{n}{\alpha}\right)\right]\left[\sum_{n=1}^{\infty} \mathrm{G}\left(\frac{n}{\alpha}\right)\right]^{-1} \leqslant \frac{2}{\pi}\,. \tag{4.3}$$

Let $h_{\mu\nu}^{(1)} \in \mathcal{H}_{\mu\nu}^{(1)}$ and $h_{\mu\nu}^{(2)} \in \mathcal{H}_{\mu\nu}^{(2)}$. Then, as noted in § 3, $h_{\mu\nu}^{(j)}$ has a Fourier series representation given by (3.8.j.$\mu\nu$) and therefore

$$\| h_{\mu\nu}^{(1)} \|^2 = ab \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_\mu(m)^2\, \beta_\nu(n)^2}{[\alpha_\mu(m)^2 + \beta_\nu(n)^2]^2}\, | a_{\mu\nu}^{(m)} |^2\,,$$

$$\| h_{\mu\nu}^{(2)} \|^2 = ab \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_\mu(m)^2\, \beta_\nu(n)^2}{[\alpha_\mu(m)^2 + \beta_\nu(n)^2]^2}\, | b_{\mu\nu}^{(n)} |^2\,,$$

and

$$\| h_{\mu\nu}^{(1)} + h_{\mu\nu}^{(2)} \|^2 = \| h_{\mu\nu}^{(1)} \|^2 + \| h_{\mu\nu}^{(2)} \|^2 + $$

$$+\, 2ab \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_\mu(m)^2\, \beta_\nu(n)^2}{[\alpha_\mu(m)^2 + \beta_\nu(n)^2]^2}\, a_{\mu\nu}^{(m)}\, b_{\mu\nu}^{(n)}\,. \tag{4.4}$$

Note that

$$2ab \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_\mu(m)^2\, \beta_\nu(n)^2}{[\alpha_\mu(m)^2 + \beta_\nu(n)^2]^2}\, a_{\mu\nu}^{(m)}\, b_{\mu\nu}^{(n)} \right| \leqslant$$

$$\leqslant 2ab \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_\mu(m)^3\, \beta_\nu(n)}{[\alpha_\mu(m)^2 + \beta_\nu(n)^2]^2}\, | a_{\mu\nu}^{(m)} |^2 \right\}^{1/2} \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_\mu(m)\, \beta_\nu(n)^3}{[\alpha_\mu(m)^2 + \beta_\nu(n)^2]^2}\, | b_{\mu\nu}^{(n)} |^2 \right\}^{1/2} \tag{4.5}$$

while, with I_ν as in Lemma 4.1,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_\mu(m)^3\, \beta_\nu(n)}{[\alpha_\mu(m)^2 + \beta_\nu(n)^2]^2}\, | a_{\mu\nu}^{(m)} |^2 = \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} \mathrm{F}\left(\frac{\beta_\nu(n)}{\alpha_\mu(m)}\right)\right] | a_{\mu\nu}^{(m)} |^2 \leqslant$$

$$\leqslant \mathrm{I}_\nu \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} \mathrm{G}\left(\frac{\beta_\nu(n)}{\alpha_\mu(m)}\right)\right] | a_{\mu\nu}^{(m)} |^2 = \mathrm{I}_\nu \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_\mu(m)^2\, \beta_\nu(n)^2}{[\alpha_\mu(m)^2 + \beta_\nu(n)^2]^2}\, | a_{\mu\nu}^{(m)} |^2 = \frac{1}{ab}\, \mathrm{I}_\nu\, \| h_{\mu\nu}^{(1)} \|^2\,,$$

and, similarly,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_\mu(m)\, \beta_\nu(n)^3}{[\alpha_\mu(m)^2 + \beta_\nu(n)^2]^2}\, | b_{\mu\nu}^{(n)} |^2 \leqslant \frac{1}{ab}\, \mathrm{I}_\mu\, \| h_{\mu\nu}^{(2)} \|^2\,.$$

Thus the right hand side of (4.5) is less than or equal to

$$2(I_\mu I_\nu)^{1/2} \| h_{\mu\nu}^{(1)} \| \| h_{\mu\nu}^{(2)} \| \leqslant (I_\mu I_\nu)^{1/2} (\| h_{\mu\nu}^{(1)} \|^2 + \| h_{\mu\nu}^{(2)} \|^2) .$$

This fact together with (4.4) gives (4.1).

Therefore Theorem 3.1 is proved provided the estimates in Lemma 4.1 are true.

5. The estimate for I_1 .

For $\alpha > 0$, let $I_1(\alpha) = \left[\sum_{n=1}^{\infty} F\left(\frac{n}{\alpha}\right)\right] \left[\sum_{n=1}^{\infty} G\left(\frac{n}{\alpha}\right)\right]^{-1}$, where $F(x) = x(1 + x^2)^{-2}$, $G(x) = xF(x)$ as in Lemma 4.1. Note that for $0 < \alpha \leqslant \frac{2}{\pi}$

$$\frac{2}{\pi} G\left(\frac{n}{\alpha}\right) - F\left(\frac{n}{\alpha}\right) = \left(\frac{2}{\pi}\frac{n}{\alpha} - 1\right) F\left(\frac{n}{\alpha}\right) \geqslant 0$$

for all $n = 1, 2, \ldots$ Thus clearly

$$I_1(\alpha) \leqslant \frac{2}{\pi} \text{ for } 0 < \alpha \leqslant \frac{2}{\pi} \cdot \tag{5.1}$$

In order to obtain an estimate for $\alpha \geqslant \frac{2}{\pi}$ we make use of the classical Euler-Maclaurin expansion, computed to the fourth derivative [3] :

$$\int_{k/\alpha}^{l/\alpha} F(x)\,dx = \frac{1}{\alpha} \sum_{n=k}^{l} F\left(\frac{n}{\alpha}\right) - \frac{1}{2\alpha}\left[F\left(\frac{k}{\alpha}\right) + F\left(\frac{l}{\alpha}\right)\right] -$$

$$- \frac{1}{12\alpha^2}\left[F'\left(\frac{l}{\alpha}\right) - F'\left(\frac{k}{\alpha}\right)\right] + \frac{1}{24\alpha^4}\int_{k/\alpha}^{l/\alpha} \Phi_4(\alpha x)\, F^{(4)}(x)\,dx ,$$

where $\Phi_4(x)$ is periodic of period one, and, for $0 < x \leqslant 1$, $\Phi_4(x) = x^2(x-1)^2$ (the fourth Bernoulli polynomial). Thus with $k = 0$ and $l = \infty$ we have

$$\int_0^\infty \frac{x}{(1+x^2)^2}\,dx = \frac{1}{\alpha}\sum_{n=0}^{\infty} F\left(\frac{n}{\alpha}\right) + \frac{1}{12\alpha^2} +$$

$$+ \frac{1}{24\alpha^4}\int_0^\infty \Phi_4(\alpha x)\,\frac{120x(x^2-3)(3x^2-1)}{(1+x^2)^6}\,dx\ .$$

The left hand side equals 1/2 and $F(0) = 0$. Thus

$$\frac{1}{\alpha}\sum_{n=1}^{\infty} F\left(\frac{n}{\alpha}\right) = \frac{1}{2} - \frac{1}{12\alpha^2} - \frac{1}{24\alpha^4} J_\alpha\ , \tag{5.2}$$

where $J_\alpha = \int_0^\infty \Phi_4(\alpha x)\,F^{(4)}(x)\,dx$. Note that $0 \leqslant \Phi_4(x) \leqslant \frac{1}{16}$, and $F^{(4)}(x) \geqslant 0$ unless $\frac{1}{\sqrt{3}} \leqslant x \leqslant \sqrt{3}$. Thus

$$J_\alpha \geqslant \frac{1}{16}\int_{1/\sqrt{3}}^{\sqrt{3}} F^{(4)}(x)\,dx = -\frac{21}{64}\ ,$$

and

$$\frac{1}{\alpha}\sum_{n=1}^{\infty} F\left(\frac{n}{\alpha}\right) \leqslant \frac{1}{2}\left(1 - \frac{1}{6\alpha^2} + \frac{7}{256\alpha^4}\right) \cdot \tag{5.3}$$

On the other hand it follows from (3.10) that

$$\frac{2}{\pi}(\sinh^2 \alpha\pi)\sum_{n=1}^{\infty}\frac{n^2}{(n^2+\alpha^2)^2} = \int_0^\pi \sinh^2 \alpha x\,dx =$$

$$= \frac{1}{2\alpha}(\sinh \alpha\pi \cosh \alpha\pi - \alpha\pi)\ ,$$

from which we obtain

$$\frac{1}{\alpha}\sum_{n=1}^{\infty} G\left(\frac{n}{\alpha}\right) = \frac{\pi}{4}\,\frac{\sinh \pi\alpha \cosh \pi\alpha - \pi\alpha}{\sinh^2 \pi\alpha} = \frac{\pi}{4}\,\frac{\sinh 2\pi\alpha - 2\pi\alpha}{\cosh 2\pi\alpha - 1} \cdot \tag{5.4}$$

Thus, from (5.3) and (5.4) :

$$I_1(\alpha) \leqslant \frac{2}{\pi}\,\frac{\cosh 2\pi\alpha - 1}{\sinh 2\pi\alpha - 2\pi\alpha}\left(1 - \frac{1}{6\alpha^2} + \frac{7}{256\alpha^4}\right) \cdot \tag{5.5}$$

Purely elementary techniques show that for $\alpha \geqslant \frac{2}{\pi}$

$$\alpha^2(2\pi\alpha - 1 + e^{-2\pi\alpha}) \leqslant 2\pi\alpha^3 \leqslant \frac{1}{11}(\cosh 2\pi\alpha - 1) \leqslant$$

$$\leqslant \left(\frac{1}{6} - \frac{7}{256\alpha^2}\right)(\cosh 2\pi\alpha - 1)\,,$$

so that

$$1 - \frac{1}{6\alpha^2} + \frac{7}{256\alpha^4} \leqslant 1 - \frac{2\pi\alpha - 1 + e^{-2\pi\alpha}}{\cosh 2\pi\alpha - 1} = \frac{\sinh 2\pi\alpha - 2\pi\alpha}{\cosh 2\pi\alpha - 1}\,.$$

From (5.5) therefore we obtain

$$I_1(\alpha) \leqslant \frac{2}{\pi} \quad \text{for} \quad \alpha \geqslant \frac{2}{\pi}\,,$$

and this fact combined with (5.1) gives equation (4.3) of Lemma 4.1 : $I_1 \leqslant \frac{2}{\pi}$.

