SIAM J. Sci. Comput.
Vol. 22, No. 5, pp. 1549–1569

A LEGENDRE SPECTRAL GALERKIN METHOD FOR THE BIHARMONIC DIRICHLET PROBLEM*

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Abstract. A Legendre spectral Galerkin method is presented for the solution of the biharmonic Dirichlet problem on a square. The solution and its Laplacian are approximated using the set of basis functions suggested by Shen, which are linear combinations of two Legendre polynomials. A Schur complement approach is used to reduce the resulting linear system to one involving the approximation of the Laplacian of the solution on the two vertical sides of the square. The Schur complement system is solved by a preconditioned conjugate gradient method or the Cholesky method. The total cost of the algorithm is $O(N^3)$. Numerical results demonstrate the spectral convergence of the method.

Key words. biharmonic Dirichlet problem, spectral Galerkin method, Schur complement matrix, preconditioned conjugate gradient method, Cholesky method

AMS subject classifications. 65N35, 65N22

PII. S1064827598342407

1. Introduction.

The aim of this work is to propose a Legendre spectral Galerkin method for the solution of the biharmonic Dirichlet problem. This problem has been the subject of several studies in recent years. Our work is related to the papers of Shen [7] and Bjørstad and Tjøstheim [3], whose approach is based on the standard variational formulation of the fourth order differential equation. Our approach is based on the mixed method of Ciarlet and Raviart [4] which gives rise to a variational formulation for two second order differential equations. A similar approach has been recently applied in the Legendre spectral collocation solution of the same problem [2].

The formulation of the biharmonic Legendre spectral Galerkin problem and the method of solution are similar to those developed in [5] for the finite element Galerkin method with piecewise Hermite bicubics.

In this study we consider the biharmonic Dirichlet problem

$$\Delta^2 u = f \text{ in } \Omega, \qquad u = \partial u/\partial n = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where Δ denotes the Laplacian, $\Omega = (-1,1) \times (-1,1)$, $\partial\Omega$ is the boundary of Ω, and $\partial/\partial n$ is the outward normal derivative on $\partial\Omega$.

We set $v = \Delta u$ and discretize a coupled pair of Poisson's equations in u and v using a Galerkin method with polynomials of degree $\leq N$. Employing a Schur complement approach, we reduce the Galerkin problem to a Schur complement system involving an approximation to v on the two vertical sides of $\partial\Omega$ and an auxiliary Galerkin problem for a related biharmonic problem with v, instead of $\partial u/\partial n$, specified on the two vertical sides of $\partial\Omega$. The Schur complement system with a symmetric positive definite matrix is solved by the preconditioned conjugate gradient (PCG) method

*Received by the editors July 27, 1998; accepted for publication (in revised form) June 22, 2000; published electronically November 17, 2000.

http://www.siam.org/journals/sisc/22-5/34240.html

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or the Cholesky method. A preconditioner for PCG is obtained from the Galerkin problem for a related biharmonic problem with v instead of $\partial u/\partial n$ specified on the two horizontal sides of $\partial\Omega$. We conjecture that the preconditioner is spectrally equivalent to the Schur complement matrix. The cost of multiplying the Schur complement matrix by a vector is $4N^2 + O(N)$ and the cost of solving a linear system with the preconditioner is $O(N)$. With the number of PCG iterations equal to $c \log N$, the dominant cost of solving the Schur complement system is $c4N^2 \log N$. The solution of the auxiliary Galerkin problem is obtained at a cost $2N^3 + O(N^2)$ using separation of variables and the solution of a simple eigenvalue problem which reduces to two symmetric eigenvalue problems with tridiagonal matrices. Hence the dominant cost of our PCG algorithm is $2N^3 + c4N^2 \log N$. The dominant cost of our Cholesky algorithm is $8N^3/3$. Both algorithms are well suited for parallel implementation since many of their steps involve independent matrix-vector multiplications. Numerical results demonstrate the spectral convergence rate of the approximations to u and v in the maximum norm.

In section 2 we introduce two polynomial spaces, the corresponding basis functions, and Galerkin matrices. We also discuss the solution of the linear system corresponding to a one-dimensional biharmonic problem. The spectral Galerkin biharmonic problem and its solution are discussed in sections 3 and 4, respectively. Numerical results are presented in section 5 and concluding remarks are given in section 6.

2. Preliminaries. For an integer $N \geq 2$, let

$$P_N^0 = \{p \in P_N : p(\pm 1) = 0\},$$

where P_k denotes the set of polynomials of degree $\leq k$ on $(-1, 1)$. (Note that the dimensions of P_N and P_N^0 are $N + 1$ and $N - 1$, respectively.) Following (2.7) of [7], we introduce the basis $\{\phi_k\}_{k=2}^N$ for P_N^0 with

$$\phi_k(x) = \frac{1}{\sqrt{4k-2}}[L_{k-2}(x) - L_k(x)], \qquad k = 2, \ldots, N,$$

where $L_k(x)$ is the kth degree Legendre polynomial normalized by $\int_{-1}^1 L_k^2(x)\,dx = 2/(2k + 1)$. Augmenting the basis $\{\phi_k\}_{k=2}^N$ for P_N^0 by

$$\phi_0(x) = \frac{\sqrt{6}}{2} L_0(x), \qquad \phi_1(x) = \frac{3\sqrt{10}}{2} L_1(x), \tag{2.1}$$

we obtain the basis $\{\phi_k\}_{k=0}^N$ for P_N.

With $(p, q) = \int_{-1}^1 (pq)(x)\,dx$, we introduce the Galerkin matrices

$$A = [(\phi_i', \phi_k')]_{i,k=2}^N\,, \qquad B = [(\phi_i, \phi_k)]_{i,k=2}^N\,, \tag{2.2}$$

$$A_t = [(\phi_i', \phi_k')]_{i=2,k=0}^{N,1}\,, \qquad B_t = [(\phi_i, \phi_k)]_{i=2,k=0}^{N,1}\,, \tag{2.3}$$

and

$$B_s = [(\phi_i, \phi_k)]_{i,k=0}^1\,, \tag{2.4}$$

where i and k are the row and column indices, respectively. It follows from Lemma 2.1 in [7] that

$$A = I_{N-1}, \qquad B = \begin{bmatrix} \times & & \times & & & & \\ & \times & & \times & & & \\ \times & & \times & & \times & & \\ & \ddots & & \ddots & & \ddots & \\ & & \times & & \times & & \times \\ & & & \times & & \times & \\ & & & & \times & & \times \end{bmatrix}, \tag{2.5}$$

where I_k is the $k \times k$ identity matrix and the symbols $\times$ denote the only nonzero matrix coefficients. The matrix B is clearly symmetric and positive definite.

Since $\phi_k(\pm 1) = 0$, $k = 2, \ldots, N$, and $\phi_0'' = \phi_1'' = 0$, using integration by parts and the orthonormality of the Legendre polynomials we obtain

$$A_t = O, \qquad B_t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \qquad B_s = \begin{bmatrix} 3 & 0 \\ 0 & 15 \end{bmatrix}. \tag{2.6}$$

For $\lambda > 0$ consider the linear system

$$\begin{aligned} R_{11}\left[\vec{u}, \vec{v}\right]^T + R_{12}\left[v_0, v_1\right]^T &= \left[\vec{g}, \vec{f}\right]^T, \\ R_{21}\left[\vec{u}, \vec{v}\right]^T + R_{22}[v_0, v_1]^T &= [g_0, g_1]^T, \end{aligned} \tag{2.7}$$

where

$$\vec{u} = [u_2, \ldots, u_N]^T, \qquad \vec{v} = [v_2, \ldots, v_N]^T,$$

$$\vec{g} = [g_2, \ldots, g_N]^T, \qquad \vec{f} = [f_2, \ldots, f_N]^T,$$

$$R_{11} = \begin{bmatrix} B + \lambda I_{N-1} & \lambda B \\ O & B + \lambda I_{N-1} \end{bmatrix}, \tag{2.8}$$

and

$$R_{12} = \begin{bmatrix} \lambda B_t \\ B_t \end{bmatrix}, \qquad R_{21} = \begin{bmatrix} B_t^T, \lambda B_t^T \end{bmatrix}, \qquad R_{22} = \lambda B_s. \tag{2.9}$$

Assume that N is even and introduce the vectors

$$\vec{u}^{(0)} = [u_2, u_4, \ldots, u_N]^T, \qquad \vec{u}^{(1)} = [u_3, u_5, \ldots, u_{N-1}]^T.$$

For $i = 0, 1$, let the vectors $\vec{v}^{(i)}$, $\vec{g}^{(i)}$, and $\vec{f}^{(i)}$ be defined in a similar way. Since B in (2.5) consists of two tridiagonal matrices B_0 and B_1 and since B_s in (2.6) is diagonal, (2.7) splits into two systems

$$\begin{aligned} A_i\left[\vec{u}^{(i)}, \vec{v}^{(i)}\right]^T + \vec{p} v_i &= \left[\vec{g}^{(i)}, \vec{f}^{(i)}\right]^T, \\ \vec{q}\left[\vec{u}^{(i)}, \vec{v}^{(i)}\right]^T + d_i v_i &= g_i \end{aligned} \tag{2.10}$$

for $i = 0, 1$, where

$$A_i = \begin{bmatrix} B_i + \lambda I_{N/2-i} & \lambda B_i \\ O & B_i + \lambda I_{N/2-i} \end{bmatrix} \tag{2.11}$$

and

$$\vec{p} = [\lambda, 0, \dots, 0, 1, 0, \dots, 0]^T, \quad \vec{q} = [1, 0, \dots, 0, \lambda, 0, \dots, 0], \quad d_0 = 3\lambda, \quad d_1 = 15\lambda. \tag{2.12}$$

