Computing 63, 97–107 (1999)

Computing

High Precision Solutions of Two Fourth Order Eigenvalue Problems

P. E. Bjørstad and **B. P. Tjøstheim,** Bergen

Received June 8, 1998; revised February 15, 1999

Abstract

We solve the biharmonic eigenvalue problem $\Delta^2 u = \lambda u$ and the buckling plate problem $\Delta^2 u = -\lambda \Delta u$ on the unit square using a highly accurate spectral Legendre–Galerkin method. We study the nodal lines for the first eigenfunction near a corner for the two problems. Five sign changes are computed and the results show that the eigenfunction exhibits a self similar pattern as one approaches the corner. The amplitudes of the extremal values and the coordinates of their location as measured from the corner are reduced by constant factors. These results are compared with the known asymptotic expansion of the solution near a corner. This comparison shows that the asymptotic expansion is highly accurate already from the first sign change as we have complete agreement between the numerical and the analytical results. Thus, we have an accurate description of the eigenfunction in the entire domain.

AMS Subject Classifications: 65N25.

Key Words: Biharmonic operator, eigenvalue, eigenfunction, plate equation, spectral method, asymptotic expansion.

1. Introduction

We consider the eigenvalue problem for the biharmonic operator:

$$\Delta^2 u = \lambda u, \qquad u \in H_0^2(\Omega), \qquad \Omega = (0,1)^2 \tag{1}$$

and the eigenvalue problem for the buckling plate problem:

$$\Delta^2 u = -\lambda \Delta u, \qquad u \in H_0^2(\Omega), \qquad \Omega = (0,1)^2. \tag{2}$$

The occurrence of nodal lines near the corners for the first eigenfunction of the biharmonic eigenvalue problem was numerically first computed by Bauer and Reiss [1][1]. Bauer and Reiss used a second order finite difference code. Bjørstad [2] developed a fast algorithm for the same discretization which allowed a detailed

[1] Their early calculations were based on a direct solution on a 32×32 grid, and an alternating direction iterative procedure with $h = 1/80$. Using extrapolation they determined the first eigenvalue to be $\lambda_1 = 1294.88$, about 0.05 smaller than the correct value.

study of the first sign change and clearly showed the structure of the nodal lines. Independently, Hackbusch and Hoffmann [8] used piecewise linear rectangular finite elements combined with a multigrid solver. They reported two sign changes, but as first reported in [5] the location of their second sign change was incorrect. About the same time Coffman [6] proved the existence of infinitely many sign changes as one approaches the corner.

The more recent code [5] used a spectral tau method which should have higher accuracy. Very recently, Wieners [13] used C^3 rectangular finite elements and furnished a numerical existence proof of nodal lines. He used the same technique to prove the existence of nodal lines also for the buckling plate problem. Owen [10] derives asymptotic estimates for the first eigenvalue on rectangular regions, he also shows that the negative part of the corresponding eigenfunction is small. Thus, to the authors' knowledge none of the previously reported methods have been accurate enough to determine more than the first sign change.

In this work, very high precision is obtained by using the spectral Legendre-Galerkin method, with a basis constructed by Shen [11], combined with a fast solution algorithm developed by the authors [3]. The Legendre–Gauss–Lobatto points used in this algorithm approach the boundary as $1/N^2$. With our algorithm we can take $N > 1000$ and study the solution within 10^{-6} of the corner. This resolution reveals five changes in the sign and shows how the alternating behavior of the eigenfunction follows a predictable pattern.

2. Asymptotic Expansion Near a Corner

Based on the theory developed by Kondrat'ev [9] the paper by Blum and Rannacher [4] study singularities near an arbitrary angle for the biharmonic problem and weakly nonlinear problems of the form

$$\Delta^2 u - \sum_{i=1}^{2} \frac{\partial}{\partial x_i} F_i(\cdot, u, \nabla u, \nabla^2 u) + F_0(\cdot, \nabla u, \nabla^2 u) = f. \tag{3}$$