Remark 5.1. – Using the estimate

$$J_\alpha \leqslant \frac{1}{16}\left[\int_0^{1/\sqrt{3}} F^{(4)}(x)\,dx + \int_{\sqrt{3}}^{\infty} F^{(4)}(x)\,dx\right] \leqslant \frac{69}{64}$$

we find from (5.2) that

$$\frac{1}{\alpha}\sum_{n=1}^{\infty} F\left(\frac{n}{\alpha}\right) \geqslant \frac{1}{2}\left(1 - \frac{1}{6\alpha^2} - \frac{23}{256\alpha^4}\right)$$

so that

$$I_1(\alpha) \geqslant \frac{2}{\pi}\,\frac{\cosh 2\pi\alpha - 1}{\sinh 2\pi\alpha - 2\pi\alpha}\left(1 - \frac{1}{6\alpha^2} - \frac{23}{256\alpha^4}\right).$$

This fact combined with (5.5) gives $\lim\limits_{\alpha\to\infty} I_1(\alpha) = \frac{2}{\pi}$; so actually $I_1 = \sup\limits_{\alpha>0} I_1(\alpha) = \frac{2}{\pi}$.

It follows from (4.1) that $\cos\theta_{11} \leqslant \frac{2}{\pi}$. But also, for every $h^{(1)} \in \mathcal{H}_{11}^{(1)}$ and $h^{(2)} \in \mathcal{H}_{11}^{(2)}$,

$$\|h^{(1)} + h^{(2)}\|^2 \geqslant (1 - \cos^2 \theta_{11}) \|h^{(1)}\|^2 .$$

For $p = 1, 2, \ldots$ define (see (3.8.1.11), (3.8.2.11))

$$h_p^{(1)} = \sum_{m=p}^{p^2} \sum_{n=1}^{\infty} \frac{(-1)^{m+n}\left(\frac{m}{a}\right)\left(\frac{n}{b}\right)}{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \left(\frac{1}{bm}\right) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$h_p^{(2)} = \sum_{m=1}^{\infty} \sum_{n=p}^{p^2} \frac{(-1)^{m+n}\left(\frac{m}{a}\right)\left(\frac{n}{b}\right)}{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \left(-\frac{2}{\pi a n}\right) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

Then $h_p^{(1)} \in \mathfrak{H}_{11}^{(1)}$, $h_p^{(2)} \in \mathfrak{H}_{11}^{(2)}$ and it can be shown that

$$\lim_{p\to\infty} \frac{\|h_p^{(1)} + h_p^{(2)}\|^2}{\|h_p^{(1)}\|^2} = 1 - \frac{4}{\pi^2} .$$

Thus

$$1 - \cos^2 \theta_{11} \leqslant 1 - \frac{4}{\pi^2} ,$$

from which we conclude $\cos\theta_{11} \geqslant \frac{2}{\pi}$. Thus

$$\cos\theta_{11} = \frac{2}{\pi} .$$

6. Estimate for I_0.

For $\alpha > 0$ let $$\mathrm{I}_0(\alpha) = \left[\sum_{n=1}^{\infty} \mathrm{F}\left(\frac{2n-1}{\alpha}\right)\right]\left[\sum_{n=1}^{\infty} \mathrm{G}\left(\frac{2n-1}{\alpha}\right)\right]^{-1},$$

where $\mathrm{F}(x) = x(1 + x^2)^{-2}$, $\mathrm{G}(x) = x\mathrm{F}(x)$ as in Lemma 4.1. We note first that for $0 < \alpha < .8$,

$$(.8)\, \mathrm{G}\left(\frac{2n-1}{\alpha}\right) - \mathrm{F}\left(\frac{2n-1}{\alpha}\right) = \left((.8)\frac{2n-1}{\alpha} - 1\right) \mathrm{F}\left(\frac{2n-1}{\alpha}\right) \geqslant 0$$

for all $n = 1, 2 \ldots$ Hence clearly

$$I_0(\alpha) \leqslant .8 \quad \text{for} \quad 0 < \alpha \leqslant .8 \, , \tag{6.1}$$

so we may restrict our attention to $\alpha \geqslant .8$.

Next note that, as in § 5, $\sum_{n=1}^{\infty} G\left(\frac{2n-1}{\alpha}\right)$ can be expressed in closed form. For it follows from (3.11) that

$$\frac{1}{2\pi}\left(\cosh\frac{\pi\alpha}{2}\right)^2 \sum_{n=1}^{\infty} \frac{(2n-1)^2}{\left[\left(\frac{2n-1}{2}\right)^2+\left(\frac{\alpha}{2}\right)^2\right]^2} = \int_0^{\pi} \left(\cosh\frac{\alpha x}{2}\right)^2 dx =$$

$$= \frac{1}{\alpha}\left(\sinh\frac{\pi\alpha}{2}\cosh\frac{\pi\alpha}{2}+\frac{\pi\alpha}{2}\right),$$

so that

$$\frac{1}{\alpha}\sum_{n=1}^{\infty} G\left(\frac{2n-1}{\alpha}\right) = \frac{\pi}{8}\,\frac{\sinh\frac{\pi\alpha}{2}\cosh\frac{\pi\alpha}{2}+\frac{\pi\alpha}{2}}{\cosh^2\frac{\pi\alpha}{2}} = \frac{\pi}{8}\,\frac{\sinh\pi\alpha+\pi\alpha}{\cosh\pi\alpha+1}\,. \tag{6.2}$$

Note that $\frac{\sinh\pi\alpha+\pi\alpha}{\cosh\pi\alpha+1}$ is monotone decreasing and larger than one for $\alpha \geqslant .8$, a fact which will be important in what follows.

In order to obtain estimates for $\sum_{n=1}^{\infty} F\left(\frac{2n-1}{\alpha}\right)$ we note that

$$\frac{1}{\alpha}\sum_{n=1}^{\infty} F\left(\frac{2n-1}{\alpha}\right) = \frac{1}{\alpha}\sum_{n=1}^{\infty} F\left(\frac{n}{\alpha}\right) - \frac{1}{2}\cdot\frac{2}{\alpha}\sum_{n=1}^{\infty} F\left(\frac{2}{\alpha}\cdot n\right).$$

Thus from (5.2) we obtain

$$\frac{1}{\alpha}\sum_{n=1}^{\infty} F\left(\frac{2n-1}{\alpha}\right) = \left(\frac{1}{2}-\frac{1}{12\alpha^2}-\frac{1}{24\alpha^4}\,J_\alpha\right) -$$

$$-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{3\alpha^2}-\frac{2}{3\alpha^2}\,J_{\alpha/2}\right),$$

or

$$\frac{1}{\alpha}\sum_{n=1}^{\infty} F\left(\frac{2n-1}{\alpha}\right) = \frac{1}{4}\left(1+\frac{1}{3\alpha^2}+\frac{4}{\alpha^4}\,\hat{J}_\alpha\right), \tag{6.3}$$

where $\hat{J}_\alpha = \int_0^\infty \hat{\Phi}(\alpha x)\, F^{(4)}(x)\, dx$, and $\hat{\Phi}(x) = \frac{1}{3}\,\Phi_4\left(\frac{x}{2}\right) - \frac{1}{24}\,\Phi_4(x)$ is periodic of period two,

$$\hat{\Phi}(x) = \begin{cases} \frac{1}{48}\, x^2 (2 - x^2), \ 0 \leqslant x \leqslant 1 \\ \frac{1}{48}\, (x-2)^2 (2 - (x-2)^2), \ 1 \leqslant x \leqslant 2 \,. \end{cases}$$

Since $F^{(4)}(x) \geqslant 0$ unless $\frac{1}{\sqrt{3}} \leqslant x \leqslant \sqrt{3}$, and $0 \leqslant \hat{\Phi}(x) \leqslant \frac{1}{48}$, therefore

$$\hat{J}_\alpha \leqslant \frac{1}{48} \left[\int_0^{1/\sqrt{3}} F^{(4)}(x)\, dx + \int_{\sqrt{3}}^\infty F^{(4)}(x)\, dx \right] = \frac{23}{64} \,,$$

so that

$$\frac{1}{\alpha} \sum_{n=1}^\infty F\left(\frac{2n-1}{\alpha}\right) \leqslant \frac{1}{4}\left(1 + \frac{1}{3\alpha^2} + \frac{23}{16\alpha^4}\right) \,,$$

and

$$I_0(\alpha) \leqslant \frac{2}{\pi}\, \frac{\cosh \pi\alpha + 1}{\sinh \pi\alpha + \pi\alpha} \left(1 + \frac{1}{3\alpha^2} + \frac{23}{16\alpha^4}\right) . \qquad (6.4)$$

If follows that for $\alpha \geqslant 2$:

$$I_0(\alpha) \leqslant \frac{2}{\pi}\left(1 + \frac{1}{12} + \frac{23}{256}\right) \leqslant .75 \qquad (6.5)$$

