On the Complete Representation of Biharmonic Functions

Roger L. Fosdick

SIAM Journal on Applied Mathematics, Volume 19, Issue 1 (Jul., 1970), 243-250.

Stable URL:
http://links.jstor.org/sici?sici=0036-1399%28197007%2919%3A1%3C243%3AOTCROB%3E2.0.CO%3B2-Q

SIAM J. APPL. MATH.
Vol. 19, No. 1, July 1970

ON THE COMPLETE REPRESENTATION OF BIHARMONIC FUNCTIONS*

ROGER L. FOSDICK†

Abstract. The emphasis of this paper is on the complete representation of biharmonic functions in terms of harmonic functions for arbitrary bounded three-dimensional domains. The main result is contained in Theorem 2, which allows for the possibility that the domain may possess inclusions (i.e., holes). Reduced forms of the representation are noted under restrictive hypotheses on the geometry of the domain. A brief analogous treatment for plane two-dimensional domains is considered in order to illustrate the significant difference that here similar reduced forms are complete without geometric restrictions on the region. However, in this case, multivalued harmonic functions are generally necessary.

1. Introduction. The representation of biharmonic functions in terms of harmonic functions has a long history; two of the earliest treatments being that of Goursat [1] in 1898, and Almansi [2] one year later. Goursat considered the case of plane two-dimensional regions, while Almansi's work dealt with three-dimensional domains. More important, however, was the fact that Goursat obtained a *complete* representation in terms of three harmonic functions while Almansi's treatment of the completeness question was valid only for regions which possess certain directional convexities. More recently, Krakowski and Charnes [3] have given several distinct reductions of Goursat's representation from three harmonic functions to two, each of which are valid in arbitrary plane two-dimensional domains.[1] Although these reductions were reported earlier by Frank and von Mises [4], their proof of completeness was restricted to regions which are either convex in one direction or star-shaped with respect to one interior point. For three dimensions, Bergman and Schiffer [5] have recorded a representation in terms of only two harmonic functions, the completeness of which was again demonstrated only for starshaped regions. Finally, a complete representation for nonperiphractic[2] three-dimensional regions in terms of four harmonic functions has been given by Fosdick [6].

In the present work we obtain in § 2 (Theorem 2) a complete representation of an arbitrary biharmonic function in terms of four harmonic functions and a linear combination of *specific* biharmonic functions, the number of which depends upon the degree to which the domain is periphractic, which is valid for general three-dimensional domains. When the region is nonperiphractic the linear combination of known biharmonic functions is not needed and the result reduces to that given earlier in [6]. In a plane two-dimensional domain the analogous representation theorem, which is quoted without proof (Theorem 3), shows that

* Received by the editors August 4, 1969.

† Aeronautical Engineering and Mechanics Department, University of Minnesota, Minneapolis, Minnesota 55455. This research was supported by the National Science Foundation under Grant GP-8728.

[1] Goursat's representation as well as the three reduced representations of Krakowski and Charnes are respectively listed in (3.1a) and (3.1b)–(3.1d) of the present paper. Timoshenko and Goodier [8, p. 185] have also recorded (3.1d).

[2] A region is *periphractic* if it possesses a boundary surface which encloses at least one point not contained in the region. Thus, a periphractic region has inclusion (i.e., holes).

three harmonic functions and a linear combination of *specific* biharmonic functions, the number of which depends upon the degree of multiply connectedness, is sufficient. While in three dimensions it is shown that the linear combination of specific biharmonic functions is necessary to preserve the completeness property (Remark 2), in two dimensions they can be dropped from the representation provided the three remaining harmonic functions are allowed to be multivalued (Theorem 4). Finally, within the present theme we briefly consider the position of Goursat's representation and the general and complete reduction to two harmonic functions of Krakowski and Charnes [3] at the conclusion of § 3. It is interesting to observe that while these reductions are indeed general and complete for arbitrary plane regions, the analogous three-dimensional reduced forms of Almansi [2] and Bergman and Schiffer [5], which are summarized at the end of § 2, have been proved only for certain convex and starshaped domains.

