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SOLVING THE BIHARMONIC EQUATION AS COUPLED FINITE DIFFERENCE EQUATIONS*

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Abstract. A technique is proposed for solving the finite difference biharmonic equation as a coupled pair of harmonic difference equations. Essentially, the method is a general block SOR method with convergence rate $O(h^{1/2})$ on a square, where h is mesh size.

1. Introduction. In [7], J. Smith presented a method for solving the biharmonic difference equation as a pair of coupled finite difference equations. He showed that for a rectangle the convergence rate of his method was $1 - K_1 h$ as $h \to 0$ for some constant K_1. Here, we propose an iteration scheme for which Smith's is a special case. Our method has optimum convergence rate $1 - \sqrt{K_2 h}$, for some constant K_2, i.e., an order of magnitude faster. (h is the mesh size.)

2. The equations. For notational simplicity, we will consider a simpler system of equations than Smith [7]. Our results, however, will be applicable to more general systems.

Consider, for $u(x, y)$:

$$\Delta^2 u \equiv u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0, \qquad 0 < x, y < 1, \tag{2.1}$$

$$\begin{aligned} u &= 0, \quad x = 0, 1 \quad \text{or} \quad y = 0, 1, \\ u_n &= 0, \quad x = 0, 1 \quad \text{or} \quad y = 0, \\ u_n &= 1, \quad y = 1, \end{aligned} \tag{2.2}$$

where u_n is the outward normal derivative on the boundary of the unit square. The equation (2.1) can be replaced by

$$\begin{aligned} \Delta u &\equiv u_{xx} + u_{yy} = v, \\ \Delta v &\equiv v_{xx} + v_{yy} = 0, \qquad 0 < x, y < 1, \end{aligned} \tag{2.3}$$

with boundary conditions (2.2). We propose to solve the finite difference analogue of (2.3) and (2.2).

Superimpose a square grid over the unit square with mesh size $h = 1/N + 1$ for some positive integer N. Let Ω be those grid points $(x, y) = (ih, jh)$ for $1 \leqq i$, $j \leqq N$ (i.e., the interior), and let $\partial\Omega$ be those points for which $i, j = 0$ or $N + 1$ (i.e., the boundary). Let u be a function where $u(x, y) = u(ih, jh) \equiv u_{ij}$. Define

$$\Delta_h u = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij}}{h^2} \tag{2.4}$$

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and

$$(2.5)\qquad \delta_h u = \begin{cases} \dfrac{u_{0,j} - u_{1,j}}{h}, & j = 1, \cdots, N, \\[2ex] \dfrac{u_{N+1,j} - u_{N,j}}{h}, & j = 1, \cdots, N, \\[2ex] \dfrac{u_{i,0} - u_{i,1}}{h}, & i = 1, \cdots, N, \\[2ex] \dfrac{u_{i,N+1} - u_{i,N}}{h}, & i = 1, \cdots, N. \end{cases}$$

To approximate the solution of (2.2) and (2.3), we approximate Δu with $\Delta_h u$. At the boundary, we assume an extra row of unknowns outside the region and then use $\Delta_h u = v$ and the boundary conditions of u to approximate boundary conditions for v. Combining these equations, we obtain, in the manner of Smith [7], the linear system

$$(2.6)\qquad \begin{aligned} Lu &= h^2 v, \\ Lv + \frac{2}{h^2}Mu &= \frac{D}{h^2} \end{aligned}$$

or

$$L^2 u + 2Mu = (L^2 + 2M)u = D,$$

where D contains the boundary conditions and where

$$(2.7)\qquad L = \begin{pmatrix} L_1 & I & \cdots & 0 \\ I & L_2 & I & \vdots \\ \vdots & & & \vdots \\ \vdots & & & I \\ 0 & & I & L_N \end{pmatrix}_{N^2 \times N^2}, \qquad L_i = \begin{pmatrix} -4 & 1 & \cdots & 0 \\ 1 & -4 & & \vdots \\ \vdots & & & \vdots \\ \vdots & & & 1 \\ 0 & \cdots & 1 & -4 \end{pmatrix}_{N \times N},$$

$$i = 1, 2, \cdots, N,$$

$$M = \begin{pmatrix} T + I & & \cdots & & 0 \\ . & T & & & . \\ . & & T & & . \\ . & & & T & . \\ 0 & & \cdots & & T + I \end{pmatrix}_{N^2 \times N^2}, \qquad T = \begin{pmatrix} 1 & & \cdots & & 0 \\ . & 0 & & & . \\ . & & & & . \\ . & & & 0 & . \\ 0 & & \cdots & & 1 \end{pmatrix}_{N \times N}.$$

We note that if (2.1) is not homogeneous, or if less of the boundary conditions are homogeneous, then extra terms will appear on the right of (2.6) (see Smith [7]).

