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THE FAST SOLUTION OF POISSON'S AND THE BIHARMONIC EQUATIONS ON IRREGULAR REGIONS*

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Abstract. We present fast methods for solving Laplace's and the biharmonic equations on irregular regions with smooth boundaries. The methods used for solving both equations make use of fast Poisson solvers on a rectangular region in which the irregular region is embedded. They also both use an integral equation formulation of the problem where the integral equations are Fredholm integral equations of the second kind. The main idea is to use the integral equation formulation to define a discontinuous extension of the solution to the rest of the rectangular region. Fast solvers are then used to compute the extended solution. Aside from solving the equations we have also been able to compute derivatives of the solutions with little loss of accuracy when the data was sufficiently smooth.

Key words. fast solver, Laplace's equation, biharmonic equation, integral equation

Introduction. We present fast methods for solving Laplace's and the biharmonic equations on irregular two-dimensional regions with sufficiently smooth boundaries. Although we have used second order accurate solvers in our experiments so far, we believe that the methods can easily be made fourth order at small additional cost. We can also compute derivatives of the solution of all points of the region to the same order of accuracy if the boundary curve is sufficiently smooth. In fact, derivatives can be computed without first finding the solution. This is especially important in applications since in general what are really needed are the derivatives of the solution, and merely differencing the solution loses accuracy.

The techniques employed in both solvers are very similar and use many of the same ideas. Each uses a fast Poisson solver on a regular region in which the irregular region is embedded. Fast Poisson solvers are methods for solving the linear systems of equations that arise from discretizing Poisson's equation on regions that are rectangular with respect to some coordinates. (See [1], [2].) These methods are efficient and require relatively low storage. For example, by using a Buneman solver one can solve a problem on a square with n equally spaced points in each direction using only $5n^2 \log_2 n$ operations and n^2+3n storage locations.

Both solvers also use integral equation formulations of the problem. In the current implementation the integral equations are Fredholm equations of the second kind. The essential idea is to use an integral equation formulation to define an extension of the solution to the rest of the regular region. The extension is discontinuous, but the discontinuities in the normal and tangential directions can be expressed in terms of the solution of the integral equation. Using these discontinuities and the shape of the boundary curve, we can find all the discontinuities in the coordinate directions and use them to compute an approximation to the discrete Laplacian at mesh points near the irregular boundary. Fast solvers can then be used to compute the extended solution.

Our method overcomes two of the common difficulties with integral equations. The first difficulty is that it is expensive to compute the solution at many points of a region by evaluating an integral. The second and more important one is that it is

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difficult to compute the solution at mesh points near the boundary because the kernel of the integral which must be evaluated becomes unbounded there. Our methods overcome these problems since we only compute the solution by evaluating an integral at the edge of the grid away from the irregular boundary. This is in contrast to our earlier work where it was necessary to evaluate integrals inside the region of interest. (See [3].) Furthermore, we can obtain the solution to full accuracy at mesh points near the irregular boundary without interpolating.

We note that fast Poisson solvers have previously been employed for solving Poisson's equation on irregular regions through the use of capacitance matrix methods. See [4], [5]. Our method, however, is better suited to exterior regions. Although by using a clever trick due to Hockney [1] it is possible to use capacitance matrix methods to solve exterior problems, both the operation count and storage requirement for the fast solver are significantly increased. Moreover, the author does not know of any implementation of a capacitance matrix method for solving the biharmonic equation on an irregular region or for computing derivatives directly.

In the case of Laplace's equation we use the standard integral equation formulation in terms of a dipole density. To solve the first biharmonic problem (u and u_n prescribed on the boundary), we use the integral equation formulation of Sherman and Lauricella [6]. We use this formulation for several reasons. First, it is very well conditioned. Also, the kernel is quite inexpensive to evaluate and no computations are needed to form the right-hand side of the equation. We note that while many numerical procedures for solving the biharmonic equation reduce the problem to a succession of Poisson problems (see for example [7]), the Sherman and Lauricella formulation reduces it to the problem of finding two independent harmonic functions and their conjugates. In fact, it reduces directly to the problem of evaluating the real and imaginary parts of two Cauchy integrals. We also note that in the cases of both Poisson's and the biharmonic equations it is relatively inexpensive to solve the integral equation. This is primarily because we use a Nystrom method with the trapezoid rule as the quadrature formula when we discretize the integral equation, and because the trapezoid rule is so accurate on periodic regions. Since the accuracy of the solution of the integral equation is the same as the accuracy of the quadrature formula, we can use very few mesh points on the boundary curve. Of course we cannot solve the integral equation so accurately or so inexpensively when there is mixed data or when the boundary curve has corners or regions of high curvature. However, the method can still be used to find the solution of the differential equation inside the region of interest once the integral equation has been solved.

The organization of this paper is as follows. In the first section we present the method we use for solving Laplace's equation and in the following section we give the details of the computational method and its operation counts. In § 3 we give the method we use to compute the derivatives and the conjugate of a harmonic function. In § 4 we present the method for solving the biharmonic problem, and in the last section we give the results of numerical experiments.

1. Solution of Laplace's equation. Suppose we want to solve $\Delta u = 0$ on an irregular region D with boundary $\partial D = (x(s), y(s))$ on which Dirichlet boundary data $u = g(s)$ is prescribed. We show later how to solve a problem where Neumann boundary data are given. We assume that both ∂D and $g(s)$ have two continuous derivatives so that the solution u has four.