In order to estimate $I_0(\alpha)$ for $\alpha < 2$ we need a better estimate for $\hat{J}_\alpha$. For this purpose we use integration by parts to calculate $\int_0^{\sqrt{3}} \hat{\Phi}(\alpha x)\, F^{(4)}(x)\, dx$ explicitly. Direct computation shows that all derivatives of $\hat{\Phi}(x)$ are periodic of period two and that all are continuous except $\hat{\Phi}'''(x)$, which has jump discontinuities at $x = 1, 3, 5, \ldots$:

$$\hat{\Phi}'(x) = \begin{cases} \frac{1}{12}\, x(1 - x^2) \,, & 0 \leqslant x \leqslant 1 \\ \frac{1}{12}\, (x-2)(1 - (x-2)^2) \,, & 1 \leqslant x \leqslant 2 \end{cases}$$

$$\hat{\Phi}''(x) = \begin{cases} \frac{1}{12}(1-3x^2), & 0 \leqslant x \leqslant 1 \\ \frac{1}{12}(1-3(x-2)^2), & 1 \leqslant x \leqslant 2 \end{cases}$$

$$\hat{\Phi}'''(x) = \begin{cases} -\frac{x}{2}, & 0 \leqslant x < 1 \\ -\frac{x-2}{2}, & 1 < x \leqslant 2 \end{cases}$$

$$\hat{\Phi}^{(4)}(x) \equiv -\frac{1}{2}$$

Moreover, for $\sqrt{3} \leqslant \alpha \leqslant 2$ and $0 \leqslant x \leqslant \sqrt{3}$, clearly $0 \leqslant \alpha x \leqslant \alpha\sqrt{3}$ and $3 < \alpha\sqrt{3} \leqslant 2\sqrt{3} < 4$. Thus

$$\int_0^{\sqrt{3}} \hat{\Phi}(\alpha x)\, F^{(4)}(x)\, dx = [\hat{\Phi}(\alpha x)\, F'''(x)]\,|_0^{\sqrt{3}} - \alpha[\hat{\Phi}'(\alpha x)\, F''(x)]\,|_0^{\sqrt{3}}$$

$$+ \alpha^2\, \hat{\Phi}''(\alpha x)\, F'(x)\,|_0^{\sqrt{3}} - \alpha^3 \left[\hat{\Phi}'''(1\,-)\, F\left(\frac{1}{\alpha}\right) - \hat{\Phi}'''(0)\, F(0) + \right.$$

$$+ \hat{\Phi}'''(3\,-)\, F\left(\frac{3}{\alpha}\right) - \hat{\Phi}'''(1\,+)\, F\left(\frac{1}{\alpha}\right) + \hat{\Phi}'''(\alpha\sqrt{3})\, F(\sqrt{3}) -$$

$$\left. - \hat{\Phi}'''(3\,+)\, F\left(\frac{3}{\alpha}\right)\right] + \alpha^4 \int_0^{\sqrt{3}} \hat{\Phi}^{(4)}(\alpha x)\, F(x)\, dx\,,$$

and it is a simple but tedious matter to compute

$$\int_0^{\sqrt{3}} \hat{\Phi}(\alpha x)\, F^{(4)}(x)\, dx = \alpha^4 \left[\frac{\alpha^2}{(\alpha^2+1)^2} + \frac{3\alpha^2}{(\alpha^2+9)^2}\right] + \frac{27}{256}\alpha^4 -$$

$$- \frac{27\sqrt{3}}{32}\alpha^3 + \frac{501}{192}\alpha^2 - \frac{45\sqrt{3}}{32}\alpha + \frac{7}{8}\,.$$

On the other hand

$$\int_{\sqrt{3}}^{\infty} \hat{\Phi}(\alpha x)\, F^{(4)}(x)\, dx \leqslant \frac{1}{48} \int_{\sqrt{3}}^{\infty} F^{(4)}(x)\, dx = \frac{1}{256}\,.$$

It therefore follows from (6.3) that for $\sqrt{3} \leqslant \alpha \leqslant 2$:

$$\frac{1}{\alpha}\sum_{n=1}^{\infty} F\left(\frac{2n-1}{\alpha}\right) \leqslant \frac{1}{4}\{4f(\alpha)+g(\alpha)\}\ , \tag{6.6}$$

where

$$f(\alpha) = \frac{\alpha^2}{(\alpha^2+1)^2} + \frac{3\alpha^2}{(\alpha^2+9)^2}$$

attains its maximum between $\alpha^2 = 1.24$ and $\alpha^2 = 1.25$, and

$$g(\alpha) = \frac{91}{64} - \frac{27\sqrt{3}}{8\alpha} + \frac{517}{48\alpha^2} - \frac{45\sqrt{3}}{8\alpha^3} + \frac{225}{64\alpha^4}$$

is monotone increasing for $\sqrt{3} \leqslant \alpha \leqslant 2$. Hence

$$\sup_{\alpha>0} f(\alpha) \leqslant \frac{1.25}{(2.24)^2} + \frac{3(1.25)}{(10.24)^2} \leqslant .2858\ , \tag{6.7}$$

$$\sup_{\alpha\geqslant\sqrt{3}} f(\alpha) \leqslant f(\sqrt{3}) = .25\ , \tag{6.8}$$

and

$$\sup_{\sqrt{3}\leqslant\alpha\leqslant 2} g(\alpha) \leqslant g(2) \leqslant .22\ , \tag{6.9}$$

Combining (6.2), (6.6), (6.8), and (6.9) we obtain

$$I_0(\alpha) \leqslant \frac{2}{\pi}\left(\frac{\cosh 7+1}{\sinh 7+7}\right)(1.22) \leqslant .78 \quad \text{for} \quad \sqrt{3} \leqslant \alpha \leqslant 2\ . \tag{6.10}$$

Finally we consider $.8 \leqslant \alpha \leqslant \sqrt{3}$. For $x > 0$ the function $\alpha^{-1}F(x/\alpha) = x\alpha^2(x^2+\alpha^2)^{-2}$ is monotone increasing in α for $0 \leqslant \alpha \leqslant x$. Thus (see (6.7)) for $\alpha \leqslant \sqrt{3}$

$$\frac{1}{\alpha}\sum_{n=1}^{\infty} F\left(\frac{2n-1}{\alpha}\right) = f(\alpha) + \sum_{n=3}^{\infty} \frac{(2n-1)\alpha^2}{((2n-1)^2+\alpha^2)^2} \leqslant .2858 + \sum_{n=3}^{\infty} \frac{(2n-1)3}{((2n-1)^2+3)^2}$$

Moreover, for any $N = 3, 4, \ldots$

$$\sum_{n=N+1}^{\infty} \frac{(2n-1)}{((2n-1)^2+3)^2} = \frac{1}{8}\sum_{n=N+1}^{\infty} \frac{n-1/2}{\left(\left(n-\frac{1}{2}\right)^2+\frac{3}{4}\right)^2} \leqslant \frac{1}{8}\int_N^{\infty} \frac{x\,dx}{\left(x^2+\frac{3}{4}\right)^2} = \frac{1}{16}\,\frac{1}{N^2+\frac{3}{4}}\ .$$

Thus for $N = 3, 4, \ldots$

$$\frac{1}{\alpha}\sum_{n=1}^{\infty} F\left(\frac{2n-1}{\alpha}\right) \leqslant .2858 + 3\sum_{n=3}^{N} \frac{2n-1}{((2n-1)^2+3)^2} + \\ + \frac{1}{16}\frac{1}{N^2+\frac{3}{4}} \cdot \qquad (6.11)$$

We therefore obtain (with $N = 10$, e.g.)

$$\frac{1}{\alpha}\sum_{n=1}^{\infty} F\left(\frac{2n-1}{\alpha}\right) \leqslant .3238 ,$$

from which it follows that

$$I_0(\alpha) \leqslant \frac{8}{\pi}\left(\frac{\cosh 5.46 + 1}{\sinh 5.46 + 5.46}\right)(.3238) \leqslant .8 , \quad .8 \leqslant \alpha \leqslant \sqrt{3} . \qquad (6.12)$$

Combining (6.1), (6.5), (6.10), and (6.12) gives

$$I_0 = \sup_{\alpha > 0} I_0(\alpha) \leqslant .8 ,$$

proving Lemma 4.1.

Remark 6.1. – Note that

$$\hat{J}_\alpha \geqslant \frac{1}{48}\int_{1/\sqrt{3}}^{\sqrt{3}} F^{(4)}(x)\, dx = -\frac{7}{64} \cdot$$

Hence from (6.3)

$$\frac{1}{\alpha}\sum_{n=1}^{\infty} F\left(\frac{2n-1}{\alpha}\right) \geqslant \frac{1}{4}\left(1 + \frac{1}{3\alpha^2} - \frac{7}{16\alpha^4}\right),$$

and

$$I_0(\alpha) \geqslant \frac{2}{\pi}\,\frac{\cosh \pi\alpha + 1}{\sinh \pi\alpha + \pi\alpha}\left(1 + \frac{1}{3\alpha^2} - \frac{7}{16\alpha^4}\right) \cdot \qquad (6.13)$$

Equations (6.4) and (6.13) show that

$$\lim_{\alpha \to \infty} I_0(\alpha) = \frac{2}{\pi} \cdot$$

However, it is not difficult to see from (6.13) that actually $I_0(\alpha) > \frac{2}{\pi}$ for sufficiently large finite α. Thus $I_0(\alpha)$ attains its maximum I_0 at some finite positive α, and $I_0 > \frac{2}{\pi}$. Obtaining an exact value for I_0 is therefore much more difficult than obtaining the exact value for I_1.

