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A GENERAL COUPLED EQUATION APPROACH FOR SOLVING THE BIHARMONIC BOUNDARY VALUE PROBLEM*

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Abstract. The biharmonic boundary value problem with Dirichlet boundary conditions is reduced to a coupled system of Poisson equations, which depends upon an arbitrary, positive coupling constant c. Since each of the Poisson equations is well-posed, the system may be solved by iteration. We show that the iterates may be represented as a linear combination of the eigenfunctions of the "Dirichlet eigenvalue problem" (a fourth order boundary value problem with the eigenvalue in the boundary condition). Convergence of the iterative scheme occurs when $0 < c < 2\nu_1$, where ν_1 is the smallest eigenvalue. By making use of an averaging scheme, convergence may be produced for any positive c. With the proper choice of c, the rate of convergence may be increased. This coupled equation approach includes the finite difference approach as a special case.

1. Introduction. In two recent papers, Julius Smith [13] has described the coupled equation approach to the biharmonic equation by finite difference methods. In this paper we discuss the coupled equation approach without restriction to finite difference methods.

In the coupled equation approach, the biharmonic problem is reduced to a system of coupled Poisson equations which depend upon an arbitrary coupling constant $c \neq 0$. This system of Poisson equations may be solved by iteration, since neither of the Poisson equations is overprescribed on the boundary.

We shall describe an iteration scheme which converges for all sufficiently small values of the coupling constant c $(0 < c < 2\nu_1)$. Here ν_1 is the smallest eigenvalue of the so-called Dirichlet eigenvalue problem (see § 3). Assuming no error in solving for the iterates, the error term may be written explicitly.

If c is too large, we show that convergence can be produced by a certain averaging procedure. In the finite difference approximations of Smith [13], the coupling constant c is no longer arbitrary. Indeed $c = 2h^{-1}$, where h is the step size. For small values of h, averaging is necessary as shown by Smith.

We examine two kinds of averaging schemes, which may be used simultaneously. Given one of the averaging constants there is an optimal choice for the other. Such an optimal choice of the averaging constants for the finite difference approach has been obtained by Ehrlich [5].

Finally we compare our results with those of Ehrlich and Smith for the special case of solution by finite difference methods.

2. Equivalent formulations of the biharmonic problem. Let G denote a bounded domain in the plane with boundary Γ. We shall denote by Δ the Laplace operator,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{2.1}$$

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The biharmonic problem for the domain G consists of determining a function u which satisfies the partial differential equation

$$\begin{aligned} \Delta\Delta u(p) &= \psi(p), && p \in G, \\ u(p) &= f(p), && p \in \Gamma, \\ \frac{\partial u(p)}{\partial n} &= g(p), && p \in \Gamma. \end{aligned} \tag{2.2}$$

We assume that ψ, f, g are given, sufficiently smooth functions and that the boundary Γ is sufficiently smooth to insure the existence of a solution to (2.2). By $\partial/\partial n$ we denote the derivative in the direction of the exterior normal.

We should like to reduce the problem of solving the biharmonic equation to the perhaps easier one of solving Poisson equations. The obvious formulation of (2.2) as a coupled system of Poisson equations is as follows:

$$\begin{aligned} \Delta u(p) &= \omega(p), && p \in G, \\ u(p) &= f(p), && p \in \Gamma, \\ \frac{\partial u(p)}{\partial n} &= g(p), && p \in \Gamma, \end{aligned} \tag{2.3a}$$

$$\Delta\omega(p) = \psi(p), \qquad p \in G. \tag{2.3b}$$

The equation (2.3a) is overprescribed on Γ, while (2.3b) has no boundary condition at all. Hence given an ω defined in G, (2.3a) cannot in general be solved. There are infinitely many solutions to (2.3b). While (2.3) is an equivalent formulation of (2.2), it definitely does not lead to a well-defined iterative scheme for the solution of (2.2).