3. The iterative scheme. Consider the following:

$$
\begin{aligned}
Lv_{m+1} + \frac{2}{h^2}M\bar{u}_m &= \frac{D}{h^2}, \\
\bar{v}_{m+1} &= \omega_2 v_{m+1} + (1-\omega_2)\bar{v}_m, \\
Lu_{m+1} &= h^2\bar{v}_{m+1}, \\
\bar{u}_{m+1} &= \omega_1 u_{m+1} + (1-\omega_1)\bar{u}_m.
\end{aligned} \tag{3.1}
$$

Smith's scheme [7] was the special case $\omega_2 = 1$. Solving (3.1) for $\bar{v}_{m+1}$ and $\bar{u}_{m+1}$ we have

$$
\begin{aligned}
\bar{v}_{m+1} &= \omega_2 L^{-1}\left\{\frac{D}{h^2} - \frac{2}{h^2}M\bar{u}_m\right\} + (1-\omega_2)\bar{v}_m, \\
\bar{u}_{m+1} &= \omega_1 L^{-1}\{\omega_2 L^{-1}D - 2\omega_2 L^{-1}M\bar{u}_m + h^2(1-\omega_2)\bar{v}_m\} + (1-\omega_1)\bar{u}_m,
\end{aligned} \tag{3.2}
$$

or in matrix notation,

$$
\begin{pmatrix} \bar{v}_{m+1} \\ \bar{u}_{m+1} \end{pmatrix}
= \begin{pmatrix} (1-\omega_2)I & -\frac{2\omega_2}{h^2}L^{-1}M \\ \omega_1(1-\omega_2)h^2L^{-1} & (1-\omega_1)I - 2\omega_1\omega_2L^{-2}M \end{pmatrix}
\begin{pmatrix} \bar{v}_m \\ \bar{u}_m \end{pmatrix}
+ \begin{pmatrix} \frac{\omega_2}{h^2}L^{-1}D \\ \omega_1\omega_2L^{-2}D \end{pmatrix}. \tag{3.3}
$$

To investigate the convergence properties of this scheme, we seek eigenvalues of the iteration matrix. If an eigenvector is $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, partitioned as above, then for eigenvalues λ we have

$$
\begin{aligned}
(1-\omega_2)x_1 - \frac{2\omega_2}{h^2}L^{-1}Mx_2 &= \lambda x_1, \\
h^2\omega_1(1-\omega_2)L^{-1}x_1 + (1-\omega_1)x_2 - 2\omega_1\omega_2L^{-2}Mx_2 &= \lambda x_2.
\end{aligned} \tag{3.4}
$$

Eliminating x_1, we have

$$((1-\omega_2-\lambda)(1-\omega_1-\lambda)I + 2\omega_1\omega_2\lambda L^{-2}M)x_2 = 0. \tag{3.5}$$

Thus, if τ is an eigenvalue of $L^{-2}M$, then the eigenvalues of our iterative method are determined by

$$\lambda^2 - [(1-\omega_1) + (1-\omega_2) - 2\omega_1\omega_2\tau]\lambda + (1-\omega_1)(1-\omega_2) = 0. \tag{3.6}$$

Similar results are obtained if we eliminate x_2 or if the equations of (3.1) are rearranged.

Let $\bar{\lambda} = \max|\lambda_i|$ be the spectral radius of a matrix whose eigenvalues are λ_i. We note that τ_i, the eigenvalues of $L^{-2}M$, are all real and nonnegative, with zero being a multiple eigenvalue.