7. Construction of the solution.

Theorem 3.1 being proved, the solution u of (0.1) can now be constructed from equation (1.6). In this equation, set $w = f - \Delta\nu$; then

$$u = \nu + G(I - P)\, w\ , \tag{7.1}$$

or, with $G^{(x,y)}(x', y') = G(x, y, x', y')$ (the Green's function corresponding to the Laplacian in $R_{a,b}$)

$$u(x, y) = \nu(x, y) + (G^{(x,y)}, (I - P)\, w)$$

for $(x, y) \in R_{a,b}$.

Since $w \in L^2(R_{a,b})$, $w = \sum_{\mu=0}^{1} \sum_{\nu=0}^{1} w_{\mu\nu}$, where $w_{\mu\nu} \in L^2_{\mu\nu}(R_{a,b})$ and each of the functions $w_{\mu\nu}$ has a Fourier expansion

$$w_{\mu\nu} \sim \sum_{m=1}^{1} \sum_{n=1}^{1} \gamma_{\mu\nu}^{mn}\, u_{\mu,m}\, \nu_{\nu,n}$$

(see equation (3.7)). If we truncate each of these expansions we obtain an approximation

$$\hat{w} = \sum_{\mu=0}^{1} \sum_{\nu=0}^{1} \hat{w}_{\mu\nu} \tag{7.2}$$

to w, where $\hat{w}_{\mu\nu} = \sum_{m=1}^{M_0} \sum_{n=1}^{N_0} \gamma_{\mu\nu}^{mn}\, u_{\mu,m}\, \nu_{\nu,n}$ is the truncation of the Fourier expansion $w_{\mu\nu} \sim \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \gamma_{\mu\nu}^{mn}\, u_{\mu,m}\, \nu_{\nu,n}$. Then $\hat{u} = \nu + G(I - P)\hat{w}$ gives an approximation to u, and by choosing M_0, N_0 sufficiently large we can insure a priori that the error

$$|u(x, y) - \hat{u}(x, y)| \leqslant \| G^{(x,y)} \| \, \| w - \hat{w} \|$$

is as small as we please.

Thus in order to approximate u it suffices to approximate $\hat{u}$. Moreover, the function v is known, and $G\hat{w}$ can be easily constructed using (7.2) and the fact that

$$G(u_{\mu,m} v_{\nu,n}) = -\pi^{-2}(\alpha_\mu(m)^2 + \beta_\nu(n)^2)^{-1} u_{\mu,m} v_{\nu,n} \,.$$

Let $h = P\hat{w}$. We therefore need only to construct a function $h' \in \mathcal{H}$ such that Gh' can be explicitly constructed and such that the error $\| h - h' \|$ can be estimated a priori. The solution u to (0.1) will then be approximated by the explicitly constructible function

$$u_0 = v + G\hat{w} - Gh' \,, \tag{7.3}$$

where $\hat{w}$ is given by (7.2), with an error

$$|u(x, y) - u_0(x, y)| \leqslant \| G^{(x,y)} \| \, \| w - \hat{w} \| + \| G^{(x,y)} \| \, \| h - h' \| \tag{7.4}$$

which can be estimated a priori. In the remainder of this section we show how such approximations h' to $h = P\hat{w}$ can be constructed.

First, note that $h = P\hat{w} = \sum_{\mu=0}^{1} \sum_{\nu=0}^{1} P_{\mu\nu} \hat{w}_{\mu\nu}$, so that it suffices to approximate each of the functions $h_{\mu\nu} = P_{\mu\nu} \hat{w}_{\mu\nu}$. Let

$$S_{\mu\nu}^K = \sum_{k=1}^{K+1} [P_{\mu\nu}^{(1)}(P_{\mu\nu}^{(2)} P_{\mu\nu}^{(1)})^{k-1} + P_{\mu\nu}^{(2)}(P_{\mu\nu}^{(1)} P_{\mu\nu}^{(2)})^{k-1} - (P_{\mu\nu}^{(2)} P_{\mu\nu}^{(1)})^k -$$

$$- (P_{\mu\nu}^{(1)} P_{\mu\nu}^{(2)})^k] = P_{\mu\nu}^{(1)} \sum_{k=0}^{K} [(P_{\mu\nu}^{(2)} P_{\mu\nu}^{(1)})^k - P_{\mu\nu}^{(2)}(P_{\mu\nu}^{(1)} P_{\mu\nu}^{(2)})^k] + \tag{7.5}$$

$$+ P_{\mu\nu}^{(2)} \sum_{k=0}^{K} [(P_{\mu\nu}^{(1)} P_{\mu\nu}^{(2)})^k - P_{\mu\nu}^{(1)}(P_{\mu\nu}^{(2)} P_{\mu\nu}^{(1)})^k] \,.$$

By (2.3) and Theorem 3.1, $S_{\mu\nu}^K \hat{w}_{\mu\nu}$ approximates $h_{\mu\nu}$ with an error

$$\| h_{\mu\nu} - S_{\mu\nu}^K \hat{w}_{\mu\nu} \| \leqslant 2(\cos\theta_{\mu\nu})^{2K+1} \| \hat{w}_{\mu\nu} \| \tag{7.6}$$

which can be approximated a priori (using the bounds for $\cos\theta_{\mu\nu}$ given in Theorem 3.1) and, by choosing K sufficiently large, can be made as small as we please.

Next, note that $S^K_{\mu\nu}\hat{w}_{\mu\nu}$ is the sum of a function in $\mathcal{H}^{(1)}_{\mu\nu}$ and a function in $\mathcal{H}^{(2)}_{\mu\nu}$ (as are all functions in $\mathcal{H}_{\mu\nu}$ by Theorem 3.1). The spaces $\mathcal{H}^{(1)}_{\mu\nu}$, $\mathcal{H}^{(2)}_{\mu\nu}$ have complete orthogonal bases $\{U^{\mu\nu}_m\}$, $\{V^{\mu\nu}_n\}$ respectively (given in Corollary 3.1). Choose $M \geqslant M_0$ and $N \geqslant N_0$, and define

$$\hat{h}_{\mu\nu} = \hat{P}^{(1)}_{\mu\nu} \sum_{k=0}^{K} [(P^{(2)}_{\mu\nu} P^{(1)}_{\mu\nu})^k - P^{(2)}_{\mu\nu}(P^{(1)}_{\mu\nu}P^{(2)}_{\mu\nu})^k]\,\hat{w}_{\mu\nu} +$$

$$+ \hat{P}^{(2)}_{\mu\nu} \sum_{k=0}^{K} [(P^{(1)}_{\mu\nu}P^{(2)}_{\mu\nu})^k - P^{(1)}_{\mu\nu}(P^{(2)}_{\mu\nu} P^{(1)}_{\mu\nu})^k]\,\hat{w}_{\mu\nu}\,, \tag{7.7}$$

where $\hat{P}^{(1)}_{\mu\nu}$, $\hat{P}^{(2)}_{\mu\nu}$ are the orthogonal projections of $L^2(R_{a,b})$ onto the finite dimensional subspaces $\hat{\mathcal{H}}^{(1)}_{\mu\nu}$, $\hat{\mathcal{H}}^{(2)}_{\mu\nu}$ spanned by $\{U^{\mu\nu}_m : m=1,\dots,M\}$, $\{V^{\mu\nu}_n : n = 1,\dots,N\}$ respectively. Then $\hat{h}_{\mu\nu}$ is an approximation to $S^K_{\mu\nu}\hat{w}_{\mu\nu}$, and

$$\| S^K_{\mu\nu}\hat{w}_{\mu\nu} - \hat{h}_{\mu\nu} \| \leqslant$$

$$\leqslant \Big\| \sum_{k=0}^{K} (P^{(1)}_{\mu\nu} - \hat{P}^{(1)}_{\mu\nu})(P^{(2)}_{\mu\nu} P^{(1)}_{\mu\nu})^k\,\hat{w}_{\mu\nu} \Big\|$$

$$+ \Big\| \sum_{k=0}^{K} (P^{(1)}_{\mu\nu} - \hat{P}^{(1)}_{\mu\nu})\,P^{(2)}_{\mu\nu}(P^{(1)}_{\mu\nu} P^{(2)}_{\mu\nu})^k\,\hat{w}_{\mu\nu} \Big\|$$

$$+ \Big\| \sum_{k=0}^{K} (P^{(2)}_{\mu\nu} - \hat{P}^{(2)}_{\mu\nu})(P^{(1)}_{\mu\nu} P^{(2)}_{\mu\nu})^k\,\hat{w}_{\mu\nu} \Big\|$$

$$+ \Big\| \sum_{k=0}^{K} (P^{(2)}_{\mu\nu} - \hat{P}^{(2)}_{\mu\nu})\,P^{(1)}_{\mu\nu}(P^{(2)}_{\mu\nu} P^{(1)}_{\mu\nu})^k\,\hat{w}_{\mu\nu} \Big\|\,. \tag{7.8}$$

We investigate separately each of the terms on the right hand side of (7.8). Elementary calculation shows that

$$(U^{\mu\nu}_m, V^{\mu\nu}_n) = ab\,\frac{\alpha_\mu(m)^2\,\beta_\nu(n)^2}{[\alpha_\mu(m)^2 + \beta_\nu(n)^2]^2}\,.$$

Note also that $(u_{\mu,m}v_{\nu,n}, U^{\mu\nu}_{m_1}) = 0$ for $m \neq m_1$ and $(u_{\mu,m}v_{\nu,n}, V^{\mu\nu}_{n_1}) = 0$ for $n \neq n_1$. (Thus, in particular, $(P^{(1)}_{\mu\nu} - \hat{P}^{(1)}_{\mu\nu})\,\hat{w}_{\mu\nu} = 0$, since $M \geqslant M_0$.)