Under the assumption that u is a classical solution of the biharmonic problems (i.e., $u \in C^4(G) \cap C^1(\bar{G})$ and u has piecewise continuous second derivatives on Γ), another equivalent formulation of (2.2) is the coupled system

$$\begin{aligned} \Delta u(p) &= \omega(p), && p \in G, \\ u(p) &= f(p), && p \in \Gamma, \end{aligned} \tag{2.4a}$$

$$\begin{aligned} \Delta\omega(p) &= \psi(p), && p \in G, \\ \omega(p) &= \Delta u(p) - c\left[\frac{\partial u(p)}{\partial n} - g(p)\right], && p \in \Gamma, \end{aligned} \tag{2.4b}$$

where c is an arbitrary nonzero constant.

In the formulation (2.4), we have avoided the necessity of overprescribing the function u on the boundary.

THEOREM 1. *Let $u \in C^4(G) \cap C^1(\bar{G})$, and assume that u has piecewise continuous second derivatives on Γ.*

(a) *If u is a solution of* (2.2), *then u is a solution of* (2.4) *for every constant c.*

(b) *If (u, ω) is a solution of* (2.4) *for any $c \neq 0$, then u is a solution of* (2.2). (*Hence by* (a), *u is a solution of* (2.4) *for every c.*)

Proof.

(a) Let u be a solution of (2.2). Then $c(\partial u/\partial n - g) = 0$ almost everywhere on Γ for Γ sufficiently smooth.

Set $\omega \equiv \Delta u$ in $\bar{G}$. Then clearly (2.4) is satisfied and

$$\Delta\omega(p) \equiv \Delta^2 u(p) = \psi(p), \qquad p \in G,$$

and

$$\omega(p) = \Delta u(p) = \Delta u(p) - c\left[\frac{\partial u(p)}{\partial n} - g(p)\right], \qquad p \in \Gamma.$$

(b) Let (u, ω) be a solution of (2.4). Then clearly

$$\Delta\omega(p) = \Delta^2 u(p) = \Psi(p) \qquad p \in G,$$

and

$$u(p) = f(p), \qquad p \in \Gamma.$$

Since $\omega = \Delta u$ almost everywhere on Γ, we have $\partial u/\partial n = g$ almost everywhere on Γ. This completes the proof.

Note that (2.4) has a unique solution only if $c \neq 0$.

Less stringent conditions may be imposed on u. These, as well as the existence of u, depend upon the smoothness of the data and the domain G. (See for example [4, p. 249].)

3. The Dirichlet eigenvalue problem. Here we consider the Dirichlet eigenvalue problem to which we shall frequently refer in later sections. In particular we shall show that its eigenvalues are positive and have no finite accumulation point and that the corresponding eigenfunctions are complete in the class of biharmonic functions which vanish on the boundary and have continuous second derivatives on the closure of the domain G.

The eigenvalue problem is to determine a scalar ν and a function $u \neq 0$ such that the boundary value problem (3.1) is satisfied.

$$\begin{aligned} \Delta^2 u(p) &= 0, & p &\in G, \\ u(p) &= 0, & p &\in \Gamma, \\ \Delta u(p) &= \nu \frac{\partial u(p)}{\partial n}, & p &\in \Gamma. \end{aligned} \tag{3.1}$$

Here $\partial/\partial n$ denotes the exterior normal derivative.

Henceforth we shall assume that the boundary Γ of G is composed of a finite number of differentiable Jordan curves. As usual we shall consider first the boundary value problem of determining a function u which satisfies

$$\begin{aligned} \Delta^2 u(p) &= 0, & p &\in G, \\ u(p) &= 0, & p &\in \Gamma, \\ \Delta u(p) &= g(p), & p &\in \Gamma, \end{aligned} \tag{3.2}$$

where g is a given continuous function on Γ.

Let $K(p, q)$ denote the Green's function for Laplace's equation in the domain G. The solution u of (3.2) may now be constructed from the Green's function $K(p, q)$ (see, e.g., [2, p. 243]).