While we recognize that the standard proof of Goursat's representation theorem using complex variables is by far the most economical,[3] our emphasis in this paper is on the completeness question for three dimensions, and we believe that the analogous treatment of the plane two-dimensional representation and its major reductions helps to clarify and emphasize the present issue.

2. Three-dimensional domains. In this section we discuss the complete representation of biharmonic functions in terms of harmonic functions in bounded open three-dimensional domains R. The closure of R will be denoted by $\bar{R}$, and the boundary ∂R is assumed to consist of a finite number of nonintersecting closed regular surfaces, the latter term being used in the sense of Kellogg [9]. Herein, such a region $\bar{R}$ will be called *regular*. In general, the region $\bar{R}$ may be periphractic, in which case ∂R will consist of several disjoint closed regular surfaces, $\partial R = \bigcup_{i=0}^{n} \partial_i R$. The *outer boundary* $\partial_0 R$ is that surface which encloses all remaining n boundary surfaces $\partial_i R$, $i = 1, 2, \cdots, n$, the latter set of which corresponds to the boundaries of the "holes" which are characteristic of a periphractic region. These boundaries will be denoted as *inner boundaries.*

We say that a real-valued function f defined on an open or closed region $G \subseteq \bar{R}$ is class $C^{N+\alpha}(G)$ and write $f \in C^{N+\alpha}(G)$, $0 < \alpha < 1$, if and only if f is N times continuously differentiable and has Nth order Hölder continuous partial derivatives with exponent α on G.

While the main representation theorem is the second theorem in this section, it is more revealing to present the result in parts. Thus, we begin with the following theorem.

THEOREM 1. *Let $\bar{R}$ be a regular three-dimensional region with boundary $\partial R = \bigcup_{i=0}^{n} \partial_i R$. The outer boundary $\partial_0 R$ as well as each inner boundary $\partial_i R$, $i = 1, 2, \cdots, n$, is assumed to be twice continuously differentiable. Further, let u have the following properties*:

(a) $u \in C^{4+\alpha}(R), u \in C^{3+\alpha}(\bar{R})$;

(b) $\displaystyle\int_{\partial D} \frac{\partial}{\partial n}(\nabla^2 u)\, dA = 0$ *for every closed regular surface* $\partial D \subset \bar{R}$.

[3] See, e.g. Mikhlin [7, § 40], or Timoshenko and Goodier [8, § 56].

Then,

(i) *u is biharmonic in R;*

(ii) *there exist harmonic functions v and $\mathbf{w}$ in R such that*

$$u = v + \mathbf{x} \cdot \mathbf{w} \quad \text{in } R. \tag{2.1}$$

For the most part, the proof of this theorem rests on the following lemma concerning the existence of a vector potential.

LEMMA 1. *Let $\bar{R}$, $\partial_0 R$ and $\partial_i R$, $i = 1, 2, \cdots, n$, meet the same hypotheses as in Theorem 1. Let $\mathbf{f}$ have the following properties:*

(a) $\mathbf{f} \in C^{N+\alpha}(R), N \geqq 1, \mathbf{f} \in C^{\alpha}(\bar{R})$;

(b) $\int_{\partial D} \mathbf{f} \cdot \mathbf{n} \, dA = 0$ *for every closed regular surface $\partial D \subset \bar{R}$.*

Then,

(i) *$\mathbf{f}$ is divergence-free in R;*

(ii) *there exist $\boldsymbol{\omega} \in C^{N+1+\alpha}(R)$, $\boldsymbol{\omega} \in C^0(\bar{R})$ such that*

$$\mathbf{f} = \nabla \times \boldsymbol{\omega} \quad \text{in } R. \tag{2.2}$$

While a proof of this lemma is essentially given by Lichtenstein [10, pp. 101–106], we remark that his smoothness hypotheses are more restrictive than those recorded here, which are sufficient.

Proof of Theorem 1. By applying condition (b) of the theorem to a surface ∂D which is reducible in R, and by making use of the divergence theorem, it follows that u is biharmonic in R.