Following McDowell [4] (see also Taylor [6]), we rewrite (3.6) as

$$\frac{1}{\omega_1\omega_2}[\lambda - (1-\omega_1)][\lambda - (1-\omega_2)] = -2\tau\lambda. \tag{3.7}$$

First, consider the case $\omega_1 = \omega_2 = \omega$ or

$$\frac{1}{\omega^2}[\lambda - (1 - \omega)]^2 = -2\tau\lambda. \tag{3.8}$$

Let

$$\begin{aligned} \sigma_1^2 &= -2\tau\lambda, \\ \sigma_2 &= \frac{1}{\omega}(\lambda - 1 + \omega) = \frac{\lambda}{\omega} + \frac{\omega - 1}{\omega}. \end{aligned} \tag{3.9}$$

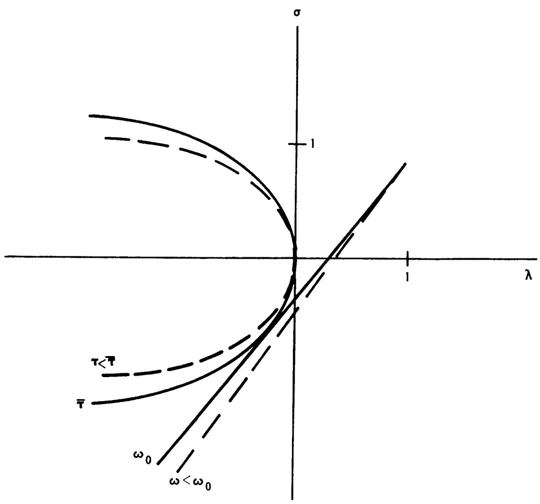

FIG. 1

In the λ, σ-plane we have Fig. 1. As ω increases from 0, the roots of (3.8) remain complex, their magnitude being $1 - \omega$, and the straight line σ_2 pivots about the point (1, 1). When the line becomes tangent to the parabola, the roots associated with τ become real. From then on these roots are the intersection of the line and parabola, one root of which grows. Thus, for a given τ we see that the minimum of the maximum $|\lambda|$ occurs at tangency or when the roots of (3.8) are equal. For each τ such that $0 \leqq \tau \leqq \bar{\tau}$, the associated parabolas (3.9) have the same vertex but have latus rectum $= 2\tau$. Hence, the minimum of maximum $|\lambda|$ is associated with $\bar{\tau}$ when the roots of (3.8) are equal. Since

$$\lambda = (1 - \omega) - \omega^2\bar{\tau} \pm \omega\sqrt{\omega^2\bar{\tau}^2 - 2(1 - \omega)\bar{\tau}}, \tag{3.10}$$

we want

$$\omega^2\bar{\tau}^2 - 2(1 - \omega)\bar{\tau} = 0. \tag{3.11}$$

Choose ω_0 such that

$$\omega_0^2\bar{\tau}^2 - 2(1 - \omega_0)\bar{\tau} = 0 \tag{3.12}$$

or

$$\omega_0 = \frac{2}{1 + \sqrt{1 + 2\bar{\tau}}} \tag{3.13}$$

with

$$\bar{\lambda} = 1 - \omega_0. \tag{3.14}$$

(The above results are also obtained from the analysis of David M. Young, Jr. [10] and from references there cited.)

Now consider $\omega_1 \neq \omega_2$. For any pair of ω's, the roots of (3.7) associated with $\tau = 0$ are $1 - \omega_1$ and $1 - \omega_2$. Thus, it is clear we need only consider ω_1, $\omega_2 > \omega_0$ (since $0 < \omega_0 < 1$). It is easy to show that the roots of (3.7) associated with $\bar{\tau}$ are real for any ω_1 and ω_2 in this range. Further, one negative root is always greater than $1 - \omega_0$ in magnitude. Thus, $\omega_1 = \omega_2 = \omega_0$ for optimum convergence and

$$\lambda = -(1 - \omega_0) = \frac{1 - \sqrt{1 + 2\bar{\tau}}}{1 + \sqrt{1 + 2\bar{\tau}}} \tag{3.15}$$

or

$$\bar{\lambda} = \frac{\sqrt{1 + 2\bar{\tau}} - 1}{\sqrt{1 + 2\bar{\tau}} + 1}. \tag{3.16}$$

For Smith's approach [7], we have $\omega_2 = 1$ and (3.7) becomes

$$\lambda_s\{\lambda_s - [(1 - \omega_1) - 2\omega_1\tau]\} = 0 \tag{3.17}$$

or

$$\lambda_s = 0,\ 1 - \omega_1 - 2\omega_1\tau. \tag{3.18}$$

The optimum ω satisfies

$$1 - \omega_1 - 2\omega_1\bar{\tau} = \omega_1 - 1 \tag{3.19}$$

or

$$\omega_1 = \frac{1}{1 + \bar{\tau}} \tag{3.20}$$

and

$$\bar{\lambda}_s = \frac{\bar{\tau}}{1 + \bar{\tau}}, \tag{3.21}$$

which are Smith's results.