Clearly,

$$h(q) = -\int_\Gamma g(p) \frac{\partial K(p, q)}{\partial n_p} ds_p \tag{3.3}$$

is a harmonic function in G which is continuous in $\bar{G}$ and which assumes the boundary values $h(q) = g(q)$ for $q \in \Gamma$. Now,

$$u(p) = \iint_G h(q)K(p, q)\, dx\, dy. \tag{3.4}$$

Obviously u is a solution of (3.2) and u is continuously differentiable in the closure of G (see, e.g., [2]). The gradient of u is then well-defined on the closure of the domain. We may now define an operator A as follows:

$$Ag(p) = \operatorname{grad} u(p) \cdot \mathbf{n}, \qquad p \in \Gamma. \tag{3.5}$$

Clearly A is defined on Γ except possibly at a finite number of points. Setting

$$f(p) \equiv Ag(p), \qquad p \in \Gamma, \tag{3.6}$$

f is at least a piecewise continuous function on Γ.

We now define the Hilbert space $L_2(\Gamma)$ with inner product

$$(g_1, g_2) = \int_\Gamma g_1(q)\overline{g_2(q)}\, ds$$

and norm

$$\|g\| = (g, g)^{1/2}.$$

With every continuous function g on Γ, we may associate a unique solution u of problem (3.2) (see construction (3.4)). We define

$$Bg(p) \equiv u(p), \qquad p \in \bar{G},$$

where $u(p)$ is the function defined by (3.3) and (3.4). Notice that

$$\Delta Bg(p) = \Delta u(p) = h(p), \qquad p \in \bar{G}.$$

Hence $\Delta Bg(p) = g(p)$, $p \in \Gamma$.

Let V denote the class of solutions to the problem (3.2). An inner product over V may be taken to be

$$[u, v] = \iint_G \Delta u(q)\overline{\Delta v(q)}\, dx\, dy, \qquad u, v \in V;$$

and the norm is then

$$\langle\!\langle u \rangle\!\rangle = [u, u]^{1/2}, \qquad u \in V.$$

LEMMA 1. *Let $g_1, g_2 \in D(A) = C(\Gamma)$. Then*

$$(Ag_1, g_2) = [Bg_1, Bg_2].$$

Proof. Clearly,

$$(Ag_1, g_2) = \int_\Gamma \frac{\partial Bg_1(q)}{\partial n} \cdot \overline{\Delta Bg_2(q)}\, ds.$$

From Green's identity and the fact that $Bg_i(q) = 0$ for $q \in \Gamma$ ($i = 1, 2$), we have

$$\begin{aligned}(Ag_1, g_2) &= \iint_G \Delta Bg_1(q)\overline{\Delta Bg_2(q)}\, dx\, dy \\ &= [Bg_1, Bg_2],\end{aligned}$$

which was to be proven.

LEMMA 2. *A is symmetric.*

Proof. Let $g_1, g_2 \in D(A)$. Then from Lemma 1, we have

$$\begin{aligned}(Ag_1, g_2) &= [Bg_1, Bg_2] = \overline{[Bg_2, Bg_1]} \\ &= \overline{(Ag_2, g_1)} = (g_1, Ag_2).\end{aligned}$$

LEMMA 3. *A is positive.*

Proof. $(Ag, g) = [Bg, Bg] = \langle\langle Bg \rangle\rangle^2 \geqq 0$.

LEMMA 4. *A is bounded.*

Proof. This follows easily from the inequalities (6.1) and (6.2), which we shall state below and discuss further in § 6.