Making use of all these facts and of the representations of $P^{(1)}_{\mu\nu}$ and $P^{(2)}_{\mu\nu}$ in terms of the orthogonal bases $\{U^{\mu\nu}_m\}$ and $\{V^{\mu\nu}_n\}$ respectively (see (2.1)), one can easily verify that

$$\| \sum_{k=0}^{K} (P^{(1)}_{\mu\nu} - \hat{P}^{(1)}_{\mu\nu})(P^{(2)}_{\mu\nu} P^{(1)}_{\mu\nu})^k \hat{w}_{\mu\nu} \|^2 =$$

$$= \sum_{k,l=1}^{K} ((P^{(1)}_{\mu\nu} - \hat{P}^{(1)}_{\mu\nu})(P^{(2)}_{\mu\nu} P^{(1)}_{\mu\nu})^k \hat{w}_{\mu\nu}, (P^{(2)}_{\mu\nu} P^{(1)}_{\mu\nu})^l \hat{w}_{\mu\nu})$$

$$= \sum_{k,l=1}^{K} \sum_{m=M+1}^{\infty} \sum_{r_l=1}^{\infty} \sum_{q_l=1}^{\infty} \cdots \sum_{r_1=1}^{\infty} \sum_{q_1=1}^{M_0} \sum_{n_k=1}^{\infty} \sum_{m_k=1}^{\infty} \cdots \sum_{n_1=1}^{\infty} \sum_{m_1=1}^{M_0}$$

$$\left\{ (\theta^{\mu\nu}_{m_1} \xi^{\mu\nu}_{m_1 n_1} \eta^{\mu\nu}_{m_2 n_1} \xi^{\mu\nu}_{m_2 n_2} \cdots \xi^{\mu\nu}_{m_k n_k} \eta^{\mu\nu}_{m n_k}) \cdot (\theta^{\mu\nu}_{q_1} \xi^{\mu\nu}_{q_1 r_1} \eta^{\mu\nu}_{q_2 r_1} \cdots \xi^{\mu\nu}_{q_l r_l}) \frac{ab}{[\alpha_\mu(m)^2 + \beta_\nu(r_l)^2]^2} \right\}, \tag{7.9}$$

where

$$\theta^{\mu\nu}_m = \alpha_\mu(m)^2 \, \| U^{\mu\nu}_m \|^{-2} (U^{\mu\nu}_m, \hat{w}_{\mu\nu}),$$

$$\xi^{\mu\nu}_{mn} = ab\, \beta_\nu(n)^4 \, \| V^{\mu\nu}_n \|^{-2} (\alpha_\mu(m)^2 + \beta_\nu(n)^2)^{-2},$$

$$\eta^{\mu\nu}_{mn} = ab\, \alpha_\mu(m)^4 \, \| U^{\mu\nu}_n \|^{-2} (\alpha_\mu(m)^2 + \beta_\nu(n)^2)^{-2}.$$

We shall derive bounds for these quantities in Lemmas 7.1, 7.2, and 7.3 below. For this purpose we shall also need the following equations :

$$\| U^{\mu\nu}_m \|^2 = \frac{\pi}{4} ab^2 \alpha_\mu(m) Q^2_\nu(\alpha_\mu(m) b),$$

$$\| V^{\mu\nu}_n \|^2 = \frac{\pi}{4} a^2 b\, \beta_\nu(n) Q^2_\mu(\beta_\nu(n) a),$$

where

$$Q^2_\mu(x) = \frac{\sinh 2\pi x + (-1)^\mu 2\pi x}{\cosh 2\pi x + (-1)^\mu}.$$

LEMMA 7.1. – *For* $m = 1, \ldots, M_0$,

$$\theta^{\mu\nu}_m \leqslant \frac{2}{\sqrt{\pi}} a^{-2} b^{-1} M_0^{3/2} C(c) \| \hat{w}_{\mu\nu} \|,$$

where $C(c) = \left(\frac{\cosh \pi c - 1}{\sinh \pi c - \pi c} \right)^{1/2}$ *and* $c = \min [a/b, b/a]$.

One checks that $Q_1^{-1}(x)$ is monotone decreasing and larger than one for $x > 0$. It is also not difficult to verify that for any $x_0 > 0$, $\sup_{x \geqslant x_0} Q_0^{-1}(x) = \max\{Q_0^{-1}(x_0), 1\}$, and that

$$Q_0^{-1}(\alpha_1(1)\,b) \leqslant Q_1^{-1}(\alpha_0(1)\,b)$$

so that $\quad \sup\limits_m Q_0^{-1}(\alpha_1(m)\,b) \leqslant \sup\limits_m Q_1^{-1}(\alpha_0(m)\,b) \leqslant C(c)$.

Similarly, $\sup\limits_n Q_0^{-1}(\beta_1(n)\,a) \leqslant \sup\limits_n Q_1^{-1}(\beta_0(n)\,b) \leqslant C(c)$. Lemma 7.1 is now immediate if one notes that for $m = 1, \ldots, M_0$,

$$\alpha_0(m) \leqslant \alpha_1(m) \leqslant M_0 a^{-1} .$$

LEMMA 7.2. – $\sum\limits_{m=1}^{\infty} \xi_{mn}^{\mu\nu} \leqslant 1$ *for every* $n = 1, 2, \ldots,$ *and* $\sum\limits_{n=1}^{\infty} \eta_{mn}^{\mu\nu} \leqslant 1$ *for every* $m = 1, 2, \ldots$.

In order to prove Lemma 7.2 we first derive the identities

$$\sum_{m=1}^{\infty} \frac{\beta_\nu(n)^3}{(\alpha_\mu(m)^2 + \beta_\nu(n)^2)^2} = \frac{\pi}{4}\, a R_\mu^2(\beta_\nu(n)\,a) ,$$

$$\sum_{n=1}^{\infty} \frac{\alpha_\mu(m)^3}{(\alpha_\mu(m)^2 + \beta_\nu(n)^2)^2} = \frac{\pi}{4}\, b R_\nu^2(\alpha_\mu(m)\,b) ,$$

where for $x > 0$,

$$R_\mu^2(x) = \frac{\sinh 2\pi x - (-1)^\mu\, 2\pi x}{\cosh 2\pi x + (-1)^\mu} - \frac{4\mu}{2\pi x}.$$

(These equations follow from the Fourier expansions

$$\cosh \gamma x \sim \frac{2}{\pi}(\sinh \gamma\pi)\left(\frac{1}{2\gamma} + \sum_{n=1}^{\infty} \frac{(-1)^n \gamma}{(\gamma^2 + n^2)} \cos nx\right) ,$$

$$\sinh \gamma x \sim \frac{2}{\pi}(\cosh \gamma\pi) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\gamma}{\gamma^2 + \left(\frac{2n-1}{2}\right)^2} \sin \frac{2n-1}{2}\, x ,$$

in the same way that (5.4) and (6.2) follow from (3.10) and (3.11).) It follows immediately that

$$\sum_{m=1}^{\infty} \xi_{mn}^{\mu\nu} = R_{\mu}^{2}(\beta_{\nu}(n)\,a)\, Q_{\mu}^{-2}(\beta_{\nu}(n)\,a) \ ,$$

$$\sum_{n=1}^{\infty} \eta_{mn}^{\mu\nu} = R_{\nu}^{2}(\alpha_{\mu}(m)\,b)\, Q_{\nu}^{-2}(\alpha_{\mu}(m)\,b) \ .$$

One verifies that the function

$$R_0^2(x)\, Q_0^{-2}(x) = (\sinh 2\pi x - 2\pi x)(\sinh 2\pi x + 2\pi x)^{-1}$$

is positive and monotone increasing for $x > 0$, and that

$$\sup_{x\geqslant 0} R_0^2(x)\, Q_0^{-2}(x) = \lim_{x\to\infty} R_0^2(x)\, Q_0^{-2}(x) = 1 \ .$$

On the other hand,

$$R_1^2(x)\, Q_1^{-2}(x) = 1 - [4(\cosh 2\pi x - 1) - 2(2\pi x)^2]\,[2\pi x(\sinh 2\pi x - 2\pi x)]^{-1}$$

is also $\leqslant 1$ for $x \geqslant 0$, so

$$\sup_{x\geqslant 0} R_1^2(x)\, Q_1^{-2}(x) = \lim_{x\to\infty} R_1^2(x)\, Q_1^{-2}(x) = 1 \ ,$$

and the lemma follows.

LEMMA 7.3. – $\displaystyle\sum_{m=M+1}^{\infty} \sum_{n=1}^{\infty} [\alpha_{\mu}(m)^2 + \beta_{\nu}(n)^2]^{-2} \leqslant \frac{\pi a^3 b}{4} M^{-2}$.