To arrive at the representation (2.1) it is sufficient to show that there exists a vector field $\mathbf{w}$ with the following properties in R;

$$\nabla^2 \mathbf{w} = 0, \tag{2.3a}$$

$$\nabla^2 (u - \mathbf{x} \cdot \mathbf{w}) = 0. \tag{2.3b}$$

Then v is defined through $v \overset{\text{def}}{=} u - \mathbf{x} \cdot \mathbf{w}$. To facilitate this proof we first define

$$\psi \overset{\text{def}}{=} \tfrac{1}{2} \nabla^2 u, \tag{2.4}$$

and observe that the hypotheses yield $\psi \in C^{2+\alpha}(R)$, $\psi \in C^{1+\alpha}(\bar{R})$, and

$$\int_{\partial D} \nabla \psi \cdot \mathbf{n} \, dA = 0$$

for every closed regular surface $\partial D \subset \bar{R}$. Hence, Lemma 1 implies the existence of a vector field $\boldsymbol{\omega} \in C^{2+\alpha}(R)$, $\boldsymbol{\omega} \in C^0(\bar{R})$ such that in R,

$$\nabla \psi = \nabla \times \boldsymbol{\omega}. \tag{2.5}$$

Finally, we define the vector field $\mathbf{w}$ through

$$4\pi \mathbf{w}(\mathbf{x}) \overset{\text{def}}{=} \nabla \times \int_R \frac{\boldsymbol{\omega}(\mathbf{y})}{\rho(\mathbf{x}, \mathbf{y})} dV(\mathbf{y}) - \nabla \int_R \frac{\psi(\mathbf{y})}{\rho(\mathbf{x}, \mathbf{y})} dV(\mathbf{y}), \tag{2.6}$$

where $\mathbf{x}$ and $\mathbf{y}$ are points in R and where $\rho(\mathbf{x}, \mathbf{y})$ is the distance between $\mathbf{x}$ and $\mathbf{y}$. From well-known properties of the Newtonian potential we remark that $\mathbf{w} \in C^{3+\alpha}(R)$, and that in R, $\nabla^2 \mathbf{w} = -\nabla \times \boldsymbol{\omega} + \nabla \psi$. Hence, with (2.5) we see that

(2.3a) is satisfied. Moreover, (2.6) yields $\nabla \cdot \mathbf{w} = \psi$ in R, and with (2.4) it readily follows that (2.3b) is satisfied to complete this proof.

Remark 1. Not all biharmonic functions satisfy hypothesis (b) of Theorem 1. Hence, this theorem does not claim a complete representation for all biharmonic functions. To substantiate this remark consider a periphractic region $\bar{R}$ with at least one closed, regular inner boundary, $\partial_1 R$, and consider the distance function $u = |\mathbf{x} - \boldsymbol{\xi}_1|$ where $\mathbf{x}$ is an arbitrary point in $\bar{R}$, and where $\boldsymbol{\xi}_1$ is a fixed point outside $\bar{R}$ and enclosed by $\partial_1 R$ (i.e., inside the "hole"). It is clear that $\nabla^4 u = 0$ in R, but it can readily be shown that condition (b) is not satisfied for any closed surface $\partial D \subset \bar{R}$ which is reducible to $\partial_1 R$. In fact, since u is biharmonic everywhere in space except at $\mathbf{x} = \boldsymbol{\xi}_1$, we need only observe that (b) is not satisfied for any *spherical surface* with center at the point $\boldsymbol{\xi}_1$. Then application of the divergence theorem in the region bounded by ∂D and the spherical surface yields the same conclusion for ∂D.

Remark 2. Theorem 1 is not true if hypothesis (b) is replaced by the hypothesis that u is biharmonic in R. Hence, condition (b) is in general necessary for the representation (2.1). To see this, we need to exhibit a biharmonic function which cannot be represented in terms of harmonic functions through (2.1). Toward this end, consider the region $\bar{R}$ and the distance function $u = |\mathbf{x} - \boldsymbol{\xi}_1|$ of the preceding remark. If we suppose that harmonic v and $\mathbf{w}$ exist such that $|\mathbf{x} - \boldsymbol{\xi}_1| = v + \mathbf{x} \cdot \mathbf{w}$ in R, then applying the Laplacian operator we reach $\nabla \cdot \mathbf{w} = 1/|\mathbf{x} - \boldsymbol{\xi}_1|$ in R. Moreover, since $\mathbf{w}$ is supposed to be harmonic in R, a well-known identity yields $\nabla \times \nabla \times \mathbf{w} = \nabla\nabla \cdot \mathbf{w}$, and we obtain the necessary result that $\nabla \times \nabla \times \mathbf{w} = \nabla(1/|\mathbf{x} - \boldsymbol{\xi}_1|)$ in R. Now, let ∂D be a closed regular surface in R which encloses the inner boundary $\partial_1 R$. Then,