Let $u \in C^2(G)$. If $u(q) = 0$, $q \in \Gamma$ and grad $u \in L_2(\Gamma)$, then there exists a positive constant α such that

$$\int_\Gamma \left| \frac{\partial u(q)}{\partial n} \right|^2 ds \leqq \alpha^2 \iint_G |\Delta u(q)|^2\, dx\, dy. \tag{3.7}$$

Hence,

$$\|Ag\| \leqq \alpha \langle\langle Bg \rangle\rangle \quad \text{for every } g \in D(A). \tag{3.8}$$

For every function v which is harmonic in G and $v \in L_2(\Gamma)$, there is a positive constant β, independent of v, such that

$$\iint_G |v(q)|^2\, dx\, dy \leqq \beta^2 \int_\Gamma |v(q)|^2\, ds.$$

Hence,

$$\langle\langle Bg \rangle\rangle \leqq \beta \|g\| \quad \text{for every } g \in D(A). \tag{3.9}$$

Then clearly from (3.8) and (3.9),

$$\|Ag\| \leqq \alpha\beta \|g\|,$$

which was to be shown. This completes the proof.

Since the domain of A, $D(A) = C(\Gamma)$, is dense in $L_2(\Gamma)$ and A is bounded, A can be extended to the entire Hilbert space $L_2(\Gamma)$ by the theorem of Hahn–Banach. Let us call this extension A'.

Since A' is symmetric, bounded and defined on the whole of $L_2(\Gamma)$, we have the following.

LEMMA 5. *A' is self-adjoint.*

Now we have only to show the following.

THEOREM 2. *A' is compact.*

Proof. We shall show that the image under the mapping A' of every uniformly bounded sequence contains a Cauchy subsequence.

Let $S = \{g_i\}_{i \in I}$ be any sequence of functions in $L_2(\Gamma)$ such that

$$\|g_i\| \leqq c \quad \text{for every } i \in I.$$

Since $C(\Gamma)$ is dense in $L_2(\Gamma)$, we may assume without loss of generality that $g_i \in C(\Gamma)$.

For each $g_i \in S$ there is a harmonic function $h_i \equiv \Delta B g_i \in C^2(G) \cap C(\bar{G})$ such that $h_i(p) = g_i(p)$, $p \in \Gamma$. From inequality (3.9),

$$\iint_G |h_i(q)|^2 \, dx \, dy = \langle\langle B g_i \rangle\rangle^2 \leqq \beta^2 \|g_i\|^2 \leqq c^2 \beta^2.$$

But $\{h_i\} = \{\Delta B g_i\}$ is a uniformly bounded sequence of harmonic functions. Hence a Cauchy sequence may be extracted, say $\{\Delta B g_{i_n}\}$. (See [4, p. 275].) Since $g_{i_n} \in D(A)$, we have

$$\|A' g_{i_n} - A' g_{i_m}\| = \|A g_{i_n} - A g_{i_m}\| \leqq \alpha \langle\langle B g_{i_n} - B g_{i_m} \rangle\rangle.$$

Hence, $f_{i_n} = A' g_{i_n}$ is a Cauchy subsequence of the image $\{f_i\}_{i \in I}$ of S.

We now state the usual theorem concerning compact, self-adjoint operators. A proof may be found in [1, p. 132].

THEOREM 3. *Let A' be a compact, self-adjoint operator. Then A' has a finite or infinite sequence of pairwise orthonormal eigenfunctions $g_1, g_2, \cdots$, which correspond to the eigenvalues $\mu_1, \mu_2, \cdots$ ($|\mu_1| \geqq |\mu_2| \geqq \cdots > 0$). The sequence of eigenfunctions has the property that every function f of the form $A'g = f$, $g \in D(A')$, satisfies*

$$(3.10) \qquad \begin{aligned} f &= \sum_{m=1}^{\infty} (f, g_m) g_m, \\ \|f\|^2 &= \sum_{m=1}^{\infty} |(f, g_m)|^2. \end{aligned}$$

From the fact that A' is a positive operator, we know that $\mu_1 \geqq \mu_2 \geqq \cdots > 0$.

