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Orthogonal spline collocation methods for biharmonic problems

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Summary. Orthogonal spline collocation methods are formulated and analyzed for the solution of certain biharmonic problems in the unit square. Particular attention is given to the Dirichlet biharmonic problem which is solved using capacitance matrix techniques.

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1. Introduction

Much attention has been devoted to biharmonic problems because of their occurrence in applications, for example, in flow and elasticity problems. Several applications involve the biharmonic Dirichlet problem:

$$
\begin{aligned}
\Delta^2 u &= f \ \text{ in } \Omega, \\
u &= g_1 \text{ on } \partial\Omega, \\
\frac{\partial u}{\partial n} &= g_2 \text{ on } \partial\Omega,
\end{aligned}
\tag{1.1}
$$

where here and throughout the paper, Δ denotes the Laplacian, $\Omega = (0,1)\times(0,1)$, $\partial\Omega$ is the boundary of Ω, and $\partial/\partial n$ is the outer normal derivative on $\partial\Omega$. In linear elasticity, u represents the Airy stress function or, as in the theory of thin plates, the vertical displacement due to an external force. In fluid mechanics, (1.1) defines the streamfunction of an incompressible two-dimensional creeping flow.

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Various methods have been developed for solving (1.1) numerically. Finite difference methods can be grouped essentially into two approaches. The first consists of the direct discretization of the biharmonic equation using a thirteen-point stencil [20,28]. Some algorithms for solving the resulting linear system have been proposed by Bjørstad [5], Buzbee and Dorr [7], and Golub [19], for example. On an $N \times N$ partition, the complexity of Golub's algorithm is $O(N^3 \log_2 N)$ with the use of fast Fourier transforms (FFTs) routines, while the complexity of Buzbee and Dorr's algorithm and that of Bjørstad are $O(N^3)$ and $O(N^2)$, respectively. The second approach is based on the so-called splitting principle [28], in which the biharmonic equation is separated into a coupled pair of Poisson equations that are discretized using the standard five-point finite difference approximation. Several iterative methods based on this approach have been presented in [15,16,22,25,26].

Some finite difference methods are very efficient, for example, that of Bjørstad [5] which is of optimal complexity $O(N^2)$. However, since all of these methods are based on the standard finite difference discretization, the global error is of second order. Higher order accuracy can be achieved using finite element methods. There are many finite element approaches which use iterative methods, such as those in [6,9,10,18,23]. Moreover, Hermite bicubic orthogonal spline collocation (OSC) methods for the biharmonic Dirichlet problem (1.1) have been considered by Cooper and Prenter [11] who proposed an alternating direction implicit OSC method, and by Sun [27] who presented a Schur complement OSC algorithm, the cost of which is $O(N^3 \log_2 N)$. These methods produce fourth order approximations although no rigorous proof of this result is provided in [11] or [27].

The purpose of this paper is to present existence, uniqueness and convergence results for OSC methods for the solution of three biharmonic problems, based on the splitting principle. In Sect. 3, we consider the first problem, which comprises the biharmonic equation in Ω with $u = g_1$ and $\Delta u = g_2$ on $\partial\Omega$. This problem becomes one of solving two nonhomogeneous Dirichlet problems for Poisson's equation. The resulting linear systems can be solved effectively with cost $O(N^2 \log_2 N)$ using the matrix decomposition algorithm of [3]. In this case, optimal order H^k-norm error estimates, $k = 0, 1, 2$, are derived. In the second problem, considered in Sect. 4, the boundary condition $\Delta u = g_2$ on the horizontal sides of $\partial\Omega$ is replaced by the condition $\partial u/\partial n = g_3$. Optimal H^1- and H^2-norm error estimates are derived and a single series OSC Fourier method is formulated for the solution of the algebraic problem. This algorithm has cost $O(N^2 \log_2 N)$. Sect. 5 is devoted to the biharmonic Dirichlet problem (1.1) and again optimal H^1- and H^2-norm error estimates are derived. The OSC linear system is solved by a direct method which is based on the capacitance matrix technique with the second biharmonic problem as the auxiliary problem. The total cost of this capacitance matrix algorithm is $O(N^3)$. We begin by introducing some basic concepts, terminology and some well known results for OSC approximation.

2. Preliminaries

Let N be a positive integer and let $\{t_n\}_{n=0}^{N}$ be a uniform partition of $[0,1]$ such that $t_n = nh,\ n = 0,\ldots,N$, where $h = 1/N$ is the stepsize. Let $\mathcal{M}_h$ be the space of piecewise Hermite cubics on $[0,1]$ defined by

$$\mathcal{M}_h = \{w \in C^1[0,1] : w|_{[t_n,t_{n+1}]} \in P_3,\ n = 0,\ldots,N-1\},$$

where P_3 denotes the set of polynomials of degree ≤ 3, and let

$$\mathcal{M}_h^0 = \{w \in \mathcal{M}_h : w(0) = w(1) = 0\},$$
$$\mathcal{M}_h^{00} = \{w \in \mathcal{M}_h^0 : w'(0) = w'(1) = 0\}.$$

Let $\{\xi_n\}_{n=1}^{2N}$ be the Gauss points in $(0,1)$ given by

$$\begin{aligned} \xi_{2m+1} &= t_m + h\frac{3-\sqrt{3}}{6}, \\ \xi_{2m+2} &= t_m + h\frac{3+\sqrt{3}}{6}, \end{aligned} \qquad m = 0,\ldots,N-1,$$

and let

$$\mathcal{G} = \{(x,y) : x,y \in \{\xi_n\}_{n=1}^{2N}\} \tag{2.1}$$

be the collection of the Gauss points in Ω.

It follows from Lemma 2.3 in [14] that each $v \in \mathcal{M}_h^0$ is uniquely defined by its values at the Gauss points $\{\xi_n\}_{n=1}^{2N}$. Therefore, in the following, $\mathcal{M}_h^0$ is regarded as a Hilbert space with the inner product $\langle \cdot,\cdot \rangle$ defined by

$$\langle w,z \rangle = \frac{h}{2}\sum_{n=1}^{2N}(wz)(\xi_n). \tag{2.2}$$

Throughout the paper, C denotes a generic positive constant. The following inequality is (2.6) of [24]:

$$\langle w,w \rangle \geq C\int_0^1 w^2(t)\,dt, \qquad w \in \mathcal{M}_h^0. \tag{2.3}$$

The following result is Lemma 3.1 in [14]:

Lemma 2.1. *For* $w,\ z\ \in \mathcal{M}_h$,

$$-\langle w'',z \rangle = \int_0^1 (w'z')(t)\,dt - w'z|_0^1 + Ch^5\sum_{n=1}^{N} w_n^{(3)} z_n^{(3)},$$

where $w_n^{(3)} \equiv w^{(3)}(t)$, $z_n^{(3)} \equiv z^{(3)}(t),\ t \in (t_{n-1},t_n)$.

Using (2.3) and Lemma 2.1 we prove the following result.

Theorem 2.1. *If* $\lambda \geq 0$*, then for any numbers* $\{f_n\}_{n=1}^{2N}$ *and* $\{g_n\}_{n=1}^{2N}$*, there exist unique* $w \in \mathcal{M}_h^{00}$ *and unique* $z \in \mathcal{M}_h$ *such that*

$$-w''(\xi_n) + \lambda w(\xi_n) + z(\xi_n) = f_n, \qquad n = 1, \ldots, 2N, \tag{2.4}$$

$$-z''(\xi_n) + \lambda z(\xi_n) = g_n, \qquad n = 1, \ldots, 2N. \tag{2.5}$$

Proof. Since in (2.4)–(2.5), the number of unknowns is equal to the number of equations, we assume that $f_n = g_n = 0$, $n = 1 \ldots, 2N$, and show that $w = z = 0$. Taking the inner product $\langle \cdot, \cdot \rangle$ with z on both sides of (2.4) and with w on both sides of (2.5), respectively, we obtain

$$\begin{aligned} \langle -w'', z\rangle + \lambda \langle w, z\rangle + \langle z, z\rangle &= 0, \\ \langle -z'', w\rangle + \lambda \langle z, w\rangle &= 0. \end{aligned} \tag{2.6}$$

It follows easily from Lemma 2.1 that $\langle -w'', z\rangle = \langle w, -z''\rangle$ and hence (2.6) gives $\langle z, z\rangle = 0$, which implies

$$z(\xi_n) = 0, \qquad n = 1, \ldots, 2N. \tag{2.7}$$

From (2.5) and (2.7), we have $z''(\xi_n) = 0$, $n = 1, \ldots, 2N$. Since on each subinterval $[t_{m-1}, t_m]$, $m = 1, \ldots, N$, $z \in P_3$, it follows from the four conditions $z(\xi_n) = z''(\xi_n) = 0$, $n = 2m-1, 2m$, that $z = 0$ on $[t_{m-1}, t_m]$ and hence $z = 0$.

From (2.4) and (2.7), we have

$$\langle -w'', w\rangle + \lambda \langle w, w\rangle = 0.$$

Since $\lambda \langle w, w\rangle \geq 0$ and since, by Lemma 2.1, $\langle -w'', w\rangle \geq \int_0^1 (w')^2(t)\, dt$, it follows that $w' = 0$ on $[0, 1]$. Together with $w(0) = 0$ this implies $w = 0$. □

For $l = 0, 1, 2, \ldots$, $H^l(\Omega)$ is the standard Sobolev space ($H^0(\Omega) = L^2(\Omega)$) equipped with the norm

$$\|v\|_{H^l(\Omega)} = \left(\sum_{0 \leq i+j \leq l} \left\| \frac{\partial^{i+j} v}{\partial x^i \partial y^j} \right\|_{L^2(\Omega)}^2 \right)^{1/2},$$

where $\|v\|_{L^2(\Omega)}^2 = \int_\Omega v^2(x, y)\, dx\, dy$. For w and z defined on $\mathcal{G}$ of (2.1), let $\langle w, z\rangle_{\mathcal{G}}$ and $\|w\|_{\mathcal{G}}$ be defined by

$$\langle w, z\rangle_{\mathcal{G}} = \frac{h^2}{4} \sum_{n=1}^{2N} \sum_{m=1}^{2N} (wz)(\xi_n, \xi_m), \qquad ||w||_{\mathcal{G}} = \langle w, w\rangle_{\mathcal{G}}^{1/2}. \tag{2.8}$$

Since $w \in \mathcal{M}_h^0 \otimes \mathcal{M}_h^0$ is uniquely determined by its values on $\mathcal{G}$, $\mathcal{M}_h^0 \otimes \mathcal{M}_h^0$ can be regarded as a Hilbert space with $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ as an inner product.

The following results follow easily from inequalities established in [24].

Lemma 2.2. *For* $w \in \mathcal{M}_h^0 \otimes \mathcal{M}_h^0$,

$$C^{-1}\|w\|_{L^2(\Omega)} \le \|w\|_{\mathcal{G}} \le C\|w\|_{L^2(\Omega)}, \tag{2.9}$$

and

$$\|w\|_{\mathcal{G}} \le C\|\Delta w\|_{\mathcal{G}}. \tag{2.10}$$

For sufficiently smooth v defined on $\overline{\Omega}$, let $v_{\mathcal{H}} \in \mathcal{M}_h \otimes \mathcal{M}_h$ be its piecewise Hermite interpolant defined by

$$\frac{\partial^{i+j}(v_{\mathcal{H}} - v)}{\partial x^i \partial y^j}(t_n, t_m) = 0, \quad n, m = 0, \dots, N, \quad i, j = 0, 1.$$

It is well known that v has a unique interpolant $v_{\mathcal{H}}$. Moreover, we have the following lemmas.

Lemma 2.3. *[8] If* $v \in H^4(\Omega)$, *then*

$$\|v - v_{\mathcal{H}}\|_{H^k(\Omega)} \le Ch^{4-k}\|v\|_{H^4(\Omega)}, \quad k = 0, 1, 2. \tag{2.11}$$

Lemma 2.4. *If* $v \in H^4(\Omega)$, *then*

$$\|v - v_{\mathcal{H}}\|_{\mathcal{G}} \le Ch^4\|v\|_{H^4(\Omega)}. \tag{2.12}$$

Moreover, if $v \in H^l(\Omega)$, $l = 4, 5$, *then*

$$\|\Delta(v - v_{\mathcal{H}})\|_{\mathcal{G}} \le Ch^{l-2}\|v\|_{H^l(\Omega)}. \tag{2.13}$$

Proof. Inequalities (2.12) and (2.13) for $l = 5$ were proved in Lemma 4.2 of [2]. Inequality (2.13) for $l = 4$ can be derived in a similar way. □

The following additional results were proved in [1].

Lemma 2.5. *For* $w \in \mathcal{M}_h^0 \otimes \mathcal{M}_h^0$,

$$\|w\|_{L^2(\Omega)} \le C\left(h\|w\|_{H^2(\Omega)} + \frac{h^2}{4}\sum_{n=1}^{N}\sum_{m=1}^{N}\left|\sum_{i=0}^{1}\sum_{j=0}^{1}\Delta w(\xi_{2n-i}, \xi_{2m-j})\right|\right). \tag{2.14}$$

If $v \in H^6(\Omega)$, *then*

$$\frac{h^2}{4}\sum_{n=1}^{N}\sum_{m=1}^{N}\left|\sum_{i=0}^{1}\sum_{j=0}^{1}\Delta(v - v_{\mathcal{H}})(\xi_{2n-i}, \xi_{2m-j})\right| \le Ch^4\|v\|_{H^6(\Omega)}. \tag{2.15}$$

To present some useful bases for $\mathcal{M}_h$, we first introduce (see, for example, [17]) the functions v_n, $s_n \in \mathcal{M}_h$, $n = 0, \dots, N$, known as the "value function" and the "scaled slope function", respectively, associated with the point t_n. These functions are defined by

$$\begin{matrix} v_n(t_m) = \delta_{n,m}, & v_n'(t_m) = 0, \\ s_n(t_m) = 0, & s_n'(t_m) = h^{-1}\delta_{n,m}, \end{matrix} \qquad n, m = 0, \dots, N,$$

where $\delta_{n,m}$ is the Kronecker delta. By ordering the v_n and s_n, we obtain two bases, $\{\phi_n\}_{n=0}^{2N+1}$ and $\{\psi_n\}_{n=0}^{2N+1}$ for $\mathcal{M}_h$, such that

$$\begin{aligned}\{\phi_0,\phi_1,\dots,\phi_{2N},\phi_{2N+1}\} = \{&v_0,v_1,\dots,v_{N-2},v_{N-1},s_0,s_1,\dots,\\ &s_{N-2},s_{N-1},s_N,v_N\},\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}\{\psi_0,\psi_1,\dots,\psi_{2N},\psi_{2N+1}\} = \{&v_0,s_0,v_1,s_1\dots,\\ &v_{N-1},s_{N-1},s_N,v_N\}.\end{aligned} \tag{2.17}$$

Note that, by removing the first and the last basis functions from (2.16) and (2.17), we obtain two bases for $\mathcal{M}_h^0$:

$$\begin{aligned}\{\phi_1,\dots,\phi_{2N}\} = \{&v_1,\dots,v_{N-2},v_{N-1},s_0,s_1,\dots,\\ &s_{N-2},s_{N-1},s_N\}\end{aligned} \tag{2.18}$$

and

$$\{\psi_1,\dots,\psi_{2N}\} = \{s_0,v_1,s_1\dots,v_{N-1},s_{N-1},s_N\}.$$

In [3], formulas were derived for $\lambda_n > 0$ and $z_n \in \mathcal{M}_h^0$, $n = 1,\dots,2N$, such that

$$-z_n''(\xi_m) = \lambda_n z_n(\xi_m), \qquad m = 1,\dots,2N, \tag{2.19}$$

and

$$\langle z_n, z_m\rangle = \delta_{n,m}, \qquad n,m = 1,\dots,2N, \tag{2.20}$$

where $\langle\cdot,\cdot\rangle$ is defined by (2.2). According to Theorem 2.1 in [3],

$$\lambda_n = \lambda_n^-, \quad n = 1,\dots,N-1, \qquad \lambda_{N+n} = \lambda_n^+, \quad n = 0,\dots,N, \tag{2.21}$$

where

$$\begin{aligned}&\lambda_n^\pm = \frac{12}{h^2}\left(\frac{8+\eta_n \pm \mu_n}{7-\eta_n}\right), \qquad \mu_n = \sqrt{43+40\eta_n-2\eta_n^2},\\ &\eta_n = \cos\frac{n\pi}{N}.\end{aligned} \tag{2.22}$$

It follows easily from (2.20) that $\{z_n\}_{n=1}^{2N}$ is a basis for $\mathcal{M}_h^0$.