It can be shown that the matrix in (2.10) is nonsingular. Moreover, A_i is nonsingular since $B_i + \lambda I_{N/2-i}$ is positive definite. Solving the first equation of (2.10) for $[\vec{u}^{(i)}, \vec{v}^{(i)}]$ and substituting it into the second equation of (2.10), we obtain

$$v_i = (g_i - \vec{s}^{(i)}[\vec{g}^{(i)}, \vec{f}^{(i)}]^T)/\delta_i, \tag{2.13}$$

where

$$\vec{s}^{(i)} = \vec{q}A_i^{-1}, \quad \delta_i = d_i - \vec{s}^{(i)}\vec{p}. \tag{2.14}$$

The vector $\vec{s}^{(i)}$ and the number δ_i can be computed with costs $O(N)$ and $O(1)$, respectively. Once the number v_i of (2.13) has been evaluated at cost $O(N)$, $\vec{u}^{(i)}$ and $\vec{v}^{(i)}$ of (2.10) can be obtained with cost $O(N)$ by solving two systems with the same tridiagonal positive definite matrix $B_i + \lambda I_{N/2-i}$. Therefore, the cost of solving (2.7) is $O(N)$.

Throughout the paper we use the symbol $\otimes$ to denote the tensor product of matrices and the tensor product of function spaces. Let I, J, K, and L be finite sets of increasing indices. Without loss of generality we assume

$$I = \{1, \dots, M_1\}, \quad K = \{1, \dots, N_1\}, \quad J = \{1, \dots, M_2\}, \quad L = \{1, \dots, N_2\}.$$

Then the matrix-vector form of

$$z_{i,j} = \sum_{k \in K} c_{i,k}^{(1)} \sum_{l \in L} c_{j,l}^{(2)} w_{k,l}, \quad i \in I, \quad j \in J, \tag{2.15}$$

is

$$\vec{z} = (C_1 \otimes C_2)\vec{w}, \tag{2.16}$$

where

$$C_1 = (c_{i,k}^{(1)})_{i \in I, k \in K}, \quad C_2 = (c_{j,l}^{(2)})_{j \in J, l \in L},$$

and

$$\vec{z} = [z_{1,1}, \dots, z_{1,M_2}, \dots, z_{M_1,1}, \dots, z_{M_1,M_2}]^T,$$
$$\vec{w} = [w_{1,1}, \dots, w_{1,N_2}, \dots, w_{N_1,1}, \dots, w_{N_1,N_2}]^T.$$

3. Biharmonic spectral Galerkin problem. Introducing $v = \Delta u$ in (1.1), we obtain the coupled problem (see Figure 1)

$$\begin{aligned} -\Delta u + v = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad \partial u/\partial n = 0 \text{ on } \partial\Omega, \\ -\Delta v = -f \text{ in } \Omega. \end{aligned} \tag{3.1}$$

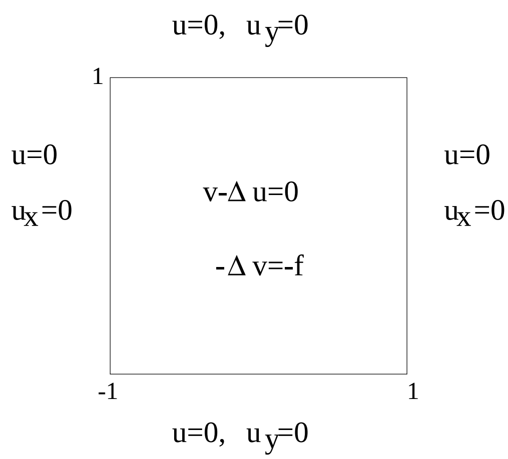

FIG. 1. *Coupled problem* (3.1).

The weak form of (3.1), obtained using Green's formula, is

$$\int_\Omega \nabla u \nabla \eta \, d\Omega + \int_\Omega v\eta \, d\Omega = 0, \qquad \eta \in H^1(\Omega),$$
$$\int_\Omega \nabla v \nabla \delta \, d\Omega = -\int_\Omega f\delta \, d\Omega, \qquad \delta \in H_0^1(\Omega).$$

Let

$$X_N = \{w \in P_N \otimes P_N : w(\alpha, \beta) = 0, \alpha, \beta = \pm 1\}, \qquad X_N^0 = P_N^0 \otimes P_N^0.$$

The corner conditions $w(\alpha, \beta) = 0$, $\alpha, \beta = \pm 1$, in the definition of X_N are motivated by the fact that

$$v(\alpha, \beta) = u_{xx}(\alpha, \beta) + u_{yy}(\alpha, \beta) = 0, \qquad \alpha, \beta = \pm 1.$$

The Legendre spectral Galerkin problem for (3.1) consists of finding $U \in X_N^0$ and $V \in X_N$ such that

$$\int_\Omega \nabla U \nabla \eta \, d\Omega + \int_\Omega V\eta \, d\Omega = 0, \qquad \eta \in X_N, \tag{3.2}$$

$$\int_\Omega \nabla V \nabla \delta \, d\Omega = -\int_\Omega f\delta \, d\Omega, \qquad \delta \in X_N^0. \tag{3.3}$$

THEOREM 3.1. *There exist unique $U \in X_N^0$ and $V \in X_N$ satisfying* (3.2)–(3.3).

Proof. Since in (3.2)–(3.3) the number of constraints is equal to the number of degrees of freedom, we assume that $f = 0$ and show that $U = V = 0$. Taking $\eta = V$ in (3.2) and $\delta = U$ in (3.3), we obtain

$$\int_\Omega \nabla U \nabla V \, d\Omega + \int_\Omega V^2 \, d\Omega = 0, \qquad \int_\Omega \nabla V \nabla U \, d\Omega = 0,$$

which gives $\int_\Omega V^2 \, d\Omega = 0$, $V = 0$. With $V = 0$ and $\eta = U$, (3.2) becomes $\int_\Omega \nabla U \nabla U \, d\Omega = 0$, which implies $U = 0$ by the Poincaré inequality. ∎

Using the basis functions $\{\phi_k\}_{k=0}^N$ of section 2 for $U \in X_N^0$ and $V \in X_N$, we have

$$U(x,y) = \sum_{k=2}^{N}\sum_{l=2}^{N} u_{k,l}\phi_k(x)\phi_l(y) \tag{3.4}$$

and

$$V(x,y) = \sum_{k=2}^{N}\sum_{l=2}^{N} v_{k,l}\phi_k(x)\phi_l(y) + \sum_{k=2}^{N}\sum_{l=0}^{1} v_{k,l}\phi_k(x)\phi_l(y) + \sum_{k=0}^{1}\sum_{l=2}^{N} v_{k,l}\phi_k(x)\phi_l(y), \tag{3.5}$$

where we used properties of ϕ_0 and ϕ_1 of (2.1) in the derivation of (3.5). Corresponding to (3.4) and (3.5) we introduce the vectors

$$\vec{u} = [u_{2,2}, \dots, u_{2,N}, \dots, u_{N,2}, \dots, u_{N,N}]^T, \tag{3.6}$$

$$\vec{v} = [v_{2,2}, \dots, v_{2,N}, \dots, v_{N,2}, \dots, v_{N,N}]^T, \tag{3.7}$$

$$\vec{v}_h = [v_{2,0}, v_{2,1}, \dots, v_{N,0}, v_{N,1}]^T, \tag{3.8}$$

$$\vec{v}_v = [v_{0,2}, \dots, v_{0,N}, v_{1,2}, \dots, v_{1,N}]^T. \tag{3.9}$$

Substituting (3.4), (3.5) into (3.2)–(3.3), taking $\eta = \phi_i(x)\phi_j(y)$, $i,j = 2,\dots,N$, $\delta = \phi_i(x)\phi_j(y)$, $i,j = 2,\dots,N$, $\eta = \phi_i(x)\phi_j(y)$, $i = 2,\dots,N$, $j = 0,1$, $\eta = \phi_i(x)\phi_j(y)$, $i = 0,1$, $j = 2,\dots,N$, and using (2.15), (2.16), (2.2)–(2.6), we obtain, respectively,

$$\begin{gathered}
(I_{N-1}\otimes B + B\otimes I_{N-1})\vec{u} + (B\otimes B)\vec{v} + (B\otimes B_t)\vec{v}_h + (B_t\otimes B)\vec{v}_v = \vec{0},\\
(I_{N-1}\otimes B + B\otimes I_{N-1})\vec{v} + (I_{N-1}\otimes B_t)\vec{v}_h + (B_t\otimes I_{N-1})\vec{v}_v = \vec{f},\\
(I_{N-1}\otimes B_t^T)\vec{u} + (B\otimes B_t^T)\vec{v} + (B\otimes B_s)\vec{v}_h + (B_t\otimes B_t^T)\vec{v}_v = \vec{0},\\
(B_t^T\otimes I_{N-1})\vec{u} + (B_t^T\otimes B)\vec{v} + (B_t^T\otimes B_t)\vec{v}_h + (B_s\otimes B)\vec{v}_v = \vec{0},
\end{gathered} \tag{3.10}$$

where

$$\vec{f} = [f_{2,2}, \dots, f_{2,N}, \dots, f_{N,2}, \dots, f_{N,N}]^T \tag{3.11}$$

and

$$f_{i,j} = -\int_\Omega f\phi_i(x)\phi_j(y)\,d\Omega.$$

Using the N-point Gauss–Legendre quadrature in the x and y coordinates, with nodes $\{\xi_k\}_{k=1}^N$ and weights $\{w_k\}_{k=1}^N$, we approximate $\vec{f}$ by

$$\vec{f} \approx -(CD\otimes CD)\left[f(\xi_1,\xi_1), \dots, f(\xi_1,\xi_N), \dots, f(\xi_N,\xi_1), \dots, f(\xi_N,\xi_N)\right]^T, \tag{3.12}$$

where C and D are given by

$$C = [\phi_i(\xi_k)]_{i=2,k=1}^{N,N}, \qquad D = \text{ diag } (w_1, \dots, w_N).$$