We embed D in some regular region R, such as a square with a uniform mesh of width h in the x and y directions. Since we want to use a fast solver on R, our objective

is to compute an approximation to the discrete Laplacian at all points of the grid. In order to do this, we use another harmonic function $\tilde{u}$ defined outside D such that the discrete Laplacian of the combined function can easily be computed. (We note that a fast Poisson solver is merely an algorithm for inverting the discrete Laplacian, i.e. the operator which at every point sums the value of a function at its four neighbors and subtracts 4 times the value at that point. Therefore if we provide the discrete Laplacian at every mesh point and apply the fast solver, then we will get back the original function values regardless of whether or not they are the values of a continuously differentiable function.)

1.1. Integral equation formulation. We first note that inside the region D, u can be represented as the integral of a double layer or dipole density function μ times a kernel which is the normal derivative of the Green's function for the Laplacian in the plane:

$$(1.1)\quad u(t)=\frac{1}{2\pi}\int_{\partial D}\frac{\partial \log r(s,t)}{\partial n_s}\mu(s)\,ds,\quad \text{where } r^2(s,t)=(x(s)-x_0)^2+(y(s)-y_0)^2,$$

$$t=(x_0,y_0).$$

By noting the singularity of the kernel at the boundary, it can be shown that the function μ is the solution of the following integral equation on the boundary:

$$(1.2)\qquad \mu(t)+\frac{1}{\pi}\int_{\partial D}\frac{\partial \log r(s,t)}{\partial n_s}\mu(s)\,ds=2g(t),\qquad t\in\partial D.$$

See for example [8, p. 299]. This is a well conditioned integral equation of the second kind and can be solved very accurately numerically. We note, in particular, that although the kernel in (1.1) becomes unbounded as the point t not on the boundary approaches a point s on the boundary, the kernel in (1.2) is bounded since s and t are constrained to lie on ∂D. (When $s=t$ on ∂D, we have $\partial \log r/\partial n_s=\frac{1}{2}\kappa(s)$ where $\kappa(s)$ is the curvature of ∂D.)

Outside D we define $\tilde{u}$ using the same formula:

$$\tilde{u}(t)=\frac{1}{2\pi}\int_{\partial D}\frac{\partial \log r(s,t)}{\partial n_s}\mu(s)\,ds.$$

The function $\tilde{u}$ is harmonic, but it is a discontinuous extension of u.

Let $\{x_i, y_j\}$ denote the mesh points of R and let U be defined on R to be the combined function

$$U_{ij}=\begin{cases}u(x_i,y_j), & (x_i,y_j)\in D,\\ \tilde{u}(x_i,y_j), & (x_i,y_j)\notin D.\end{cases}$$

Since u and $\tilde{u}$ are harmonic, the 5-point discrete Laplacian of U, $\Delta_h U_{ij}=1/h^2[U_{i+1j}+U_{i-1,j}+U_{ij+1}+U_{ij-1}-4U_{ij}]$, will be 0 up to terms of second order at those mesh points of R which have their 4 neighbors on the same side of the irregular boundary. Let B denote the set of other points, that is the set of irregular mesh points. At points of B even though the analytic Laplacian of u and of $\tilde{u}$ is 0, the discrete Laplacian, $\Delta_h U$, will be nonzero. If we could find values for $\Delta_h U$ at those points and the values of $\tilde{u}$ on ∂R, then we could apply a fast Poisson solver on R and we would be done. Fortunately, we can compute an approximation to the discrete Laplacian without solving explicitly for u or $\tilde{u}$ anywhere. This is because we only need to evaluate the jump discontinuities between u and $\tilde{u}$ and the jump discontinuities in their

derivatives at the irregular boundary ∂D in order to compute such an approximation, and because we can compute these jumps in terms of the density μ.

1.2. Evaluation of discontinuities. We now show how to compute these discontinuities.

For example, it is known that the discontinuity between u and $\tilde{u}$ at a point on the boundary is equal to the value of the density at that point. Therefore, the discontinuity between their tangential derivatives is equal to the derivative of the density.

$$u_s - \tilde{u}_s = \frac{d\mu}{ds}.$$

It is also known that there is no discontinuity between their normal derivatives.

$$u_n = \tilde{u}_n.$$

Using these two facts and the direction of the curve, we may compute the discontinuities between u_x and $\tilde{u}_x$ and between u_y and $\tilde{u}_y$:

$$u_x - \tilde{u}_x = \frac{\mu'(s)x'(s)}{x'(s)^2 + y' - (s)^2}, \qquad u_y - \tilde{u}_y = \frac{\mu'(s)y'(s)}{x'(s)^2 + y'(s)^2}.$$

By differentiating the above equations and using higher derivatives of the boundary curve, it is also possible to derive formulas for discontinuities in higher order derivatives of u.

It is also possible to see how to compute these discontinuities by using the fact that the function U is the real part of a Cauchy integral with the same density function μ. A Cauchy integral is any integral of the form:

$$\int_C \frac{b(\zeta)}{\zeta - z}\, d\zeta$$

where C is a closed curve and z is any point not on C. The density function $b(\zeta)$ need not be the boundary values of an analytic function; all that is necessary is that $b(\zeta)$ be Holder continuous.

Let

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{\mu(\zeta)}{\zeta - z}\, d\zeta. \tag{1.3}$$

The kernel in (1.1) is the real part of the kernel in (1.3):

$$\operatorname{Re} \frac{1}{2\pi i} \frac{d\zeta/ds}{\zeta - z} = \frac{1}{2\pi} \frac{y'(s)[x(s) - x(t)] - x'(s)[y(s) - y(t)]}{(x(s) - x(t))^2 + (y(s) - y(t))^2}\, ds = \frac{1}{2\pi} \frac{\partial \log r(s,t)}{\partial n_s}\, ds,$$

where $\zeta(s) = x(s) + iy(s)$, $z = x(t) + iy(t)$.