In addition to the bases $\{\phi_n\}_{n=1}^{2N}$ and $\{z_n\}_{n=1}^{2N}$, we introduce the basis $\{\theta_n\}_{n=1}^{2N}$ for $\mathcal{M}_h^0$ defined by

$$\theta_n(\xi_m) = \delta_{n,m}, \qquad n,m = 1,\dots,2N. \tag{2.23}$$

Then any $w \in \mathcal{M}_h^0$ can be represented in the forms

$$w = \sum_{n=1}^{2N} w_n^z z_n = \sum_{n=1}^{2N} w_n^\phi \phi_n = \sum_{n=1}^{2N} w_n^\theta \theta_n, \tag{2.24}$$

where w_n^z, w_n^ϕ, and w_n^θ are the Fourier coefficients of w with respect to the corresponding basis functions. It follows from (2.24), (2.20), and (2.23) that

$$w_n^z = \langle w, z_n \rangle, \quad w_n^\theta = w(\xi_n), \quad n = 1, \ldots, 2N. \tag{2.25}$$

Moreover, if

$$\mathbf{w}^z = [w_1^z, \ldots, w_{2N}^z]^{\mathrm{T}}, \quad \mathbf{w}^\phi = [w_1^\phi, \ldots, w_{2N}^\phi]^{\mathrm{T}}, \quad \mathbf{w}^\theta = [w_1^\theta, \ldots, w_{2N}^\theta]^{\mathrm{T}}, \tag{2.26}$$

then it was shown in [4] that

$$\mathbf{w}^\phi = Z\mathbf{w}^z, \tag{2.27}$$

$$\mathbf{w}^z = \frac{h}{2} Z^{\mathrm{T}} B_\phi^{\mathrm{T}} B_\phi \mathbf{w}^\phi, \tag{2.28}$$

and

$$\mathbf{w}^z = \frac{h}{2} Z^{\mathrm{T}} B_\phi^{\mathrm{T}} \mathbf{w}^\theta, \tag{2.29}$$

where

$$B_\phi = (b_{mn})_{m,n=1}^{2N}, \qquad b_{mn} = \phi_n(\xi_m), \tag{2.30}$$

and

$$Z = 6\sqrt{3} \left[\begin{array}{c|c|c|c} S\Lambda_\alpha^- & \mathbf{0} & S\Lambda_\alpha^+ & \mathbf{0} \\ \hline \tilde{C}\Lambda_\beta^- & \multicolumn{3}{c}{C\Lambda_\beta^+} \end{array} \right], \tag{2.31}$$

$$S = \left(\sin \frac{mn\pi}{N} \right)_{m,n=1}^{N-1}, \quad C = \left(\cos \frac{mn\pi}{N} \right)_{m,n=0}^{N},$$
$$\tilde{C} = \left(\cos \frac{mn\pi}{N} \right)_{m=0,n=1}^{N,N-1}.$$

The diagonal matrices $\Lambda_\alpha^\pm$, $\Lambda_\beta^\pm$ in Z are defined by

$$\Lambda_\alpha^\pm = \mathrm{diag}(\alpha_1^\pm, \ldots, \alpha_{N-1}^\pm),$$

$$\Lambda_\beta^- = \mathrm{diag}(\beta_1^-, \ldots, \beta_{N-1}^-), \qquad \Lambda_\beta^+ = \mathrm{diag}(1, \beta_1^+, \ldots, \beta_{N-1}^+, 1/\sqrt{3}),$$

where

$$\alpha_n^\pm = (5 + 4\eta_n \mp)\nu_n^\pm, \qquad \beta_n^\pm = 18 \sin\left(\frac{n\pi}{N}\right) \nu_n^\pm,$$

$$\nu_n^\pm = [27(1+\eta_n)(8+\eta_n \mp \mu_n)^2 + (1-\eta_n)(11 + 7\eta_n \mp 4\mu_n)^2]^{-1/2},$$

and μ_n and η_n are as in (2.22).

3. Biharmonic problem I

3.1. OSC scheme

In this section, we consider Biharmonic Problem I (BPI):

$$\begin{aligned} \Delta^2 u &= f \quad \text{in } \Omega, \\ u &= g_1 \quad \text{on } \partial\Omega, \\ \Delta u &= g_2 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

Introducing $v = \Delta u$, we obtain the equivalent decoupled problem

$$\begin{aligned} -\Delta u &= -v \quad \text{in } \Omega, \\ u &= g_1 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.2}$$

$$\begin{aligned} -\Delta v &= -f \quad \text{in } \Omega, \\ v &= g_2 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.3}$$

Thus we can solve (3.1) by sequentially solving two nonhomogeneous Dirichlet problems for Poisson's equation, (3.3) and then (3.2).

The piecewise Hermite bicubic OSC method for solving (3.2)–(3.3) consists in finding $u_h, v_h \in \mathcal{M}_h \otimes \mathcal{M}_h$ such that

$$\begin{aligned} -\Delta u_h(\xi) &= -v_h(\xi), \ \xi \in \mathcal{G}, \\ -\Delta v_h(\xi) &= -f(\xi), \ \ \xi \in \mathcal{G}, \end{aligned} \tag{3.4}$$

where the boundary coefficients in u_h and v_h are determined by piecewise Hermite cubic interpolation as in [2]. Specifically, using $\{\phi_i\}_{i=0}^{2N+1}$ and $\{\psi_j\}_{j=0}^{2N+1}$ of (2.16) and (2.17) as the bases for $\mathcal{M}_h$, we write $u_h(x,y)$ and $v_h(x,y)$ in the form

$$u_h(x,y) = \bar{u}_h(x,y) + \tilde{u}_h(x,y), \qquad v_h(x,y) = \bar{v}_h(x,y) + \tilde{v}_h(x,y), \tag{3.5}$$

where

$$\bar{u}_h(x,y) = \sum_{i=1}^{2N}\sum_{j=1}^{2N} u_{i,j}\phi_i(x)\psi_j(y),$$

$$\begin{aligned} \tilde{u}_h(x,y) = &\sum_{i=1}^{2N} u_{i,0}\phi_i(x)\psi_0(y) + \sum_{i=1}^{2N} u_{i,2N+1}\phi_i(x)\psi_{2N+1}(y) \\ &+ \sum_{j=0}^{2N+1} u_{0,j}\phi_0(x)\psi_j(y) + \sum_{j=0}^{2N+1} u_{2N+1,j}\phi_{2N+1}(x)\psi_j(y), \end{aligned}$$

with $\bar{v}_h(x,y)$ and $\tilde{v}_h(x,y)$ defined similarly. Using the boundary conditions in (3.2) and (3.3), the coefficients in $\tilde{u}_h(x,y)$ and $\tilde{v}_h(x,y)$ are selected so that

$$u_h(x,y) = u_{\mathcal{H}}(x,y), \qquad v_h(x,y) = v_{\mathcal{H}}(x,y), \qquad (x,y) \in \partial\Omega. \tag{3.6}$$

Substituting (3.5) into (3.4), we obtain

$$\begin{aligned} -\Delta\bar{u}_h(\xi) + \bar{v}_h(\xi) &= \Delta\tilde{u}_h(\xi) - \tilde{v}_h(\xi), \, \xi \in \mathcal{G}, \\ -\Delta\bar{v}_h(\xi) &= \Delta\tilde{v}_h(\xi) - f(\xi), \;\; \xi \in \mathcal{G}, \end{aligned} \tag{3.7}$$

where $\bar{u}_h, \bar{v}_h \in \mathcal{M}_h^0 \otimes \mathcal{M}_h^0$ and the right hand sides are known. Hence existence and uniqueness of the approximate solutions u_h and v_h of (3.4) follow from those for $\bar{u}_h$ and $\bar{v}_h$ in (3.7). Moreover, $\bar{v}_h$ and then $\bar{u}_h$ of (3.7), and hence v_h and u_h of (3.4), can be computed using the matrix decomposition algorithm of [3].

3.2. Convergence analysis

3.2.1. Additional lemmas

We need the following lemmas.

Lemma 3.1. *For $w,\ z \in \mathcal{M}_h^0 \otimes \mathcal{M}_h^0$, we have*

$$\langle -\Delta w, z\rangle_{\mathcal{G}} = \langle w, -\Delta z\rangle_{\mathcal{G}}. \tag{3.8}$$

Proof. Since $w,\ z \in \mathcal{M}_h^0 \otimes \mathcal{M}_h^0$, we have

$$w(a,y) = z(a,y) = 0, \qquad a = 0,1, \qquad y \in [0,1], \tag{3.9}$$

and

$$w(x,b) = z(x,b) = 0, \qquad x \in [0,1], \qquad b = 0,1. \tag{3.10}$$

In order to prove (3.8), it suffices to show that

$$\langle -w_{xx}, z\rangle_{\mathcal{G}} = \langle w, -z_{xx}\rangle_{\mathcal{G}}, \qquad \langle -w_{yy}, z\rangle_{\mathcal{G}} = \langle w, -z_{yy}\rangle_{\mathcal{G}}. \tag{3.11}$$

Using (2.8), (2.2), Lemma 2.1, and (3.9) for z, we have

$$\begin{aligned} \langle -w_{xx}, z\rangle_{\mathcal{G}} &= -\frac{h}{2}\sum_{m=1}^{2N} \langle w_{xx}(\cdot,\xi_m), z(\cdot,\xi_m)\rangle \\ &= \frac{h}{2}\sum_{m=1}^{2N}\left[\int_0^1 (w_x z_x)(x,\xi_m)\,dx - (w_x z)(x,\xi_m)\Big|_{x=0}^{x=1}\right. \\ &\qquad \left. + Ch^5 \sum_{n=1}^{N} w_{n,m}^{(3,0)} z_{n,m}^{(3,0)}\right] \\ &= \frac{h}{2}\sum_{m=1}^{2N}\left[\int_0^1 (w_x z_x)(x,\xi_m)\,dx + Ch^5\sum_{n=1}^{N} w_{n,m}^{(3,0)} z_{n,m}^{(3,0)}\right], \end{aligned} \tag{3.12}$$

where

$$w_{n,m}^{(3,0)} = w_{xxx}(x, \xi_m), \qquad z_{n,m}^{(3,0)} = z_{xxx}(x, \xi_m), \qquad x \in (t_{n-1}, t_n).$$

Since the right-hand side of (3.12) is symmetric with respect to w and z, the first equation of (3.11) follows easily. A similar argument using (3.10) gives the second equation in (3.11). □

Lemma 3.2. *For* $w \in \mathcal{M}_h^0 \otimes \mathcal{M}_h^0$,

$$\|w\|_{H^2(\Omega)} \le C\|\Delta w\|_{\mathcal{G}}. \tag{3.13}$$

Proof. From (8.24) in [21] we have

$$\|w\|_{H^2(\Omega)}^2 \le C\|\Delta w\|_{L^2(\Omega)}^2, \qquad w \in H^2(\Omega),\ w = 0 \ \text{ on } \ \partial\Omega,$$

and from the Cauchy-Schwarz inequality we have

$$\|\Delta w\|_{L^2(\Omega)}^2 \le 2\left(\|w_{xx}\|_{L^2(\Omega)}^2 + \|w_{yy}\|_{L^2(\Omega)}^2\right).$$

Thus, since

$$\|\Delta w\|_{\mathcal{G}}^2 = \|w_{xx}\|_{\mathcal{G}}^2 + 2\left\langle w_{xx}, w_{yy}\right\rangle_{\mathcal{G}} + \|w_{yy}\|_{\mathcal{G}}^2,$$

it suffices to show that

$$\|w_{xx}\|_{L^2(\Omega)}^2 \le C\|w_{xx}\|_{\mathcal{G}}^2, \quad \|w_{yy}\|_{L^2(\Omega)}^2 \le C\|w_{yy}\|_{\mathcal{G}}^2, \quad w \in \mathcal{M}_h^0 \otimes \mathcal{M}_h^0, \tag{3.14}$$

and

$$\left\langle w_{xx}, w_{yy}\right\rangle_{\mathcal{G}} \ge 0, \qquad w \in \mathcal{M}_h^0 \otimes \mathcal{M}_h^0. \tag{3.15}$$

Using (2.8), the exactness of 2-point Gauss quadrature for polynomials of degree ≤ 3, the fact that $w_{xx}(x,\cdot) \in \mathcal{M}_h^0$, $x \in (t_{n-1}, t_n)$, and (2.3), we have

$$\begin{aligned}\|w_{xx}\|_{\mathcal{G}}^2 &= \frac{h}{2}\sum_{m=1}^{2N}\frac{h}{2}\sum_{n=1}^{2N} w_{xx}^2(\xi_n, \xi_m) = \frac{h}{2}\sum_{m=1}^{2N}\int_0^1 w_{xx}^2(x, \xi_m)dx \\ &= \int_0^1 \frac{h}{2}\sum_{m=1}^{2N} w_{xx}^2(x, \xi_m)dx \ge C\int_\Omega w_{xx}^2(x,y)\,dx\,dy.\end{aligned}$$

Thus interchanging the roles of x and y, we obtain (3.14).