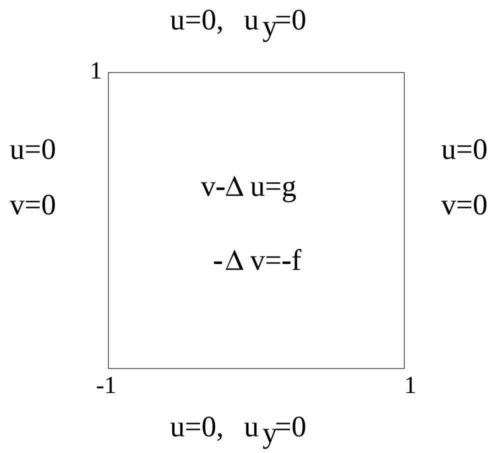

FIG. 2. *Auxiliary problem* (4.1).

The computation of the right-hand side of (3.12) requires $O(N^3)$ operations. The vector $\vec{f}$ can be also approximated in a slightly different and more efficient way using the approach described in [6].

Equations (3.10) can be written as

$$\begin{aligned} S_{11}\left[\vec{u},\vec{v},\vec{v}_h\right]^T + S_{12}\vec{v}_v &= \left[\vec{0},\vec{f},\vec{0}\right]^T, \\ S_{21}\left[\vec{u},\vec{v},\vec{v}_h\right]^T + S_{22}\vec{v}_v &= \vec{0}, \end{aligned} \tag{3.13}$$

where

$$S_{11} = \begin{bmatrix} I_{N-1}\otimes B + B\otimes I_{N-1} & B\otimes B & B\otimes B_t \\ O & I_{N-1}\otimes B + B\otimes I_{N-1} & I_{N-1}\otimes B_t \\ I_{N-1}\otimes B_t^T & B\otimes B_t^T & B\otimes B_s \end{bmatrix}, \tag{3.14}$$

$$S_{12} = \begin{bmatrix} B_t\otimes B \\ B_t\otimes I_{N-1} \\ B_t\otimes B_t^T \end{bmatrix}, \tag{3.15}$$

$$S_{21} = \begin{bmatrix} B_t^T\otimes I_{N-1}, B_t^T\otimes B, B_t^T\otimes B_t \end{bmatrix}, \tag{3.16}$$

$$S_{22} = B_s\otimes B. \tag{3.17}$$

In the remainder of the paper we describe an algorithm for solving (3.13) assuming that $\vec{f}$ or its approximation is given.

4. Solving the biharmonic linear system.

4.1. Inverse of S_{11}. The matrix S_{11} of (3.14) arises in the spectral Galerkin method for the auxiliary problem (see Figure 2)

$$\begin{aligned} -\Delta u + v = g \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \qquad \partial u/\partial n = 0 \text{ on } \partial\Omega_h, \\ -\Delta v = -f \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega_v, \end{aligned} \tag{4.1}$$

where $\partial\Omega_h$ is the union of the horizontal sides of $\partial\Omega$ and $\partial\Omega_v$ is the union of the vertical sides of $\partial\Omega$. The weak form of (4.1) is

$$\int_\Omega \nabla u \nabla \eta \, d\Omega + \int_\Omega v\eta \, d\Omega = \int_\Omega g\eta \, d\Omega, \qquad \eta \in H^1(\Omega), \qquad \eta = 0 \text{ on } \partial\Omega_v,$$
$$\int_\Omega \nabla v \nabla \delta \, d\Omega = -\int_\Omega f\delta \, d\Omega, \qquad \delta \in H_0^1(\Omega).$$

The spectral Galerkin problem for (4.1) consists of finding $U \in X_N^0$ and $V \in P_N^0 \otimes P_N$ such that

$$\begin{aligned} \int_\Omega \nabla U \nabla \eta \, d\Omega + \int_\Omega V\eta \, d\Omega &= \int_\Omega g\eta \, d\Omega, \qquad \eta \in P_N^0 \otimes P_N, \\ \int_\Omega \nabla V \nabla \delta \, d\Omega &= -\int_\Omega f\delta \, d\Omega, \qquad \delta \in X_N^0. \end{aligned} \tag{4.2}$$

LEMMA 4.1. *The matrix S_{11} of* (3.14) *is nonsingular.*

Proof. It can be shown, using an approach similar to the proof of Theorem 3.1, that (4.2) has a unique solution. □

Since B of (2.2) is symmetric, it is orthogonally similar to a diagonal matrix, that is, there exist real $Z = (z_{i,j})_{i,j=2}^N$ and real

$$\Lambda = \operatorname{diag}(\lambda_2, \ldots, \lambda_N) \tag{4.3}$$

such that

$$Z^T Z = I_{N-1}, \qquad Z^T B Z = \Lambda. \tag{4.4}$$

It follows from the structure of B in (2.5) that the computation of Λ and Z satisfying (4.3) and (4.4) reduces to solving two symmetric eigenvalue problems with tridiagonal matrices. With the use of the QR algorithm for evaluating eigenvalues and the inverse iteration for evaluating the corresponding eigenvectors, Λ and Z can be precomputed with cost $O(N^2)$. Moreover, the coefficients of Z are such that

$$z_{i,j} = 0 \quad \text{if} \quad i+j \text{ is odd}. \tag{4.5}$$

Let

$$W = \begin{bmatrix} Z \otimes I_{N-1} & O & O \\ O & Z \otimes I_{N-1} & O \\ O & O & Z \otimes I_2 \end{bmatrix}. \tag{4.6}$$

Then using (3.14) and (4.4), we obtain

$$W^T S_{11} W = G,$$

where

$$G = \begin{bmatrix} I_{N-1} \otimes B + \Lambda \otimes I_{N-1} & \Lambda \otimes B & \Lambda \otimes B_t \\ O & I_{N-1} \otimes B + \Lambda \otimes I_{N-1} & I_{N-1} \otimes B_t \\ I_{N-1} \otimes B_t^T & \Lambda \otimes B_t^T & \Lambda \otimes B_s \end{bmatrix}. \tag{4.7}$$

Since W is orthogonal, it follows that

$$S_{11}^{-1} = W G^{-1} W^T. \tag{4.8}$$

Let the vectors $\vec{u}$, $\vec{v}$, $\vec{v}_h$, and $\vec{f}$ have the forms given in (3.6), (3.7), (3.8), and (3.11), respectively, and let

$$\vec{g} = [g_{2,2}, \ldots, g_{2,N}, \ldots, g_{N,2}, \ldots, g_{N,N}]^T, \qquad \vec{g}_h = [g_{2,0}, g_{2,1}, \ldots, g_{N,0}, g_{N,1}]^T. \tag{4.9}$$

In the following, the vectors $\vec{u}'$, $\vec{v}'$, $\vec{v}_h'$, $\vec{f}'$, $\vec{g}'$, and $\vec{g}_h'$ have the same forms as $\vec{u}$, $\vec{v}$, $\vec{v}_h$, $\vec{f}$, $\vec{g}$, and $\vec{g}_h$, respectively. The components of the primed vectors are denoted by the primed letters corresponding to the unprimed vectors. For example,

$$\vec{u}' = [u_{2,2}', \ldots, u_{2,N}', \ldots, u_{N,2}', \ldots, u_{N,N}']^T, \qquad \vec{v}_h' = [v_{2,0}', v_{2,1}', \ldots, v_{N,0}', v_{N,1}']^T.$$

Consider the computation of

$$[\vec{u}', \vec{v}', \vec{v}_h']^T = G^{-1}[\vec{g}', \vec{f}', \vec{g}_h']^T. \tag{4.10}$$

It follows from (4.7) that (4.10) is equivalent to the system

$$\begin{aligned}
(I_{N-1} \otimes B + \Lambda \otimes I_{N-1})\vec{u}' + (\Lambda \otimes B)\vec{v}' + (\Lambda \otimes B_t)\vec{v}_h' &= \vec{g}', \\
(I_{N-1} \otimes B + \Lambda \otimes I_{N-1})\vec{v}' + (I_{N-1} \otimes B_t)\vec{v}_h' &= \vec{f}', \\
(I_{N-1} \otimes B_t^T)\vec{u}' + (\Lambda \otimes B_t^T)\vec{v}' + (\Lambda \otimes B_s)\vec{v}_h' &= \vec{g}_h',
\end{aligned}$$

which, from (4.3), becomes

$$\begin{aligned}
R_{11}^{(k)} \left[\vec{u}_{k,\cdot}', \vec{v}_{k,\cdot}'\right]^T + R_{12}^{(k)} \left[v_{k,0}', v_{k,1}'\right]^T &= \left[\vec{g}_{k,\cdot}', \vec{f}_{k,\cdot}'\right]^T, \\
R_{21}^{(k)} \left[\vec{u}_{k,\cdot}', \vec{v}_{k,\cdot}'\right]^T + R_{22}^{(k)} [v_{k,0}', v_{k,1}']^T &= [g_{k,0}', g_{k,1}']^T
\end{aligned} \tag{4.11}$$

for $k = 2, \ldots, N$, where the $R_{ij}^{(k)}$ are given by (2.8), (2.9) with λ replaced by λ_k, and

$$\vec{u}_{k,\cdot}' = [u_{k,2}', \ldots, u_{k,N}']^T, \qquad \vec{v}_{k,\cdot}' = [v_{k,2}', \ldots, v_{k,N}']^T,$$

$$\vec{g}_{k,\cdot}' = [g_{k,2}', \ldots, g_{k,N}']^T, \qquad \vec{f}_{k,\cdot}' = [f_{k,2}', \ldots, f_{k,N}']^T.$$