LEMMA 6. *Let μ_m be an eigenvalue of A', and let g_m be an eigenfunction corresponding to μ_m. Then $D_m = \mu_m^{-1/2} B g_m$ is an eigenfunction of the Dirichlet eigenvalue problem, and $\nu_m = \mu_m^{-1}$ is the eigenvalue to which it corresponds.*

Proof. Since Bg_m is a solution of (3.2), we have

$$\Delta^2 D_m(p) = \mu_m^{-1/2} \Delta^2 B g_m(p) = 0, \qquad p \in G,$$

$$D_m(p) = \mu_m^{-1/2} B g_m(p) = 0, \qquad p \in \Gamma.$$

Now,

$$\begin{aligned} \Delta D_m(p) &= \mu_m^{-1/2} \Delta B g_m(p) = \mu_m^{-1/2} g_m(p) \\ &= \mu_m^{-3/2} A' g_m(p) = \mu_m^{-3/2} \frac{\partial B g_m(p)}{\partial n} = \mu_m^{-1} \frac{\partial D_m(p)}{\partial n}, \qquad p \in \Gamma. \end{aligned}$$

Hence,

$$\Delta D_m(p) = \mu_m^{-1} \frac{\partial D_m(p)}{\partial n}, \qquad p \in \Gamma.$$

THEOREM 4. *Let $\varphi \in V$, that is let φ be a solution of* (3.2) *such that g is continuous on Γ. Then*

$$(3.11) \qquad \varphi = \sum_{m=1}^{\infty} \frac{(A'g, g_m)}{\sqrt{\mu_m}} D_m, \qquad \langle\!\langle \varphi \rangle\!\rangle^2 = \sum_{m=1}^{\infty} \frac{|(A'g, g_m)|^2}{\mu_m}.$$

Proof. From a simple calculation we have

$$\left\langle\!\!\left\langle \varphi - \sum_{m=1}^{N} \frac{(A'g, g_m)}{\sqrt{\mu_m}} D_m \right\rangle\!\!\right\rangle^2 = \left\langle\!\!\left\langle Bg - \sum_{m=1}^{N} \frac{(A'g, g_m)}{\mu_m} Bg_m \right\rangle\!\!\right\rangle^2$$
$$= \left(A'g - \sum_{m=1}^{N} (A'g, g_m) g_m, g - \sum_{m=1}^{N} \frac{(A'g, g_m)}{\mu_m} g_m \right)$$
$$\leqq \left\| A'g - \sum_{m=1}^{N} (A'g, g_m) g_m \right\| \cdot \left\| g - \sum_{m=1}^{N} \frac{(A'g, g_m)}{\mu_m} g_m \right\|.$$

The factor on the right is bounded while the factor on the left approaches zero as $N \to \infty$ from Theorem 3. Hence (3.11) follows. This completes the proof.

Hence every function in V may be represented as a linear combination of the orthonormal eigenfunctions D_m of the Dirichlet eigenvalue problem.

As has been pointed out by the editors, Theorem 4 (the main result of this section) is contained in a more general development by Ercolano and Schechter [6].

4. An iterative scheme. In this section we present an iterative scheme for the coupled system. This scheme converges for a proper choice of the coupling constant c and an essentially arbitrary starting function $\omega^{(0)}$.

Let $\omega^{(0)}$ be given such that

$$\Delta\omega^{(0)}(p) = \psi(p), \qquad p \in G.$$

Then the sequences $\{u^{(k)}\}$ and $\{\omega^{(k)}\}$ are defined by

$$(4.1a) \qquad \Delta u^{(k)}(p) = \omega^{(k-1)}(p), \quad p \in G, \qquad u^{(k)}(p) = f(p), \quad p \in \Gamma,$$

$$(4.1b) \qquad \Delta\omega^{(k)}(p) = \psi(p), \quad p \in G, \qquad \omega^{(k)}(p) = \Delta u^{(k)} - c\left[\frac{\partial u^{(k)}(p)}{\partial n} - g(p)\right], \quad p \in \Gamma,$$

$$k = 1, 2, \cdots.$$

Our goal is to give conditions on c for which (4.1) converges.