It follows from Lemma 2.1, applied with respect to x and y directions, that the operators $-w_{xx}$, $-w_{yy}$ from $\mathcal{M}_h^0 \otimes \mathcal{M}_h^0$ into $\mathcal{M}_h^0 \otimes \mathcal{M}_h^0$ are self-adjoint and nonnegative definite with respect to $\langle\cdot,\cdot\rangle_{\mathcal{G}}$. Moreover, it is easy to verify that they commute. Therefore, the operator $(w_{xx})_{yy}$ is nonnegative definite which implies (3.15). □

3.2.2. H^k-error estimates

We now prove the following convergence result for the OSC scheme (3.5)–(3.7).

Theorem 3.1. *Let $u \in H^{8-k}(\Omega)$, $k = 0, 1, 2$, be the solution of BPI (3.1). Let $u_h, v_h \in \mathcal{M}_h \otimes \mathcal{M}_h$ of (3.5) satisfy (3.6) and (3.7). Then*

$$\|u - u_h\|_{H^k(\Omega)} \le Ch^{4-k} \|u\|_{H^{8-k}(\Omega)}, \quad k = 0, 1, 2. \tag{3.16}$$

Proof. Let $u_{\mathcal{H}}$ and $v_{\mathcal{H}}$ be the piecewise Hermite bicubic interpolants of u and $v = \Delta u$, respectively, and let

$$w = u_h - u_{\mathcal{H}}, \quad z = v_h - v_{\mathcal{H}}. \tag{3.17}$$

Then it follows from (3.17), the first equation of (3.4), and $v = \Delta u$ that

$$\begin{aligned} -\Delta w(\xi) + z(\xi) &= -\Delta(u_h - u_{\mathcal{H}})(\xi) + (v_h - v_{\mathcal{H}})(\xi) \\ &= \Delta u_{\mathcal{H}}(\xi) - v_{\mathcal{H}}(\xi) \\ &= -\Delta(u - u_{\mathcal{H}})(\xi) + (v - v_{\mathcal{H}})(\xi), \quad \xi \in \mathcal{G}. \end{aligned} \tag{3.18}$$

Similarly from (3.17), the second equation of (3.4), and $f = \Delta v$, we have

$$-\Delta z(\xi) = -f(\xi) + \Delta v_{\mathcal{H}}(\xi) = -\Delta(v - v_{\mathcal{H}})(\xi), \quad \xi \in \mathcal{G}. \tag{3.19}$$

Equations (3.17) and (3.6) imply that $w, z \in \mathcal{M}_h^0 \otimes \mathcal{M}_h^0$. Therefore, taking the inner product $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ with $-\Delta w$ on both sides of (3.18), we obtain

$$\|\Delta w\|_{\mathcal{G}}^2 + \langle z, -\Delta w \rangle_{\mathcal{G}} = \langle -\Delta(u - u_{\mathcal{H}}) + v - v_{\mathcal{H}}, -\Delta w \rangle_{\mathcal{G}}. \tag{3.20}$$

Taking the inner product $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ with w on both sides of (3.19), we have

$$\langle -\Delta z, w \rangle_{\mathcal{G}} = \langle -\Delta(v - v_{\mathcal{H}}), w \rangle_{\mathcal{G}}. \tag{3.21}$$

Hence from (3.20), (3.21), and Lemma 3.1, we obtain

$$\|\Delta w\|_{\mathcal{G}}^2 = \langle -\Delta(u - u_{\mathcal{H}}) + v - v_{\mathcal{H}}, -\Delta w \rangle_{\mathcal{G}} + \langle \Delta(v - v_{\mathcal{H}}), w \rangle_{\mathcal{G}}.$$

Using the Cauchy-Schwarz inequality and (2.10), we have

$$\|\Delta w\|_{\mathcal{G}} \le C(\|\Delta(u - u_{\mathcal{H}})\|_{\mathcal{G}} + \|v - v_{\mathcal{H}}\|_{\mathcal{G}} + \|\Delta(v - v_{\mathcal{H}})\|_{\mathcal{G}}).$$

Thus from this inequality, (3.13), (2.12), and (2.13), we obtain

$$\|w\|_{H^2(\Omega)} \le Ch^{4-k} \|u\|_{H^{8-k}(\Omega)}, \quad k = 1, 2. \tag{3.22}$$

For $k = 1, 2$, (3.16) follows now from the triangle inequality, (2.11) with $k = 1, 2$, (3.17), and (3.22).

To prove (3.16) for $k = 0$, we use (2.14), (3.18), the triangle inequality, and the Cauchy-Schwarz inequality to obtain

$$\|w\|_{L^2(\Omega)} \le C\Bigg(h\|w\|_{H^2(\Omega)} + \frac{h^2}{4}\sum_{n=1}^{N}\sum_{m=1}^{N}\left|\sum_{i=0}^{1}\sum_{j=0}^{1}\Delta(u-u_{\mathcal{H}})(\xi_{2n-i},\xi_{2m-j})\right| + \|v-v_{\mathcal{H}}\|_{\mathcal{G}} + \|z\|_{\mathcal{G}}\Bigg). \tag{3.23}$$

Using (2.9), (2.14), (3.13), and (3.19), we also have

$$\|z\|_{\mathcal{G}} \le C\left(h\|\Delta(v-v_{\mathcal{H}})\|_{\mathcal{G}} + \frac{h^2}{4}\sum_{n=1}^{N}\sum_{m=1}^{N}\left|\sum_{i=0}^{1}\sum_{j=0}^{1}\Delta(v-v_{\mathcal{H}})(\xi_{2n-i},\xi_{2m-j})\right|\right). \tag{3.24}$$

Finally (3.16) for $k = 0$, follows from the triangle inequality, (2.11) with $k = 0$, (3.17), (3.23), (3.24), (3.22) with $k = 1$, (2.12), (2.13) with $l = 5$, and (2.15). □

4. Biharmonic problem II

4.1. OSC scheme: existence, uniqueness, convergence

Consider Biharmonic Problem II (BPII):

$$\begin{aligned} \Delta^2 u &= f \quad \text{in } \Omega,\\ u &= g_1 \quad \text{on } \partial\Omega,\\ \frac{\partial u}{\partial n} &= g_2 \quad \text{on } \partial\Omega_1,\\ \Delta u &= g_3 \quad \text{on } \partial\Omega_2, \end{aligned} \tag{4.1}$$

where $\partial\Omega_1$ is the union of the horizontal sides of $\partial\Omega$ and $\partial\Omega_2$ is the union of the vertical sides of $\partial\Omega$. By introducing $v = \Delta u$, we obtain the coupled problem

$$\begin{aligned} -\Delta u &= -v \quad \text{in } \Omega,\\ u &= g_1 \quad \text{on } \partial\Omega,\\ \frac{\partial u}{\partial n} &= g_2 \quad \text{on } \partial\Omega_1, \end{aligned} \tag{4.2}$$

and

$$-\Delta v = -f \quad \text{in } \Omega, \qquad v = g_3 \quad \text{on } \partial\Omega_2. \tag{4.3}$$

We seek OSC solutions $u_h, v_h \in \mathcal{M}_h \otimes \mathcal{M}_h$ of (4.2)–(4.3) defined in the following way. Since $u_h, v_h \in \mathcal{M}_h \otimes \mathcal{M}_h$, they are of the form

$$\begin{aligned} u_h(x,y) &= \sum_{i=0}^{2N+1} \sum_{j=0}^{2N+1} u_{i,j}\phi_i(x)\psi_j(y), \\ v_h(x,y) &= \sum_{i=0}^{2N+1} \sum_{j=0}^{2N+1} v_{i,j}\phi_i(x)\psi_j(y), \end{aligned} \tag{4.4}$$

where the bases $\{\phi_i\}_{i=0}^{2N+1}$ and $\{\psi_j\}_{j=0}^{2N+1}$ for $\mathcal{M}_h$ are defined in (2.16) and (2.17).

The nonhomogeneous boundary conditions for u and v in (4.2) and (4.3) are approximated using piecewise Hermite cubic interpolation. The coefficients $u_{i,0}, u_{i,2N+1}$, $i = 1, \ldots, 2N$, and $u_{0,j}, u_{2N+1,j}$, $j = 0, \ldots, 2N+1$, in (4.4) which correspond to the boundary condition $u = g_1$ on $\partial\Omega$ are determined in exactly the same way as those in Sect. 3.1. Now we describe the approximation of the boundary condition $\partial u/\partial n = g_2$ on $\partial\Omega_1$ to determine the coefficients $u_{i,1}, u_{i,2N}$, $i = 1, \ldots, 2N$, and the approximation of the boundary condition $v = g_3$ on $\partial\Omega_2$ to determine the coefficients $v_{0,j}$, $v_{2N+1,j}$, $j = 0, \ldots, 2N+1$. On the bottom side of $\partial\Omega$, we require that

$$\begin{aligned} [(u_h)_y + g_2](t_n, 0) &= 0, \quad n = 1, \ldots, N-1, \\ [(u_h)_y + g_2]_x(t_n, 0) &= 0, \quad n = 0, \ldots, N. \end{aligned}$$

Substituting $u_h(x,y)$ of (4.4) into these equations, we obtain formulas for the coefficients $u_{i,1}$, $i = 1, \ldots, 2N$:

$$\begin{aligned} u_{n,1} &= -hg_2(t_n, 0), \quad n = 1, \ldots, N-1, \\ u_{N+n,1} &= -h^2(g_2)_x(t_n, 0), \quad n = 0, \ldots, N. \end{aligned}$$

On the top side of $\partial\Omega$, we require

$$\begin{aligned} [(u_h)_y - g_2](t_n, 1) &= 0, \quad n = 1, \ldots, N-1, \\ [(u_h)_y - g_2]_x(t_n, 1) &= 0, \quad n = 0, \ldots, N, \end{aligned}$$

from which, we obtain formulas for the coefficients $u_{i,2N}$, $i = 1, \ldots, 2N$:

$$\begin{aligned} u_{n,2N} &= hg_2(t_n, 1), \quad n = 1, \ldots, N-1, \\ u_{N+n,2N} &= h^2(g_2)_x(t_n, 1), \quad n = 0, \ldots, N. \end{aligned}$$

On vertical sides of $\partial\Omega$, we require that

$$(v_h - g_3)(a, t_n) = 0, \quad (v_h - g_3)_y(a, t_n) = 0, \quad a = 0, 1, \quad n = 0, \ldots, N.$$

By substituting $v_h(x,y)$ of (4.4) into these equations, we obtain formulas for the coefficients $v_{0,j}$, $j = 0,\ldots,2N+1$:

$$v_{0,2n} = g_3(0,t_n), \qquad v_{0,2n+1} = h(g_3)_y(0,t_n), \qquad n = 0,\ldots,N-1,$$
$$v_{0,2N} = h(g_3)_y(0,t_N),\ v_{0,2N+1} = g_3(0,t_N),$$

and the coefficients $v_{2N+1,j}$, $j = 0,\ldots,2N+1$:

$$v_{2N+1,2n} = g_3(1,t_n), \qquad v_{2N+1,2n+1} = h(g_3)_y(1,t_n),\ \ n = 0,\ldots,N-1,$$
$$v_{2N+1,2N} = h(g_3)_y(1,t_N),\ v_{2N+1,2N+1} = g_3(1,t_N).$$

Thus, we determine $16N+8$ coefficients $u_{i,0}$, $u_{i,1}$, $u_{i,2N}$, $u_{i,2N+1}$, $i = 1,\ldots,2N$, $u_{0,j}, u_{2N+1,j}$, $j = 0,\ldots,2N+1$, and $v_{0,j}, v_{2N+1,j}$, $j = 0,\ldots, 2N+1$.

If we rewrite $u_h(x,y)$ and $v_h(x,y)$ of (4.4) in the form

$$\begin{aligned} u_h(x,y) &= \bar{u}_h(x,y) + \tilde{u}_h(x,y), \\ v_h(x,y) &= \bar{v}_h(x,y) + \tilde{v}_h(x,y), \end{aligned} \tag{4.5}$$

where

$$\bar{u}_h(x,y) = \sum_{i=1}^{2N} \sum_{j=2}^{2N-1} u_{i,j}\phi_i(x)\psi_j(y), \tag{4.6}$$

$$\begin{aligned} \tilde{u}_h(x,y) = &\sum_{i=1}^{2N} u_{i,0}\phi_i(x)\psi_0(y) + \sum_{i=1}^{2N} u_{i,1}\phi_i(x)\psi_1(y) \\ &+ \sum_{i=1}^{2N} u_{i,2N}\phi_i(x)\psi_{2N}(y) + \sum_{i=1}^{2N} u_{i,2N+1}\phi_i(x)\psi_{2N+1}(y) \\ &+ \sum_{j=0}^{2N+1} u_{0,j}\phi_0(x)\psi_j(y) + \sum_{j=0}^{2N+1} u_{2N+1,j}\phi_{2N+1}(x)\psi_j(y), \end{aligned} \tag{4.7}$$

and

$$\bar{v}_h(x,y) = \sum_{i=1}^{2N} \sum_{j=0}^{2N+1} v_{i,j}\phi_i(x)\psi_j(y), \tag{4.8}$$

$$\tilde{v}_h(x,y) = \sum_{j=0}^{2N+1} v_{0,j}\phi_0(x)\psi_j(y) + \sum_{j=0}^{2N+1} v_{2N+1,j}\phi_{2N+1}(x)\psi_j(y), \tag{4.9}$$

then all of the coefficients in $\tilde{u}_h(x,y)$ and $\tilde{v}_h(x,y)$ are known. Thus we need to determine $\bar{u}_h(x,y)$ and $\bar{v}_h(x,y)$ of (4.6) and (4.8), which contain $8N^2$ unknown coefficients. These coefficients are obtained by requiring that

$$\begin{aligned} -\Delta\bar{u}_h(\xi) + \bar{v}_h(\xi) &= \Delta\tilde{u}_h(\xi) - \tilde{v}_h(\xi),\ \xi \in \mathcal{G}, \\ -\Delta\bar{v}_h(\xi) &= \Delta\tilde{v}_h(\xi) - f(\xi),\ \ \xi \in \mathcal{G}, \end{aligned} \tag{4.10}$$

where the right hand sides are known. Equations (4.10) impose $8N^2$ constraints, which is the same as the number of unknown coefficients in $\bar{u}_h(x,y)$ and $\bar{v}_h(x,y)$.

Now we prove the existence and uniqueness for $\bar{u}_h$ and $\bar{v}_h$ of (4.10). To do so, we require the following lemma.

Lemma 4.1. *If* $w \in \mathcal{M}_h^0 \otimes \mathcal{M}_h^{00}$ *and* $z \in \mathcal{M}_h^0 \otimes \mathcal{M}_h$, *then*

$$\langle -\Delta w, z\rangle_{\mathcal{G}} = \langle w, -\Delta z\rangle_{\mathcal{G}}.$$

Proof. Since $w \in \mathcal{M}_h^0 \otimes \mathcal{M}_h^{00}$ and $z \in \mathcal{M}_h^0 \otimes \mathcal{M}_h$, we have

$$w(a,y) = z(a,y) = 0, \quad a = 0,1, \quad y \in [0,1], \tag{4.11}$$

and

$$w(x,b) = w_y(x,b) = 0, \quad x \in [0,1], \quad b = 0,1. \tag{4.12}$$

The proof is then identical to that of Lemma 3.1 except that (4.11) and (4.12) are used instead of (3.9) and (3.10). □

Using Lemma 4.1, we prove the following theorem.