Now, Smith [7] has shown that $\bar{\tau} = 1/h\sigma_h$, where

$$\sigma_h = \min \frac{h^2 \sum_{\Omega} (\Delta_h u)^2}{h \sum_{\partial\Omega} (\delta_h u)^2}. \tag{3.22}$$

In the Appendix, we show that

$$2.95 + 4.37h \leqq \sigma_h \leqq 4 + 8h$$

for small h. Thus, for small h,

$$\frac{0.25}{h} \leqq \bar{\tau} \leqq \frac{0.339}{h},$$

and one can show that

$$1 - \sqrt{8h} \leqq \bar{\lambda} \leqq 1 - \sqrt{5.9h}$$

and

$$1 - 4h \leqq \bar{\lambda}_s \leqq 1 - 2.95h.$$

The method proposed here is then an order of magnitude faster. Smith [8] is able to obtain this rate of convergence but has to resort to a Chebyshev scheme to do so.

Appendix. In [3], Kuttler shows that

$$\sigma_h = \min \frac{h \sum_{\partial\Omega} u^2}{h^2 \sum_{\Omega} u^2}, \tag{A.1}$$

where $\Delta_h u = 0$ in Ω. (The continuous analogue of this result is due to Fichera [1].) To obtain an upper bound for σ_h, let $u \equiv 1$. Then from (A.1), we have

$$\sigma_h \leqq \frac{h(4N + 4)}{h^2 N^2} = 4\left(\frac{N + 1}{N}\right)^2 = \frac{4}{(1 - h)^2}. \tag{A.2}$$

To obtain a lower bound, we note first the discrete analogue of the maximum principle [3, Theorem 4], [2].

THEOREM. *If u is defined over Ω and $\partial\Omega$, and if $\Delta_h u \geqq 0$ ($\leqq 0$) in Ω, then u assumes its maximum (minimum) value on $\partial\Omega$.*

What follows is the finite difference analogue of the continuous analysis of Payne [5] (see also Theorem 9 [3]). Let u be the minimizing function of (A.1). Define H such that

$$\begin{aligned} \Delta_h H &= 0 \quad \text{in } \Omega, \\ H &= u^2 \quad \text{on } \partial\Omega. \end{aligned} \tag{A.3}$$

Now, we observe that since $\Delta_h u = 0$ in Ω,

$$\Delta_h u^2 = \Delta_h u^2 - u\Delta_h u \geqq 0 \tag{A.4}$$

by Cauchy's inequality (see also [3]). Thus, we have

$$\begin{aligned} \Delta_h(H - u^2) &= -\Delta_h u^2 \leqq 0 \quad \text{in } \Omega, \\ H - u^2 &= 0 \qquad\qquad\quad \text{on } \partial\Omega, \end{aligned}$$

and by the maximum principle,

$$H - u^2 \geqq 0 \quad \text{in } \Omega. \tag{A.5}$$

Define

$$\text{(A.6)} \qquad \begin{aligned} \Delta_h \varphi &= -2 \quad \text{in } \Omega, \\ \varphi &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

(This is the discrete analogue of the stress function [9, p. 116].) From (A.5) and (A.6), we have

$$\text{(A.7)} \qquad \sum_{\Omega} u^2 \leqq \sum_{\Omega} H = -\frac{1}{2}\sum_{\Omega} H\Delta_h\varphi.$$

We now apply the discrete form of Green's identity (see, e.g., [3]):

$$\text{(A.8)} \qquad h^2 \sum_{\Omega} \varphi\Delta_h H - h^2 \sum_{\Omega} H\Delta_h\varphi = h \sum_{\partial\Omega} \varphi\delta_h H - h \sum_{\partial\Omega} H\delta_h\varphi$$

and obtain, using (A.3) and (A.6),

$$\text{(A.9)} \qquad \sum_{\Omega} u^2 \leqq -\frac{1}{2}\sum_{\Omega} H\Delta_h\varphi = -\frac{1}{2h}\sum_{\partial\Omega} H\delta_h\varphi.$$