Therefore we have $u(z) = \operatorname{Re} f(z)$ for $z \in D$, and $\tilde{u}(z) = \operatorname{Re} f(z)$ for $z \notin D$. In order to find the jump between u and $\tilde{u}$ we use the known discontinuities of Cauchy integrals across the boundary curve and the fact that Cauchy integrals are analytic functions. See [9].

For example, on ∂D,

$$2f(z)=\begin{cases}\mu(z)+\dfrac{1}{\pi i}\displaystyle\int_{\partial D}\frac{\mu(\zeta)}{\zeta-z}\,d\zeta & \text{if } z \text{ approaches the boundary from inside } D,\\ -\mu(z)+\dfrac{1}{\pi i}\displaystyle\int_{\partial D}\frac{\mu(\zeta)}{\zeta-z}\,d\zeta & \text{if } z \text{ approaches the boundary from outside } D.\end{cases}$$

Therefore there is a discontinuity of magnitude $\mu(z)$ in f as we cross ∂D. Since $\mu(z)$ is a real function, it follows that $u(z)-\tilde{u}(z)=\mu(z)$ if $z\in\partial D$.

To compute discontinuities in derivatives we note that:

$$\frac{d}{dz}\int_{\partial D}\frac{\mu(\zeta)}{\zeta-z}\,d\zeta=\int_{\partial D}\frac{d}{dz}\frac{\mu(\zeta)}{\zeta-z}\,d\zeta=\int_{\partial D}\frac{\mu'(\zeta)}{\zeta-z}\,d\zeta.$$

The second equality follows by integration by parts. Therefore the derivative of a Cauchy integral is another Cauchy integral with density function equal to the derivative of the original density function. It follows that $f'(z)$ has a discontinuity of magnitude $\mu'(z)$ across ∂D. Now we use the fact that $f(z)$ is analytic and $u_x(z)=\operatorname{Re} f'(z)$.

Therefore

$$u_x(z)-\tilde{u}_x(z)=\operatorname{Re}\mu'(z)=\operatorname{Re}\frac{d\mu/ds}{dz/ds}=\frac{\mu'(s)x'(s)}{x'(s)^2+y'(s)^2}$$

and

$$u_y(z)-\tilde{u}_y(z)=-\operatorname{Im}\mu'(z)=\frac{\mu'(s)y'(s)}{x'(s)^2+y'(s)^2}.$$

Similarly $\operatorname{Re} f''(z)=u_{xx}(z)$ and so $u_{xx}-\tilde{u}_{xx}=\operatorname{Re}\mu''(z)$, and $u_{yy}-\tilde{u}_{yy}=-\operatorname{Re}\mu''(z)$.

These discontinuities can be used to compute an approximation to the discrete Laplacian at mesh points near the boundary of D.

1.3. Evaluation of the discrete Laplacian. Now we show how to compute an approximation to the discrete Laplacian of U at points of B. In the derivation we require our solution to have four continuous derivatives.

Suppose, for example, p is in D, but its neighbor to the right, p_E, is not. Let p^* be the point on the line between p and p_E which intersects ∂D, let h_1 be the distance between p and p^*, and let $h_0=h-h_1$.

Writing the Taylor series expansion for u about p evaluated at p^*, we have

$$u(p^*)=u(p)+h_1u_x(p)+\tfrac{1}{2}h_1^2u_{xx}(p)+O(h^3),\quad\text{so}$$

$$\begin{aligned}u(p)&=u(p^*)-h_1u_x(p)-\tfrac{1}{2}h_1^2u_{xx}(p)+O(h^3)\\&=u(p^*)+h_2u_x(p^*)+h_2(u_x(p)-u_x(p^*))-hu_x(p)-\tfrac{1}{2}h_1^2u_{xx}(p)+O(h^3)\end{aligned}\tag{1.4}$$

(since $-h_1=h_2-h$).

Now we write the Taylor series expansion for $\tilde{u}$ about p_E evaluated at p^*,

$$\tilde{u}(p^*)=\tilde{u}(p_E)-h_2\tilde{u}_x(p_E)+\tfrac{1}{2}h_2^2\tilde{u}_{xx}(p_E)+O(h^3).$$

We note that the x derivatives of $\tilde{u}$ exist at the boundary point p^*. This follows because both the boundary curve ∂D and the boundary data $g(s)$ were assumed to be sufficiently smooth. See [10].

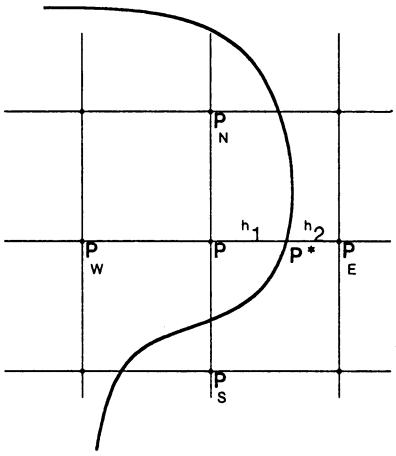

FIG. 1

Therefore we can write

$$\begin{aligned} h_2\tilde{u}_x(p_E) &= h_2\tilde{u}_x(p^*) + h_2^2\tilde{u}_{xx}(p_E) + O(h^3), \quad \text{so} \\ \tilde{u}(p_E) &= \tilde{u}(p^*) + h_2\tilde{u}_x(p^*) + \tfrac{1}{2}h_2^2\tilde{u}_{xx}(p_E) + O(h^3). \end{aligned} \tag{1.5}$$