Since B is positive definite, it follows from the second equation in (4.4) that Λ is positive definite and hence, by (4.3), $\lambda_k > 0$, $k = 2, \ldots, N$. Clearly (4.11) is of the same form as (2.7) with λ replaced by λ_k. Hence it follows from the discussion in section 2 that $[\vec{u}', \vec{v}', \vec{v}_h']^T$ of (4.10) can be computed with cost $O(N^2)$. Also it follows from (2.13) that in the special case of $\vec{g}' = \vec{f}' = \vec{0}$, $\vec{v}_h'$ of (4.10) can be obtained with cost $O(N)$.

4.2. Description of the algorithm. Since S_{11} is nonsingular (see Lemma 4.1), eliminating $[\vec{u}, \vec{v}, \vec{v}_h]^T$ from (3.13), and using (4.8), (4.6), we obtain

$$S\vec{v}_v = -S_{21}WG^{-1}\left[\vec{0}, (Z^T \otimes I_{N-1})\vec{f}, \vec{0}\right]^T, \tag{4.12}$$

where S is the $2(N-1) \times 2(N-1)$ Schur complement matrix given by

$$S = S_{22} - S_{21}S_{11}^{-1}S_{12}. \tag{4.13}$$

(S is the Schur complement of S_{11} in $\left[\begin{smallmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{smallmatrix}\right]$.) It also follows from (3.13), (4.8), and (4.6) that

$$[\vec{u}, \vec{v}, \vec{v}_h]^T = WG^{-1}\{[\vec{0}, (Z^T \otimes \vec{f}, \vec{0})] - W^T S_{12}\vec{v}_v\}. \tag{4.14}$$

Using (4.12), (4.14), and (4.6) we arrive at the following algorithm for solving (3.13).

ALGORITHM.

Step 1. Compute $\vec{f}' = (Z^T \otimes I_{N-1})\vec{f}$.

Step 2. Compute $[\vec{u}', \vec{v}', \vec{v}_h']^T = G^{-1}[\vec{0}, \vec{f}', \vec{0}]^T$.

Step 3. Compute $\vec{r} = -S_{21}[\vec{u}, \vec{v}, \vec{v}_h]^T$, where $\vec{u} = (Z \otimes I_{N-1})\vec{u}'$, $\vec{v} = (Z \otimes I_{N-1})\vec{v}'$, $\vec{v}_h = (Z \otimes I_2)\vec{v}_h'$.

Step 4. Solve $S\vec{v}_v = \vec{r}$ for $\vec{v}_v$.

Step 5. Compute $\vec{g}' = (Z^T \otimes I_{N-1})\vec{g}$, $\vec{f}'' = (Z^T \otimes I_{N-1})\vec{f}$, $\vec{g}_h' = (Z^T \otimes I_2)\vec{g}_h$, where $[\vec{g}, \vec{f}, \vec{g}_h]^T = S_{12}\vec{v}_v$.

Step 6. Compute $[\vec{g}'', \vec{f}''', \vec{g}_h'']^T = [\vec{0}, \vec{f}', \vec{0}]^T - [\vec{g}', \vec{f}'', \vec{g}_h']^T$.

Step 7. Compute $[\vec{u}', \vec{v}', \vec{v}_h']^T = G^{-1}[\vec{g}'', \vec{f}''', \vec{g}_h'']^T$.

Step 8. Compute $\vec{u} = (Z \otimes I_{N-1})\vec{u}'$, $\vec{v} = (Z \otimes I_{N-1})\vec{v}'$,$\vec{v}_h = (Z \otimes I_2)\vec{v}_h'$.

It follows from (4.5) that the cost of Step 1 is $N^3 + O(N^2)$ and the cost of Step 8 is $2N^3 + O(N^2)$. However, if the approximation to $v = \Delta u$ is not required, then the cost of Step 8 is only $N^3 + O(N^2)$. It follows from the discussion in section 4.1 that the costs of Steps 2 and 7 are $O(N^2)$ each. Clearly, the cost of Step 6 is also $O(N^2)$.

Let $\vec{u}$, $\vec{v}$, and $\vec{v}_h$ be of the forms given in (3.6), (3.7), and (3.8), respectively, and let, for $k = 2, \dots, N$,

$$\vec{u}_{k,\cdot} = [u_{k,2}, \dots, u_{k,N}]^T, \qquad \vec{v}_{k,\cdot} = [v_{k,2}, \dots, v_{k,N}]^T.$$

Then it follows from (3.16) and (2.6) that

$$S_{21}[\vec{u}, \vec{v}, \vec{v}_h]^T = [\vec{u}_{2,\cdot}, \vec{u}_{3,\cdot}]^T + [B\vec{v}_{2,\cdot}, B\vec{v}_{3,\cdot}]^T + [v_{2,0}, v_{2,1}, 0, \dots, 0, v_{3,0}, v_{3,1}, 0, \dots, 0]^T. \tag{4.15}$$

Hence the cost of Step 3 is $O(N^2)$.

Let $\vec{v}_v$, $\vec{f}$, and $\vec{g}$, $\vec{g}_h$ be of the forms given in (3.9), (3.11), and (4.9), respectively. Let

$$\vec{v}_v^{(0)} = [v_{0,2}, \dots, v_{0,N}]^T, \qquad \vec{v}_v^{(1)} = [v_{1,2}, \dots, v_{1,N}]^T, \tag{4.16}$$

and let, for $k = 2, \dots, N$,

$$\vec{f}_{k,\cdot} = [f_{k,2}, \dots, f_{k,N}]^T, \qquad \vec{g}_{k,\cdot} = [g_{k,2}, \dots, g_{k,N}]^T.$$

Then it follows from (3.15) and (2.6) that for

$$[\vec{g}, \vec{f}, \vec{g}_h] = S_{12}\vec{v}_v, \tag{4.17}$$

we have

$$\vec{g}_{2,\cdot} = B\vec{v}_v^{(0)}, \quad \vec{g}_{3,\cdot} = B\vec{v}_v^{(1)}, \qquad \vec{f}_{2,\cdot} = \vec{v}_v^{(0)}, \quad \vec{f}_{3,\cdot} = \vec{v}_v^{(1)}, \qquad \vec{f}_{k,\cdot} = \vec{g}_{k,\cdot} = \vec{0}, \quad k = 4, \dots, N, \tag{4.18}$$

$$g_{2,0} = v_{0,2}, \quad g_{2,1} = v_{0,3} \qquad g_{3,0} = v_{1,2}, \quad g_{3,1} = v_{1,3}, \qquad g_{k,0} = g_{k,1} = 0, \quad k = 4, \dots, N. \tag{4.19}$$

Hence the cost of Step 5 is $O(N^2)$.

In the next subsection we discuss in more detail Step 4.

4.3. Solving systems with S.

THEOREM 4.1. *The matrix S of* (4.13) *is symmetric and positive definite.*

Proof. For $n = 1, 2$, and arbitrary

$$\vec{v}_v^{(n)} = [v_{0,2}^{(n)}, \dots, v_{0,N}^{(n)}, v_{1,2}^{(n)}, \dots, v_{1,N}^{(n)}]^T, \tag{4.20}$$

using (4.13), we obtain

$$(S\vec{v}_v^{(1)}, \vec{v}_v^{(2)})_{R^{2N-2}} = (S_{22}\vec{v}_v^{(1)}, \vec{v}_v^{(2)})_{R^{2N-2}} - (S_{21}S_{11}^{-1}S_{12}\vec{v}_v^{(1)}, \vec{v}_v^{(2)})_{R^{2N-2}}. \tag{4.21}$$

Let

$$\vec{u}^{(n)} = [u_{2,2}^{(n)}, \dots, u_{2,N}^{(n)}, \dots, u_{N,2}^{(n)}, \dots, u_{N,N}^{(n)}]^T, \tag{4.22}$$

$$\vec{v}^{(n)} = [v_{2,2}^{(n)}, \dots, v_{2,N}^{(n)}, \dots, v_{N,2}^{(n)}, \dots, v_{N,N}^{(n)}]^T, \tag{4.23}$$

$$\vec{v}_h^{(n)} = [v_{2,0}^{(n)}, v_{2,1}^{(n)}, \dots, v_{N,0}^{(n)}, v_{N,1}^{(n)}]^T$$

be such that

$$S_{11}[\vec{u}^{(n)}, \vec{v}^{(n)}, \vec{v}_h^{(n)}]^T + S_{12}\vec{v}_v^{(n)} = \vec{0}. \tag{4.24}$$