Theorem 4.1. *There exist unique functions* $\bar{u}_h(x,y)$ *of the form (4.6) and* $\bar{v}_h(x,y)$ *of the form (4.8) satisfying (4.10).*

Proof. Clearly, $\bar{u}_h \in \mathcal{M}_h^0 \otimes \mathcal{M}_h^{00}$ and $\bar{v}_h \in \mathcal{M}_h^0 \otimes \mathcal{M}_h$. Since the number of unknown coefficients in $\bar{u}_h(x,y)$ and $\bar{v}_h(x,y)$ is equal to the number of equations in (4.10), it suffices to show that if

$$-\Delta\bar{u}_h(\xi) + \bar{v}_h(\xi) = 0, \quad \xi \in \mathcal{G}, \tag{4.13}$$

$$-\Delta\bar{v}_h(\xi) = 0, \quad \xi \in \mathcal{G}, \tag{4.14}$$

then $\bar{u}_h = \bar{v}_h = 0$. Taking the inner product $\langle\cdot,\cdot\rangle_{\mathcal{G}}$ with $\bar{v}_h$ on both sides of (4.13), we obtain

$$\langle -\Delta\bar{u}_h, \bar{v}_h\rangle_{\mathcal{G}} + \langle \bar{v}_h, \bar{v}_h\rangle_{\mathcal{G}} = 0. \tag{4.15}$$

Similarly, taking the inner product $\langle\cdot,\cdot\rangle_{\mathcal{G}}$ with $\bar{u}_h$ on both sides of (4.14), we obtain

$$\langle -\Delta\bar{v}_h, \bar{u}_h\rangle_{\mathcal{G}} = 0. \tag{4.16}$$

From (4.15), (4.16), and Lemma 4.1, we have $\langle \bar{v}_h, \bar{v}_h\rangle_{\mathcal{G}} = 0$, which implies

$$\bar{v}_h(\xi) = 0, \quad \xi \in \mathcal{G}. \tag{4.17}$$

Thus by (4.13), $\Delta\bar{u}_h(\xi) = 0,\ \xi \in \mathcal{G}$. Since $\bar{u}_h \in \mathcal{M}_h^0 \otimes \mathcal{M}_h^0$, (2.10) and (2.9) imply that $\bar{u}_h = 0$.

From (4.17) and the fact that $\bar{v}_h \in \mathcal{M}_h^0 \otimes \mathcal{M}_h$, we see that $\bar{v}_h(x,\xi_m) = 0$, $x \in [0,1]$, $m = 1,\dots,2N$. Then $(\bar{v}_h)_{xx}(\xi) = 0,\ \xi \in \mathcal{G}$, and hence (4.14) implies that $(\bar{v}_h)_{yy}(\xi) = 0,\ \xi \in \mathcal{G}$. For fixed n, $n = 1,\dots,2N$, let $w(y) = \bar{v}_h(\xi_n, y)$, $y \in [0,1]$. Since on each subinterval $[t_{m-1}, t_m]$, $m = 1, \dots, N$, w is cubic and $w = w'' = 0$ at two Gauss points in $[t_{m-1}, t_m]$, it follows that $w = 0$ on $[t_{m-1}, t_m]$, $m = 1, \dots, N$, and hence $\bar{v}_h(\xi_n, y) = 0$, $y \in [0,1]$. Therefore, $\bar{v}_h = 0$ on all vertical lines of Ω passing through $\xi \in \mathcal{G}$. In particular, $\bar{v}_h = 0$ at the Gauss points on the horizontal sides of $\partial\Omega$. Since $\bar{v}_h = 0$ at the corner points of Ω, it follows that $\bar{v}_h = 0$ on the horizontal sides of $\partial\Omega$. Thus $\bar{v}_h = 0$ on $\partial\Omega$, and hence (4.17) and (2.9) imply that $\bar{v}_h = 0$. □

The following result gives error bounds for the OSC scheme (4.10).

Theorem 4.2. *Let $u \in H^{8-k}(\Omega)$, $k = 1, 2$, be the solution of BPII (4.1). Let $u_h, v_h \in \mathcal{M}_h \otimes \mathcal{M}_h$ of (4.5)–(4.9) be solutions of (4.10), where $\tilde{u}_h$, $\tilde{v}_h$ are obtained by approximating the boundary conditions in (4.2) and (4.3) using piecewise Hermite cubic interpolation. Then*

$$\|u - u_h\|_{H^k(\Omega)} \le Ch^{4-k}\|u\|_{H^{8-k}(\Omega)}, \quad k = 1, 2. \tag{4.18}$$

Proof. It follows from the way the coefficients in $\tilde{u}_h$ and $\tilde{v}_h$ are selected that $w = u_h - u_{\mathcal{H}} \in \mathcal{M}_h^0 \otimes \mathcal{M}_h^{00}$ and $z = v_h - v_{\mathcal{H}} \in \mathcal{M}_h^0 \otimes \mathcal{M}_h$. Therefore, the derivation of (4.18) is identical to that of (3.16) with $k = 1, 2$, except that Lemma 4.1 is used instead of Lemma 3.1. □

4.2. Single series OSC Fourier method

In this section, we present the single series OSC Fourier method for solving (4.10). First we rewrite (4.10) in a different form by moving some known coefficients of $\tilde{u}_h(x, y)$ from the right hand side to the left hand side; the rationale for this will become clear later. To this end, we write $\tilde{u}_h(x, y)$ of (4.7) as

$$\tilde{u}_h(x, y) = \tilde{u}_h^{\mathrm{I}}(x, y) + \tilde{u}_h^{\mathrm{II}}(x, y), \tag{4.19}$$

where

$$\begin{aligned} \tilde{u}_h^{\mathrm{I}}(x, y) = &\sum_{i=1}^{2N} u_{i,0}\phi_i(x)\psi_0(y) + \sum_{i=1}^{2N} u_{i,1}\phi_i(x)\psi_1(y) \\ &+ \sum_{i=1}^{2N} u_{i,2N}\phi_i(x)\psi_{2N}(y) + \sum_{i=1}^{2N} u_{i,2N+1}\phi_i(x)\psi_{2N+1}(y), \end{aligned} \tag{4.20}$$

and

$$\tilde{u}_h^{\mathrm{II}}(x, y) = \sum_{j=0}^{2N+1} u_{0,j}\phi_0(x)\psi_j(y) + \sum_{j=0}^{2N+1} u_{2N+1,j}\phi_{2N+1}(x)\psi_j(y). \tag{4.21}$$

With $\bar{u}_h(x,y)$ as in (4.6) and

(4.22) $$\hat{u}_h(x,y) = \bar{u}_h(x,y) + \tilde{u}_h^{\mathrm{I}}(x,y) = \sum_{i=1}^{2N} \sum_{j=0}^{2N+1} u_{i,j}\phi_i(x)\psi_j(y),$$

where the coefficients $u_{i,0}$, $u_{i,1}$, $u_{i,2N}$, $u_{i,2N+1}$, $i = 1, \ldots, 2N$, are known, (4.10) can be rewritten in the form

(4.23) $$\begin{aligned} -\Delta\hat{u}_h(\xi) + \bar{v}_h(\xi) &= F(\xi), \quad \xi \in \mathcal{G}, \\ -\Delta\bar{v}_h(\xi) &= G(\xi), \quad \xi \in \mathcal{G}, \end{aligned}$$

where

(4.24) $$F(\xi) = \Delta\tilde{u}_h^{\mathrm{II}}(\xi) - \tilde{v}_h(\xi), \qquad G(\xi) = \Delta\tilde{v}_h(\xi) - f(\xi), \qquad \xi \in \mathcal{G},$$

and $\bar{v}_h(x,y)$ and $\tilde{v}_h(x,y)$ are given by (4.8) and (4.9), respectively. Note that F and G are known. Moreover, $\hat{u}_h,\ \bar{v}_h \in \mathcal{M}_h^0 \otimes \mathcal{M}_h$, which was the purpose of the reformulation (4.23) of (4.10).

Since $\{\phi_i\}_{i=1}^{2N}$ of (2.18) and $\{z_i\}_{i=1}^{2N}$ of (2.19)–(2.20) are bases for $\mathcal{M}_h^0$, and $\{\psi_j\}_{j=0}^{2N+1}$ of (2.17) is a basis for $\mathcal{M}_h$, from (4.22) and (4.8), we may write

(4.25) $$\hat{u}_h(x,y) = \sum_{i=1}^{2N} \sum_{j=0}^{2N+1} u_{i,j}\phi_i(x)\psi_j(y) = \sum_{i=1}^{2N} z_i(x)u_i^z(y),$$

and

(4.26) $$\bar{v}_h(x,y) = \sum_{i=1}^{2N} \sum_{j=0}^{2N+1} v_{i,j}\phi_i(x)\psi_j(y) = \sum_{i=1}^{2N} z_i(x)v_i^z(y),$$

where $u_i^z, v_i^z \in \mathcal{M}_h$. Substituting (4.25) and (4.26) into the first equation of (4.23) with $\xi = (\xi_n, \xi_m) \in \mathcal{G}$ and using (2.19), we obtain

(4.27) $$\sum_{i=1}^{2N} z_i(\xi_n)\{\lambda_i u_i^z(\xi_m) - [u_i^z]''(\xi_m) + v_i^z(\xi_m)\} = F(\xi_n, \xi_m), \quad n, m = 1, \ldots, 2N,$$

where $\{\lambda_i\}_{i=1}^{2N}$ are given in (2.21)–(2.22). For fixed ξ_m, we take the inner product $\langle\cdot,\cdot\rangle$ of both sides of (4.27) with z_k, $k = 1, \ldots, 2N$, and use (2.20) to obtain

(4.28) $$-[u_k^z]''(\xi_m) + \lambda_k u_k^z(\xi_m) + v_k^z(\xi_m) = F_k^z(\xi_m), \quad k, m = 1, \ldots, 2N,$$

where

(4.29) $$F_k^z(\xi_m) = \langle F(\cdot, \xi_m), z_k\rangle.$$

Similarly, from the second equation of (4.23), we obtain

$$-[v_k^z]''(\xi_m)+\lambda_k v_k^z(\xi_m)=G_k^z(\xi_m),\qquad k,m=1,\dots,2N, \tag{4.30}$$

where

$$G_k^z(\xi_m)=\langle G(\cdot,\xi_m),z_k\rangle. \tag{4.31}$$

Since the coefficients $u_{i,0},u_{i,1},u_{i,2N},u_{i,2N+1}, i=1,\dots,2N$, of $\hat{u}_h$ are known, we also expect u_k^z to contain some known coefficients. In fact, if we express u_k^z, $v_k^z \in \mathcal{M}_h$, $k=1,\dots,2N$, in the form

$$u_k^z(y)=\sum_{j=0}^{2N+1}u_{k,j}^{z,\psi}\psi_j(y),\qquad v_k^z(y)=\sum_{j=0}^{2N+1}v_{k,j}^{z,\psi}\psi_j(y), \tag{4.32}$$

then, for each $k=1,\dots,2N$, the coefficients $u_{k,0}^{z,\psi},\ u_{k,1}^{z,\psi},\ u_{k,2N}^{z,\psi},\ u_{k,2N+1}^{z,\psi}$ are known. For example, from (4.32), and properties of the basis functions $\{\psi_j\}_{j=0}^{2N+1}$, we have

$$u_k^z(0)=\sum_{j=0}^{2N+1}u_{k,j}^{z,\psi}\psi_j(0)=u_{k,0}^{z,\psi},\qquad k=1,\dots,2N. \tag{4.33}$$

On the other hand, from (4.25) and (4.33), we obtain

$$\hat{u}_h(x,0)=\sum_{i=1}^{2N}u_{i,0}\phi_i(x)=\sum_{i=1}^{2N}u_{i,0}^{z,\psi}z_i(x). \tag{4.34}$$

Let

$$\mathbf{u}_0=[u_{1,0},\dots,u_{2N,0}]^{\mathrm{T}},\qquad \mathbf{u}_0^{z,\psi}=[u_{1,0}^{z,\psi},\dots,u_{2N,0}^{z,\psi}]^{\mathrm{T}}, \tag{4.35}$$

where the vector $\mathbf{u}_0$ is known. On comparing (4.34) with (2.24) and (4.35) with (2.26), it follows from (2.28) that the vector $\mathbf{u}_0^{z,\psi}$ can be computed using the relation

$$\mathbf{u}_0^{z,\psi}=\frac{h}{2}Z^{\mathrm{T}}B_\phi^{\mathrm{T}}B_\phi\mathbf{u}_0, \tag{4.36}$$

where B and Z are given in (2.30) and (2.31), respectively. Again, from (4.32), we have

$$[u_k^z]'(0)=\sum_{j=0}^{2N+1}u_{k,j}^{z,\psi}\psi_j'(0)=u_{k,1}^{z,\psi}h^{-1},\qquad k=1,\dots,2N. \tag{4.37}$$

On the other hand, from (4.25) and (4.37), we obtain

$$(\hat{u}_h)_y(x,0)=h^{-1}\sum_{i=1}^{2N}u_{i,1}\phi_i(x)=h^{-1}\sum_{i=1}^{2N}u_{i,1}^{z,\psi}z_i(x).$$

Let

$$\mathbf{u}_1 = [u_{1,1}, \ldots, u_{2N,1}]^{\mathrm{T}}, \qquad \mathbf{u}_1^{z,\psi} = [u_{1,1}^{z,\psi}, \ldots, u_{2N,1}^{z,\psi}]^{\mathrm{T}}, \tag{4.38}$$

where $\mathbf{u}_1$ is known. Then the vector $\mathbf{u}_1^{z,\psi}$ can be computed using the relation

$$\mathbf{u}_1^{z,\psi} = \frac{h}{2} Z^{\mathrm{T}} B_\phi^{\mathrm{T}} B_\phi \mathbf{u}_1. \tag{4.39}$$

Similarly, let

$$\mathbf{u}_{2N} = [u_{1,2N}, \ldots, u_{2N,2N}]^{\mathrm{T}}, \qquad \mathbf{u}_{2N}^{z,\psi} = [u_{1,2N}^{z,\psi}, \ldots, u_{2N,2N}^{z,\psi}]^{\mathrm{T}}, \tag{4.40}$$

and

$$\begin{aligned} \mathbf{u}_{2N+1} &= [u_{1,2N+1}, \ldots, u_{2N,2N+1}]^{\mathrm{T}}, \\ \mathbf{u}_{2N+1}^{z,\psi} &= [u_{1,2N+1}^{z,\psi}, \ldots, u_{2N,2N+1}^{z,\psi}]^{\mathrm{T}}, \end{aligned} \tag{4.41}$$

where $\mathbf{u}_{2N}$ and $\mathbf{u}_{2N+1}$ are known. Then we have

$$\mathbf{u}_{2N}^{z,\psi} = \frac{h}{2} Z^{\mathrm{T}} B_\phi^{\mathrm{T}} B_\phi \mathbf{u}_{2N}, \qquad \mathbf{u}_{2N+1}^{z,\psi} = \frac{h}{2} Z^{\mathrm{T}} B_\phi^{\mathrm{T}} B_\phi \mathbf{u}_{2N+1}. \tag{4.42}$$

We are ready to describe the following method.