This lemma is immediately verified :

$$\sum_{m=M+1}^{\infty} \sum_{n=1}^{\infty} [\alpha_{\mu}(m)^2 + \beta_{\nu}(n)^2]^{-2} \leqslant \sum_{m=M+1}^{\infty} \sum_{n=1}^{\infty} [\alpha_0(m)^2 + \beta_0(n)^2]^{-2}$$

$$\leqslant \sum_{m=M+1}^{\infty} \int_{-1/2}^{\infty} \left[\alpha_0(m)^2 + \left(\frac{x}{b}\right)^2\right]^{-2} dx$$

$$\leqslant 2 \sum_{m=M+1}^{\infty} \int_{0}^{\infty} \left[\alpha_0(m)^2 + \left(\frac{x}{b}\right)^2\right]^{-2} dx \leqslant \frac{\pi b}{2} \sum_{m=M+1}^{\infty} \alpha_0(m)^{-3}$$

$$\leqslant \frac{\pi a^3 b}{2} \int_M^\infty x^{-3} dx = \frac{\pi a^3 b}{4} \, M^{-2} .$$

Applying the estimates in Lemmas 7.1, 7.2, and 7.3 to equation (7.9), one obtains that

$$\| \sum_{k=0}^{K} (P_{\mu\nu}^{(1)} - \hat{P}_{\mu\nu}^{(1)}) (P_{\mu\nu}^{(2)} P_{\mu\nu}^{(1)})^k \hat{w}_{\mu\nu} \| \leqslant K M_0^{3/2} C(c) M^{-1} \| \hat{w}_{\mu\nu} \| .$$

In an entirely analogous fashion one finds that

$$\| \sum_{k=0}^{K} (P_{\mu\nu}^{(1)} - \hat{P}_{\mu\nu}^{(1)}) P_{\mu\nu}^{(2)} (P_{\mu\nu}^{(1)} P_{\mu\nu}^{(2)})^k \hat{w}_{\mu\nu} \| \leqslant \frac{a}{b} (K + 1) N_0^{3/2} C(c) M^{-1} \| \hat{w}_{\mu\nu} \| ,$$

$$\| \sum_{k=0}^{K} (P_{\mu\nu}^{(2)} - \hat{P}_{\mu\nu}^{(2)}) (P_{\mu\nu}^{(1)} P_{\mu\nu}^{(2)})^k \hat{w}_{\mu\nu} \|^2 \leqslant K N_0^{3/2} C(c) N^{-1} \| \hat{w}_{\mu\nu} \| ,$$

$$\| \sum_{k=0}^{K} (P_{\mu\nu}^{(2)} - \hat{P}_{\mu\nu}^{(2)}) P_{\mu\nu}^{(1)} (P_{\mu\nu}^{(2)} P_{\mu\nu}^{(1)})^k \hat{w}_{\mu\nu} \|^2 \leqslant \frac{b}{a} (K + 1) M_0^{3/2} C(c) N^{-1} \| \hat{w}_{\mu\nu} \| ,$$

so that

$$\| S_{\mu\nu}^K \hat{w}_{\mu\nu} - \hat{h}_{\mu\nu} \| \leqslant [\Lambda_1 M^{-1} + \Lambda_2 N^{-1}] \| \hat{w}_{\mu\nu} \| \tag{7.10}$$

where

$$\Lambda_1 = C(c) (K + 1) \left(\left(\frac{b}{a}\right)^{1/2} M_0^{3/2} + \left(\frac{a}{b}\right)^{1/2} N_0^{3/2} \right) \left(\frac{a}{b}\right)^{1/2} , \tag{7.11.1}$$

$$\Lambda_2 = C(c) (K + 1) \left(\left(\frac{b}{a}\right)^{1/2} M_0^{3/2} + \left(\frac{a}{b}\right)^{1/2} N_0^{3/2} \right) \left(\frac{b}{a}\right)^{1/2} , \tag{7.11.2}$$

and $C(c)$ is given in Lemma 7.1.

Thus the function $\hat{h}_{\mu\nu}$ defined by (7.7) gives an approximation to $h_{\mu\nu}$, and the error (see (7.6) and (7.10)),

$$\| h_{\mu\nu} - \hat{h}_{\mu\nu} \| \leqslant [2 (\cos \theta_{\mu\nu})^{2K+1} + \Lambda_1 M^{-1} + \Lambda_2 N^{-1}] \cdot \| \hat{w}_{\mu\nu} \|$$

can be estimated a priori. Note that $\hat{h}_{\mu\nu}$ belongs to the $M + N$ dimensional subspace $\mathcal{K}_{\mu\nu}$ of $\mathcal{H}_{\mu\nu}$ spanned by the vectors $U_1^{\mu\nu}, \ldots, U_M^{\mu\nu}$, $V_1^{\mu\nu}, \ldots, V_N^{\mu\nu}$.

Let $Q_{\mu\nu}$ be the projection of $L^2(R_{a,b})$ onto $\mathcal{K}_{\mu\nu}$, and (recall $M \geqslant M_0$, $N \geqslant N_0$) $h'_{\mu\nu} = Q_{\mu\nu} \hat{w}_{\mu\nu} = Q_{\mu\nu} h_{\mu\nu}$. Then

$$\| h_{\mu\nu} - h'_{\mu\nu} \| = \inf \{ \| h_{\mu\nu} - k_{\mu\nu} \| : k_{\mu\nu} \in \mathcal{K}_{\mu\nu} \} \leqslant \| h_{\mu\nu} - \hat{h}_{\mu\nu} \| ,$$

so $h'_{\mu\nu}$ is an approximation to $h_{\mu\nu}$ with an error which satisfies the a priori estimate

$$\| h_{\mu\nu} - h'_{\mu\nu} \| \leqslant 2(\cos\theta_{\mu\nu})^{2K+1} \| \hat{w}_{\mu\nu} \| + [\Lambda_1 M^{-1} + \Lambda_2 N^{-1}] \| \hat{w}_{\mu\nu} \| . \tag{7.12}$$

Moreover, $h'_{\mu\nu}$ can be computed explicitly, for

$$h'_{\mu\nu} = \sum_{m=1}^{M} s_m^{\mu\nu} U_m^{\mu\nu} + \sum_{n=1}^{N} t_n^{\mu\nu} V_n^{\mu\nu} , \tag{7.13}$$

where the coefficients $s_m^{\mu\nu}$, $t_n^{\mu\nu}$ are so chosen that $\| \hat{w}_{\mu\nu} - h'_{\mu\nu} \|^2 = \inf\{ \| \hat{w}_{\mu\nu} - k_{\mu\nu} \|^2 : k_{\mu\nu} \in \mathcal{K}_{\mu\nu} \}$; i.e., so that the function

$$X(s_1^{\mu\nu}, \ldots, s_M^{\mu\nu}, t_1^{\mu\nu}, \ldots, t_N^{\mu\nu}) = \| \hat{w}_{\mu\nu} - \sum_{m=1}^{M} s_m^{\mu\nu} U_m^{\mu\nu} - \sum_{n=1}^{N} t_n^{\mu\nu} V_n^{\mu\nu} \|^2$$

attains its minimum. This occurs precisely when $\dfrac{\partial X}{\partial s_m^{\mu\nu}} = \dfrac{\partial X}{\partial t_n^{\mu\nu}} = 0$ for $m = 1, \ldots, M$ and $n = 1, \ldots, N$; i.e., if and only if

$$(U_m^{\mu\nu}, h'_{\mu\nu}) = s_m^{\mu\nu} \| U_m^{\mu\nu} \|^2 + \sum_{n=1}^{N} t_n^{\mu\nu} (U_m^{\mu\nu}, V_n^{\mu\nu}) = (\hat{w}_{\mu\nu}, U_m^{\mu\nu}), \quad m = 1, \ldots, M , \tag{7.14.1}$$

$$(V_n^{\mu\nu}, h'_{\mu\nu}) = t_n^{\mu\nu} \| V_n^{\mu\nu} \|^2 + \sum_{m=1}^{M} s_m^{\mu\nu} (U_m^{\mu\nu}, V_n^{\mu\nu}) = (\hat{w}_{\mu\nu}, V_n^{\mu\nu}) , \quad n = 1, \ldots, N . \tag{7.14.2}$$

These equations can be solved explicitly for $s_1^{\mu\nu}, \ldots, s_M^{\mu\nu}, t_1^{\mu\nu}, \ldots, t_N^{\mu\nu}$; (7.13) then gives $h'_{\mu\nu}$ explicitly.

Furthermore, having constructed $h'_{\mu\nu}$, $Gh'_{\mu\nu}$ can be constructed explicitly using (7.13) and the equations :

$$(GU_m^{11})(x, y) = \frac{\pi}{2}\frac{b}{a} \frac{(-1)^m a}{2\pi \sinh^2 \dfrac{m\pi b}{a}} \left\{ b \cosh \frac{m\pi b}{a} \sinh \frac{m\pi y}{a} - y \sinh \frac{m\pi b}{a} \cosh \frac{m\pi y}{a} \right\} \sin \frac{m\pi x}{a}$$

$$(\mathrm{GV}_n^{11})(x, y) = \frac{\pi}{2}\frac{a}{b}\frac{(-1)^n b}{2\pi \sinh^2 \frac{n\pi a}{b}}\left\{a \sinh \frac{n\pi x}{b} \cosh \frac{n\pi a}{b} - \right.$$

$$\left. - x \sinh \frac{n\pi a}{b} \cosh \frac{n\pi x}{b}\right\} \sin \frac{n\pi y}{b} \,.$$

$$(\mathrm{GU}_m^{10})(x, y) = \frac{\pi}{2}\frac{b}{a}\frac{(-1)^m a}{2\pi \cosh^2 \frac{m\pi b}{a}}\left\{b \sinh \frac{m\pi b}{a} \cosh \frac{m\pi y}{a} - \right.$$

$$\left. - y \sinh \frac{m\pi y}{a} \cosh \frac{m\pi b}{a}\right\} \sin \frac{m\pi x}{a} \,.$$

$$(\mathrm{GV}_n^{01})(x, y) = \frac{\pi}{2}\frac{a}{b}\frac{(-1)^n b}{2\pi \cosh^2 \frac{n\pi a}{b}}\left\{a \sinh \frac{n\pi a}{b} \cosh \frac{n\pi x}{b} - \right.$$

$$\left. - x \sinh \frac{n\pi x}{b} \cosh \frac{n\pi a}{b}\right\} \sin \frac{n\pi y}{b} \,.$$

$$(\mathrm{GU}_m^{01})(x, y) = \frac{\pi}{2}\frac{b}{a}\frac{(-1)^m a}{2\pi \sinh^2 \frac{2m-1}{2}\frac{\pi b}{a}}\left\{b \sinh \frac{2m-1}{2}\frac{\pi y}{a} \cosh \frac{2m-1}{2}\frac{\pi b}{a} - \right.$$

$$\left. - y \sinh \frac{2m-1}{2}\frac{\pi b}{a} \cosh \frac{2m-1}{2}\frac{\pi y}{a}\right\} \cos \frac{2m-1}{2}\frac{\pi x}{a} \,.$$