Existence and uniqueness of $\vec{u}^{(n)}$, $\vec{v}^{(n)}$, and $\vec{v}_h^{(n)}$ follow from the nonsingularity of S_{11} (see Lemma 4.1). Then (4.21), (4.24), (3.17), and (3.16) give

$$\begin{aligned}(S\vec{v}_v^{(1)}, \vec{v}_v^{(2)})_{R^{2N-2}} &= (S_{22}\vec{v}_v^{(1)}, \vec{v}_v^{(2)})_{R^{2N-2}} + (S_{21}[\vec{u}^{(1)}, \vec{v}^{(1)}, \vec{v}_h^{(1)}]^T, \vec{v}_v^{(2)})_{R^{2N-2}} \\ &= ((B_s \otimes B)\vec{v}_v^{(1)}, \vec{v}_v^{(2)})_{R^{2N-2}} + ((B_t^T \otimes I_{N-1})\vec{u}^{(1)}, \vec{v}_v^{(2)})_{R^{2N-2}} \\ &\quad + ((B_t^T \otimes B)\vec{v}^{(1)}, \vec{v}_v^{(2)})_{R^{2N-2}} + ((B_t^T \otimes B_t)\vec{v}_h^{(1)}, \vec{v}_v^{(2)})_{R^{2N-2}}.\end{aligned} \tag{4.25}$$

Equation (4.24) is the matrix-vector form of the spectral Galerkin problem

$$\begin{aligned}\int_\Omega \nabla U^{(n)} \nabla \eta \, d\Omega + \int_\Omega V^{(n)} \eta \, d\Omega = 0, \quad \eta \in P_N^0 \otimes P_N, \\ \int_\Omega \nabla V^{(n)} \nabla \delta \, d\Omega = 0, \quad \delta \in X_N^0,\end{aligned} \tag{4.26}$$

where

$$U^{(n)}(x, y) = \sum_{k=2}^{N} \sum_{l=2}^{N} u_{k,l}^{(n)} \phi_k(x)\phi_l(y), \tag{4.27}$$

$$V^{(n)}(x, y) = V_i^{(n)}(x, y) + V_h^{(n)}(x, y) + V_v^{(n)}(x, y), \tag{4.28}$$

and

$$V_i^{(n)}(x, y) = \sum_{k=2}^{N} \sum_{l=2}^{N} v_{k,l}^{(n)} \phi_k(x)\phi_l(y), \tag{4.29}$$

$$V_h^{(n)}(x, y) = \sum_{k=2}^{N} \sum_{l=0}^{1} v_{k,l}^{(n)} \phi_k(x)\phi_l(y), \tag{4.30}$$

$$V_v^{(n)}(x,y) = \sum_{k=0}^{1}\sum_{l=2}^{N} v_{k,l}^{(n)}\phi_k(x)\phi_l(y). \tag{4.31}$$

Since $V_i^{(2)} + V_h^{(2)} \in P_N^0 \otimes P_N$ and $U^{(1)} \in X_N^0$, it follows from (4.28) and (4.26) that

$$\begin{aligned}\int_\Omega V^{(1)}V^{(2)}\,d\Omega &= \int_\Omega V^{(1)}V_v^{(2)}\,d\Omega + \int_\Omega V^{(1)}\left(V_i^{(2)} + V_h^{(2)}\right)\,d\Omega\\ &= \int_\Omega V^{(1)}V_v^{(2)}\,d\Omega - \int_\Omega \nabla U^{(1)}\nabla\left(V^{(2)} - V_v^{(2)}\right)\,d\Omega\\ = \int_\Omega V_i^{(1)}V_v^{(2)}\,d\Omega + \int_\Omega V_h^{(1)}V_v^{(2)}\,d\Omega &+ \int_\Omega V_v^{(1)}V_v^{(2)}\,d\Omega + \int_\Omega \nabla U^{(1)}\nabla V_v^{(2)}\,d\Omega.\end{aligned} \tag{4.32}$$

Using (4.29), (4.31), (2.15), (2.16), (2.3), and (2.2) we have

$$\begin{aligned}\int_\Omega V_i^{(1)}V_v^{(2)}\,d\Omega &= \int_\Omega\left[\sum_{k=2}^{N}\sum_{l=2}^{N} v_{k,l}^{(1)}\phi_k(x)\phi_l(y)\right]\left[\sum_{i=0}^{1}\sum_{j=2}^{N} v_{i,j}^{(2)}\phi_i(x)\phi_j(y)\right]\,d\Omega\\ &= \sum_{i=0}^{1}\sum_{j=2}^{N}\left[\sum_{k=2}^{N}\int_{-1}^{1}\phi_i(x)\phi_k(x)\,dx\sum_{l=2}^{N}\int_{-1}^{1}\phi_j(y)\phi_l(y)\,dy v_{k,l}^{(1)}\right] v_{i,j}^{(2)}\\ &= ((B_t^T\otimes B)\vec{v}^{(1)}, \vec{v}_v^{(2)})_{R^{2N-2}}.\end{aligned} \tag{4.33}$$

In a similar way, using (4.30), (4.31), (4.27), (2.15), (2.16), and (2.2)–(2.4), we obtain

$$\int_\Omega V_h^{(1)}V_v^{(2)}\,d\Omega = ((B_t^T\otimes B_t)\vec{v}_h^{(1)}, \vec{v}_v^{(2)})_{R^{2N-2}}, \tag{4.34}$$

$$\int_\Omega V_v^{(1)}V_v^{(2)}\,d\Omega = ((B_s\otimes B)\vec{v}_v^{(1)}, \vec{v}_v^{(2)})_{R^{2N-2}}, \tag{4.35}$$

$$\int_\Omega \nabla U^{(1)}V_v^{(2)}\,d\Omega = ((B_t^T\otimes I_{N-1})\vec{u}^{(1)}, \vec{v}_v^{(2)})_{R^{2N-2}}. \tag{4.36}$$

Hence, it follows from (4.25) and (4.32)–(4.36) that

$$(S\vec{v}_v^{(1)}, \vec{v}_v^{(2)})_{R^{2N-2}} = \int_\Omega V^{(1)}V^{(2)}\,d\Omega, \tag{4.37}$$

which implies

$$(S\vec{v}_v^{(1)}, \vec{v}_v^{(2)})_{R^{2N-2}} = (S\vec{v}_v^{(2)}, \vec{v}_v^{(1)})_{R^{2N-2}}, \qquad \vec{v}_v^{(1)}, \vec{v}_v^{(2)} \in R^{2N-2}.$$

This proves the symmetry of S.

For $\vec{v}_v^{(1)} = \vec{v}_v^{(2)}$, (4.37) gives

$$(S\vec{v}_v^{(1)}, \vec{v}_v^{(1)})_{R^{2N-2}} = \int_\Omega [V^{(1)}]^2\,d\Omega.$$

This and the linear independence of the basis functions in $V^{(1)}$ of (4.28)–(4.31) show that S is positive definite. $\square$

It follows from Theorem 4.1 that the PCG method is a good candidate for solving a linear system with S. Therefore, in the following, we discuss the matrix-vector multiplication involving S, the selection of a preconditioner, and the solution of a linear system with this preconditioner.

It follows from (4.13) that in order to multiply by S, we have to multiply by S_{22} and $S_{21}S_{11}^{-1}S_{12}$. Assume that N is even. Let $\vec{v}_v$ and $\vec{v}_v^{(0)}$, $\vec{v}_v^{(1)}$ be as in (3.9) and (4.16), respectively. Then it follows from (3.17) and (2.6) that

$$S_{22}\vec{v}_v = [3B\vec{v}_v^{(0)}, 15B\vec{v}_v^{(1)}]^T. \tag{4.38}$$

Using (4.17)–(4.19), (4.8), (4.10), (4.15), (4.6), (4.5), and (4.11) it can be shown that

$$S_{21}S_{11}^{-1}S_{12}\vec{v}_v = [\vec{u}_{2,\cdot}, \vec{u}_{3,\cdot}]^T + [B\vec{v}_{2,\cdot}, B\vec{v}_{3,\cdot}]^T + [v_{2,0}, v_{2,1}, 0, \dots, 0, v_{3,0}, v_{3,1}, 0, \dots, 0]^T, \tag{4.39}$$

where

$$[\vec{u}_{2,\cdot}, \vec{v}_{2,\cdot}, v_{2,0}, v_{2,1}]^T = \sum_{k=1}^{N/2} z_{2,2k}^2 [R^{(2k)}]^{-1}[B\vec{v}_v^{(0)}, \vec{v}_v^{(0)}, v_{0,2}, v_{0,3}]^T, \tag{4.40}$$

$$[\vec{u}_{3,\cdot}, \vec{v}_{3,\cdot}, v_{3,0}, v_{3,1}]^T = \sum_{k=1}^{N/2-1} z_{3,2k+1}^2 [R^{(2k+1)}]^{-1}[B\vec{v}_v^{(1)}, \vec{v}_v^{(1)}, v_{1,2}, v_{1,3}]^T, \tag{4.41}$$

$$R^{(k)} = \begin{bmatrix} R_{11}^{(k)} & R_{12}^{(k)} \\ R_{21}^{(k)} & R_{22}^{(k)} \end{bmatrix},$$

and $R_{ij}^{(k)}$ are given by (2.8), (2.9) with λ replaced by λ_k. We next analyze (4.40). The analysis of (4.41) is similar. Introducing

$$[\vec{u}^{(2k)}, \vec{v}^{(2k)}, v_0^{(2k)}, v_1^{(2k)}]^T = [R^{(2k)}]^{-1}[B\vec{v}_v^{(0)}, \vec{v}_v^{(0)}, v_{0,2}, v_{0,3}]^T$$

and using (4.40), we obtain

$$[\vec{u}_{2,\cdot}, \vec{v}_{2,\cdot}]^T = \sum_{k=1}^{N/2} z_{2,2k}^2 [R_{11}^{(2k)}]^{-1}\left\{[B\vec{v}_v^{(0)}, \vec{v}_v^{(0)}]^T - R_{12}^{(2k)}[v_0^{(2k)}, v_1^{(2k)}]^T\right\}, \tag{4.42}$$

$$v_{2,j} = \sum_{k=1}^{N/2} z_{2,2k}^2 v_j^{(2k)}, \qquad j = 0, 1.$$

Further, (2.8) and (4.4) imply that

$$[R_{11}^{(2k)}]^{-1} = (I_2 \otimes Z)D_{2k}^{-1}(I_2 \otimes Z^T),$$

where

$$D_{2k} = \begin{bmatrix} \Lambda + \lambda_{2k}I_{N-1} & \lambda_{2k}\Lambda \\ O & \Lambda + \lambda_{2k}I_{N-1} \end{bmatrix}.$$