Single series OSC Fourier method for solving (4.10)

1. Determine $F(\xi_n, \xi_m)$, $G(\xi_n, \xi_m)$, $n, m = 1, \ldots, 2N$, using (4.24).
2. Find $F_k^z(\xi_m)$, $G_k^z(\xi_m)$, $k, m = 1, \ldots, 2N$, of (4.29) and (4.31).
3. Compute $\mathbf{u}_0^{z,\psi}$, $\mathbf{u}_1^{z,\psi}$, $\mathbf{u}_{2N}^{z,\psi}$, and $\mathbf{u}_{2N+1}^{z,\psi}$ using (4.36), (4.39), and (4.42).
4. For $k = 1, \ldots, 2N$, find the coefficients $\{u_{k,j}^{z,\psi}\}_{j=2}^{2N-1}$ and $\{v_{k,j}^{z,\psi}\}_{j=0}^{2N+1}$ in u_k^z and v_k^z of (4.32), so that (4.28) and (4.30) are satisfied. (Notice that for each $k = 1, \ldots, 2N$, the four coefficients $u_{k,0}^{z,\psi},\ u_{k,1}^{z,\psi},\ u_{k,2N}^{z,\psi},\ u_{k,2N+1}^{z,\psi}$ are known from Step 3.)
5. Compute the coefficients $\{u_{i,j}\}_{i=1,j=2}^{2N,2N-1}$ and $\{v_{i,j}\}_{i=1,j=0}^{2N,2N+1}$ of $\bar{u}_h(x,y)$, $\bar{v}_h(x,y)$ of (4.6), (4.8) using $\{u_{k,j}^{z,\psi}\}_{k=1,j=2}^{2N,2N-1}$ and $\{v_{k,j}^{z,\psi}\}_{k=1,j=0}^{2N,2N+1}$ obtained in Step 4.

We now describe in more detail the implementation of the single series OSC Fourier method.

Step 1. Assuming that the values of $f(\xi)$, $\xi \in \mathcal{G}$, have been determined, we can directly compute $F(\xi_n, \xi_m)$ and $G(\xi_n, \xi_m)$, $n, m = 1, \ldots, 2N$, from (4.24), since $\tilde{u}_h^{\mathrm{II}}(x,y)$ of (4.21), and $\tilde{v}_h(x,y)$ of (4.9) are known. Hence the cost of Step 1 is $O(N)$.

Step 2. We use $F(\xi_n, \xi_m)$ and $G(\xi_n, \xi_m)$, $n, m = 1, \ldots, 2N$, obtained in Step 1 to compute $F_k^z(\xi_m)$ and $G_k^z(\xi_m)$, $k, m = 1, \ldots, 2N$, defined by (4.29) and (4.31). For fixed m, $m = 1, \ldots, 2N$, let $w \in \mathcal{M}_h^0$ be such that

$$w(\xi_n) = F(\xi_n, \xi_m), \qquad n = 1, \ldots, 2N. \tag{4.43}$$

Then (4.29) gives

$$F_n^z(\xi_m) = \langle F(\cdot,\xi_m), z_n\rangle = \langle w, z_n\rangle, \quad n = 1,\dots,2N. \tag{4.44}$$

If

$$\begin{aligned}\mathbf{F}_m^{z,\theta} &= [F_1^z(\xi_m),\dots,F_{2N}^z(\xi_m)]^{\mathrm{T}},\\ \mathbf{F}_m^{\theta,\theta} &= [F(\xi_1,\xi_m),\dots,F(\xi_{2N},\xi_m)]^{\mathrm{T}},\end{aligned} \quad m = 1,\dots,2N, \tag{4.45}$$

then on comparing (4.43), (4.44) with (2.25) and (4.45) with (2.26), it follows from (2.29) that

$$\mathbf{F}_m^{z,\theta} = \frac{h}{2} Z^{\mathrm{T}} B_\phi^{\mathrm{T}} \mathbf{F}_m^{\theta,\theta}, \quad m = 1,\dots,2N. \tag{4.46}$$

Similarly, if

$$\begin{aligned}\mathbf{G}_m^{z,\theta} &= [G_1^z(\xi_m),\dots,G_{2N}^z(\xi_m)]^{\mathrm{T}},\\ \mathbf{G}_m^{\theta,\theta} &= [G(\xi_1,\xi_m),\dots,G(\xi_{2N},\xi_m)]^{\mathrm{T}},\end{aligned} \quad m = 1,\dots,2N, \tag{4.47}$$

then

$$\mathbf{G}_m^{z,\theta} = \frac{h}{2} Z^{\mathrm{T}} B_\phi^{\mathrm{T}} \mathbf{G}_m^{\theta,\theta}, \quad m = 1,\dots,2N. \tag{4.48}$$

Since there are at most four nonzero elements in each column of the matrix B_ϕ, the matrix-vector multiplications involving the matrix B_ϕ^{T} in (4.46) and (4.48) require $O(N)$ arithmetic operations for each m. It follows from the form of Z in (2.31) that FFT routines can be used to perform multiplications by the matrix Z^{T} in (4.46) and (4.48) with cost $O(N\log_2 N)$ for each m. Thus the cost of Step 2 is $O(N^2\log_2 N)$.

Step 3. This step involves computing four vectors using the formulas (4.36), (4.39), and (4.42). The matrix-vector multiplications involving the matrix B_ϕ and B_ϕ^{T} require $O(N)$ arithmetic operations, since there are at most four nonzero elements in each row and column of B_ϕ. As in Step 2, FFTs can be used to perform multiplications by Z^{T}. Consequently, the cost of Step 3 is $O(N\log_2 N)$.

Step 4. For $k = 1,\dots,2N$, we need to solve one dimensional OSC problem

$$\begin{aligned}-[u_k^z]''(\xi_m) + \lambda_k u_k^z(\xi_m) + v_k^z(\xi_m) &= F_k^z(\xi_m),\\ -[v_k^z]''(\xi_m) + \lambda_k v_k^z(\xi_m) &= G_k^z(\xi_m),\end{aligned} \quad m = 1,\dots,2N, \tag{4.49}$$

where $u_k^z(0)$, $[u_k^z]'(0)$, $u_k^z(1)$, $[u_k^z]'(1)$ are specified (see (4.33), (4.37)) and $\{\lambda_k\}_{k=1}^{2N}$ are given in (2.21)–(2.22). Since each $\lambda_k > 0$, it follows from Theorem 2.1 that (4.49) is uniquely solvable for $u_k^z, v_k^z \in \mathcal{M}_h$ with specified values of $u_k^z(0)$, $[u_k^z]'(0)$, $u_k^z(1)$, $[u_k^z]'(1)$.

We now show how to solve the one dimensional OSC problem (4.49) for each $k = 1,\dots,2N$. We introduce $(4N+4)$-vectors

$$\mathbf{w}_k^z = [u_{k,0}^{z,\psi}, u_{k,1}^{z,\psi}, v_{k,0}^{z,\psi}, v_{k,1}^{z,\psi}, \dots, u_{k,2N+1}^{z,\psi}, u_{k,2N}^{z,\psi}, v_{k,2N+1}^{z,\psi}, v_{k,2N}^{z,\psi}]^{\mathrm{T}}$$

and

$$\mathbf{f}_k^z = [u_{k,0}^{z,\psi}, u_{k,1}^{z,\psi}, F_k^z(\xi_1), F_k^z(\xi_2), G_k^z(\xi_1), G_k^z(\xi_2), \ldots, \\ F_k^z(\xi_{2N-1}), F_k^z(\xi_{2N}), G_k^z(\xi_{2N-1}), G_k^z(\xi_{2N}), u_{k,2N+1}^{z,\psi}, u_{k,2N}^{z,\psi}]^{\mathrm{T}},$$

where $\{u_{k,j}^{z,\psi}\}_{j=0}^{2N+1}$ and $\{v_{k,j}^{z,\psi}\}_{j=0}^{2N+1}$ in $\mathbf{w}_k^z$ are the coefficients of u_k^z and v_k^z in (4.32) with the four known coefficients $u_{k,0}^{z,\psi}$, $u_{k,1}^{z,\psi}$, $u_{k,2N}^{z,\psi}$, and $u_{k,2N+1}^{z,\psi}$. Then (4.49) can be written in the matrix-vector form

$$B_k \mathbf{w}_k^z = \mathbf{f}_k^z, \tag{4.50}$$

where the B_k is the almost block diagonal matrix of the form

$$\begin{bmatrix}
1&0&0&0&&&&&&&&&&&&\\
0&1&0&0&&&&&&&&&&&&\\
\times&\times&\times&\times&\times&\times&\times&\times&&&&&&&&\\
\times&\times&\times&\times&\times&\times&\times&\times&&&&&&&&\\
\times&\times&\times&\times&\times&\times&\times&\times&&&&&&&&\\
\times&\times&\times&\times&\times&\times&\times&\times&&&&&&&&\\
&&&&\times&\times&\times&\times&\times&\times&\times&\times&&&&\\
&&&&\times&\times&\times&\times&\times&\times&\times&\times&&&&\\
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&&&&&&&&\times&\times&\times&\times&\times&\times&\times&\times\\
&&&&&&&&\times&\times&\times&\times&\times&\times&\times&\times\\
&&&&&&&&&&&&1&0&0&0\\
&&&&&&&&&&&&0&1&0&0
\end{bmatrix}. \tag{4.51}$$

Since (4.50) is the matrix-vector representation of the uniquely solvable problem (4.49), it follows that B_k is nonsingular. Moreover, the linear system (4.50) can be solved directly using COLROW [12, 13] with cost $O(N)$. Consequently, the total cost of Step 4 is $O(N^2)$.

Step 5. We need to compute the coefficients $\{u_{i,j}\}_{i=1,j=2}^{2N,2N-1}$ and $\{v_{i,j}\}_{i=1,j=0}^{2N,2N+1}$ of $\bar{u}_h(x,y)$ and $\bar{v}_h(x,y)$ of (4.6) and (4.8). To this end, we substitute $u_k^z(y)$ of (4.32) into (4.25) to obtain

$$\sum_{j=0}^{2N+1} \left[\sum_{i=1}^{2N} u_{i,j} \phi_i(x) \right] \psi_j(y) = \sum_{j=0}^{2N+1} \left[\sum_{i=1}^{2N} u_{i,j}^{z,\psi} z_i(x) \right] \psi_j(y). \tag{4.52}$$

Since $\{\psi_j\}_{j=0}^{2N+1}$ is a basis for $\mathcal{M}_h$, (4.52) implies that

$$\sum_{i=1}^{2N} u_{i,j}\phi_i(x) = \sum_{i=1}^{2N} u_{i,j}^{z,\psi} z_i(x), \qquad j = 2, \ldots, 2N-1. \tag{4.53}$$

Similarly, by substituting $v_k^z(y)$ of (4.32) into (4.26), we obtain

$$\sum_{j=0}^{2N+1}\left[\sum_{i=1}^{2N} v_{i,j}\phi_i(x)\right]\psi_j(y) = \sum_{j=0}^{2N+1}\left[\sum_{i=1}^{2N} v_{i,j}^{z,\psi} z_i(x)\right]\psi_j(y),$$

which implies that

$$\sum_{i=1}^{2N} v_{i,j}\phi_i(x) = \sum_{i=1}^{2N} v_{i,j}^{z,\psi} z_i(x), \qquad j = 0, \ldots, 2N+1. \tag{4.54}$$

Thus, for fixed j, finding $\{u_{i,j}\}_{i=1}^{2N}$ and $\{v_{i,j}\}_{i=1}^{2N}$ corresponds to finding Fourier coefficients with respect to the basis $\{\phi_i\}_{i=1}^{2N}$ from the Fourier coefficients $\{u_{i,j}^{z,\psi}\}_{i=1}^{2N}$ and $\{v_{i,j}^{z,\psi}\}_{i=1}^{2N}$ with respect to the basis $\{z_i\}_{i=1}^{2N}$. Introducing the $2N$-vectors

$$\begin{aligned} \mathbf{u}_j &= [u_{1,j}, \ldots, u_{2N,j}]^{\mathrm{T}}, \\ \mathbf{u}_j^{z,\psi} &= [u_{1,j}^{z,\psi}, \ldots, u_{2N,j}^{z,\psi}]^{\mathrm{T}}, \end{aligned} \qquad j = 2, \ldots, 2N-1, \tag{4.55}$$

and

$$\begin{aligned} \mathbf{v}_j &= [v_{1,j}, \ldots, v_{2N,j}]^{\mathrm{T}}, \\ \mathbf{v}_j^{z,\psi} &= [v_{1,j}^{z,\psi}, \ldots, v_{2N,j}^{z,\psi}]^{\mathrm{T}}, \end{aligned} \qquad j = 0, \ldots, 2N+1, \tag{4.56}$$

and comparing (4.53), (4.54) with (2.24), and (4.55), (4.56) with (2.26), we obtain from (2.27)

$$\begin{aligned} \mathbf{u}_j &= Z\mathbf{u}_j^{z,\psi}, \quad j = 2, \ldots, 2N-1, \\ \mathbf{v}_j &= Z\mathbf{v}_j^{z,\psi}, \quad j = 0, \ldots, 2N+1. \end{aligned} \tag{4.57}$$

Since FFT routines can be used in (4.57) to perform multiplications by the matrix Z, the cost of Step 5 is $O(N^2 \log_2 N)$. Therefore the total cost of the single series OSC Fourier method is $O(N^2 \log_2 N)$.

4.3. A special case

In this section, we consider a special case of the OSC scheme (4.10), which will be used in Sect. 5. We assume that in (4.10), $f(\xi) = 0$, $\xi \in \mathcal{G}$, $\tilde{u}_h = 0$, and all the coefficients in $\tilde{v}_h$ of (4.9) are zero except that one of the coefficients $v_{0,j}$ or $v_{2N+1,j}$, $j = 2, \ldots, 2N-1$, is equal to one. Also, we assume that we only need to know the coefficients $u_{N,j}$ and $u_{2N,j}$,

$j = 2, \ldots, 2N-1$, in $\bar{u}_h(x, y)$ of (4.6). We show how to modify the single series OSC Fourier method so that the total cost of finding the desired coefficients of $\bar{u}_h$ is reduced to $O(N^2)$.