Subtracting (1.4) from (1.5), we find:

$$\begin{aligned} \tilde{u}(p_E) - u(p) = (\tilde{u}(p^*) - u(p^*)) + h_2(\tilde{u}_x(p^*) - u_x(p^*)) + \tfrac{1}{2}h_2^2\tilde{u}_{xx}(p_E) \\ - h_2(u_x(p) - u_x(p^*)) + \tfrac{1}{2}h_1^2 u_{xx}(p) + hu_x(p) + O(h^3). \end{aligned} \tag{1.6}$$

Noting that $u_x(p) - u_x(p^*) = -h_1 u_{xx}(p) + O(h^2)$, we have

$$-h_2(u_x(p) - u_x(p^*)) = h_1 h_2 u_{xx}(p) + O(h^3).$$

Substituting this, then adding and subtracting $\frac{1}{2}h_2^2 u_{xx}(p)$ and regrouping terms on the right-hand side of (1.6), we find:

$$\begin{aligned} \tilde{u}(p_E) - u(p) = (\tilde{u}(p^*) - u(p^*)) + h_2(\tilde{u}_x(p^*) - u_x(p^*)) + \tfrac{1}{2}(h_1 + h_2)^2 u_{xx}(p) \\ + \tfrac{1}{2}h_2^2(\tilde{u}_{xx}(p_E) - u_{xx}(p)) + hu_x(p) + O(h^3). \end{aligned} \tag{1.7}$$

But $\frac{1}{2}h_2^2 u_{xx}(p) = \frac{1}{2}h_2^2 u_{xx}(p^*) + O(h^3)$ and $\frac{1}{2}h_2^2\tilde{u}_{xx}(p) = \frac{1}{2}h_2^2\tilde{u}_{xx}(p^*) + O(h^3)$. It follows that

$$\begin{aligned} \tilde{u}(p_E) - u(p) &= (\tilde{u}(p^*) - u(p^*)) + h_2(\tilde{u}_x(p^*) - u_x(p^*)) + \tfrac{1}{2}h_2^2(\tilde{u}_{xx}(p^*) - u_{xx}(p^*)) \\ &\quad + hu_x(p) + \tfrac{1}{2}h^2 u_{xx}(p) + O(h^3) \\ &= \{\text{known quantities}\} + hu_x(p) + \tfrac{1}{2}h^2 u_{xx}(p) + O(h^3), \end{aligned} \tag{1.8}$$

where the known quantities can be computed in terms of the density function and the distances of the irregular mesh points from the boundary.

Now let p_W be the mesh point to the left of p. If p_W is in D, then

$$U(p_W) - U(p) = -hu_x(p) + h^2 u_{xx}(p) + O(h^3);$$

if not then

$$U(p_W) - U(p) = \{\text{known quantities}\} - hu_x(p) + h^2 u_{xx}(p) + O(h^3).$$

In any case,

$$U(p_W) + U(p_E) - 2U(p) = \{\text{known quantities}\} + h^2 u_{xx}(p) + O(h^3).$$

Let p_N be the point above p, and let p_S be the point below p. By the same arguments as above we have:

$$U(p_N) + U(p_S) - 2U(p) = \{\text{known quantities}\} + h^2 u_{yy}(p) + O(h^3).$$

It follows that $h^2 \Delta_h U(p) = \{\text{known quantities}\} + O(h^3)$ since $u_{xx}(p) + u_{yy}(p) = 0$.

The same is true if p is a point outside D since $\tilde{u}$ is harmonic. (It is important to notice that we do not assume that either of the harmonic functions u or $\tilde{u}$ can be extended beyond ∂D. If, however, they could both be extended one mesh width, then the formulas we would obtain for the discrete Laplacian clearly agree with those we have obtained.)

We note that we can also retain fourth order Taylor series terms in our derivation. For example (1.8) can be replaced by

$$\begin{aligned}\tilde{u}(p_E) - u(p) = {} & (\tilde{u}(p^*) - u(p^*)) + h_2(\tilde{u}_x(p^*) - u_x(p^*)) + \tfrac{1}{2}h_2^2(\tilde{u}_{xx}(p^*) - u_{xx}(p^*)) \\ & + \tfrac{1}{6}h_2^3(\tilde{u}_{xxx}(p^*) - u_{xxx}(p^*)) + hu_x(p) + \tfrac{1}{2}h^2 u_{xx}(p) + \tfrac{1}{3}h^3 u_{xxx}(p) \\ & + O(h^4).\end{aligned} \tag{1.8'}$$

By so doing we instead obtain a fourth order accurate approximation to $h^2 \Delta_h U(p)$ at points of B.

This guarantees the accuracy of the solution we obtain after applying a fast solver.

Let μ^* be the solution of the integral equation obtained by some numerical procedure, and for mesh points $(x_i, y_j) \in B$ define the mesh function m_{ij} to be the value of the discrete Laplacian we get by our procedure using μ^* and its derivatives. Using the function μ^*, we can also compute an approximation v_{ij}^* to $\tilde{u}$ at points of ∂R at the edge of the rectangle.

We define v_{ij} to the solution of the following equations:

$$\begin{aligned}\Delta_h v_{ij} &= \begin{cases} 0, & (x_i, y_j) \in R - B, \\ m_{ij}, & (x_i, y_j) \in B, \end{cases} \\ v_{ij} &= v_{ij}^*, \qquad (x_i, y_j) \in \partial R.\end{aligned}$$

If μ^* and v^* are sufficiently accurate, then v_{ij} will be a second order accurate approximation to U.