Hence (4.42) becomes

$$[\vec{u}_{2,\cdot}, \vec{v}_{2,\cdot}]^T = (I_2 \otimes Z) \left\{ D_0 [Z^T B \vec{v}_v^{(0)}, Z^T \vec{v}_v^{(0)}]^T - \sum_{k=1}^{N/2} z_{2,2k}^2 \left(v_0^{(2k)} \vec{p}^{(2k)} + v_1^{(2k)} \vec{q}^{(2k)} \right) \right\}, \tag{4.43}$$

where

$$D_0 = \sum_{k=1}^{N/2} z_{2,2k}^2 D_{2k}^{-1}, \qquad [\vec{p}^{(2k)}, \vec{q}^{(2k)}] = D_{2k}^{-1} (I_2 \otimes Z^T) R_{12}^{(2k)}. \tag{4.44}$$

The presence of B in (4.38), $I_2 \otimes Z$ in (4.43), and (4.4) suggest that we introduce

$$\hat{S} = (I_2 \otimes \Lambda^{-1/2} Z^T) S (I_2 \otimes Z \Lambda^{-1/2}), \quad \hat{r} = (I_2 \otimes \Lambda^{-1/2} Z^T) \vec{r}, \quad \hat{v}_v = (I_2 \otimes \Lambda^{1/2} Z^T) \vec{v}_v, \tag{4.45}$$

and solve the system

$$\hat{S} \hat{v}_v = \hat{r} \tag{4.46}$$

instead of $S\vec{v}_v = \vec{r}$. This approach for computing $\vec{v}_v$ involves one additional multiplication by $I_2 \otimes \Lambda^{-1/2} Z^T$ to obtain $\hat{r}$ and one additional multiplication by $I_2 \otimes Z \Lambda^{-1/2}$ to recover $\vec{v}_v$ from $\hat{v}_v$. However, multiplication of a vector by $\hat{S}$ takes only $4N^2 + O(N)$ operations provided that additional diagonal matrices related to D_0, D_1, and additional vectors related to $\vec{p}^{(2k)}$, $\vec{q}^{(2k)}$, $k = 1, \ldots, N/2$, $\vec{p}^{(2k+1)}$, $\vec{q}^{(2k+1)}$, $k = 1, \ldots, N/2 - 1$, are precomputed with cost $O(N^2)$. (Here D_1 and $\vec{p}^{(2k+1)}$, $\vec{q}^{(2k+1)}$ are counterparts of D_0 and $\vec{p}^{(2k)}$, $\vec{q}^{(2k)}$ for (4.41).)

In the remainder of this section we select a preconditioner for $\hat{S}$ and discuss the solution of a linear system with this preconditioner. First, to select the preconditioner for S of (4.13), we consider the linear system

$$\begin{aligned} (I_{N-1} \otimes B + B \otimes I_{N-1}) \vec{u} + (B \otimes B) \vec{v} + (B_t \otimes B) \vec{v}_v &= \vec{0}, \\ (I_{N-1} \otimes B + B \otimes I_{N-1}) \vec{v} + (B_t \otimes I_{N-1}) \vec{v}_v &= \vec{0}, \\ (B_t^T \otimes I_{N-1}) \vec{u} + (B_t^T \otimes B) \vec{v} + (B_s \otimes B) \vec{v}_v &= \vec{g}_v, \end{aligned} \tag{4.47}$$

where

$$\vec{g}_v = [g_{0,2}, \ldots, g_{0,N}, g_{1,2}, \ldots, g_{1,N}]^T.$$

Note that the matrix in (4.47) is the same as S_{11} of (3.14) but with the roles of the x and y coordinates interchanged. The matrix of (4.47) arises in the spectral Galerkin method for the auxiliary problem (see Figure 3)

$$\begin{aligned} -\Delta u + v = g \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \qquad \partial u / \partial n = 0 \text{ on } \partial\Omega_v, \\ -\Delta v = -f \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega_h, \end{aligned} \tag{4.48}$$

which only differs from (4.1) in the roles of the x and y coordinates. Equations (4.47) can be written as

$$\begin{aligned} P_{11} [\vec{u}, \vec{v}]^T + P_{12} \vec{v}_v &= \vec{0}, \\ P_{21} [\vec{u}, \vec{v}]^T + P_{22} \vec{v}_v &= \vec{g}_v, \end{aligned} \tag{4.49}$$

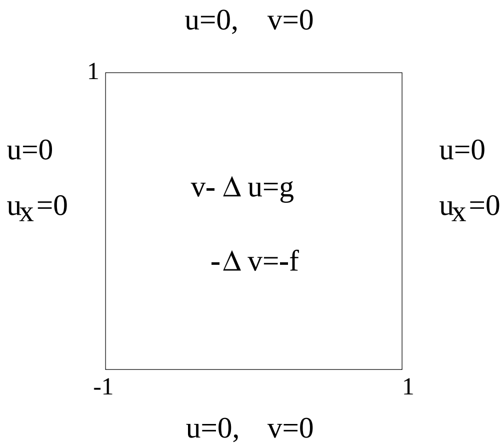

FIG. 3. *Auxiliary problem* (4.48).

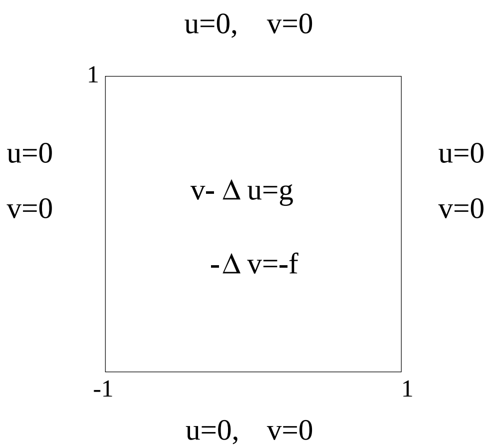

FIG. 4. *Decoupled problem* (4.52).

where

$$P_{11} = \begin{bmatrix} I_{N-1} \otimes B + B \otimes I_{N-1} & B \otimes B \\ O & I_{N-1} \otimes B + B \otimes I_{N-1} \end{bmatrix}, \tag{4.50}$$

and

$$P_{12} = \begin{bmatrix} B_t \otimes B \\ B_t \otimes I_{N-1} \end{bmatrix}, \qquad P_{21} = \begin{bmatrix} B_t^T \otimes I_{N-1}, B_t^T \otimes B \end{bmatrix}, \qquad P_{22} = \begin{bmatrix} B_s \otimes B \end{bmatrix}. \tag{4.51}$$

The matrix P_{11} arises in the spectral Galerkin method for the decoupled problem (see Figure 4)

$$\begin{aligned} -\Delta u + v &= g \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \\ -\Delta v &= -f \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega. \end{aligned} \tag{4.52}$$

The weak form of (4.52) is

$$\int_\Omega \nabla u \nabla \eta \, d\Omega + \int_\Omega v\eta \, d\Omega = \int_\Omega g\eta \, d\Omega, \qquad \eta \in H_0^1(\Omega),$$

$$\int_\Omega \nabla v \nabla \delta \, d\Omega = -\int_\Omega f\delta \, d\Omega, \qquad \delta \in H_0^1(\Omega),$$

and the spectral Galerkin problem consists of finding $U, V \in X_N^0$ such that

$$(4.53)\qquad \begin{aligned} \int_\Omega \nabla U \nabla \eta \, d\Omega + \int_\Omega V\eta \, d\Omega &= \int_\Omega g\eta \, d\Omega, \qquad && \eta \in X_N^0, \\ \int_\Omega \nabla V \nabla \delta \, d\Omega &= -\int_\Omega f\delta \, d\Omega, \qquad && \delta \in X_N^0. \end{aligned}$$

LEMMA 4.2. *The matrix* P_{11} *of* (4.50) *is nonsingular.*

Proof. It is easy to show that (4.53) has a unique solution. ◻

Since P_{11} is nonsingular, eliminating $[\vec{u}, \vec{v}]^T$ from (4.49), we obtain

$$(4.54)\qquad P\vec{v}_v = \vec{g}_v,$$

where the $2(N-1) \times 2(N-1)$ Schur complement matrix

$$(4.55)\qquad P = P_{22} - P_{21} P_{11}^{-1} P_{12}.$$

(P is the Schur complement of P_{11} in $\left[\begin{smallmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{smallmatrix}\right]$.)