Their analysis can be applied to the biharmonic eigenvalue problem. Using the general expansion formula developed by Blum and Rannacher it is simple to show that in the special case of a rectangle with Dirichlet boundary conditions u has an expansion near the corner of the form

$$u(x, y) = cr^z \psi(\theta) + \bar{u}(x, y)$$

where (r, θ) are local polar coordinates with origin at a corner, c is a complex constant, while $\psi(\theta)$ and $\bar{u}(x, y)$ are smooth functions. The theory developed in [4] further shows that this expansion holds when the equation is changed in lower order terms, ie., the biharmonic operator determines the asymptotic behavior alone. We should therefore expect to see the same behavior for both the biharmonic eigenfunction defined in (1) and for the buckling plate problem (2). In order to distinguish between the two cases we will subscript ω and the constant $a\psi(\pi/4)$ in (4) with index 1 and 2 respectively. The complex exponent z can be shown to be the unique complex root ($\mathrm{Im}(z) \neq 0$) with real part $2 \leq \mathrm{Re}(z) \leq 4$ of the equation

$$\sin((z-1)\pi/2) + (z-1) = 0.$$

This equation is easy to solve and we find

$$z = \alpha + i\beta = 3.739593356 + 1.119024534i.$$

We now consider this expansion along the line $x = y$, (that is, $\theta = \pi/4$) where it takes the form

$$u(r) = ar^{\alpha}\cos(\beta \log r + \omega)\psi(\pi/4) + \bar{u}(r/\sqrt{2}, r/\sqrt{2}). \tag{4}$$

Let r_n be the distance to a local extremal value of this function, where we let n decrease with increasing radius r. If we neglect $\bar{u}$ when determining the local extremal points then we readily find that

$$u'(r) = \psi(\pi/4)ar^{\alpha-1}(\alpha\cos(\beta\log r + \omega) - \beta\sin(\beta\log r + \omega)) = 0 \tag{5}$$

thus

$$r_n = e^{\frac{\tan^{-1}(\alpha/\beta) - \omega - (n-1)\pi}{\beta}} \qquad n = 1, 2 \ldots \tag{6}$$

and the ratio

$$\frac{r_n}{r_{n+1}} = e^{\pi/\beta} = 16.56742776. \tag{7}$$

The ratio of two consecutive extremal values must be

$$\frac{-u(r_n)}{u(r_{n+1})} = (\frac{r_n}{r_{n+1}})^{\alpha} = 36267.54987. \tag{8}$$

Table 1. Computation of the first eigenvalue of the biharmonic problem

N	$\Delta^2 u = \lambda u$
14	1294.93398
100	1294.93397959171283
1000	1294.9339795917128081703026 48
2000	1294.933979591712808170302647974 4
3000	1294.933979591712808170302647974 4
4000	1294.933979591712808170302647974 4
5000	1294.933979591712808170302647974 4

Table 2. Computation of the first eigenvalue for the buckling plate problem

N	$\Delta^2 u = -\lambda \Delta u$
14	52.3446913
100	52.344691168416546
1000	52.34469116841654453870533075 2
2000	52.344691168416544538705330750368
3000	52.344691168416544538705330750366
4000	52.344691168416544538705330750365
5000	52.344691168416544538705330750365

3. An Accurate Numerical Method

Using the discretization developed in [11] the basis for the biharmonic problem is given by:

$$\phi_k(x) = d_k(L_k(x) - \frac{2(2k+5)}{2k+7}L_{k+2}(x) + \frac{2k+3}{2k+7}L_{k+4}(x)) \; k = 1, \ldots, N \quad (9)$$