THEOREM. *Suppose*

$$\frac{d^i \mu^*(s)}{ds^i} - \frac{d^i \mu(s)}{ds^i} = O(h^{4-i}), \qquad 0 \leqq i \leqq 4,$$

and

$$v_{ij}^* - \tilde{u}(x_i, y_j) = O(h^2), \qquad (x_i, y_j) \in \partial R.$$

Then

$$v_{ij} - U(x_i, y_j) = O(h^2), \qquad (x_i, y_j) \in R.$$

Proof. Since u and $\tilde{u}$ are harmonic, we have

$$\Delta_h v_{ij} - \Delta_h(U)_{ij} = O(h^2), \qquad (x_i, y_j) \in R - B.$$

By the above derivation and our assumptions on μ^*, it follows that terms which are computed using μ^* are second order accurate. Using this and the fact that u and $\tilde{u}$ are harmonic, we have

$$\Delta_h v_{ij} - g_{ij} = O(h^2), \qquad (x_i, y_j) \in B.$$

Now we use the fact that if R is a rectangle with sides a and b, and $\Delta_h W_{ij} = f_{ij}$, then

$$\max_R |W_{ij}| \leqq \tfrac{1}{4}(a^2 + b^2) \max_R |\Delta_h W_{ij}| + \max_{\partial R} |W_{ij}|.$$

If we let $W_{ij} = U(x_i, y_j) - v_{ij}$ the result follows.

Therefore, if we know the solution of the integral equation accurately enough, we can compute a second order accurate solution. Fortunately, however, the integral equation can be solved to almost machine accuracy with the trapezoid rule as the quadrature formula if the boundary data is smooth enough, and by using splines we can obtain accurate values for the derivatives of the density function.

The only place we need compute an integral is at the edge of the grid where we need boundary values for the fast solver.

At this point it is important to note that in practice we have found that it is not necessary to compute a fourth order accurate approximation to $\Delta_h U_{ij}$ at points of B in order to obtain a second order accurate solution. In fact, we have found that there was no noticeable difference in the solution if we only retained the terms involving the first derivative of the density when we computed g. Part of the reason for this is of course that points of B are sparse in R. Let $(G_h)_{ij} = |m_{ij} - (\Delta_h U)_{ij}|$ and

$$\|G_h\| = \sum_{(i,j) \in R} [h^2 |(G_h)_{ij}|]^{1/2}.$$

If we assume that $h^2 \Delta_h U$ is third order accurate on B, then, since there are only $O(1/h)$ points in B it follows that $\|G_h\| = O(h^{3/2})$.

2. Method of computation. There are seven steps.

1) First we embed the irregular region D in a rectangle R. In general we choose the rectangle so no point of the irregular region is closer than three mesh widths from the boundary of the rectangle. Although we can in theory even allow part of the boundary of D to coincide with the boundary of R, this will mean that some other points of the boundary of R will be very close to the boundary of D. At such points it will be more expensive to comput $\tilde{u}$ accurately since the kernal of the integral that must be evaluated will be large.

2) Then we find the irregular mesh points and their distances to the boundary in the x and y directions. The method we have developed to find the irregular mesh points requires $O(n)$ operations where $n = 1/h$.

3) Next we solve the integral equation. When we have smooth data (as we have had in our experiments), we do this by using a Nystrom method. That is, we first replace the integral by a sum at a set of points on the boundary. This gives rise to a dense linear system of equations:

$$\mu(t_i) + \sum w_i K(i, j) \mu(t_j) = 2g(t_i), \qquad i = 1, \cdots, n. \tag{2.1}$$

The points used as nodes for solving the integral equation are completely independent of the mesh points used for the fast solver. In our experiments we chose the points

equally spaced with respect to the parameter used on the boundary, and we used the trapezoid rule as the quadrature formula. We did this because the trapezoid rule is so accurate for computing integrals on periodic regions. We can see why this is so by examining the Euler–Maclaurin summation formula. This fact guarantees the accuracy of the solution of (2.1). Anselone [11, p. 297] has proved a result that shows that if n is sufficiently large, then the accuracy of the solution of a Fredholm integral equation of the second kind with a unique solution is the same as the accuracy of the quadrature formula. Therefore we can use surprisingly few mesh points on the boundary curve. This is very important when we solve the integral equation for the biharmonic equation since the system of equations is larger there.

We note that in cases where the boundary data is not smooth enough, and near portions of the curve with high curvature, using the trapezoid rule with a Nystrom method will not be sufficiently accurate. In such cases a Galerkin method with a basis that contains appropriate singular functions will probably be necessary.

In our experiments so far we have solved the system of equations by using Gaussian elimination. Large systems, however, should probably be solved with iterative methods.

4) Next we interpolate the values of the density with a spline. If we use quintic splines, we have sixth order accurate values for the dipole density at intermediate points. Therefore, if h is the mesh width on the square, and k is the mesh width on ∂D, we only need choose $k^6 = O(h^3)$ or $k = O(h^{1/2})$. It follows that if m is the number of mesh points on the boundary curve, then the number of operations needed to solve the integral equation, that is $m^3/3 + m^2/2$, is $O(n^{3/2})$. However, if we are solving many problems on the same region, then the number of operations needed for each additional calculation is only the number needed for backsolving, which is only $O(n)$. At this point however only programs using cubic splines have been implemented.

5) Next we compute the discrete Laplacian at the irregular mesh points. So far we have only used the approximations such as (1.8) which are derived by using Taylor series. If the nature of a singularity, say near a corner, is known, then it may be possible to use this information to get a more accurate approximation to the Laplacian.