THEOREM 4.2. *The matrix* P *is symmetric and positive definite.*

Proof. Proof of this theorem is similar to that of Theorem 4.1. For $n = 1, 2$, and arbitrary $\vec{v}_v^{(n)}$ of the form (4.20), let $\vec{u}^{(n)}$ of the form (4.22) and $\vec{v}^{(n)}$ of the form (4.23) be such that

$$(4.56)\qquad P_{11}[\vec{u}^{(n)}, \vec{v}^{(n)}]^T + P_{12}\vec{v}_v^{(n)} = \vec{0}.$$

Then (4.55), (4.56), and (4.51) give

$$(4.57)\qquad \begin{aligned} (P\vec{v}_v^{(1)}, \vec{v}_v^{(2)})_{R^{2N-2}} &= (P_{22}\vec{v}_v^{(1)}, \vec{v}_v^{(2)})_{R^{2N-2}} + (P_{21}[\vec{u}^{(1)}, \vec{v}^{(1)}]^T, \vec{v}_v^{(2)})_{R^{2N-2}} \\ &= ((B_s \otimes B)\vec{v}_v^{(1)}, \vec{v}_v^{(2)})_{R^{2N-2}} + ((B_t^T \otimes I_{N-1})\vec{u}^{(1)}, \vec{v}_v^{(2)})_{R^{2N-2}} \\ &\quad + ((B_t^T \otimes B)\vec{v}^{(1)}, \vec{v}_v^{(2)})_{R^{2N-2}}. \end{aligned}$$

Equation (4.56) is the matrix-vector form of the spectral Galerkin problem

$$(4.58)\qquad \begin{aligned} \int_\Omega \nabla U^{(n)} \nabla \eta \, d\Omega + \int_\Omega V^{(n)} \eta \, d\Omega &= 0, \quad && \eta \in X_N^0, \\ \int_\Omega \nabla V^{(n)} \nabla \delta \, d\Omega &= 0, \quad && \delta \in X_N^0, \end{aligned}$$

where $U^{(n)}(x, y)$ is of the from (4.27),

$$(4.59)\qquad V^{(n)}(x, y) = V_i^{(n)}(x, y) + V_v^{(n)}(x, y),$$

and $V_i^{(n)}(x, y)$, $V_v^{(n)}(x, y)$ are of the forms (4.29), (4.31), respectively. Since $V_i^{(2)}, U^{(1)} \in X_N^0$, it follows from (4.59) and (4.58) that

$$(4.60)\qquad \begin{aligned} \int_\Omega V^{(1)} V^{(2)} \, d\Omega &= \int_\Omega V^{(1)} V_v^{(2)} \, d\Omega + \int_\Omega V^{(1)} V_i^{(2)} \, d\Omega \\ &= \int_\Omega V^{(1)} V_v^{(2)} \, d\Omega - \int_\Omega \nabla U^{(1)} \nabla \left(V^{(2)} - V_v^{(2)} \right) \, d\Omega \\ &= \int_\Omega V_i^{(1)} V_v^{(2)} \, d\Omega + \int_\Omega V_v^{(1)} V_v^{(2)} \, d\Omega + \int_\Omega \nabla U^{(1)} \nabla V_v^{(2)} \, d\Omega. \end{aligned}$$

TABLE 1
$\kappa_2(\hat{P}^{-1/2}\hat{S}\hat{P}^{-1/2})$ and $\kappa_2(\hat{S}_i)$, $i = 1, 2, 3, 4$.

N	16	32	64	128
$\kappa_2(\hat{P}^{-1/2}\hat{S}\hat{P}^{-1/2})$	1.65	1.67	1.68	1.68
$\kappa_2(\hat{S}_1)$	28.68	104.91	400.84	1566.64
$\kappa_2(\hat{S}_2)$	24.39	90.99	350.59	1375.56
$\kappa_2(\hat{S}_3)$	13.93	53.43	209.14	827.35
$\kappa_2(\hat{S}_4)$	13.57	52.54	206.82	820.78

Hence, it follows from (4.57), (4.60), (4.33), (4.35), and (4.36) that

$$(P\vec{v}_v^{(1)}, \vec{v}_v^{(2)})_{R^{2N-2}} = \int_\Omega V^{(1)}V^{(2)}\,d\Omega,$$

which implies the symmetry and positive definiteness of P. $\square$

We take P of (4.55) as a preconditioner for S and

$$\hat{P} = (I_2 \otimes \Lambda^{-1/2}Z^T)P(I_2 \otimes Z\Lambda^{-1/2})$$

as a preconditioner for $\hat{S}$ of (4.45). For arbitrary $\vec{g}_v$, the solution of (4.54) can be obtained by solving (4.49) for $\vec{v}_v$. Equivalently, we compute $\vec{v}_h$ of

$$[\vec{u}, \vec{v}, \vec{v}_h]^T = S_{11}^{-1}[\vec{0}, \vec{0}, \vec{g}_h]^T.$$

(Note that the roles of the x and y coordinates are interchanged, that is, for $k = 2, \ldots, N$, the components $g_{k,0}$, $g_{k,1}$ of $\vec{g}_h$ are equal to the components $g_{0,k}$, $g_{1,k}$ of $\vec{g}_v$ and the components $v_{0,k}$, $v_{1,k}$ of $\vec{v}_v$ are equal to the components $v_{k,0}$, $v_{k,1}$ of $\vec{v}_h$.) It follows from (4.8), (4.6), and the discussion in section 4.1 of the special case of (4.10) that the cost of solving a linear system with $\hat{P}$ is $O(N)$.

With the preconditioner $\hat{P}$, the convergence rate of the PCG method applied to a linear system with $\hat{S}$ depends on $\kappa_2(\hat{P}^{-1/2}\hat{S}\hat{P}^{-1/2})$, where, for a symmetric and positive definite matrix M, $\kappa_2(M) = \lambda_{max}(M)/\lambda_{min}(M)$. A careful analysis shows that $\hat{S}$ splits into four symmetric positive definite matrices $\hat{S}_i$, $i = 1, 2, 3, 4$, and $\hat{P}$ splits into four positive definite diagonal matrices $\hat{P}_i$, $i = 1, 2, 3, 4$. ($\hat{S}_i$, $\hat{P}_i$, $i = 1, 2$, are of order $N/2 \times N/2$ and $\hat{S}_i$, $\hat{P}_i$, $i = 3, 4$, are of order $(N/2-1) \times (N/2-1)$.) Since our preconditioning of $\hat{S}$ by $\hat{P}$ is equivalent to preconditioning each $\hat{S}_i$ by $\hat{P}_i$, we have

$$\kappa_2(\hat{P}^{-1/2}\hat{S}\hat{P}^{-1/2}) = \max_{i=1,2,3,4} \lambda_{max}(\hat{P}_i^{-1/2}\hat{S}_i\hat{P}_i^{-1/2}) / \min_{i=1,2,3,4} \lambda_{min}(\hat{P}_i^{-1/2}\hat{S}_i\hat{P}_i^{-1/2}).$$

This last formula was used to compute $\kappa_2(\hat{P}^{-1/2}\hat{S}\hat{P}^{-1/2})$ numerically for several values of N. Based on the results presented in Table 1, we conjecture that $\kappa_2(\hat{P}^{-1/2}\hat{S}\hat{P}^{-1/2})$ is bounded from above by a positive constant which is independent of N while $\kappa(\hat{S}_i)$, $i = 1, 2, 3, 4$, grows quadratically with N.

Since the matrix $\hat{S}$ of (4.45) is symmetric and positive definite, the system (4.46), equivalently the corresponding systems with $\hat{S}_i$, $i = 1, 2, 3, 4$, can also be solved by Cholesky's method. The columns of $\hat{S}$, equivalently the columns of $\hat{S}_i$, $i = 1, 2, 3, 4$, can be obtained using (4.38) and (4.39) with one component of $\vec{v}_v$ equal to 1 and all remaining components equal to 0. All four matrices $\hat{S}_i$ can be formed with cost $N^3/2 + O(N^2)$ and factored out with cost $N^3/6 + O(N^2)$. Clearly, the cost of the solution stage is $O(N^2)$.

4.4. Cost and memory requirements of the algorithm. We now discuss the cost and memory requirements of solving (3.13) using the algorithm of section 4.2.

The preprocessing stage consists of computing

Λ, Z of (4.3), (4.4) using the LAPACK [1] routines dsteqr and dstein;

$\vec{s}^{(i)}$, δ_i of (2.14) and the factorization of $B_i + \lambda I_{N/2-i}$ for $\lambda = \lambda_k$, $k = 2, \dots, N$, and $i = 0, 1$;

the diagonal matrices related to D_0, D_1 and the vectors related to $\vec{p}^{(2k)}$, $\vec{q}^{(2k)}$, $k = 1, \dots, N/2$, $\vec{p}^{(2k+1)}$, $\vec{q}^{(2k+1)}$, $k = 1, \dots, N/2 - 1$, (cf. (4.44)).

The cost of these computations is $O(N^2)$ and the memory requirements are $11N^2/2$.

In the case of the Cholesky method for solving (4.46), the preprocessing stage also includes the computation of the matrices $\hat{S}_i$, $i = 1, 2, 3, 4$, and their factorizations. The cost of these computations is $2N^3/3$ and the memory requirements are N^2.