THEOREM 5. *The iterative scheme* (4.1) *converges for arbitrary $\omega^{(0)}$ such that $\Delta\omega^{(0)} = \psi$ if and only if $0 < c < 2\nu_1$, where ν_1 is the smallest eigenvalue of the Dirichlet eigenvalue problem.*

Proof. Let B_k denote the biharmonic function $B_k(p) \equiv u^{(k)}(p) - u(p), p \in G$. From (3.11) it is clear that B_k may be represented by

$$B_k = \sum_{m=1}^{\infty} \alpha_m^{(k)} D_m. \tag{4.2}$$

Set $H_k(p) \equiv \omega^{(k)}(p) - \omega(p), p \in G$. Then H_k is harmonic in G. For a sufficiently smooth boundary Γ,

$$\omega^{(k)}(p) = \Delta u(p) + \Delta B_k(p) - c\frac{\partial B_k(p)}{\partial n}, \qquad p \in \Gamma. \tag{4.3}$$

Since

$$\Delta u = \Delta u^{(k)} - \Delta B_k = \omega^{(k-1)} - H_{k-1} = \omega,$$

we have

$$\Delta B_k = H_{k-1} \quad \text{for every } k.$$

Now,

$$\begin{aligned} \Delta^2 B_{k+1}(p) &= 0, & p \in G,\\ B_{k+1}(p) &= 0, & p \in \Gamma,\\ \Delta B_{k+1}(p) &= H_k(p) = \Delta B_k(p) - c\frac{\partial B_k(p)}{\partial n}, & p \in \Gamma. \end{aligned} \tag{4.4}$$

But on Γ,

$$\sum_{m=1}^{\infty} \alpha_m^{(k+1)} \Delta D_m = \sum_{m=1}^{\infty} \alpha_m^{(k)} \left(1 - \frac{c}{v_m}\right) \Delta D_m.$$

Using induction it follows immediately (taking $\alpha_m \equiv \alpha_m^{(1)}$) that

$$\alpha_m^{(k+1)} = (1 - cv_m^{-1})^k \alpha_m \tag{4.5}$$

and

$$B_{k+1} = \sum_{m=1}^{\infty} (1 - cv_m^{-1})^k \alpha_m D_m. \tag{4.6}$$

Then

$$\langle\!\langle B_{k+1} \rangle\!\rangle^2 = \sum_{m=1}^{\infty} |1 - cv_m^{-1}|^{2k} |\alpha_m|^2. \tag{4.7}$$

Hence, $\langle\!\langle B_{k+1} \rangle\!\rangle \to 0$ as $k \to \infty$ if and only if $|1 - cv_m^{-1}| < 1$ for each m.
Since $0 < v_1 \leqq v_2 \leqq \cdots$, this is true only for

$$0 < c < 2v_1,$$

which was to be proven. This completes the proof.

Using continuous methods to solve the system (4.1), the choice of c is, subject to the conditions of Theorem 5, arbitrary. For certain finite difference approximations, c is fixed and inversely proportional to the step size. This we shall discuss in § 7.

Notice that although there is convergence for $0 < c < 2\nu_1$, there is no finite rate of convergence, since

$$|1 - c\nu_m^{-1}| \to 1 \quad \text{as } m \to \infty.$$

This motivates the next section.

5. Averaging to produce a convergent scheme. In the last paragraph, we saw that for certain values of the constant c the iterative scheme is divergent. In this section we shall describe an averaging scheme which produces convergence. As a matter of fact, we shall describe two such schemes. When used together one may be used to speed-up convergence of the other.