Assume $v_{0,k} = 1$, where $k = 2l$ or $2l+1$ with $l = 1, \ldots, N-1$. Then (4.24) and (4.9) yield

$$\begin{aligned} F(\xi_n, \xi_m) &= -\phi_0(\xi_n)\psi_k(\xi_m), \\ G(\xi_n, \xi_m) &= \phi_0''(\xi_n)\psi_k(\xi_m) + \phi_0(\xi_n)\psi_k''(\xi_m), \end{aligned} \qquad n, m = 1, \ldots, 2N.$$

Since $\psi_{2l} = v_l$ and $\psi_{2l+1} = s_l$ (cf. (2.17)), it is easy to see that only the 8 values of $F(\xi_n, \xi_m)$, and the 8 values of $G(\xi_n, \xi_m)$ corresponding to $n = 1, 2$, and $m = 2l-1, 2l, 2l+1, 2l+2$, are nonzero. Therefore, Step 1 of the single series OSC Fourier method requires $O(1)$ arithmetic operations. Further, in (4.45), there are only four nonzero vectors $\mathbf{F}_m^{\theta,\theta}$, namely,

$$\mathbf{F}_m^{\theta,\theta} = [F(\xi_1, \xi_m), F(\xi_2, \xi_m), 0, \ldots, 0]^{\mathrm{T}}, \quad m = 2l-1, 2l, 2l+1, 2l+2.$$

Similarly, in (4.47), there are only four nonzero vectors $\mathbf{G}_m^{\theta,\theta}$,

$$\mathbf{G}_m^{\theta,\theta} = [G(\xi_1, \xi_m), G(\xi_2, \xi_m), 0 \ldots, 0]^{\mathrm{T}}, \quad m = 2l-1, 2l, 2l+1, 2l+2.$$

Thus (4.46) and (4.48) imply that only four nonzero vectors $\mathbf{F}_{2l-1}^{z,\theta}$, $\mathbf{F}_{2l}^{z,\theta}$, $\mathbf{F}_{2l+1}^{z,\theta}$, $\mathbf{F}_{2l+2}^{z,\theta}$, and four nonzero vectors $\mathbf{G}_{2l-1}^{z,\theta}$, $\mathbf{G}_{2l}^{z,\theta}$, $\mathbf{G}_{2l+1}^{z,\theta}$, $\mathbf{G}_{2l+2}^{z,\theta}$ need to be computed in Step 2. This can be done without the use of FFTs and consequently Step 2 requires only $O(N)$ arithmetic operations. Since $\tilde{u}_h = 0$, it follows from (4.19), (4.20), (4.35)–(4.36), (4.38)–(4.39) and (4.40)–(4.42) that $\mathbf{u}_0^{z,\psi} = \mathbf{u}_1^{z,\psi} = \mathbf{u}_{2N}^{z,\psi} = \mathbf{u}_{2N+1}^{z,\psi} = 0$, and hence Step 3 need not be performed. As in the general case, Step 4 requires $O(N^2)$ arithmetic operations. In the general case, Step 5 is performed using FFTs. In the special case, the coefficients $u_{N,j}$ and $u_{2N,j}$, $j = 2, \ldots, 2N-1$, are computed directly without using FFT routines. It is easy to see from (4.55) and (4.57) that in order to obtain $u_{N,j}$, $j = 2, \ldots, 2N-1$, we need to multiply each $\mathbf{u}_j^{z,\psi}$, $j = 2, \ldots, 2N-1$, by the N^{th} row of Z. Similarly, to obtain $u_{2N,j}$, $j = 2, \ldots, 2N-1$, we need to multiply each $\mathbf{u}_j^{z,\psi}$, $j = 2, \ldots, 2N-1$, by the $2N^{th}$ row of Z. This requires $O(N^2)$ arithmetic operations. Since the case of $v_{2N+1,k} = 1$, where $k = 2l$ or $2l+1$ with $l = 1, \ldots, N-1$, can be treated in a similar way, the total cost of the modified single series OSC Fourier method for the special case is $O(N^2)$.

5. Biharmonic problem III

5.1. OSC scheme: existence, uniqueness, convergence

We are now ready to solve the biharmonic Dirichlet problem (1.1), which we call Biharmonic Problem III (BPIII). We solve the corresponding OSC problem by employing the capacitance matrix method with the OSC BPII as the auxiliary problem. A similar approach involving a finite difference approximation of BPIII was used by Buzbee and Dorr [7]. As before, introducing $v = \Delta u$, we obtain from (1.1) the coupled problem:

$$\begin{aligned} -\Delta u &= -v \quad \text{in } \Omega, \\ u &= g_1 \quad \text{on } \partial\Omega, \\ \frac{\partial u}{\partial n} &= g_2 \quad \text{on } \partial\Omega, \end{aligned} \tag{5.1}$$

and

$$-\,\Delta v = -f \text{ in } \Omega. \tag{5.2}$$

Note that in contrast to (4.2)–(4.3), no boundary conditions are imposed on v.

The OSC solutions $u_h, v_h \in \mathcal{M}_h \otimes \mathcal{M}_h$ of (5.1)–(5.2) are defined as follows. Since $u_h, v_h \in \mathcal{M}_h \otimes \mathcal{M}_h$, they can be written in the form

$$\begin{aligned} u_h(x,y) &= \sum_{i=0}^{2N+1} \sum_{j=0}^{2N+1} u_{i,j}\phi_i(x)\psi_j(y), \\ v_h(x,y) &= \sum_{i=0}^{2N+1} \sum_{j=0}^{2N+1} v_{i,j}\phi_i(x)\psi_j(y), \end{aligned} \tag{5.3}$$

where the bases $\{\phi_i\}_{i=0}^{2N+1}$ and $\{\psi_j\}_{j=0}^{2N+1}$ for $\mathcal{M}_h$ are defined in (2.16) and (2.17). As before, we determine in advance certain coefficients of (5.3) corresponding to the two boundary conditions in (5.1) using piecewise Hermite cubic interpolation. The coefficients $u_{i,0}, u_{i,2N+1}, i = 1, \ldots, 2N$, and $u_{0,j}, u_{2N+1,j}, j = 0, \ldots, 2N+1$, which correspond to the boundary condition $u = g_1$ on $\partial\Omega$, and the coefficients $u_{i,1}$, $u_{i,2N}$, $i = 1, \ldots, 2N$, which correspond to the boundary condition $\partial u/\partial n = g_2$ on the horizontal sides of $\partial\Omega$ are determined in exactly the same way as in Sect. 4.1. In particular, the coefficients $u_{N,0}$, $u_{N,1}$, $u_{N,2N}$, $u_{N,2N+1}$, $u_{2N,0}$, $u_{2N,1}$, $u_{2N,2N}$, and $u_{2N,2N+1}$ are determined by this process. The boundary condition $\partial u/\partial n = g_2$ on the vertical sides of $\partial\Omega$ are approximated in the following way to determine the coefficients $u_{N,j}, u_{2N,j}, j = 2, \ldots, 2N-1$. On the left hand side of $\partial\Omega$, we require that

$$[(u_h)_x + g_2](0, t_n) = 0, \quad [(u_h)_x + g_2]_y(0, t_n) = 0, \quad n = 1, \ldots, N-1.$$

Substituting $u_h(x,y)$ of (5.3) into these equations, we obtain explicitly the coefficients $u_{N,j}$, $j=2,\ldots,2N-1$:

$$u_{N,2n} = -hg_2(0,t_n), \quad u_{N,2n+1} = -h^2(g_2)_y(0,t_n), \quad n=1,\ldots,N-1.$$

On the right hand side of $\partial\Omega$, we require that

$$[(u_h)_x - g_2](1,t_n) = 0, \quad [(u_h)_x - g_2]_y(1,t_n) = 0, \quad n=1,\ldots,N-1,$$

from which, we obtain the coefficients $u_{2N,j}$, $j=2,\ldots,2N-1$:

$$u_{2N,2n} = hg_2(1,t_n), \quad u_{2N,2n+1} = h^2(g_2)_y(1,t_n), \quad n=1,\ldots,N-1.$$

Thus we determine $16N$ coefficients $u_{i,0}$, $u_{i,1}$, $u_{i,2N}$, $u_{i,2N+1}$, $i=1,\ldots,2N$, $u_{0,j}, u_{2N+1,j}$, $j=0,\ldots,2N+1$, and $u_{N,j}$, $u_{2N,j}$, $j=2,\ldots,2N-1$.

If we split $u_h(x,y)$ of (5.3) in the form

$$u_h(x,y) = \bar{u}_h(x,y) + \tilde{u}_h(x,y), \tag{5.4}$$

where

$$\bar{u}_h(x,y) = \sum_{i=1,i\neq N}^{2N-1} \sum_{j=2}^{2N-1} u_{i,j}\phi_i(x)\psi_j(y), \tag{5.5}$$

and

$$\begin{aligned}
\tilde{u}_h(x,y) = & \sum_{i=1}^{2N} u_{i,0}\phi_i(x)\psi_0(y) + \sum_{i=1}^{2N} u_{i,1}\phi_i(x)\psi_1(y) \\
& + \sum_{i=1}^{2N} u_{i,2N}\phi_i(x)\psi_{2N}(y) + \sum_{i=1}^{2N} u_{i,2N+1}\phi_i(x)\psi_{2N+1}(y) \\
& + \sum_{j=0}^{2N+1} u_{0,j}\phi_0(x)\psi_j(y) + \sum_{j=0}^{2N+1} u_{2N+1,j}\phi_{2N+1}(x)\psi_j(y) \\
& + \sum_{j=2}^{2N-1} u_{N,j}\phi_N(x)\psi_j(y) + \sum_{j=2}^{2N-1} u_{2N,j}\phi_{2N}(x)\psi_j(y),
\end{aligned} \tag{5.6}$$

then all coefficients of $\tilde{u}_h(x,y)$ are known. Thus the total number of unknown coefficients in $\bar{u}_h(x,y)$ of (5.5) and $v_h(x,y)$ of (5.3) is $8N^2+8$. However, if we collocate the first equation of (5.1) and equation (5.2) at all $\xi \in \mathcal{G}$, we obtain only $8N^2$ equations. Notice that for BPII, we did not have such a problem. For BPIII, we need to impose 8 additional equations or determine 8 additional coefficients independently. Since we use the substitution $v = \Delta u$, one obvious choice is to impose four corner conditions on v_h:

$$v_h(a,b) = \Delta u(a,b), \qquad a,b=0,1. \tag{5.7}$$

Substituting v_h of (5.3) into (5.7), and using $u = g_1$ on $\partial\Omega$ to determine $\Delta u(a,b)$, we obtain

$$v_{0,0} = \Delta g_1(0,0), \qquad v_{2N+1,0} = \Delta g_1(1,0), \tag{5.8}$$
$$v_{0,2N+1} = \Delta g_1(0,1), \; v_{2N+1,2N+1} = \Delta g_1(1,1).$$

In order to find another four equations, we first note that for $v = \Delta u$, we have

$$v_y = (\Delta u)_y = \Delta u_y.$$

Hence, we impose four corner conditions on $(v_h)_y$:

$$(v_h)_y(a,b) = \Delta u_y(a,b), \qquad a, b = 0, 1. \tag{5.9}$$

Substituting $v_h(x,y)$ of (5.3) into (5.9), using $u = g_1$ on the vertical sides of $\partial\Omega$ to determine $u_{yyy}(a,b)$ and using $-u_y = g_2$ on the bottom side of $\partial\Omega$ and $u_y = g_2$ on the top side of $\partial\Omega$ to determine $u_{yxx}(a,b)$, we obtain

$$\begin{aligned} v_{0,1} &= h[(g_1)_{yyy} - (g_2)_{xx}](0,0), \\ v_{2N+1,1} &= h[(g_1)_{yyy} - (g_2)_{xx}](1,0), \\ v_{0,2N} &= h[(g_1)_{yyy} + (g_2)_{xx}](0,1), \\ v_{2N+1,2N} &= h[(g_1)_{yyy} + (g_2)_{xx}](1,1). \end{aligned} \tag{5.10}$$

In (5.9), we selected corner conditions for $(v_h)_y$ rather than $(v_h)_x$ because the values $v_{0,1}$, $v_{0,2N}$, $v_{2N+1,1}$, and $v_{2N+1,2N}$ are known explicitly in the OSC scheme for the BPII. A finite difference approach could be used to approximate the partial derivatives on the right hand sides of (5.8) and (5.10). However, since at this point $u_h(x,y)$ of (5.4) is known on $\partial\Omega$, it is possible, in place of (5.7), to impose the following four corner conditions on v_h:

$$v_h(a,b) = \Delta u_h(a,b), \qquad a, b = 0, 1. \tag{5.11}$$

Substituting $u_h(x,y)$ of (5.3) into (5.11) and using (2.16) and (2.17), we obtain

$$\begin{aligned} v_{0,0} = {} & 2u_{0,0}v_0''(0) + (u_{1,0} + u_{0,2})v_1''(0) + (u_{N,0} + u_{0,1})s_0''(0) \\ & + (u_{N+1,0} + u_{0,3})s_1''(0), \end{aligned}$$

and similar formulas for $v_{2N+1,0}$, $v_{0,2N+1}$, and $v_{2N+1,2N+1}$. All of the right hand sides in these formulas are known, because the involved $u_{i,j}$ are coefficients of the known $\tilde{u}_h(x,y)$ of (5.6). Similarly, since $(u_h)_y$ is known on the horizontal sides of $\partial\Omega$, in place of (5.9), we can impose the following four corner conditions:

$$(v_h)_y(a,b) = \Delta(u_h)_y(a,b), \qquad a, b = 0, 1. \tag{5.12}$$

Substituting $u_h(x,y)$ of (5.3) into (5.12) and using (2.16) and (2.17), we obtain

$$\begin{aligned} v_{0,1} &= u_{0,1}v_0''(0) + u_{1,1}v_1''(0) + u_{N,1}s_0''(0) + u_{N+1,1}s_1''(0) \\ &\quad + h[u_{0,0}v_0'''(0) + u_{0,1}s_0'''(0) + u_{0,2}v_1'''(0) + u_{0,3}s_1'''(0)], \end{aligned}$$

and similar formulas for $v_{2N+1,1}$, $v_{0,2N}$, and $v_{2N+1,2N}$. Again, all of the right hand sides in these formulas are known. The nice thing about this approximation is that, in contrast to (5.8) and (5.10), no partial derivatives of g_1 and g_2 or their approximations are needed.