From the discussions in sections 4.2 and 4.3, it follows that the cost of the remaining portion of the PCG algorithm for finding $\vec{u}$ is

$$2N^3 + (\text{number of PCG iterations}) \times 4N^2, \tag{4.61}$$

where $4N^2$ is the cost of one PCG iteration. We select $\vec{0}$ as the initial guess for the PCG method. Our numerical tests indicate that the number of PCG iterations should be an integer multiple of $\log N$ if the accuracy of the PCG algorithm is to be comparable with that of the Cholesky algorithm. Hence the total cost of the PCG algorithm for finding $\vec{u}$ is given by (4.61) with the second term dominating the first one for small values of N.

The cost of the remaining portion of the Cholesky algorithm for finding $\vec{u}$ is $2N^3$ which gives the total cost $8N^3/3$ of this algorithm.

The memory requirements for the remaining portions of the PCG and Cholesky algorithms are $3N^2$.

For both the PCG and Cholesky algorithms, the additional cost of computing the vector $\vec{v}$ in Step 8 is $N^3 + O(N^2)$.

5. Numerical results. We solved (1.1) with

$$f(x, y) = 128\pi^4[\cos(4\pi x)\cos(4\pi y) - \sin^2(2\pi x)\cos(4\pi y) - \cos(4\pi x)\sin^2(2\pi y)].$$

The exact solution of this problem, which was also considered by Shen [7] and Bjørstad and Tjøstheim [3], is $u = \sin^2(2\pi x)\sin^2(2\pi y)$.

Our PCG and Cholesky algorithms were implemented in Fortran 77 and numerical experiments were carried out on an IBM RS6000 (processor type: Power PC 604/100MHz), for which the LINPACK TPP benchmark in MFLOPS is 56.4.

The number of PCG iterations in Step 4 of our PCG algorithm was taken to be $3\log N$. In Table 2 we present the maximum absolute error in u and $v = \Delta u$ on a uniform $(0.02) \times (0.02)$ grid for different values of N. Comparable errors were obtained for the Cholesky algorithm. The exponential convergence achieved is shown in Figure 5 where we present the graph of the logarithm of the maximum absolute error versus N.

Following the format of Table 3 in [7], in Table 3 we present the CPU times in seconds (excluding preprocessing times) for computing $\vec{u}$ by the PCG and Cholesky algorithms. The preprocessing times are given separately in parentheses. As expected from the discussion in section 4.4, preprocessing for the Cholesky algorithm takes

TABLE 2
Maximum absolute error for numerical example.

N	16	20	24	28	32	128	512
$\|u-U\|_\infty$	0.21(−1)	0.22(−3)	0.13(−5)	0.34(−8)	0.51(−11)	0.30(−12)	0.62(−12)
$\|v-V\|_\infty$	0.60(+1)	0.77(−1)	0.53(−3)	0.15(−5)	0.23(−8)	0.93(−10)	0.12(−9)

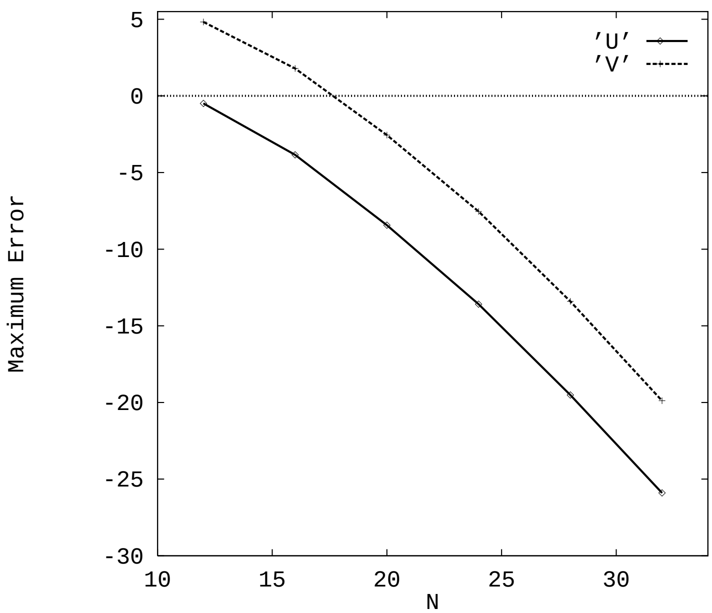

FIG. 5. *Logarithm of the maximum absolute error versus N.*

TABLE 3
Execution time for PCG and Cholesky algorithms.

N	16	32	64	128	512
PCG: CPU(Pre-P)	0.0096 (0.0012)	0.034 (0.012)	0.176 (0.076)	1.076 (0.44)	54.840 (24.844)
Chol: CPU (Pre-P)	0.0072 (0.0052)	0.026 (0.020)	0.134 (0.116)	0.892 (0.70)	48.224 (40.674)

more time than preprocessing for the PCG algorithm, whereas the CPU time for the Cholesky algorithm is smaller than the CPU time for the PCG algorithm. Also, for large values of N the total execution time (CPU + Pre-P) for the PCG algorithm is smaller than that for the Cholesky algorithm.

6. Concluding remarks. As shown in (3.9) of [7], the Legendre spectral Galerkin method based on the standard weak form

$$\int_\Omega \Delta u \Delta \eta \, d\Omega = \int_\Omega f\eta \, d\Omega, \qquad \eta \in H_0^2(\Omega),$$

of (1.1) leads to the linear system

$$(I_{N-3} \otimes B + 2C \otimes C + B \otimes I_{N-3})\vec{u} = \vec{f}, \tag{6.1}$$

where B and C are the Galerkin matrices corresponding to the derivatives of order zero and two in the space $\{p \in P_N^0 : p'(\pm 1) = 0\}$. Shen [7] solves (6.1) using a capacitance matrix approach with the auxiliary matrix

$$I_{N-3} \otimes B + 2C \otimes C + \tilde{B} \otimes I_{N-3}.$$

The matrix $\tilde{B}$, a modification of B, is such that C and $\tilde{B}$ commute and hence diagonalization in the x direction is possible. The capacitance matrix of [7], which is neither symmetric nor positive definite, is formed explicitly and then factored out using Gauss elimination. The cost of Shen's algorithm is $4N^3 + O(N^2)$ and this count does not include formation and factorization of the capacitance matrix. Both formation and factorization require $O(N^3)$ operations, where the constant multiple of N^3 is not given in [7].

In [3], Bjørstad and Tjøstheim solve (6.1) using the Sherman–Morrison formula with the auxiliary matrix

$$I_{N-3} \otimes B + 2C \otimes C + C^2 \otimes I_{N-3}.$$

Their counterpart of our Schur complement matrix S is symmetric and it is preconditioned by another symmetric matrix. Numerical tests indicate that both matrices are positive definite. The dominant cost of the PCG algorithm of [3] is given by (4.61), where $4N^2$ should be replaced with $24N^2$ [8]. The cost of the Cholesky algorithm in [3] is $16N^3/3 + O(N^2)$.

It follows from section 4.4 that our algorithms, based on discretizing the coupled problem (3.1), are more efficient with respect to operation counts. Our Schur complement matrices S of (4.13) and P of (4.55) are closely related to boundary value problems (3.1), (4.1), and (4.48), (4.52), respectively. This allows us to prove that they are both symmetric and positive definite. Moreover, if required, we also obtain an approximation to $v = \Delta u$ with an additional cost of $N^3 + O(N^2)$.

The algorithms of [3] and [7] were developed for the fourth order problem of the form

$$\Delta^2 u - \beta \Delta u + \alpha u = f \text{ in } \Omega, \qquad u = \partial u/\partial n = 0 \text{ on } \partial\Omega, \tag{6.2}$$

where α and β are nonnegative constants. Our approach, based on introducing $v = \Delta u$, and the corresponding PCG and Cholesky algorithms generalize, with the same dominant costs, to (6.2). In this more general case, the blocks S_{11}, S_{12} of (3.14), (3.15) are to be replaced, respectively, by

$$S_{11} = \begin{bmatrix} I_{N-1} \otimes B + B \otimes I_{N-1} & B \otimes B & B \otimes B_t \\ -\beta(B \otimes B) & (I_{N-1} + \alpha B) \otimes B + B \otimes I_{N-1} & (I_{N-1} + \alpha B) \otimes B_t \\ I_{N-1} \otimes B_t^T & B \otimes B_t^T & B \otimes B_s \end{bmatrix},$$

$$S_{12} = \begin{bmatrix} B_t \otimes B \\ B_t \otimes (I_{N-1} + \alpha B) \\ B_t \otimes B_t^T \end{bmatrix}.$$

Similarly, A_i of (2.11) and $\vec{p}$ of (2.12) are to replaced with

$$A_i = \begin{bmatrix} B_i + \lambda I_{N/2-i} & \lambda B_i \\ -\beta\lambda B_i & (1 + \alpha\lambda) B_i + \lambda I_{N/2-i} \end{bmatrix},$$

$$\vec{p} = [\lambda, 0, \dots, 0, 1 + \alpha\lambda, 0, \dots, 0]^T.$$

The new matrix A_i reduces to a block tridiagonal matrix with 2×2 blocks.

Our approach allows also for an efficient treatment of some other boundary conditions, such as $u = \Delta u = 0$ on $\partial\Omega$ or $u = 0$ on $\partial\Omega$, $\partial u/\partial n = 0$ on the horizontal sides of $\partial\Omega$ and $\Delta u = 0$ on the vertical sides of $\partial\Omega$. In fact, as discussed in section 4.1, the last set of boundary conditions gives rise to a linear system with matrix S_{11} of (3.14) and hence its solution, for the approximation to u, can be obtained at a cost of only $2N^3 + O(N^2)$.

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