$$\int_{\partial D} (\nabla \times \nabla \times \mathbf{w}) \cdot \mathbf{n}\, dA = \int_{\partial D} \nabla \frac{1}{|\mathbf{x} - \boldsymbol{\xi}_1|} \cdot \mathbf{n}\, dA,$$

where $\mathbf{n}$ is the unit normal on one side of ∂D. Since the left-hand side of this equation is identically zero due to Stokes' theorem, we reach the conclusion that the right-hand side must also vanish. However, analogous to the previous remark, it can be shown that this integral cannot vanish since ∂D encloses the point $\boldsymbol{\xi}_1$, which supplies a contradiction to the assumption that $|\mathbf{x} - \boldsymbol{\xi}_1|$ admits the representation (2.1).

The preceding remarks suggest the following theorem.

THEOREM 2. *Let $\bar{R}$, $\partial_0 R$ and $\partial_i R$, $i = 1, 2, \cdots, n$, meet the same hypotheses as in Theorem 1. Let u have the following properties*:

(1a) $u \in C^{4+\alpha}(R), u \in C^4(\bar{R})$;

(b) *u is biharmonic in R.*

Then, there exist constants k_i, $i = 1, 2, \cdots, n$, and harmonic functions v and $\mathbf{w}$ in R such that

$$u = v + \mathbf{x} \cdot \mathbf{w} + \sum_{i=1}^{n} k_i |\mathbf{x} - \boldsymbol{\xi}_i|, \tag{2.7}$$

where $\boldsymbol{\xi}_i$, $i = 1, 2, \cdots, n$, represents an arbitrary fixed point not in $\bar{R}$, but enclosed by the inner boundary $\partial_i R$, $i = 1, 2, \cdots, n$, for each choice of index i.

Proof. Let $\{k_i\}$ be a set of n constants and choose a set of n fixed points $\{\xi_i\}$ as in the theorem statement. Then, the function $f \stackrel{\text{def}}{=} u - \sum_{i=1}^{n} k_i |\mathbf{x} - \xi_i|$ clearly satisfies the hypotheses (a) and (b) of the present theorem. Further, a straightforward calculation yields

$$\int_{\partial D} \frac{\partial}{\partial n}(\nabla^2 f)\, dA = \int_{\partial D} \frac{\partial}{\partial n}(\nabla^2 u)\, dA - 2 \sum_{i=1}^{n} k_i \int_{\partial D} \frac{\partial}{\partial n} \frac{1}{|\mathbf{x} - \xi_i|}\, dA \tag{2.8}$$

for every closed regular surface $D \subset \bar{R}$, where $\mathbf{n}$ is the outer unit normal to the region of space enclosed by ∂D. Now, since $\nabla^2 u$ and $1/|\mathbf{x} - \xi_i|$, $i = 1, 2, \cdots, n$, are harmonic in R, it readily follows from (2.8) that

$$\int_{\partial D} \frac{\partial}{\partial n}(\nabla^2 f)\, dA = 0 \tag{2.9}$$

for every closed regular surface ∂D which is reducible to a point in R. In addition, if in (2.8) we set $\partial D = \partial_j R$ where $\partial_j R$ is any one of the *inner* boundaries of $\bar{R}$, then in the region enclosed by $\partial_j R$, $1/|\mathbf{x} - \xi_i|$ is harmonic for all i except $i = j$, and we reach

$$\int_{\partial_j R} \frac{\partial}{\partial n}(\nabla^2 f)\, dA = \int_{\partial_j R} \frac{\partial}{\partial n}(\nabla^2 u)\, dA - 2k_j \int_{\partial_j R} \frac{\partial}{\partial n} \frac{1}{|\mathbf{x} - \xi_j|}\, dA. \tag{2.10}$$