We rewrite $v_h(x,y)$ of (5.3) as

$$v_h(x,y) = \bar{v}_h(x,y) + \tilde{v}_h(x,y), \tag{5.13}$$

where

$$\begin{aligned} \bar{v}_h(x,y) &= \sum_{i=1}^{2N}\sum_{j=0}^{2N+1} v_{i,j}\phi_i(x)\psi_j(y) + \sum_{j=2}^{2N-1} v_{0,j}\phi_0(x)\psi_j(y) \\ &\quad + \sum_{j=2}^{2N-1} v_{2N+1,j}\phi_{2N+1}(x)\psi_j(y), \end{aligned} \tag{5.14}$$

and

$$\begin{aligned} \tilde{v}_h(x,y) &= \sum_{j=0,1,2N,2N+1} v_{0,j}\phi_0(x)\psi_j(y) \\ &\quad + \sum_{j=0,1,2N,2N+1} v_{2N+1,j}\phi_{2N+1}(x)\psi_j(y), \end{aligned} \tag{5.15}$$

where the coefficients of $\tilde{v}_h(x,y)$ are obtained using (5.7) and (5.9) or (5.11) and (5.12). Thus the total number of unknown coefficients in $\bar{u}_h(x,y)$ of (5.5) and $\bar{v}_h(x,y)$ of (5.14) is $8N^2$. Finally, the OSC scheme for (5.1)–(5.2) has the form

$$\begin{aligned} -\Delta\bar{u}_h(\xi) + \bar{v}_h(\xi) &= \Delta\tilde{u}_h(\xi) - \tilde{v}_h(\xi),\ \xi \in \mathcal{G}, \\ -\Delta\bar{v}_h(\xi) &= \Delta\tilde{v}_h(\xi) - f(\xi),\ \ \xi \in \mathcal{G}, \end{aligned} \tag{5.16}$$

where the right hand sides are known. Equations (5.16) give $8N^2$ constraints which are matched by the same number of unknown coefficients in $\bar{u}_h(x,y)$ and $\bar{v}_h(x,y)$.

Next, we give the existence and uniqueness result for $\bar{u}_h$ and $\bar{v}_h$ of (5.16). First, we prove two lemmas.

Lemma 5.1. *For $w \in \mathcal{M}_h^{00} \otimes \mathcal{M}_h^{00}$ and $z \in \mathcal{M}_h \otimes \mathcal{M}_h$,*

$$\langle -\Delta w, z\rangle_{\mathcal{G}} = \langle w, -\Delta z\rangle_{\mathcal{G}}.$$

Proof. Since $w \in \mathcal{M}_h^{00} \otimes \mathcal{M}_h^{00}$, we have

$$w(a,y) = w_x(a,y) = 0, \quad a = 0,1, \quad y \in [0,1], \tag{5.17}$$

and

$$w(x,b) = w_y(x,b) = 0, \quad x \in [0,1], \quad b = 0,1. \tag{5.18}$$

The proof is then identical to that of Lemma 3.1 except that (5.17) and (5.18) are used instead of (3.9) and (3.10). □

Lemma 5.2. *If* $z \in \mathcal{M}_h \otimes \mathcal{M}_h$ *satisfies*

$$z(a,b) = z_y(a,b) = 0, \quad a,b = 0,1, \tag{5.19}$$

then

$$\langle z_{xx}, z_{yy}\rangle_{\mathcal{G}} = \langle z, z_{xxyy}\rangle_{\mathcal{G}}. \tag{5.20}$$

Proof. Applying Lemma 2.1 with respect to y to the left-hand side of (5.20), we have

$$\begin{aligned}\langle z_{xx}, z_{yy}\rangle_{\mathcal{G}} &= \frac{h}{2}\sum_{n=1}^{2N}\langle z_{yy}(\xi_n,\cdot), z_{xx}(\xi_n,\cdot)\rangle \\ &= -\frac{h}{2}\sum_{n=1}^{2N}\int_0^1 (z_y z_{xxy})(\xi_n,y)\,dy \\ &\quad +\frac{h}{2}\sum_{n=1}^{2N}(z_y z_{xx})(\xi_n,y)|_{y=0}^{y=1} - \frac{h}{2}\sum_{n=1}^{2N} Ch^5 \sum_{m=1}^{N} z_{n,m}^{(0,3)} z_{n,m}^{(2,3)},\end{aligned} \tag{5.21}$$

where

$$z_{n,m}^{(0,3)} = z_{yyy}(\xi_n,y), \quad z_{n,m}^{(2,3)} = z_{xxyyy}(\xi_n,y), \quad y \in (t_{m-1},t_m). \tag{5.22}$$

To rewrite the second term on the right-hand side in (5.21), we use Lemma 2.1 with respect to x to obtain, for $y = 0,1$,

$$\begin{aligned}\frac{h}{2}\sum_{n=1}^{2N}(z_y z_{xx})(\xi_n,y) &= \langle z_{xx}(\cdot,y), z_y(\cdot,y)\rangle \\ &= -\int_0^1 (z_x z_{yx})(x,y)\,dx + (z_x z_y)(x,y)|_{x=0}^{x=1} \\ &\quad -Ch^5 \sum_{n=1}^{N} z_{n,y}^{(3,0)} z_{n,y}^{(3,1)},\end{aligned} \tag{5.23}$$

where

$$z_{n,y}^{(3,0)} = z_{xxx}(x,y), \quad z_{n,y}^{(3,1)} = z_{yxxx}(x,y), \quad x \in (t_{n-1},t_n). \tag{5.24}$$

Using again Lemma 2.1 with respect to x, we rewrite the first term on the right-hand side in (5.23) to obtain, for $y = 0, 1$,

$$
\begin{aligned}
-\int_0^1 (z_x z_{yx})(x,y)\,dx &= \frac{h}{2}\sum_{n=1}^{2N}(z_{yxx}z)(\xi_n,y) - (z_{yx}z)(x,y)|_{x=0}^{x=1} \\
&\quad + Ch^5 \sum_{n=1}^{N} z_{n,y}^{(3,1)} z_{n,y}^{(3,0)},
\end{aligned} \tag{5.25}
$$

where $z_{n,y}^{(3,1)}$ and $z_{n,y}^{(3,0)}$ are given in (5.24). Substituting (5.25) into (5.23) and using (5.19), we have

$$
\frac{h}{2}\sum_{n=1}^{2N}(z_y z_{xx})(\xi_n,y) = \frac{h}{2}\sum_{n=1}^{2N}(z_{yxx}z)(\xi_n,y), \qquad y = 0, 1. \tag{5.26}
$$

Applying Lemma 2.1 with respect to y to the right-hand side of (5.20), we also obtain

$$
\begin{aligned}
\langle z, z_{xxyy}\rangle_{\mathcal{G}} &= \frac{h}{2}\sum_{n=1}^{2N}\langle z_{xxyy}(\xi_n,\cdot), z(\xi_n,\cdot)\rangle \\
&= -\frac{h}{2}\sum_{n=1}^{2N}\int_0^1 (z_{xxy}z_y)(\xi_n,y)\,dy + \frac{h}{2}\sum_{n=1}^{2N}(z_{xxy}z)(\xi_n,y)|_{y=0}^{y=1} \\
&\quad -\frac{h}{2}\sum_{n=1}^{2N} Ch^5 \sum_{m=1}^{N} z_{n,m}^{(2,3)} z_{n,m}^{(0,3)},
\end{aligned} \tag{5.27}
$$

where $z_{n,m}^{(2,3)}$ and $z_{n,m}^{(0,3)}$ are defined in (5.22). Comparing the right-hand sides of (5.21) and (5.27), and using (5.26), we obtain (5.20). □

We then have the following theorem.

Theorem 5.1. *There exist unique functions $\bar{u}_h(x,y)$ of the form (5.5) and $\bar{v}_h(x,y)$ of the form (5.14) satisfying (5.16).*

Proof. Clearly, $\bar{u}_h \in \mathcal{M}_h^{00} \otimes \mathcal{M}_h^{00}$, $\bar{v}_h \in \mathcal{M}_h \otimes \mathcal{M}_h$, and

$$
\bar{v}_h(a,b) = (\bar{v}_h)_y(a,b) = 0, \qquad a, b = 0, 1. \tag{5.28}
$$

Since the number of unknown coefficients in $\bar{u}_h(x,y)$ and $\bar{v}_h(x,y)$ is equal to the number of equations in (5.16), it suffices to show that if

$$
-\Delta\bar{u}_h(\xi) + \bar{v}_h(\xi) = 0, \qquad \xi \in \mathcal{G}, \tag{5.29}
$$

$$
-\Delta\bar{v}_h(\xi) = 0, \qquad \xi \in \mathcal{G}, \tag{5.30}
$$

then $\bar{u}_h = \bar{v}_h = 0$. Taking the inner product $\langle\cdot,\cdot\rangle_{\mathcal{G}}$ with $\bar{v}_h$ on both sides of (5.29), we obtain

$$
\langle -\Delta\bar{u}_h, \bar{v}_h\rangle_{\mathcal{G}} + \langle \bar{v}_h, \bar{v}_h\rangle_{\mathcal{G}} = 0. \tag{5.31}
$$

Similarly, by taking the inner product $\langle\cdot,\cdot\rangle_{\mathcal{G}}$ with $\bar{u}_h$ on both sides of (5.30), we obtain

$$\langle -\Delta\bar{v}_h, \bar{u}_h\rangle_{\mathcal{G}} = 0. \tag{5.32}$$

Using (5.31), (5.32), and Lemma 5.1, we have $\langle\bar{v}_h, \bar{v}_h\rangle_{\mathcal{G}} = 0$, which implies

$$\bar{v}_h(\xi) = 0, \qquad \xi \in \mathcal{G}. \tag{5.33}$$

Thus it follows from (5.29) that $\Delta\bar{u}_h(\xi) = 0, \xi \in \mathcal{G}$. Since $\bar{u}_h \in \mathcal{M}_h^0 \otimes \mathcal{M}_h^0$, (2.10) and (2.9) imply that $\bar{u}_h = 0$.

To show that $\bar{v}_h = 0$, we use (5.30), (5.28), Lemma 5.2 , and (5.33) to obtain

$$\begin{aligned} 0 &= \|\Delta\bar{v}_h\|_{\mathcal{G}}^2 = \|(\bar{v}_h)_{xx}\|_{\mathcal{G}}^2 + 2\langle(\bar{v}_h)_{xx}, (\bar{v}_h)_{yy}\rangle_{\mathcal{G}} + \|(\bar{v}_h)_{yy}\|_{\mathcal{G}}^2 \\ &= \|(\bar{v}_h)_{xx}\|_{\mathcal{G}}^2 + 2\langle\bar{v}_h, (\bar{v}_h)_{xxyy}\rangle_{\mathcal{G}} + \|(\bar{v}_h)_{yy}\|_{\mathcal{G}}^2 = \|(\bar{v}_h)_{xx}\|_{\mathcal{G}}^2 + \|(\bar{v}_h)_{yy}\|_{\mathcal{G}}^2. \end{aligned}$$

Hence

$$(\bar{v}_h)_{xx}(\xi) = (\bar{v}_h)_{yy}(\xi) = 0, \qquad \xi \in \mathcal{G},$$

which along with (5.33) implies that $\bar{v}_h = 0$ on horizontal and vertical lines passing through $\xi \in \mathcal{G}$. This and $\bar{v}_h(a,b) = 0$, $a, b = 0, 1$, imply in turn that $\bar{v}_h = 0$ on $\partial\Omega$. Therefore, using in addition (5.33) and (2.9), we conclude that $\bar{v}_h = 0$. □

We now prove the following theorem.

Theorem 5.2. *Let $u \in H^{8-k}(\Omega)$, $k = 1, 2$, be the solution of the biharmonic problem (1.1). Let $u_h \in \mathcal{M}_h \otimes \mathcal{M}_h$ of (5.4)–(5.6), $v_h \in \mathcal{M}_h \otimes \mathcal{M}_h$ of (5.13)–(5.15) be solutions of (5.16), where $\tilde{u}_h(x,y)$ of (5.6) is obtained by approximating the boundary conditions $u = g_1$ and $\partial u/\partial n = g_2$ using piecewise Hermite cubic interpolation and $\tilde{v}_h(x,y)$ of (5.15) is obtained using (5.7) and (5.9) or (5.11) and (5.12). Then*

$$\|u - u_h\|_{H^k(\Omega)} \le Ch^{4-k}\|u\|_{H^{8-k}(\Omega)}, \qquad k = 1, 2. \tag{5.34}$$

Proof. It follows from the way in which the coefficients in $\tilde{u}_h$ are selected that $w = u_h - u_{\mathcal{H}} \in \mathcal{M}_h^{00} \otimes \mathcal{M}_h^{00}$. Moreover $z = v_h - v_{\mathcal{H}} \in \mathcal{M}_h \otimes \mathcal{M}_h$ regardless of how the coefficients in $\tilde{v}_h$ are selected. Therefore, the derivation of (5.34) is identical to that of (3.16) with $k = 1, 2$, except that Lemma 5.1 is used instead of Lemma 3.1. □

5.2. Capacitance matrix method for OSC biharmonic problem III

In order to solve (5.16) efficiently, we employ the capacitance matrix method to obtain a linear system of smaller dimension which can be solved by Gauss elimination. Our auxiliary problem is the OSC BPII (4.10) which can be solved very efficiently using the single series OSC Fourier method. We first rewrite the scheme (5.16) by splitting $\tilde{u}_h(x,y)$ of (5.6) in the form

$$\tilde{u}_h(x,y) = \tilde{u}_h^{\mathrm{I}}(x,y) + \tilde{u}_h^{\mathrm{II}}(x,y), \tag{5.35}$$

where

$$\begin{aligned}\tilde{u}_h^I(x,y) = &\sum_{i=1}^{2N} u_{i,0}\phi_i(x)\psi_0(y) + \sum_{i=1}^{2N} u_{i,1}\phi_i(x)\psi_1(y) \\ &+ \sum_{i=1}^{2N} u_{i,2N}\phi_i(x)\psi_{2N}(y) + \sum_{i=1}^{2N} u_{i,2N+1}\phi_i(x)\psi_{2N+1}(y) \\ &+ \sum_{j=0}^{2N+1} u_{0,j}\phi_0(x)\psi_j(y) + \sum_{j=0}^{2N+1} u_{2N+1,j}\phi_{2N+1}(x)\psi_j(y),\end{aligned}$$

and

$$\tilde{u}_h^{\mathrm{II}}(x,y) = \sum_{j=2}^{2N-1} u_{N,j}\phi_N(x)\psi_j(y) + \sum_{j=2}^{2N-1} u_{2N,j}\phi_{2N}(x)\psi_j(y). \tag{5.36}$$

We substitute (5.35) into (5.16) and move the known term $\Delta\tilde{u}_h^{\mathrm{II}}$ to the left hand side. The equivalent form of (5.16) becomes

$$\begin{aligned}-\Delta u_h^{(3)}(\xi) + v_h^{(3)}(\xi) &= \Delta\tilde{u}_h^I(\xi) - \tilde{v}_h(\xi) \equiv F^{(3)}(\xi), \ \xi \in \mathcal{G}, \\ -\Delta v_h^{(3)}(\xi) &= \Delta\tilde{v}_h(\xi) - f(\xi) \equiv G^{(3)}(\xi), \quad \xi \in \mathcal{G},\end{aligned} \tag{5.37}$$

where $u_h^{(3)} = \bar{u}_h + \tilde{u}_h^{\mathrm{II}}$ and $v_h^{(3)} = \bar{v}_h$. By (5.5), (5.36), and (5.14),

$$u_h^{(3)}(x,y) = \sum_{i=1}^{2N}\sum_{j=2}^{2N-1} u_{i,j}\phi_i(x)\psi_j(y), \tag{5.38}$$

$$\begin{aligned}v_h^{(3)}(x,y) = &\sum_{i=1}^{2N}\sum_{j=0}^{2N+1} v_{i,j}\phi_i(x)\psi_j(y) + \sum_{j=2}^{2N-1} v_{0,j}\phi_0(x)\psi_j(y) \\ &+ \sum_{j=2}^{2N-1} v_{2N+1,j}\phi_{2N+1}(x)\psi_j(y).\end{aligned} \tag{5.39}$$

Note that the coefficients $u_{N,j}$ and $u_{2N,j}$, $j = 2,\ldots,2N-1$, in (5.38) are known.