Let ε and δ be two real constants, and let $u^{(0)}$ and $\omega^{(0)}$ be given such that $\Delta^2 u^{(0)} = \Delta\omega^{(0)} = \psi$ in G and $u^{(0)} = f$ on Γ. For $k = 1, 2, 3, \cdots,$

$$\begin{aligned} &\Delta\bar{u}^{(k)}(p) = \omega^{(k-1)}(p), && p \in G, \\ (5.1a)\qquad &\bar{u}^{(k)}(p) = f(p), && p \in \Gamma, \\ &u^{(k)}(p) = \varepsilon u^{(k-1)}(p) + (1-\varepsilon)\bar{u}^{(k)}(p), && p \in G, \end{aligned}$$

$$\begin{aligned} &\Delta\bar{\omega}^{(k)}(p) = \psi(p), && p \in G, \\ (5.1b)\qquad &\bar{\omega}^{(k)}(p) = \Delta u^{(k)}(p) - c\left[\frac{\partial u^{(k)}(p)}{\partial n} - g(p)\right], && p \in \Gamma, \\ &\omega^{(k)}(p) = \delta\omega^{(k-1)}(p) + (1-\delta)\bar{\omega}^{(k)}(p), && p \in G. \end{aligned}$$

When $\varepsilon = 0$, $\delta \neq 0$, we shall call the scheme (5.1) an iteration of the first kind. Similarly for $\varepsilon \neq 0$, $\delta = 0$, we have an iteration of the second kind. We shall show that for properly chosen ε and δ the scheme (5.1) converges. Furthermore, given a δ for which an iteration of the first kind converges, there is an $\varepsilon \neq 0$ which speeds-up the convergence in a sense to be defined.

LEMMA 7. *Let $u^{(0)}$ and $\omega^{(0)}$ be given such that*

$$(5.2)\qquad \begin{aligned} \Delta^2 u^{(0)} &= \Delta\omega^{(0)} = \psi \quad \text{in } G, \\ u^{(0)} &= f \quad \text{on } \Gamma. \end{aligned}$$

Assume that the functions $B_0 := u^{(0)} - u$ and $H_0 := \omega^{(0)} - \omega$ may be represented by

$$(5.3)\qquad \begin{aligned} B_0 &= \sum_{m=1}^{\infty} \alpha_m^{(0)} D_m, \\ H_0 &= \sum_{m=1}^{\infty} \beta_m^{(0)} \Delta D_m, \end{aligned}$$

where the D_m's are the orthonormalized eigenfunctions of the Dirichlet eigenvalue problem. Then

(5.4)
$$B_k \equiv u^{(k)} - u = \sum_{m=1}^{\infty} \alpha_m^{(k)} D_m,$$
$$H_k \equiv \omega^{(k)} - \omega = \sum_{m=1}^{\infty} \beta_m^{(k)} \Delta D_m,$$

where

(5.5)
$$\alpha_m^{(k)} = \varepsilon \alpha_m^{(k-1)} + (1 - \varepsilon)\beta_m^{(k-1)},$$
$$\beta_m^{(k)} = \delta \beta_m^{(k-1)} + (1 - \delta)(1 - c\nu_m^{-1})\alpha_m^{(k)}.$$

Proof. Assume that (5.4) and (5.5) hold for $k = r$. Since

$$\Delta \bar{u}^{(r+1)} = \omega^{(r)} \quad \text{in } G,$$

we have

$$\Delta \bar{B}_{r+1} = \Delta(\bar{u}^{(r+1)} - u) = \omega^{(r)} - \omega = H_r.$$

Hence,

(5.6)
$$\bar{B}_{r+1} = \sum_{m=1}^{\infty} \beta_m^{(r)} D_m.$$

Now,

$$B_{r+1} = \varepsilon B_r + (1 - \varepsilon)\bar{B}_{r+1} = \sum_{m=1}^{\infty} [\varepsilon \alpha_m^{(r)} + (1 - \varepsilon)\beta_m^{(r)}] D_m.$$

It then follows that

$$\alpha_m^{(r+1)} = \varepsilon \alpha_m^{(r)} + (1 - \varepsilon)\beta_m^{(r)}.$$