The last integral on the right can be conveniently evaluated as -4π by enclosing $\partial_j R$ in a spherical surface with center at ξ_j and by appealing to the divergence theorem in the region bounded by this spherical surface and $\partial_j R$. Thus, if we define the set of constants $\{k_i\}$ through

$$k_j \stackrel{\text{def}}{=} \frac{1}{8\pi} \int_{\partial_j R} \frac{\partial}{\partial n}(\nabla^2 u)\, dA, \tag{2.11}$$

where here we have taken $\mathbf{n}$ to be the unit normal to $\partial_j R$ in the outer direction to R (i.e., opposite to that used in (2.8)–(2.10)), it follows from (2.10) that

$$\int_{\partial_j R} \frac{\partial}{\partial n}(\nabla^2 f)\, dA = 0, \qquad j = 1, 2, \cdots, n. \tag{2.12}$$

Finally, since f satisfies (a) and (b) of this theorem and since any closed regular surface in $\bar{R}$ is reducible to a linear combination of inner boundaries, we reach

$$\int_{\partial D} \frac{\partial}{\partial n}(\nabla^2 f)\, dA = 0 \tag{2.13}$$

for every closed regular surface $\partial D \subset \bar{R}$. With this much shown, we observe that f meets the hypotheses of Theorem 1 and thus admits the representation (2.1). By recalling the earlier definition of f, the proof is completed.

Remark 3. The analysis in Remark 2 shows that the series in (2.7) cannot be combined with either of the harmonic functions v or $\mathbf{w}$.

Remark 4. Since conditions (b) of Theorems 1 and 2 are equivalent if and only if $\bar{R}$ is nonperiphractic, the series in (2.7) is needed only for the representation of biharmonic functions in periphractic regions.

The work of Almansi [2] and Bergman and Schiffer [5] shows that for non-periphractic regions, appropriate restrictions on the geometry of the domain will permit further reduction of the representation (2.1). The specific results are summarized in the following two statements:

(a) (Almansi). *If u is biharmonic in R and if $\bar{R}$ is convex with respect to the direction of a fixed unit vector* $\mathbf{e}$, *then u admits the representation $u = q + (\mathbf{x} \cdot \mathbf{e})h$, where g and h are harmonic in R*;

(b) (Bergman and Schiffer). *If u is biharmonic in R and if $\bar{R}$ is starshaped with respect to an interior point $Q \in R$, then u admits the representation $u = g + r^2 h$, where r is the distance measured from Q and where g and h are harmonic in R.*

3. Two-dimensional domains. For a plane two-dimensional region A, Goursat's representation of a biharmonic function in terms of three harmonic functions, two of which are conjugate, and the representations of Krakowski and Charnes [3] in terms of two harmonic functions are complete and general only if the harmonic functions are permitted to be multivalued in the region of representation. These results are summarized in the following statement: *If u is biharmonic in A, then u admits any one of the representations*;

$$\text{(3.1)} \qquad \begin{array}{ll} \text{(a) } u = \operatorname{Re}\{\phi(z) + \bar{z}\psi(z)\}, & \text{(b) } u = g + x_1 h, \\ \text{(c) } u = g + x_2 h, & \text{(d) } u = g + r^2 h. \end{array}$$

Here, g and h are harmonic in A, ϕ and ψ are analytic functions of the complex variable $z = x_1 + ix_2$, $\bar{z}$ is the conjugate of z, and $r^2 = x_1^2 + x_2^2$, where x_1 and x_2 are the coordinates of a point in A. Although these representations are well known and most economically established with the aid of complex variables, it is revealing and of comparative interest to see how a treatment analogous to that given in § 2 can yield such results. Certain proofs will be omitted in order to avoid being repetitive.

Here, we let A denote a bounded open plane region with closure $\bar{A}$ and boundary ∂A which is such that $\bar{A}$ is *regular* in the sense that ∂A consists of a finite number of nonintersecting closed regular curves, the latter term being used after Kellogg [9]. In general, $\bar{A}$ may be periphractic (i.e., multiply connected in the present situation), in which case ∂A will consist of several disjoint closed regular curves, $\partial A = \bigcup_{i=0}^{n} \partial_i A : \partial_0 A$ is the outer boundary which encloses the remaining n inner boundary curves $\partial_i A$, $i = 1, 2, \cdots, n$.