Now we rewrite the scheme (4.10) by splitting $\tilde{v}_h(x,y)$ of (4.9) in the form

$$\tilde{v}_h(x,y) = \tilde{v}_h^{\mathrm{I}}(x,y) + \tilde{v}_h^{\mathrm{II}}(x,y), \tag{5.40}$$

where

$$\tilde{v}_h^{\mathrm{I}}(x,y) = \sum_{j=2}^{2N-1} v_{0,j}\phi_0(x)\psi_j(y) + \sum_{j=2}^{2N-1} v_{2N+1,j}\phi_{2N+1}(x)\psi_j(y), \tag{5.41}$$

and

$$\begin{aligned}\tilde{v}_h^{\mathrm{II}}(x,y) = &\sum_{j=0,1,2N,2N+1} v_{0,j}\phi_0(x)\psi_j(y)\\ &+ \sum_{j=0,1,2N,2N+1} v_{2N+1,j}\phi_{2N+1}(x)\psi_j(y).\end{aligned}$$

We substitute (5.40) into (4.10) and move the terms $-\tilde{v}_h^I$ and $\Delta\tilde{v}_h^{\mathrm{I}}$ to the left hand sides to obtain

$$\begin{aligned}-\Delta u_h^{(2)}(\xi) + v_h^{(2)}(\xi) &= \Delta\tilde{u}_h(\xi) - \tilde{v}_h^{\mathrm{II}}(\xi) \equiv F^{(2)}(\xi), \xi \in \mathcal{G},\\ -\Delta v_h^{(2)}(\xi) &= \Delta\tilde{v}_h^{\mathrm{II}}(\xi) - f(\xi) \equiv G^{(2)}(\xi), \ \ \xi \in \mathcal{G},\end{aligned} \tag{5.42}$$

where $u_h^{(2)} = \bar{u}_h$ and $v_h^{(2)} = \bar{v}_h + \tilde{v}_h^I$. By (4.6), (4.8), and (5.41),

$$u_h^{(2)}(x,y) = \sum_{i=1}^{2N}\sum_{j=2}^{2N-1} u_{i,j}\phi_i(x)\psi_j(y), \tag{5.43}$$

$$\begin{aligned}v_h^{(2)}(x,y) = &\sum_{i=1}^{2N}\sum_{j=0}^{2N+1} v_{i,j}\phi_i(x)\psi_j(y) + \sum_{j=2}^{2N-1} v_{0,j}\phi_0(x)\psi_j(y)\\ &+ \sum_{j=2}^{2N-1} v_{2N+1,j}\phi_{2N+1}(x)\psi_j(y).\end{aligned} \tag{5.44}$$

Note that $v_{0,j}$ and $v_{2N+1,j}$, $j = 2, \ldots, 2N-1$, in (5.44) are known. Now we see that $u_h^{(3)}(x,y)$ of (5.38) and $u_h^{(2)}(x,y)$ in (5.43) are given in terms of the same basis functions. Similarly, $v_h^{(3)}(x,y)$ of (5.39) and $v_h^{(2)}(x,y)$ of (5.44) are given in terms of the same basis functions. Moreover, the left hand sides of (5.37) and (5.42) have the same form.

Substituting (5.38) and (5.39) into (5.37) and (5.43) and (5.44) into (5.42), we obtain the linear systems

$$A\mathbf{w}^{(3)} = \mathbf{f}^{(3)}, \tag{5.45}$$

and

$$B\mathbf{w}^{(2)} = \mathbf{f}^{(2)}, \tag{5.46}$$

respectively, where

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad B = \begin{bmatrix} A_1 \\ B_2 \end{bmatrix}, \tag{5.47}$$

are $(8N^2+4N-4) \times (8N^2+4N-4)$ matrices, A_1 is an $8N^2 \times (8N^2 + 4N-4)$ submatrix corresponding to the left hand sides of (5.37) and (5.42), and where A_2 and B_2 are $(4N-4) \times (8N^2+4N-4)$ submatrices. All elements in each row of A_2 and B_2 are 0 except one which equals 1. In A_2, the ones correspond to the known coefficients $u_{N,j}$ and $u_{2N,j}$, $j = 2, \ldots, 2N-1$, of $u_h^{(3)}(x,y)$ in (5.38), and the ones in B_2 correspond to the known coefficients $v_{0,j}$ and $v_{2N+1,j}$, $j = 2, \ldots, 2N-1$, of $v_h^{(2)}(x,y)$ in (5.44). The vectors $\mathbf{w}^{(3)}$ and $\mathbf{w}^{(2)}$ are $(8N^2+4N-4)$-vectors corresponding to the $8N^2+4N-4$ coefficients (including $4N-4$ known coefficients) of $u_h^{(3)}$, $v_h^{(3)}$, and $u_h^{(2)}$ and $v_h^{(2)}$. These coefficients are ordered in the same way in $\mathbf{w}^{(3)}$ and $\mathbf{w}^{(2)}$. The right hand side $\mathbf{f}^{(3)}$ of (5.45) is an $(8N^2+4N-4)$-vector containing $F^{(3)}(\xi_n,\xi_m)$, $G^{(3)}(\xi_n,\xi_m)$, $n,m = 1,\ldots,2N$, and $4N-4$ known coefficients $u_{N,j}$ and $u_{2N,j}$, $j = 2,\ldots,2N-1$, as the last components. Also the right hand side $\mathbf{f}^{(2)}$ of (5.46) is an $(8N^2+4N-4)$-vector containing $F^{(2)}(\xi_n,\xi_m)$, $G^{(2)}(\xi_n,\xi_m)$, $n,m = 1,\ldots,2N$, and $4N-4$ known coefficients $v_{0,j}$ and $v_{2N+1,j}$, $j = 2,\ldots,2N-1$, as the last components. $F^{(3)}(\xi_n,\xi_m)$, $G^{(3)}(\xi_n,\xi_m)$, and $F^{(2)}(\xi_n,\xi_m)$, $G^{(2)}(\xi_n,\xi_m)$, $n,m = 1,\ldots,2N$, are ordered in the same way in $\mathbf{f}^{(3)}$ and $\mathbf{f}^{(2)}$.

Note that matrices A and B in (5.47) differ only in the last $4N-4$ rows, and the system (5.46), which is a matrix-vector representation of the OSC approximation of BPII, can be solved using the single series OSC Fourier method described in Sect. 4.2. We employ the capacitance matrix method to solve the system (5.45). We rewrite the $(8N^2+4N-4)$-vector $\mathbf{f}^{(3)}$ of (5.45) in the form

$$\mathbf{f}^{(3)} = [\mathbf{b}_1, \quad \mathbf{b}_2]^{\mathrm{T}}, \tag{5.48}$$

where $\mathbf{b}_1$ contains $F^{(3)}(\xi_n,\xi_m)$ and $G^{(3)}(\xi_n,\xi_m)$, $n,m = 1,\ldots,2N$, and $\mathbf{b}_2$ contains the values of $4N-4$ known coefficients $u_{N,j}$ and $u_{2N,j}$, $j = 2,\ldots,2N-1$, of $u_h^{(3)}(x,y)$ in (5.38). The algorithm is then defined in four phases.

Phase 1. Form the $(4N-4)\times(4N-4)$ capacitance matrix

$$D \;=\; A_2 B^{-1} \begin{bmatrix} 0 \\ I_{4N-4} \end{bmatrix}, \tag{5.49}$$

where 0 is an $8N^2 \times (4N-4)$ zero matrix and I_{4N-4} is the identity matrix of order $4N-4$.

Phase 2. Solve the system

$$B\boldsymbol{\beta} \;=\; \begin{bmatrix} \mathbf{b_1} \\ \mathbf{0} \end{bmatrix},$$

where $\mathbf{b}_1$ is the first subvector of $\mathbf{f}^{(3)}$ in (5.48) and $\mathbf{0}$ is a $(4N-4)$-vector of zeros.
Phase 3. Solve the system

$$D\boldsymbol{\alpha} \;=\; \mathbf{b}_2 \;-\; A_2\boldsymbol{\beta}, \tag{5.50}$$

where $\mathbf{b}_2$ is the second subvector of $\mathbf{f}^{(3)}$ in (5.48).
Phase 4. Solve the system

$$B\mathbf{w}^{(3)} \;=\; \begin{bmatrix} \mathbf{b}_1 \\ \boldsymbol{\alpha} \end{bmatrix}$$

for the desired solution $\mathbf{w}^{(3)}$ of (5.45).

We discuss the algorithm in more detail after first proving the following result.

Theorem 5.3. *The capacitance matrix D given by (5.49) is nonsingular.*

Proof. We show that $\boldsymbol{\alpha} = \mathbf{0}$ is the only solution to $D\boldsymbol{\alpha} = \mathbf{0}$. It follows from (5.49) and (5.47) that $D\boldsymbol{\alpha} = \mathbf{0}$ is equivalent to

$$A_2\mathbf{w}^{(3)} = \mathbf{0}, \qquad A_1\mathbf{w}^{(3)} = \mathbf{0}, \qquad \boldsymbol{\alpha} = B_2\mathbf{w}^{(3)}, \tag{5.51}$$

where the components of $\mathbf{w}^{(3)}$ can be identified with the coefficients of $u_h^{(3)}(x,y)$ and $v_h^{(3)}(x,y)$ in (5.38) and (5.39). The first equation of (5.51), the structure of A_2, and (5.36) imply that $\tilde{u}_h^{\mathrm{II}} = 0$ in $u_h^{(3)} = \bar{u}_h + \tilde{u}_h^{\mathrm{II}}$. Thus the second equation of (5.51), which is equivalent to (5.37) with $F^{(3)}(\xi) = G^{(3)}(\xi) = 0$, $\xi \in \mathcal{G}$, gives

$$-\Delta\bar{u}_h(\xi) + \bar{v}_h(\xi) = 0, \qquad -\Delta\bar{v}_h(\xi) = 0, \qquad \xi \in \mathcal{G}.$$

Theorem 5.1 implies $\bar{u}_h = \bar{v}_h = 0$ and hence $\mathbf{w}^{(3)} = \mathbf{0}$. Therefore $\boldsymbol{\alpha} = \mathbf{0}$ by the third equation in (5.51). □

In the algorithm, we form the capacitance matrix D of (5.49) explicitly. To this end, in Phase 1 of the capacitance matrix algorithm, we need to solve, for $i = 1, \ldots, 4N-4$, a system of the form

$$B\boldsymbol{\beta}_i = \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_i \end{bmatrix}, \tag{5.52}$$

where $\mathbf{0}$ is an $8N^2$-vector of zeros and $\mathbf{e}_i$ is the i^{th} column of the identity matrix I_{4N-4}. Solving each system (5.52) is equivalent to solving the OSC

BPII (4.10) with the right hand side $f(\xi) = 0$, $\xi \in \mathcal{G}$, $\tilde{u}_h = 0$, and all the coefficients in $\tilde{v}_h(x,y)$ of (4.9) zero except one of $v_{0,j}$, $v_{2N+1,j}$, $j = 2,\ldots,2N-1$, which is equal to one. By (5.49), observe also that, for each $i = 1,\ldots,4N-4$, the multiplication of A_2 by an $(8N^2+4N-4)$-vector $\boldsymbol{\beta}_i$ simply corresponds to taking from the vector $\boldsymbol{\beta}_i$ the coefficients $u_{N,j}$ and $u_{2N,j}$, $j = 2,\ldots,2N-1$, which are the coefficients of $\bar{u}_h(x,y)$ in (4.6). Thus each column of D is computed using the modification of the single series OSC Fourier method for solving the special case of OSC BPII described in Sect. 4.3. Computation of a column in D requires the solution of problems (4.49) for $k = 1,\ldots,2N$. For each k we factor the matrix of the form (4.51) only once and then use this factorization in the computation of each column of D. Thus, we perform $2N$ factorizations, the cost of each factorization being $O(N)$, [12]. According to [12] the cost of the solution phase of COLROW for an almost block diagonal matrix (4.51) of size $4N+4$ with 4×8 blocks is $44N$. Thus the cost of Step 4 of the modified single series OSC Fourier method is $88N^2$ for each $i = 1,\ldots,4N-4$. The costs of Step 1 and 2 are $O(1)$ and $O(N)$, respectively. There is no need to perform Step 3, while the cost of Step 5 is $8N^2$. Therefore the total cost of forming the capacitance matrix D is $384N^3 + O(N^2)$.

Phase 2 of the capacitance matrix algorithm involves the OSC solution of a BPII. This can be done with cost $O(N^2 \log_2 N)$ using the single series OSC Fourier method described in Sect. 4.2. In Phase 3, as we have just observed, the multiplication of A_2 by $\boldsymbol{\beta}$ amounts to retrieving part of the solution from $\boldsymbol{\beta}$. To solve the linear system (5.50) with the dense $4(N-1) \times 4(N-1)$ coefficient matrix D, we use Gauss elimination. Hence the cost of Phase 3 is $64N^3/3 + O(N^2)$. In the final phase, Phase 4, we solve another BPII, which requires $O(N^2 \log_2 N)$ arithmetic operations. Hence the total cost of the OSC capacitance method for solving the biharmonic Dirichlet problem (1.1) is then $(384 + 64/3)N^3 + O(N^2 \log_2 N)$.

Sun [27] presented a Schur complement algorithm for solving the biharmonic Dirichlet problem using OSC. The total cost of his algorithm is $O(N^3 \log_2 N)$. In Phase 3 of his algorithm, Gauss elimination is used to solve a linear system $D\boldsymbol{\beta} = \mathbf{b}$, where D is a $16(N-1) \times 16(N-1)$ matrix. Therefore the cost of this phase is $4096N^3/3 + O(N^2)$. In comparison, Phase 3 of our algorithm requires $64N^3/3 + O(N^2)$ operations which is approximately 64 times less expensive. Forming the matrix D in Sun's algorithm requires $256N^3 \log_2 N + O(N^3)$ operations, where the coefficient of N^3 is not given. Forming D in our algorithm takes $384N^3 + O(N^2)$ operations.

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