Letting $|\delta_h\varphi|_{\partial\Omega} = \max_{\partial\Omega} |\delta_h\varphi|$ and using (A.3), we have

$$\text{(A.10)} \qquad \sum_{\Omega} u^2 \leqq \frac{|\delta_h\varphi|_{\partial\Omega}}{2h} \sum_{\partial\Omega} u^2.$$

Combining this with (A.1), we finally have

$$\text{(A.11)} \qquad \sigma_h \geqq \frac{2}{|\delta_h\varphi|_{\partial\Omega}}.$$

We now consider solving (A.6). It is not difficult to verify that

$$\text{(A.12)} \quad \varphi_{ij} = \frac{4}{(N+1)^2} \sum_{m,n=1}^{N} \frac{\left(\sum_{l=1}^{N} \sin\frac{m\pi l}{N+1}\right)\left(\sum_{k=1}^{N} \sin\frac{n\pi k}{N+1}\right)}{2 - \cos\frac{m\pi}{N+1} - \cos\frac{n\pi}{N+1}} \sin\frac{n\pi i}{N+1} \sin\frac{m\pi j}{N+1}$$

is the solution. However, finding a *good* upper bound for $\varphi_{0,j}$, $\varphi_{N+1,j}$, $\varphi_{i,0}$ and $\varphi_{i,N+1}$ does not appear promising. Instead, we solve a series of difference equations which are direct analogues of a series of differential equations considered by Sokolnikoff [9, pp. 114–131].

Define

$$\text{(A.13)} \qquad \psi_{ij} = \varphi_{ij} + \frac{h^2}{2}(i^2 + j^2).$$

Then it can be verified that

$$
\begin{aligned}
&\Delta_h\psi = 0, && i,j = 1, \cdots, N,\\
&\psi_{0,j} = \frac{h^2j^2}{2}, && j = 1, \cdots, N,\\
\text{(A.14)}\qquad &\psi_{N+1,j} = \frac{1}{2} + \frac{h^2j^2}{2}, && j = 1, \cdots, N,\\
&\psi_{i,0} = \frac{h^2i^2}{2}, && i = 1, \cdots, N,\\
&\psi_{i,N+1} = \frac{1}{2} + \frac{h^2i^2}{2}, && i = 1, \cdots, N.
\end{aligned}
$$

Define

$$
\text{(A.15)}\qquad f_{ij} = \frac{\psi_{i-1,j} - 2\psi_{i,j} + \psi_{i+1,j}}{h^2} + 1, \qquad \begin{array}{l} j = 0, 1, \cdots, N+1, \\ i = 1, \cdots, N. \end{array}
$$

Using (A.14), we can also write

$$
\text{(A.16)}\qquad f_{ij} = -\left(\frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{h^2}\right) + 1, \qquad \begin{array}{l} i = 0, \cdots, N + 1, \\ j = 1, \cdots, N. \end{array}
$$

One can show that

$$
\begin{aligned}
&\Delta_h f = 0, && i,j = 1, \cdots, N,\\
\text{(A.17)}\qquad &f_{0,j} = f_{N+1,j} = 0, && j = 1, \cdots, N,\\
&f_{i,0} = f_{i,N+1} = 2, && i = 1, \cdots, N,
\end{aligned}
$$

where the first equation follows from (A.15) and (A.16), the second equation follows from (A.16) applied at $i = 0, N + 1$ and the third follows from (A.15) at $j = 0, N + 1$.

The solution of (A.17) is obtainable by separation of variables and is

$$
\text{(A.18)}\qquad f_{ij} = \frac{4}{N+1}\sum_{n=1}^{N} A_n \sin\frac{n\pi i}{N+1}(\sinh j\alpha_n + \sinh (N + 1 - j)\alpha_n),
$$

where

$$
\text{(A.19)}\qquad A_n = \frac{\sum_{k=1}^{N} \sin\frac{n\pi k}{N + \text{v}}}{\sinh (N + 1)\alpha_n} = \frac{\sin^2\frac{n\pi}{2}\cot\frac{n\pi}{2(N + 1)}}{\sinh (N + 1)\alpha_n}
$$

and where α_n satisfies

$$
\text{(A.20)}\qquad \cosh \alpha_n = 2 - \cos\frac{n\pi}{N + 1}.
$$

From (A.15) and (A.16), it is easy to show

$$\text{(A.21)}\qquad \begin{aligned}\sum_{i=1}^{N} f_{ij} &= N + \frac{\psi_{0,j} - \psi_{1,j} + \psi_{N+1,j} - \psi_{N,j}}{h^2},\\ \sum_{j=1}^{N} f_{ij} &= N + \frac{\psi_{i,1} - \psi_{i,0} + \psi_{i,N} - \psi_{i,N+1}}{h^2}.\end{aligned}$$