The following theorem, given without proof, represents the analogue of the main Theorem 2 in the plane.

THEOREM 3. *Let A be a regular plane region with boundary $\partial A = \bigcup_{i=0}^{n} \partial_i A$. The outer boundary $\partial_0 A$ as well as each inner boundary $\partial_i A$, $i = 1, 2, \cdots, n$, is assumed to be twice continuously differentiable. Let u have the following properties*:

(a) $u \in C^{4+\alpha}(A), u \in C^4(\bar{A})$;

(b) *u is biharmonic in A.*

Then, there exist constants $k_i, i = 1, 2, \cdots, n$, and harmonic functions v and $\mathbf{w}$ *in A such that*

$$\text{(3.2)} \qquad u = v + \mathbf{x} \cdot \mathbf{w} + \sum_{i=1}^{n} k_i |\mathbf{x} - \xi_i|^2 [\log |\mathbf{x} - \xi_i| - 1],$$

where ξ_i, $i = 1, 2, \cdots, n$, *represents an arbitrary fixed point not in* $\bar{A}$, *but enclosed by the inner boundary* $\partial_i A$, $i = 1, 2, \cdots, n$, *for each choice of index i.*

As with the three-dimensional representation given in Theorem 2, the question also arises here as to whether the series in (3.2) is necessary for the representation to remain complete. Although for three dimensions we found that the series in (2.7) was indeed necessary, the following lemma shows that such is not the case for the representation in the plane.

LEMMA 2. *Let* $\{\xi_i\}$, $i = 1, 2, \cdots, n$, *denote a set of points as described in Theorem* 3 *and let* $\{k_i\}$, $i = 1, 2, \cdots, n$, *be a set of constants. Then, there exist harmonic functions* v_0 *and* $\mathbf{w}_0$ *such that*

$$\sum_{i=1}^{n} k_i|\mathbf{x} - \xi_i|^2 [\log|\mathbf{x} - \xi_i| - 1] = v_0 + \mathbf{x}\cdot\mathbf{w}_0, \tag{3.3}$$

everywhere except at the points $\mathbf{x} = \xi_i$, $i = 1, 2, \cdots, n$.

Proof. It follows by direct calculation that the functions defined by

$$\mathbf{w}_0 \overset{\text{def}}{=} \sum_{i=1}^{n} k_i\{(\mathbf{x} - \xi_i)[\log|\mathbf{x} - \xi_i| - 1] - (\mathbf{x} - \xi_i)\times \mathbf{a}_3\theta_i\}, \tag{3.4}$$

$$v_0 \overset{\text{def}}{=} -\sum_{i=1}^{n} k_i\xi_i\cdot\{(\mathbf{x} - \xi_i)[\log|\mathbf{x} - \xi_i| - 1] - (\mathbf{x} - \xi_i)\times \mathbf{a}_3\theta_i\}, \tag{3.5}$$

where $\mathbf{a}_3$ is a unit vector perpendicular to the plane of $\bar{A}$ and in the right-hand sense relative to two fixed orthonormal vectors $\mathbf{a}_1$, $\mathbf{a}_2$ in the plane and where θ_i is given by

$$\theta_i \overset{\text{def}}{=} \tan^{-1}\frac{(\mathbf{x} - \xi_i)\cdot\mathbf{a}_2}{(\mathbf{x} - \xi_i)\cdot\mathbf{a}_1}, \qquad i = 1, 2, \cdots, n, \tag{3.6}$$

are harmonic everywhere except at the points $\mathbf{x} = \xi_i$. Further it is clear that (3.3) is satisfied by these functions. Thus, the lemma is proved, which in combination with Theorem 3 yields the following theorem.

THEOREM 4. *Under the same hypotheses as in Theorem* 3, *u admits the complete representation*

$$u = v + \mathbf{x}\cdot\mathbf{w} \quad in\ A, \tag{3.7}$$

where v *and* $\mathbf{w}$ *are harmonic in* A.