6) After this we compute the values U at the edge of the grid. They are easy to obtain since we know the outside function $\tilde{u}$ as an integral:

$$\tilde{u}(z) = \frac{1}{2\pi} \int_{\partial D} \frac{\partial \log r(s,t)}{\partial n_s} \mu(s)\, ds.$$

Since the points at which we evaluate the integral are not near the irregular boundary and since the region of integration is periodic, we again use the trapezoid rule. Each integral requires $O(n)$ operations, so the total number necessary for this operation is $O(n^2)$.

7) Finally we apply the fast solver. This requires $O(n^2 \log_2 n)$ operations.

In order to solve the inhomogeneous problem

$$\Delta u = g \quad \text{in } D, \qquad u = h \quad \text{on } \partial D,$$

we first solve $\Delta w = g$ in the rectangle R with arbitrary Dirichlet boundary data (e.g. $w = 0$) prescribed on ∂R. This is done with a high order Poisson solver. Our experiments have shown fourth order accuracy to be sufficient. Then a high order interpolation formula is used to obtain values of w on ∂D. Next the homogeneous problem $\Delta v = g$ in D, $v = h - w$ on ∂D is solved. Finally we set $u = w - v$.

3. Extensions.

3.1. Computation of derivatives. One important property of these methods is that we can easily compute the derivatives of a harmonic function without computing

the function itself. For example, we can compute u_x if we know the density accurately enough. This is because u_x is the real part of a Cauchy integral with density μ:

$$u_x(z) = \operatorname{Re} f'(z) = \operatorname{Re} \frac{1}{2\pi i} \int \frac{\mu'(\zeta)}{\zeta - z}\, ds$$

$$= \frac{1}{2\pi} \int_{\partial D} \frac{\mu'(s)}{x(s)^2 + y(s)^2} \left[\frac{\partial \log r(s,t)}{\partial n_s} x'(s) - \frac{\partial \log r(s,t)}{\partial s} y'(s) \right] ds, \qquad z \in D.$$

In the same way as before, this formula defines another harmonic function $\tilde{u}_x$ outside D. The discrete Laplacian of the combined function:

$$U_x = \begin{cases} u_x(x_i, y_j), & (x_i, y_j) \in D, \\ \tilde{u}_x(x_i, y_j), & (x_i, y_j) \notin D \end{cases}$$

is the same as the discrete Laplacian we would obtain if we were computing U, except that where we would use the real part of the mth derivative of μ before, now we use the real part of the $(m+1)$st derivative.

Of course we need to know μ more accurately to get the same accuracy in the solution, and the boundary terms are slightly more expensive to calculate. We compute derivatives in this way because the density function is known so accurately and therefore we still have very accurate values after differentiating it. In cases where this is not so we believe that it is usually preferable to perform the extra work to solve the integral equation more accurately instead of differencing the solution.

3.2. Computation of conjugate functions. We can also compute the conjugate function at small additional cost. That is because by using the Cauchy Riemann equations, we can express the discontinuities in the conjugate function v in terms of the discontinuities in u. For example we note that

$$v_x - \tilde{v}_x = -(u_y - \tilde{u}_y) = \frac{\mu'(s) y'(s)}{x'(s)^2 + y'(s)^2}.$$

We can also use the fact that the conjugate function v is the imaginary part of the same Cauchy integral that determines u to compute the discontinuities in the derivatives of the conjugate function:

$$v(z) = \operatorname{Im} f(z) = \operatorname{Im} \frac{1}{2\pi i} \int_{\partial D} \frac{\mu(\zeta)}{\zeta - z}\, ds = \frac{1}{2\pi} \int_{\partial D} \frac{\partial \log r(s,t)}{\partial s} \mu(s)\, ds. \tag{3.1}$$

In order to find the boundary values, we compute the third integral in (3.1). In this way we can compute the real and imaginary parts of any Cauchy integral. This fact is used when we solve the biharmonic equation. We note that the ability to find the conjugate of a harmonic function allows us to solve Neumann problems. This follows because the integral of Neumann data for u is equal to the Dirichlet boundary data of the conjugate function v. That is, $\int u_n\, ds = -\int v_s\, ds = -v(s)$. So, when we want to solve a Neumann problem we first integrate the data and use it as Dirichlet data for the conjugate function.

3.3. Other extensions. Also, although no experiments have been performed so far, we believe that we can obtain 4th order accurate solutions. If we use a higher order accurate approximation to the Laplacian, there will of course be more points with nonzero Laplacian, and the discrete Laplacian at a given point can depend on values of the dipole density at points which are farther away than before.

In addition, since we can use any integral equation formulation in terms of Cauchy integrals, we can also solve exterior, interface, and modified Dirichlet problems. Work has already been completed on solving interface problems.

We also note that even though the solution to three-dimensional problems cannot be expressed in terms of Cauchy integrals, these techniques are nevertheless applicable. This is because the solution of the Dirichlet problem can still be expressed as the integral of a double layer density function where the density is again the solution of a Fredholm integral equation of the second kind,

$$u=\frac{1}{4\pi}\int_S \mu(s,t)\frac{\partial}{\partial n}\frac{1}{r}\,dS, \quad \text{where } \mu(s,t)+\frac{1}{2\pi}\int_S \mu(s,t)\frac{\partial}{\partial n}\frac{1}{r}\,dS=2g.$$

As in the two-dimensional case, it is possible to compute discontinuities between the interior and exterior potential functions. For example, using the values of $\mu(s,t)$, we can find the discontinuities in the two tangential derivatives. We also know that there is no discontinuity in the normal direction. In the same way as before, we can use these discontinuities to find the discontinuities in the first derivatives in the coordinate directions. By differentiating again and using the properties of the boundary surface, we can obtain discontinuities in higher order derivatives. Once we know the discontinuities, we can use them to approximate the three-dimensional discrete Laplacian.