where L_k are Legendre polynomials and $d_k = (2(2k+1)^2(2k+3))^{-1/2}$. Let

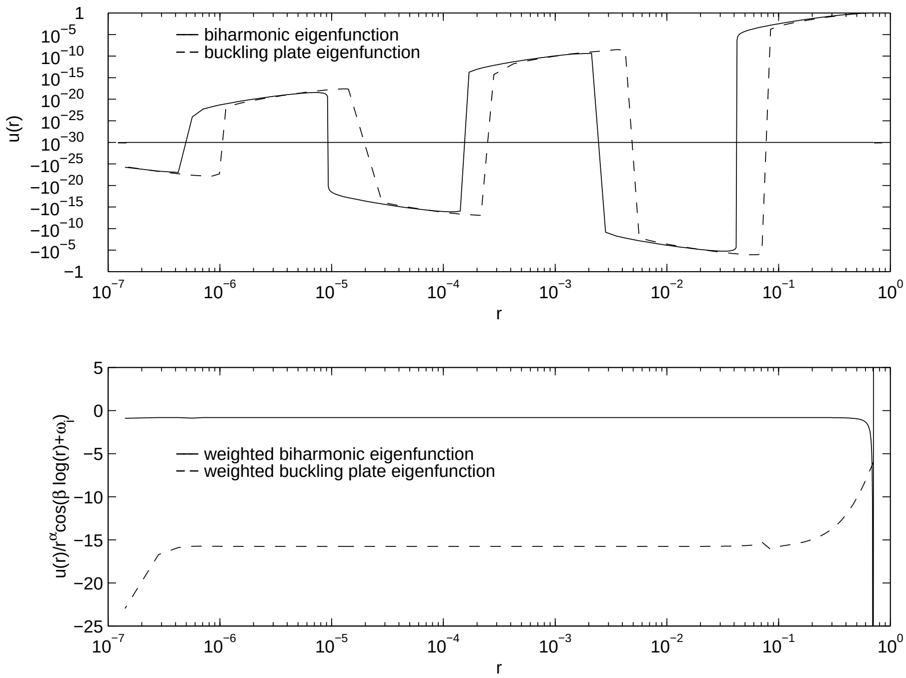

Figure 1. The first biharmonic eigenfunction and the buckling plate eigenfunction. The functions are plotted in a log-log scale on top, while the lower graph scales by the asymptotic expansion given by (4)

$D = diag(d_k)$, $k = 1, \ldots, N$ and let the matrices B and C be defined as

$$b_{kj} = (\phi_j(x), \phi_k(x)) \; c_{kj} = (\phi_j^{'}(x), \phi_k^{'}(x)). \tag{10}$$

It is shown in [11] that the nonzero elements of B and C are given by:

$$\begin{aligned}
b_{kk} &= d_k^2(e_k + h_k^2 e_{k+2} + g_k^2 e_{k+4}) \\
b_{kk+2} &= d_k d_{k+2}(h_k e_{k+2} + g_k h_{k+2} e_{k+4}) \\
b_{k+2k} &= b_{kk+2} \\
b_{kk+4} &= d_k d_{k+4} g_k e_{k+4} \\
b_{k+4k} &= b_{kk+4}
\end{aligned}$$

$$\begin{aligned}
c_{kk} &= -2(2(k-1)+3) d_k^2 h_k \\
c_{kk+2} &= -2(2(k-1)+3) d_k d_{k+2} \\
c_{k+2k} &= c_{kk+2}
\end{aligned}$$

where $e_k = \frac{2}{2k-1}$, $g_k = \frac{2k+1}{2k+5}$ and $h_k = -(1 + g_k) \quad k = 1, \ldots, N$.

The matrix B can be split in two parts B_1 and B_2 by ordering the odd and even components, the same can be done for C and D. The discrete counterpart to $\Delta^2 u = \lambda u$ is given by

$$Mu_N = \lambda(B \otimes B)u_N, \tag{11}$$

where $M = (I_N \otimes B) + (B \otimes I_N) + 2(C \otimes C)$. We refer to [3] for more details of the efficient solver that we use in the iterative scheme below. It can be proved that first eigenfunction has all the symmetries of the domain (cf. [7]). The discrete spectrum splits with the matrix splitting and studies show that the spectrum associated with $M_1 = (I_{\frac{N}{2}} \otimes B_1) + (B_1 \otimes I_{\frac{N}{2}}) + 2(C_1 \otimes C_1)$ contains the first eigenvalue. Thus it is only necessary to solve this system, saving computational time and reducing memory requirements. Since we are only concerned about the first eigenvalue and its corresponding eigenfunction we will denote this pair (λ, u) without any identifying index. The first eigenpair (λ, u) is computed using the Rayleigh quotient iteration:

An initial estimate, $\lambda^{(0)}$, is chosen.
Pick x_0 with $\|x_0\|_2 = 1$.
Then for $k = 0, 1, \ldots$
$\lambda^{(k)} := \frac{x_k^T M_1 x_k}{x_k^T (B_1 \otimes B_1) x_k}$
Solve $(M_1 - \lambda^{(k)}(B_1 \otimes B_1))z_{k+1} = (B_1 \otimes B_1)x_k$
$x_{k+1} := z_{k+1}/\|z_{k+1}\|_2$
end

Table 3. Computed extremal values for the first biharmonic eigenfunction

	$N = 3000$	$N = 4000$	$N = 5000$
u_1	$-1.697912879810476 \cdot 10^{-5}$	$-1.697912879810476 \cdot 10^{-5}$	$-1.697912879810476 \cdot 10^{-5}$
u_2	$4.681610062754766 \cdot 10^{-10}$	$4.681610062754766 \cdot 10^{-10}$	$4.681610062754765 \cdot 10^{-10}$
u_3	$-1.290853694317260 \cdot 10^{-14}$	$-1.290853694319487 \cdot 10^{-14}$	$-1.290853694319680 \cdot 10^{-14}$
u_4	$3.559233569291186 \cdot 10^{-19}$	$3.559251855971790 \cdot 10^{-19}$	$3.559252581823539 \cdot 10^{-19}$
u_5	$-1.527481389963724 \cdot 10^{-24}$	$-8.332122807525830 \cdot 10^{-24}$	$-9.748609616111923 \cdot 10^{-24}$

The rate of convergence is cubic, and only three iterations were needed. The system $(M_1 - \lambda^{(k)}(B_1 \otimes B_1))z_{k+1} = (B_1 \otimes B_1)x_k$ is solved using the direct solver described in [3], the corresponding iterative solver could also have been used. All the problems of the form $M_1 x = y$ tested in [12] with analytically known solutions showed high accuracy which did not deteriorate when N was increased. The same behavior has also been observed for problems of the form $(M_1 - \alpha(B_1 \otimes B_1))x = y$ where the analytical solution is known.

However, the product $(B_1 \otimes B_1)x_k$, when computed directly, caused a dramatic increase in the quantity $| \lambda^{(3)} - \lambda^{(2)} |$ for large values of N.

Table 4. Computed locations of the extremal points for the first biharmonic eigenfunction

	$N = 3000$	$N = 4000$	$N = 5000$
r_1	0.0326294729775	0.0326294729775	0.0326294729775
r_2	0.0019694919355	0.0019694919355	0.0019694919355
r_3	0.0001188773517	0.0001188773517	0.0001188773517
r_4	0.0000071753303	0.0000071753303	0.0000071753652
r_5	0.0000002394873	0.0000004279764	0.0000004327667

Table 5. Ratios of consecutive extremal values for the first biharmonic eigenfunction

	N=3000	N=4000	N=5000
$\lvert u_1/u_2 \rvert$	$\underline{36267}.71254$	$\underline{36267}.71254$	$\underline{36267}.71254$
$\lvert u_2/u_3 \rvert$	$\underline{36267.549}79$	$\underline{36267.549}79$	$\underline{36267.549}79$
$\lvert u_3/u_4 \rvert$	$\underline{36267}.74330$	$\underline{36267.5}5696$	$\underline{36267.549}57$
$\lvert u_4/u_5 \rvert$	233013.2199	42717.22751	$\underline{36}510.36119$
$\lvert u_n/u_{n+1} \rvert$	36267.54987	36267.54987	36267.54987

This error was reduced using the decomposition $B_1 = D_1 \hat{B}_1 D_1$ and $C_1 = D_1 \hat{C}_1 D_1$. Written on this form this diagonal transformation can be performed before and after the Rayleigh quotient iteration. (The elements of D_1 are defined after (9).) Next, consider the buckling plate problem. This problem can be studied using the same technique. Also in this case the first eigenvalue belongs to the M_1 part of the problem. The Laplace operator $-\Delta$ has the discrete representation $R_1 = (B_1 \otimes C_1) + (C_1 \otimes B_1)$. At each iteration we must compute $\lambda^{(k)} = \frac{x_k^T M_1 x_k}{x_k^T R_1 x_k}$ and solve the equation

$$(M_1 - \lambda^{(k)} R_1) z_{k+1} = R_1 x_k. \tag{12}$$

Also in this case a diagonal transformation was applied before and after the iteration in order to improve the numerical accuracy.