$$(\mathrm{GV}_n^{10})(x, y) = \frac{\pi}{2}\frac{a}{b}\frac{(-1)^n b}{2\pi \sinh^2 \frac{2n-1}{2}\frac{\pi a}{b}}\left\{a \sinh \frac{2n-1}{2}\frac{\pi x}{b} \cosh \frac{2n-1}{2}\frac{\pi a}{b} - \right.$$

$$\left. - x \sinh \frac{2n-1}{2}\frac{\pi a}{b} \cosh \frac{2n-1}{2}\frac{\pi x}{b}\right\} \cos \frac{2n-1}{2}\frac{\pi y}{b} \,.$$

$$(\mathrm{GU}_m^{00})(x, y) = \frac{\pi}{2}\frac{b}{a}\frac{(-1)^m a}{2\pi \cosh^2 \frac{2m-1}{2}\frac{\pi b}{a}}\left(b \sinh \frac{(2m-1)}{2}\frac{\pi b}{a} \cosh \frac{2m-1}{2}\frac{\pi y}{a} - \right.$$

$$\left. - y \sinh \frac{2m-1}{2}\frac{\pi y}{a} \cosh \frac{2m-1}{2}\frac{\pi b}{a}\right) \cos \frac{2m-1}{2}\frac{\pi x}{a} \,.$$

$$(\mathrm{GV}_n^{00})(x, y) = \frac{\pi}{2}\frac{a}{b}\frac{(-1)^n b}{2\pi \cosh^2 \frac{2n-1}{2}\frac{\pi a}{b}}\left(a \sinh \frac{2n-1}{2}\frac{\pi a}{b} \cosh \frac{2n-1}{2}\frac{\pi x}{b} - \right.$$

$$\left. - x \sinh \frac{2n-1}{2}\frac{\pi x}{b} \cosh \frac{2n-1}{2}\frac{\pi a}{b}\right) \cos \frac{2n-1}{2}\frac{\pi y}{b} \,.$$

Finally, let $h' = \sum_{\mu=0}^{1} \sum_{\nu=0}^{1} h'_{\mu\nu}$, where $h'_{\mu\nu}$ is given by (7.13) with coefficients which satisfy (7.14). Then both h' and Gh' are explicitly computable, and h' gives an approximation to $h = P\hat{w}$ (see (7.12)) with an error which can be made, a priori, arbitrarily small. As noted earlier, (7.3) then gives an explicitly computable approximation to u and (7.4) the corresponding error estimate.

Remark 7.1. – In the construction of $h'_{\mu\nu}$ (see (7.13) and (7.12)) the choice of M and N determines how small the quantity

$$[\Lambda_1 M^{-1} + \Lambda_2 N^{-1}] \, \| \hat{w}_{\mu\nu} \|$$

will be. Suppose one requires

$$[\Lambda_1 M^{-1} + \Lambda_2 N^{-1}] \, \| \hat{w}_{\mu\nu} \| < \varepsilon \, . \tag{7.15}$$

There are infinitely many choices of M and N which imply (7.15). We wish to find that choice which will make construction of $h'_{\mu\nu}$ simplest ; i.e., such that M + N is smallest. Put another way, for the sum M + N fixed, we wish to find that choice of M and N which will make the left hand side of (7.15) smallest.

If (see (7.11)) one minimizes the function $(a/b)^{1/2} x^{-1} + (b/a)^{1/2} y^{-1}$ subject to the constraint $x + y = L$, one finds the minimum is $(ab)^{-1/2}(\sqrt{a}+\sqrt{b})^2 L^{-1}$ and this is attained when $x = \sqrt{a}(\sqrt{a}+\sqrt{b})^{-1}L$, $y = \sqrt{b}(\sqrt{a}+\sqrt{b})^{-1}L$. Thus, for fixed L, the "best" choice for M and N such that $L \leqslant M + N \leqslant L + 1$ would be the smallest integers such that $M \geqslant \sqrt{a}(\sqrt{a}+\sqrt{b})^{-1}L$ and $N \geqslant \sqrt{b}(\sqrt{a}+\sqrt{b})^{-1}L$. In this case the left hand side of (7.15) becomes

$$C(c)\,(K+1)\,[(b/a)^{1/2} M_0^{3/2} + (a/b)^{1/2} N_0^{3/2}]\,(ab)^{-1/2}(\sqrt{a}+\sqrt{b})^2 L^{-1},$$

and the error estimate (7.12) takes the form

$$\| h_{\mu\nu} - h'_{\mu\nu} \| \leqslant 2(\cos\theta_{\mu\nu})^{2K+1} \, \| \hat{w}_{\mu\nu} \| + C(c)\,(K+1)\,(ab)^{-1/2} \left[\left(\frac{b}{a}\right)^{1/2} M_0^{3/2} + \left(\frac{a}{b}\right)^{1/2} N_0^{3/2} \right] (\sqrt{a}+\sqrt{b})^2 L^{-1} \, . \tag{7.16}$$

In particular the preceding considerations show that it is advantageous to choose M and N so that their ratio is approximately $(a/b)^{1/2}$. Of course we must also always choose $M \geqslant M_0$ and $N \geqslant N_0$.

Remark 7.2. – Solving equations (7.14) requires solving M + N linear equations in M + N unknowns. However, the number of equations can be reduced considerably. For instance if $M \geqslant N$, we have

$$s_m^{\mu\nu} = \| U_m^{\mu\nu} \|^{-1} \left\{ (\hat{w}_{\mu\nu}, U_m^{\mu\nu}) - \sum_{n=1}^{N} t_n^{\mu\nu} (U_m^{\mu\nu}, V_n^{\mu\nu}) \right\}, \quad m = 1, \dots, M,$$

where $t_1^{\mu\nu}, \dots, t_N^{\mu\nu}$ satisfy the N equations

$$t_n^{\mu\nu} \| V_n^{\mu\nu} \|^2 - \sum_{k=1}^{N} \sum_{m=1}^{M} \| U_m^{\mu\nu} \|^{-1} (U_m^{\mu\nu}, V_k^{\mu\nu}) (U_m^{\mu\nu}, V_n^{\mu\nu}) t_k^{\mu\nu} =$$

$$= (\hat{w}_{\mu\nu}, V_n^{\mu\nu}) - \sum_{m=1}^{M} \| U_m^{\mu\nu} \|^{-1} (U_m^{\mu\nu}, V_n^{\mu\nu}) (\hat{w}_{\mu\nu}, U_m^{\mu\nu}) .$$

Note also that the above formulas (as well as (7.14) are unchanged if $\hat{w}_{\mu\nu}$ is replaced by $\hat{w}$.

Remark 7.3. – It follows from (7.1) that

$$\Delta u = (I - P) w + \Delta v = w - h + \Delta v ,$$

while (7.3) implies that

$$\Delta u_0 = \hat{w} - h' + \Delta v .$$

Thus

$$\| \Delta (u - u_0) \| = \| \Delta u - \Delta u_0 \| \leqslant \| w - \hat{w} \| + \| h - h' \| ,$$

and, since $\| \Delta u \| \leqslant \| (1 - P) w \| + \| \Delta v \| \leqslant \| w \| + \| \Delta v \|$,

$$| \| \Delta u \|^2 - \| \Delta u_0 \|^2 | = | \| \Delta u + \Delta (u_0 - u) \|^2 - \| \Delta u \|^2 | \leqslant$$

$$\leqslant 2 | (\Delta u , \Delta (u_0 - u)) | + \| \Delta (u_0 - u) \|^2$$

$$\leqslant 2 (\| \Delta u \| + \| \Delta (u_0 - u) \|) \| \Delta (u_0 - u) \|$$

$$\leqslant 2 (\| w \| + \| \Delta v \| + \| \Delta (u - u_0) \|) \| \Delta (u - u_0) \|$$

can be estimated a priori using (7.12) or (7.16). Therefore $\|\Delta u_0\|^2$ gives an approximation to the energy $\|\Delta u\|^2$ of system (0.1) and the error can be estimated a priori.

8. A posteriori estimates.

As mentioned in the introduction, the a priori error estimates derived in § 7 are so large as to be mainly of theoretical interest only. However, the method used to derive these estimates makes no use of the cancellations inherent in the definition of $S^K_{\mu\nu}$ (see (7.5)). Thus it is extremely likely that the a priori bound (7.12) obtained for $\|h_{\mu\nu} - h'_{\mu\nu}\|$ is much larger than the actual error ; i.e., that M and N could actually be chosen much smaller than (7.12) would indicate in order to insure that $\|h_{\mu\nu} - h'_{\mu\nu}\|$ is small.

It thus becomes all the more important from a practical point of view that an alternative method is available for approximating the solution u which makes use of quite accurate *a posteriori* estimates.

Specifically, according to (7.1) the solution u is given by the formula

$$u = v + \mathrm{G}w - \mathrm{GP}w \ ,$$

and as seen in § 7, v is a known function and $\mathrm{G}w$ can be easily approximated. Thus in order to approximate u it suffices to approximate $h = \mathrm{P}w$ by an explicitly constructible function h' such that $\mathrm{G}h'$ is also explicitly constructible. (Note that here we use the function w itself to define h rather than the approximating function $\hat{w}$ as in § 7.)