Finally we note that it is also possible to use the method presented to find the solution of a differential equation when the solution can be expressed as the sum of single and double layer density functions. For example, one can use Green's third identity to solve Laplace's equation.

4. The biharmonic equation. The method we use for solving the biharmonic problem is essentially the same as the method we use for solving Poisson's equation. The reason we can take advantage of this method is that by using the correct integral equation formulation, the problem reduces to finding two harmonic functions and their conjugates. It in fact reduces directly to finding the real and imaginary parts of two Cauchy integrals.

This way of reducing the problem is due to Soviet mathematicians who used it to solve problems in elasticity. It relies heavily on complex variable theory, and consequently is only applicable to two dimensional problems. The idea behind it is the following.

Any two dimensional biharmonic function $W(z)$ can be expressed as

$$W(z)=\operatorname{Re}\{\bar{z}\varphi(z)+\chi(z)\}$$

where φ and χ are analytic functions. See [9]. The functions $\varphi(z)$ and $\psi(z)=\chi'(z)$ are known as the Goursat functions. For a given biharmonic function W the Goursat functions are not uniquely determined. However, all the physically meaningful quantities can be expressed in terms of certain of their derivatives which are uniquely determined. For example, the stress functions satisfy the equations:

$$\sigma_x+\sigma_y=4\operatorname{Re}\{\varphi'(z)\}, \qquad \sigma_x-\sigma_y+2i\tau_{xy}=2[\bar{z}\varphi''(z)+\psi'(z)]$$

and the displacements satisfy the equation:

$$k\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)}=2\mu(u_x+iu_y)$$

where k and μ are elastic constants. What we compute are the Goursat functions; we do not actually find $W(z)$. In order to compute them we use the fact that they can

be expressed as Cauchy integrals:

$$\varphi(z)=\frac{1}{2\pi i}\int_{\partial D}\frac{\omega(t)}{t-z}\,dt, \qquad \psi(z)=\frac{1}{2\pi i}\int_{\partial D}\frac{\omega(t)-t\omega'(t)}{t-z}\,dt.$$

As before we can compute $\varphi(z)$ and $\psi(z)$ if we know the density function ω accurately enough. The integral formulation defining ω depends on the boundary conditions which are given. In the first biharmonic problem, W_x and W_y (or, equivalently W and W_n) are given. We use the following equation due to Sherman and Lauricella [6]:

$$\omega(t)+\frac{1}{\pi}\int\omega(s)\,d\theta-\frac{1}{\pi}\int\omega(s)\,e^{2i\theta}\,d\theta+\frac{b}{\pi i}\operatorname{Re}\int\frac{\omega(s)}{(s-a)^2}\,ds=f(t),$$

where

$$f(t)=W_x+iW_y,$$

$$\theta(s,t)=\arctan\left(\frac{y(s)-y(t)}{x(s)-x(t)}\right),$$

$$b=\frac{1}{t-a}-\frac{1}{(\bar{t}-\bar{a})^2}+\frac{t-a}{(\bar{t}-\bar{a})^2}$$

and a is any point inside the region D. See [9, pp. 255–257].

It is a system of two Fredholm integral equations of the second kind. It is also valid without the last integral if the right-hand side satisfies certain compatibility conditions. ($\operatorname{Re}\int f(t)\,dt=0$.) However, it is retained to keep the problem well conditioned. With it the integral equation can be solved for any right-hand side. We note that the kernel in the first term, that is $d\theta$, is the same as the kernel used for solving Poisson's equation. The second kernel is bounded, and when integrated along the boundary gives 0. Since the point a can be chosen away from the boundary, the third integrand is also bounded and easy to compute.

We now consider the expense of generating the discrete matrix equation. If we have p points on the boundary, we get a $2p\times 2p$ linear system of equations to solve. Each element of the matrix requires $4\frac{1}{4}$ multiplications or divisions to generate. (No trigonometric functions need be evaluated.) Therefore the total number of operations necessary to generate the matrix is $17p^2$. The number is this small because of the symmetry in the kernels and because the kernel of the third integral can be computed just once for a given value of s and the coefficient can be computed once for a given value of t. As in the case of Poisson's equation, the Fredholm integral equations are of the second kind and well conditioned. Consequently the trapezoid rule is highly accurate and we can keep p relatively small.

Once we have the density ω, we evaluate the relevant Cauchy integrals and their derivatives depending on whether we want displacements or stresses.

We note that other biharmonic problems can be formulated in terms of Cauchy integrals and can therefore be solved by these methods. For example it is possible to solve exterior problems and the problem of an elastic plane with an infinite series of regularly spaced identical holes.

5. Results of numerical experiments. In this section we report on results of numerical experiments which were carried out on the CDC 7600 computer at the Lawrence Berkeley Laboratory.

In Table 1 we give results of experiments where we prescribed Dirichlet boundary data and where we computed the harmonic function u, its conjugate harmonic function v, and u_x. The irregular regions were either circles of radius .35 or ellipses with semiaxes .18 and .35. In all cases the region was embedded in a unit square and a Buneman solver was used as the fast solver. The running time noted includes the time needed to set up the problem, to generate the data and to find the irregular mesh points. Here n is the number of mesh points in each direction on the square and p is the number of nodes on the boundary curve.