Remark 5. In general, the representation (3.7) will not be complete unless the harmonic functions v and $\mathbf{w}$ are allowed to be multivalued in A.

In the remainder of this section we shall show how the four representations of (3.1) emerge as corollaries from (3.7). Toward this end we introduce the complex variable $z = x_1 + ix_2$ with conjugate $\bar{z}$ and define the analytic functions

$$f_1 \overset{\text{def}}{=} w_1 + iw_1', \qquad f_2 \overset{\text{def}}{=} w_2 + iw_2', \tag{3.8}$$

where $\mathbf{w}$ is the harmonic vector function in (3.7) and where $\mathbf{w}'$ denotes the conjugate harmonic of $\mathbf{w}$. To obtain (3.1c) we first observe that $\mathrm{Re}\,(zf_1) = x_1w_1 - x_2w_1'$, and add and subtract this quantity to the right-hand side of (3.7) to reach

$$u = v + \mathrm{Re}\,(zf_1) + x_2(w_2 + w_1'). \tag{3.9}$$

Thus, by defining the harmonic functions g and h through

$$g \overset{\text{def}}{=} v + \operatorname{Re}(zf_1), \qquad h \overset{\text{def}}{=} w_2 + w_1', \tag{3.10}$$

the validity of (3.1c) is demonstrated. In a similar manner, by adding and subtracting the expression $\operatorname{Re}(izf_2)$ to the right-hand side of (3.7), we also establish the representation (3.1b) as a consequence of Theorem 4. To reach Goursat's representation (3.1a) we observe that (3.7) may be written in terms of $z, \bar{z}$ and the analytic functions f_1 and f_2 as

$$u = v + \tfrac{1}{2}\operatorname{Re}[z(f_1 - if_2)] + \tfrac{1}{2}\operatorname{Re}[\bar{z}(f_1 + if_2)]. \tag{3.11}$$

Thus, by identifying $v + \frac{1}{2}\operatorname{Re}[z(f_1 - if_2)]$ with the real part of an analytic function ϕ and by noting that $f_1 + if_2$ is analytic, ψ, say, we obtain the first of (3.1). Finally, as was shown by Krakowski and Charnes [3] and Timoshenko and Goodier [8], (3.1d) follows directly from Goursat's representation by writing (3.1a) as

$$u = \operatorname{Re}\{\phi + (z\bar{z}/z)\psi\}$$

and observing that $z\bar{z} = r^2$.

REFERENCES

[1] E. GOURSAT, *Sur l'equation* $\Delta^2 u = 0$, Bull. Soc. Math. France, 26 (1898), pp. 236–237.

[2] EMILIO ALMANSI, *Sull' integrazione dell' equazione differenziale* $\nabla^{2n} = 0$, Am. Mat. Ser. (3), 2 (1899), pp. 1–51.

[3] M. KRAKOWSKI AND A. CHARNES, *Stokes' paradox and biharmonic flows*, Rep. 37, Carnegie Institute of Technology, Dept. of Mathematics, Pittsburgh, Penna., 1953.

[4] PHILIPP FRANK AND RICHARD V. MISES, eds., *Die Differential- und Integralgleichungen der Mechanik and Physik*, vol. 1, Dover, New York, 1961.

[5] STEFAN BERGMAN AND M. SCHIFFER, *Kernel Functions and Elliptic Differential Equations in Mathematical Physics*, Academic Press, New York, 1953.

[6] R. L. FOSDICK, *On the vector potential and the representation of a polyharmonic function in n-dimensions*, J. Math. Mech., 14 (1965), pp. 573–588.

[7] S. G. MIKHLIN, *Integral Equations*, Pergamon Press, New York, 1957.

[8] S. TIMOSHENKO AND J. N. GOODIER, *Theory of Elasticity*, 2nd ed., McGraw-Hill, New York, 1951.

[9] O. D. KELLOGG, *Foundations of Potential Theory*, Springer, Berlin, 1929.

[10] L. LICHTENSTEIN, *Grundlagen der Hydromechanik*, Springer, Berlin, 1929.