Table 6. Ratios between consecutive extremal points for the first biharmonic eigenfunction

	N=3000	N=4000	N=5000
r_1/r_2	$\underline{16.5674}5701$	$\underline{16.5674}5701$	$\underline{16.5674}5701$
r_2/r_3	$\underline{16.5674277}5$	$\underline{16.5674277}5$	$\underline{16.5674277}5$
r_3/r_4	$\underline{16.567}50941$	$\underline{16.5674277}9$	$\underline{16.56742}802$
r_4/r_5	21.18577379	$\underline{16}.76579822$	$\underline{16.5}8008884$
r_n/r_{n+1}	16.56742776	16.56742776	16.56742776

4. Computational Results

All computations were performed on a Cray Origin 2000 with quadruple precision. The eigenfunctions were normalized such that the maximum value of the eigenfunction equals 1. The computed eigenfunction is symmetric with respect to the line $x = y$, and the extremal points of the eigenfunction lies on this line. By using very accurate starting points the extrema were efficiently computed using an elementary iterative method along the line $x = y$.

To the author's knowledge the most accurate bounds for the first eigenvalue are given in [14], where the lower bound is 1294.933940 and the upper bound is 1294.933988. The approximation of the first eigenvalue given in [5] is 1294.93761 which is larger than the current upper bound. Our computation has $\lambda^{(3)}$ inside these bounds for $N \geq 14$, see Table 1. We note that our computation provide an upper bound (since we use a Galerkin method) as long as roundoff errors can be neglected. This is clearly seen in Table 1 as the computed value decreases in a monotone way when we increase N.

In Fig. 1, we plot the computed eigenfunctions for both the problems considered along the diagonal line $x = y$. In the first graph the functions are plotted from the corner ($r = 0$) to the center ($r = \sqrt{2}/2$) using a logarithmic scale to resolve the small oscillations. The function values are plotted on a log-scale that displays both positive and negative values starting from 10^{-30}. Note that this scaling greatly distorts the shape of the functions, but enables us to directly read off the values that describe the eigenfunctions. The second graph shows the same functions scaled by the asymptotic expansion (4). We have used the values $\omega_1 = 5.109947$ and $\omega_2 = 4.436918$ as determined from the extremal values of the functions. We further estimate the values $a_1\psi_1(\pi/4) = -0.8051070$ and $a_2\psi_2(\pi/4) = -15.76212$ in agreement with the second graph, that is, we have a complete and accurate determination of the expansion from (4).

We note that there is very good agreement between the formula (4) and the actual eigenfunction well inside the location of the first extremal point. The buckling plate function deviates slightly more from the model near the end points compared to the biharmonic case. This deviation near the corner end point is due to numerical errors.

Table 7. Computed extremal values for the first eigenfunction for the buckling plate problem

	$N = 3000$	$N = 4000$	$N = 5000$
u_1	$-1.17054642119349 \cdot 10^{-4}$	$-1.17054642119349 \cdot 10^{-4}$	$-1.17054642119349 \cdot 10^{-4}$
u_2	$3.26414244011922 \cdot 10^{-9}$	$3.26414244011922 \cdot 10^{-9}$	$3.26414244011922 \cdot 10^{-9}$
u_3	$-9.00054558016984 \cdot 10^{-14}$	$-9.00054558017012 \cdot 10^{-14}$	$-9.00054558017014 \cdot 10^{-14}$
u_4	$2.48170769549679 \cdot 10^{-18}$	$2.48170802554547 \cdot 10^{-18}$	$2.48170802642496 \cdot 10^{-18}$
u_5	$-5.93638671361257 \cdot 10^{-18}$	$-6.82641834748896 \cdot 10^{-23}$	$-6.83051144901473 \cdot 10^{-23}$

Table 8. Computed locations of the extremal values for the first eigenfunction for the buckling plate problem