TABLE 1

n	p	u	Region	Max error in u	Max error in v	Max error in u_x	CPU time in sec.
16	12	x^2-y^2	circle	$.51 \cdot 10^{-2}$	$.32 \cdot 10^{-2}$	$.98 \cdot 10^{-2}$	.022
	16	x^2-y^2	ellipse	$.58 \cdot 10^{-2}$	$.38 \cdot 10^{-2}$	$.76 \cdot 10^{-2}$	.024
	24	$e^x \cos y$	circle	$.72 \cdot 10^{-2}$	$.71 \cdot 10^{-2}$	$.97 \cdot 10^{-2}$	.031
	30	$e^x \cos y$	circle	$.63 \cdot 10^{-2}$	$.53 \cdot 10^{-2}$	$.87 \cdot 10^{-2}$	.038
32	24	x^2-y^2	circle	$.64 \cdot 10^{-4}$	$.93 \cdot 10^{-4}$	$.44 \cdot 10^{-3}$	.079
	24	$e^x \cos y$	ellipse	$.47 \cdot 10^{-3}$	$.45 \cdot 10^{-4}$	$.71 \cdot 10^{-3}$	.079
	50	x^2-y^2	ellipse	$.91 \cdot 10^{-4}$	$.68 \cdot 10^{-4}$	$.84 \cdot 10^{-4}$	.223
64	30	x^2-y^2	ellipse	$.73 \cdot 10^{-5}$	$.72 \cdot 10^{-5}$	$.96 \cdot 10^{-5}$	.095
	30	$e^x \cos y$	ellipse	$.84 \cdot 10^{-5}$	$.44 \cdot 10^{-4}$	$.87 \cdot 10^{-4}$	.095
	50	$e^{7x} \cos 7y$	ellipse	$.81 \cdot 10^{-4}$	$.39 \cdot 10^{-3}$	$.84 \cdot 10^{-3}$	.434

In all cases the solution of the integral equation was interpolated with a cubic spline. Better results could of course have been obtained by using higher order splines.

Our main purpose throughout has been to study the effectiveness of our method of using the fast Poisson solver to compute the potential functions once we have solved the integral equation. Since we knew that the fast Poisson solver is second order accurate, we often therefore worked with $u = x^2 - y^2$, for which there is no discretization error. We note, however, that even though u has no discretization error in this case, $\tilde{u}$ does. This explains why we could not achieve greater accuracy in computing this function.

It appears that we obtained second order accuracy in computing our test functions and their conjugates by taking p equal to $\frac{1}{2}$ the number of interior irregular mesh points. Adding extra points on the boundary curve did not seem to change the accuracy in the examples given. However, if we had used regions which have portions with high curvature, then we could not have solved the integral equations as accurately as we did, and therefore could not have obtained as high accuracy in the interior. For example, we did some experiments on extremely eccentric ellipses. For certain boundary data, in particular $u = x$, the exact solution of the integral equation is known. We note that we could not achieve more than 1 digit of accuracy on such problems by using the trapezoid rule with 10 mesh points as the quadrature formula. By using more accurate quadrature formulas, and in particular by integrating the kernel exactly, we were able to do better.

We also note that we achieved slightly less accuracy in computing derivatives than in computing the harmonic function itself. This is of course due to the fact that values of the derivative of the density are less accurate than values of the density.

In Table 2 we give results of experiments on the biharmonic equation. We computed φ, φ', and ψ. We note that we could get roughly the same degree of accuracy

TABLE 2

n	p	W	Region	Max error in φ	Max error in ψ	Max error in φ'	CPU time in sec.
16	12	x^2+y^2+x	circle	$.58 \cdot 10^{-3}$	$.82 \cdot 10^{-3}$	$.54 \cdot 10^{-2}$	.033
	12	x^3+x^2y+x	ellipse	$.27 \cdot 10^{-2}$	$.78 \cdot 10^{-2}$	$.61 \cdot 10^{-2}$	.035
	40	x^2+y^2+x	circle	$.15 \cdot 10^{-3}$	$.95 \cdot 10^{-3}$	$.82 \cdot 10^{-3}$	.084
32	24	x^3+x^2y+x	ellipse	$.78 \cdot 10^{-3}$	$.87 \cdot 10^{-3}$	$.42 \cdot 10^{-2}$	.275
	40	x^3+x^2y+x	circle	$.63 \cdot 10^{-3}$	$.71 \cdot 10^{-3}$	$.95 \cdot 10^{-3}$	.378
64	50	x^2+y^2+x	circle	$.28 \cdot 10^{-5}$	$.74 \cdot 10^{-4}$	$.92 \cdot 10^{-4}$	.628
	50	x^3+x^2y+x	ellipse	$.38 \cdot 10^{-4}$	$.44 \cdot 10^{-4}$	$.93 \cdot 10^{-4}$	.628
	60	$x^3+x^2y+e^x \cos y$	ellipse	$.79 \cdot 10^{-4}$	$.12 \cdot 10^{-3}$	$.23 \cdot 10^{-3}$	.823

in computing the Goursat functions as we could in computing the harmonic functions we tested when we used the same n and p. In future experiments we plan to use higher order methods, and therefore to be able to compute higher order derivatives of the Goursat functions.

Finally we remark on some experiments which we have begun on regions with corners. If the solution to the problem does not have a singularity, then we can still achieve essentially the same accuracy in the solution as we can in solving the integral equation. If, however, there is a discontinuity in, say, a first derivative of the solution, then first of all our method of solving the integral equation is not adequate. Second, the 5-point discrete Laplacian is not accurate enough. We have begun experimenting with other methods of solving the integral equation. In particular we have been trying Galerkin methods with a basis that contains the appropriate singular functions. Once we know the coefficient of the leading term of the singularity, we use it to get a more accurate approximation to the discrete Laplacian. Preliminary results are encouraging.

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