Using this and (A.13), we have

$$\text{(A.22)}\qquad \begin{aligned}\sum_{i=1}^{N} f_{ij} &= 2N + \frac{\varphi_{0,j} - \varphi_{1,j} + \varphi_{N+1,j} - \varphi_{N,j}}{h^2},\\ \sum_{j=1}^{N} f_{ij} &= \frac{\varphi_{i,1} - \varphi_{i,0} + \varphi_{i,N} - \varphi_{i,N+1}}{h^2}\end{aligned}$$

or

$$\text{(A.23)}\qquad \begin{aligned}\sum_{i=1}^{N} f_{ij} &= 2N + \frac{\delta_h\varphi_{0,j} + \delta_h\varphi_{N+1,j}}{h},\\ \sum_{j=1}^{N} f_{ij} &= -\frac{\delta_h\varphi_{i,0} - \delta_h\varphi_{i,N+1}}{h}.\end{aligned}$$

From the symmetry of φ, we note that

$$\begin{aligned}\delta_h\varphi_{0,j} = \delta_h\varphi_{N+1,j} &= \left(\sum_{i=1}^{N} f_{ij} - 2N\right)\frac{h}{2},\\ -\delta_h\varphi_{i,0} = -\delta_h\varphi_{i,N+1} &= \frac{h}{2}\sum_{j=1}^{N} f_{ij}.\end{aligned}$$

Thus, since our region is a square, we have

$$\begin{aligned}|\delta_n\varphi|_{\partial\Omega} &= \max_j |\delta_h\varphi_{0,j}| \leqq \max_j \frac{h}{2}\left(2N - \sum_{i=1}^{N} f_{ij}\right)\\ &\leqq \max_j \frac{h}{2}\left(2N - \frac{4}{N+1}\sum_{n=1}^{N} \frac{\sinh j\alpha_n + \sinh (N+1-j)\alpha_n}{\sinh (N+1)\alpha_n}\left(\sum_{k=1}^{N} \sin\frac{n\pi k}{N+1}\right)^2\right).\end{aligned}$$

The j which maximizes the right side is $j = (N+1)/2$ (if N is odd) or $j = N/2$ (if N is even).

For N odd, we have

$$\text{(A.24)}\qquad |\delta_h\varphi|_{\partial\Omega} \leqq \frac{h}{2}\left(2N - \frac{4}{N+1}\sum_{n=1}^{N} \frac{1}{\cosh\frac{(N+1)\alpha_n}{2}}\left(\sum_{k=1}^{N} \sin\frac{n\pi k}{N+1}\right)^2\right).$$

Since the summation has only positive terms, we continue the inequality by dropping all terms after the first and have, in terms of h,

$$|\delta_h \varphi|_{\partial\Omega} \leqq \frac{h}{2}\left(\frac{2(1-h)}{h} - 4h\frac{\cot^2(\pi h/2)}{\cosh(\alpha_1/(2h))}\right). \tag{A.25}$$

For small h, one can show that $\alpha_1 = \pi h + O(h^3)$. Thus we have $(\cosh \alpha_1/(2h))^{-1} \approx (\cosh \pi/2 + O(h^2))^{-1}$. Indeed, this can also be shown to be true for N even. Thus, we have

$$\begin{aligned}
|\delta_h \varphi|_{\partial\Omega} &\leqq 1 - h - 2h^2\left(\operatorname{sech}\frac{\pi}{2} + O(h^2)\right)\cot^2\frac{\pi h}{2} \\
&\leqq 1 - h - 2h^2\left(\operatorname{sech}\frac{\pi}{2} + O(h^2)\right)\left(\frac{4}{\pi^2 h^2} - \frac{2}{3} + O(h^2)\right) \\
&\leqq 1 - h - \frac{8}{\pi^2}\operatorname{sech}\frac{\pi}{2} + O(h^2) \\
&\leqq 0.677 - h
\end{aligned}$$

for sufficiently small h. Finally,

$$4 + 8h \approx \frac{4}{(1-h)^2} \geqq \sigma_h \geqq \frac{2}{0.677 - h} \approx 2.95 + 4.37h.$$

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