	$N = 3000$	$N = 4000$	$N = 5000$
r_1	0.0593622581801	0.0593622581801	0.0593622581801
r_2	0.0035937907192	0.0035937907192	0.0035937907192
r_3	0.0002169214273	0.0002169214274	0.0002169214274
r_4	0.0000130932495	0.0000130932465	0.0000130932465
r_5	0.0000007827203	0.0000007895344	0.0000007904724

Table 9. Ratios between consecutive extremal points for the first eigenfunction for the buckling plate problem

	N=3000	N=4000	N=5000
$\mid r_1/r_2 \mid$	$\underline{16.5}1800642$	$\underline{16.5}1800642$	$\underline{16.5}1800642$
$\mid r_2/r_3 \mid$	$\underline{16.567}24633$	$\underline{16.567}24631$	$\underline{16.567}24632$
$\mid r_3/r_4 \mid$	$\underline{16.56742}490$	$\underline{16.56742}856$	$\underline{16.56742}858$
$\mid r_4/r_5 \mid$	$\underline{16}.72787282$	$\underline{16.5}8350288$	$\underline{16.56}382390$
$\mid r_n/r_{n+1} \mid$	16.56742776	16.56742776	16.56742776

The minimum value after the first sign change turned out to be very close to the corresponding value observed by Bjørstad [2]. Our results for the first sign change also correspond well with the observations made in [14]. In [8] two sign changes are reported, but the second sign change does not coincide with our results. In [8] a 128×128 uniform grid was used, and a second sign change was reported at $(2/128, 3/128)$. We find the second sign change at approximately $(1/550, 1/550)$ well outside the resolution of a 128×128 grid.

Finally, Table 5 and 6 show the computed values of the ratios in (7) and (8) and compare these with the theoretical limit given on the last line of each table. We observe a remarkable agreement of up to eight figures already for the second ratio. From Table 5 it is seen that the ratio $\mid u_4/u_5 \mid$ does not approximate the asymptotic ratio $\mid u_n/u_{n+1} \mid$ as well as the other ratios. As mentioned in Section 3 problems similar to the eigenvalue problem with known solutions have been solved very accurately. We therefore believe that the roundoff error when computing $u(r)$ can be neglected and that the the ratio $\mid u_4/u_5 \mid$ is further from the asymptotic ratio due to lack of resolution as one moves closer to the corner. It should be noted, however, how much the ratio improves from $N = 3000$ to $N = 5000$. The same behavior can be observed in Table 6.

We next consider the buckling plate problem. This eigenvalue problem showed a very similar behavior. Also in this case three Rayleigh quotient iterations proved sufficient. In [13] Wieners reports an approximation $\lambda = 52.3446911$ for the first eigenvalue. As seen from Table 2 our approximation is very close to Wieners approximation even for small values of N. The first sign change is detected further from the corner and the amplitude of the extremal values are larger than in the biharmonic case. This also corresponds well with the observations made in [13]. Tables 7–10 present the same information about the buckling plate problem as previously discussed for the biharmonic operator. From Tables [9] and [10] it is seen that the ratios converge towards the same values as for the biharmonic case in accordance with the prediction made in Section 2. As for the biharmonic case the last ratios have lower accuracy due to lack of resolution.

Table 10. Ratios of consecutive extremal values for the first eigenfunction for the buckling plate problem

	$N = 3000$	$N = 4000$	$N = 5000$
$\mid u_1/u_2 \mid$	35860.76413	35860.76413	35860.76413
$\mid u_2/u_3 \mid$	36266.05089	36266.05089	36266.05089
$\mid u_3/u_4 \mid$	36267.54432	36267.54431	36267.54431
$\mid u_4/u_5 \mid$	41805.02072	36354.46729	36332.68233
$\mid u_n/u_{n+1} \mid$	36267.54987	36267.54987	36267.54987

In conclusion, we note that the asymptotic form (4) is an accurate approximation of all the oscillations of the two eigenfunctions. With the numerical method presented in this note combined with the asymptotic form given in Section 2 we feel that one has the necessary tools to completely determine all properties of the eigenfunctions considered.

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P.E. Bjørstad, B.P. Tjøstheim
Institute for Informatics
University of Bergen
5020 Bergen, Norway
e-mail: petter@ii.uib.no, bjornpt@ii.uib.no