For any choice of M and N, such an approximating function $h' = \sum_{\mu=0}^{1} \sum_{\nu=0}^{1} h'_{\mu\nu}$ can be chosen as in § 7 by taking $h'_{\mu\nu}$ to be the projection of w onto the M + N dimensional subspace $\mathcal{K}_{\mu\nu}$ of $\mathcal{H}_{\mu\nu}$ spanned by $U_1^{\mu\nu}, \ldots, U_M^{\mu\nu}, V_1^{\mu\nu}, \ldots, V_N^{\mu\nu}$. $h'_{\mu\nu}$ is given by (7.13), where the coefficients are determined by solving the linear system (7.14) with $\hat{w}_{\mu\nu}$ replaced by $w_{\mu\nu}$ (see also Remark 7.2). The resulting error $\|h_{\mu\nu} - h'_{\mu\nu}\| = \|\mathrm{P}_{\mu\nu}(w - h'_{\mu\nu})\|$ can then be approximated *a posteriori* using the inequalities

$$\max[\| P^{(1)}_{\mu\nu}(w - h'_{\mu\nu}) \| \, , \, \| P^{(2)}_{\mu\nu}(w - h'_{\mu\nu}) \|] \leqslant$$
$$\leqslant \| h_{\mu\nu} - h'_{\mu\nu} \| \leqslant$$
$$\leqslant (1 - \cos^2 \theta_{\mu\nu})^{-1} [\| P^{(1)}_{\mu\nu}(w - h'_{\mu\nu}) \| + \| P^{(2)}_{\mu\nu}(w - h'_{\mu\nu}) \|] \, , \quad (8.1)$$

which follow from (2.4) and Theorem 3.1.

Suppose then one wishes to choose $h'_{\mu\nu}$ so that $\| h_{\mu\nu} - h'_{\mu\nu} \| < \varepsilon$. The approximation procedure would be as follows : For a first approximation, one could, for example, choose M and N sufficiently large that $\| (P^{(j)}_{\mu\nu} - \hat{P}^{(j)}_{\mu\nu}) w \| < \varepsilon$ for $j = 1, 2$, where $\hat{P}^{(1)}_{\mu\nu}$ and $\hat{P}^{(2)}_{\mu\nu}$ are the orthogonal projections of $L^2(R_{a,b})$ onto the spaces spanned by $\{U^{\mu\nu}_m : m = 1, \ldots, M\}$ and by $\{V^{\mu\nu}_n : n = 1, \ldots, N\}$, respectively. (It seems likely heuristically that M and N should be taken at least this large.) Find the function $h'_{\mu\nu}$ corresponding to this choice of M and N, compute $\| P^{(j)}_{\mu\nu}(w - h'_{\mu\nu}) \|$ (which can be done to any desired accuracy), and use (8.1) to evaluate the error a posteriori.

If (8.1) does not show that $\| h_{\mu\nu} - h'_{\mu\nu} \| < \varepsilon$, then M and N are replaced by larger integers $M' = M + M_1$, $N' = N + N_1$, and $h'_{\mu\nu}$ is replaced by a new approximation $h''_{\mu\nu}$ corresponding to M' and N'. (In view of Remark 7.1 it would seem reasonable to choose these new values M' and N' so that the quotient $M'N'^{-1}$ is approximately equal to $(a^{1/2} b^{-1/2}$.) This procedure is continued until the desired accuracy is insured.

In following this procedure, it would perhaps be advantageous to apply the Gram-Schmidt orthonormalization process to the basis vectors $U^{\mu\nu}_1, \ldots, U^{\mu\nu}_M, V^{\mu\nu}_1, \ldots, V^{\mu\nu}_N$ to obtain an orthonormal basis $W^{\mu\nu}_1, \ldots, W^{\mu\nu}_{M+N}$ for $\mathcal{K}_{\mu\nu}$. Although this process entails considerable effort, the expression for $h'_{\mu\nu}$ in terms of the new basis is immediate :

$$h'_{\mu\nu} = \sum_{m=1}^{M+N} (w \, , W^{\mu\nu}_m) \, W^{\mu\nu}_m \, .$$

More important, if the resulting a posteriori error bound shows that M and N are too small, then rather than starting all over from the beginning using the larger values $M' = M + M_1$, $N' = N + N_1$, one need only to *continue* the orthonormalization procedure to obtain (from the augmented basis $U^{\mu\nu}_1, \ldots, U^{\mu\nu}_M, V^{\mu\nu}_1, \ldots, V^{\mu\nu}_N, U^{\mu\nu}_{M+1}, \ldots,$ $U^{\mu\nu}_{M+M_1}, V^{\mu\nu}_{N+1}, \ldots, V^{\mu\nu}_{N+N_1}$) an orthonormal basis $W^{\mu\nu}_1, \ldots, W^{\mu\nu}_{M'+N'}$

for the new space $\mathcal{K}'_{\mu\nu}$ corresponding to M′ and N′, and the improved approximation $h''_{\mu\nu} = \sum_{m=1}^{M'+N'} (w, W_m^{\mu\nu}) W_m^{\mu\nu}$ to $h_{\mu\nu}$. In this way very little of the effort expended at the initial stage (using M and N) has been wasted.

Remark 8.1. – As noted in the introduction, other ways are known for decomposing $\mathcal{H}_{\mu\nu}$ in case $\mu = \nu$. Let $\mathcal{C}$ be the subspace of constant functions on $R_{a,b}$ and, considering the elements of $\mathcal{H}$ as functions of $z = x + iy$, define

$$\hat{\mathcal{H}}_{\mu\nu}^{(1)} = \{h \in \mathcal{H}_{\mu\nu} : h \text{ can be analytically extended to the band } |y| < b \text{ periodically of period } 2a\},$$

$$\hat{\mathcal{H}}_{\mu\nu}^{(2)} = \{h \in \mathcal{H}_{\mu\nu} : h \text{ can be analytically extended to the band } |x| < a \text{ periodically of period } 2b\} \cap \mathcal{C}^{\perp}.$$

It was shown in [3] that for $(\mu, \nu) = (0,0)$ or $(1,1)$, that

$$\mathcal{H}_{\mu\nu} = \hat{\mathcal{H}}_{\mu\nu}^{(1)} + \hat{\mathcal{H}}_{\mu\nu}^{(2)}$$

and that $\cos \hat{\theta}_{\mu\nu} = \frac{2}{\pi}$, where $\hat{\theta}_{\mu\nu}$ is the minimal angle between $\hat{\mathcal{H}}_{\mu\nu}^{(1)}$ and $\hat{\mathcal{H}}_{\mu\nu}^{(2)}$. Moreover, complete orthogonal bases $\{\hat{U}_m^{\mu\nu}\}$ and $\{\hat{V}_n^{\mu\nu}\}$ for $\hat{\mathcal{H}}_{\mu\nu}^{(1)}$ and $\hat{\mathcal{H}}_{\mu\nu}^{(2)}$ are known, so that the techniques of sections 7 and 8 could be applied to the decomposition $\mathcal{H}_{\mu\nu} = \hat{\mathcal{H}}_{\mu\nu}^{(1)} + \hat{\mathcal{H}}_{\mu\nu}^{(2)}$ for $\mu = \nu$. It would seem, in fact, that this would lead to some improvement in our results, at least for the case $\mu = \nu = 0$, since those error bounds which involve $\cos \theta_{\mu\nu}$ would surely be improved if $\cos \hat{\theta}_{00} = \frac{2}{\pi}$ were substituted for $\cos \theta_{00} \leqslant .8$.

The drawback in this approach is that the basis functions

$$\hat{U}_m^{00}(x, y) = \cos \frac{m\pi x}{a} \cosh \frac{m\pi y}{a}, \quad m = 0, 1, \ldots,$$

$$\hat{V}_n^{00}(x, y) = \cosh \frac{n\pi x}{b} \cos \frac{n\pi y}{b}, \quad n = 1, 2, \ldots,$$

are not as convenient for our purposes as are $\{U_m^{00}\}$ and $\{V_n^{00}\}$. This is due to the fact that the functions $G\hat{U}_m^{00}$, $G\hat{V}_n^{00}$ cannot be given explicitly in closed form in terms of elementary functions.

Appendix.

In this appendix we give the results of numerical calculations performed, for each of the four parity cases, on a clamped plate $R_{a,b}$ with $a = 2\pi$, $b = \pi$. The computations were programmed (and the calculations supervised) by Professor R. G. Hetherington of the University of Kansas computer science department.

We took as data $F \equiv 1$ in the positive quarter and $F \equiv \pm 1$, depending on the parity, in the other quarters of $R_{a,b}$. To establish the approximate solution $\hat{u}$ we used the projection onto the subspace $\mathcal{H}_{\mu\nu}$ generated by $U_m^{\mu\nu}$, $m = 1, \ldots, 12$, and $V_n^{\mu\nu}$, $n = 1, \ldots, 8$. The value of $\hat{u}$ was calculated at the points $(0, 0)$, $(0, \pi/2)$, $(\pi, 0)$, $(\pi, \pi/2)$ for parity $(0, 0)$, at the points $(0, \pi/2)$, $(\pi, \pi/2)$ for parity $(0, 1)$, at the points $(\pi, 0)$, $(\pi, \pi/2)$ for parity $(1, 0)$, and at the point $(\pi, \pi/2)$ for parity $(1, 1)$. The results are listed below :

Parity $(0, 0)$:
$$3.82 < u(0, 0) < 3.88$$
$$2.11 < u(0, \pi/2) < 2.17$$
$$3.03 < u(\pi, 0) < 3.10$$
$$1.76 < u(\pi, \pi/2) < 1.82$$
$$107.8600 < \|\Delta u\|^2 < 107.8608$$

Parity $(0, 1)$:
$$0.36 < u(0, \pi/2) < 0.41$$
$$0.53 < u(\pi, \pi/2) < 0.59$$
$$18.3511 < \|\Delta u\|^2 < 18.3516$$

Parity $(1, 0)$:
$$2.42 < u(\pi, 0) < 2.47$$
$$1.42 < u(\pi, \pi/2) < 1.47$$
$$65.1830 < \|\Delta u\|^2 < 65.1834$$

Parity $(1, 1)$:
$$0.44 < u(\pi, \pi/2) < 0.48$$
$$14.3069 < \|\Delta u\|^2 < 14.3072 \,.$$

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