On Γ we have

$$\bar{H}_{r+1} = \Delta B_{r+1} - c\frac{\partial B_{r+1}}{\partial n} = \sum_{m=1}^{\infty} [\alpha_m^{(r+1)} - c\nu_m^{-1}\alpha_m^{(r+1)}] \Delta D_m,$$

since $\partial D_m/\partial n = \nu_m^{-1} \Delta D_m$ on Γ. Then

$$H_{r+1} = \delta H_r + (1 - \delta)\bar{H}_{r+1} = \sum_{m=1}^{\infty} [\delta \beta_m^{(r)} + (1 - \delta)(1 - c\nu_m^{-1})\alpha_m^{(r+1)}] \Delta D_m.$$

Hence,

$$\beta_m^{(r+1)} = \delta \beta_m^{(r)} + (1 - \delta)(1 - c\nu_m^{-1})\alpha_m^{(r+1)}.$$

Clearly (5.4) and (5.5) hold for $r = 1$ if B_0 and H_0 may be represented as in (5.3). Hence it follows that (5.4) and (5.5) hold for all k. This completes the proof.

One should note that conditions (5.2) are not essential; without them, however, the proofs would be similar but much more cumbersome.

The following consequences of Lemma 7 may be noted.

COROLLARY 1. *If* $\varepsilon = 0, \delta \neq 0$, *then*

$$\alpha_m^{(k+1)} = \beta_m^{(k)} = [\delta + (1 - \delta)(1 - cv_m^{-1})]^k \beta_m^{(0)}. \tag{5.7}$$

COROLLARY 2. *If* $\varepsilon \neq 0, \delta = 0$, *then*

$$\begin{aligned} \alpha_m^{(k+1)} &= [\varepsilon + (1 - \varepsilon)(1 - cv_m^{-1})]^k(\varepsilon\alpha_m^{(0)} + (1 - \varepsilon)\beta_m^{(0)}), \\ \beta_m^{(k+1)} &= (1 - cv_m^{-1})\alpha_m^{(k)}. \end{aligned} \tag{5.8}$$

In the remainder of this paragraph, we shall use the following notation:

(i) $A(\varepsilon, \delta, \eta) = \varepsilon + \delta + (1 - \varepsilon)(1 - \delta)\eta$.

(ii) $z_1 = z_1(\varepsilon, \delta, \eta)$, $z_2 = z_2(\varepsilon, \delta, \eta)$ are zeros of the polynomial

$$p(\lambda) = \lambda^2 - A(\varepsilon, \delta, \eta)\lambda + \varepsilon\delta.$$

(iii) $R(\varepsilon, \delta, \eta) = \max\{|z_1|, |z_2|\}$.

(iv) $\eta_m = 1 - cv_m^{-1}$.

(v) If $f = f(\eta)$, then we write

$$f_m \equiv f(\eta_m).$$

LEMMA 8. *Let* $\varepsilon \cdot \delta \neq 0$. *Then the constants* $\alpha_m^{(k)}$ *and* $\beta_m^{(k)}$ *may be represented by*

$$\alpha_m^{(k)} = \begin{cases} c_{1m}z_{1m}^k + c_{2m}z_{2m}^k & \text{if } A_m^2 \neq 4\varepsilon\delta, \\ c_{1m}A_m^k + kc_{2m}A_m^{k-1}, & \text{otherwise}, \end{cases} \tag{5.9}$$

$$\beta_m^{(k)} = \begin{cases} d_{1m}z_{1m}^k + d_{2m}z_{2m}^k & \text{if } A_m^2 \neq 4\varepsilon\delta, \\ d_{1m}A_m^k + kd_{2m}A_m^{k-1}, & \text{otherwise}. \end{cases} \tag{5.10}$$

The constants $c_{1m}, c_{2m}, d_{1m}, d_{2m}$ are determined from $\alpha_m^{(0)}$ and $\beta_m^{(0)}$, which are given, and from $\alpha_m^{(1)}$ and $\beta_m^{(1)}$, which may be obtained from (5.5).

Proof. The proof follows from the fact that both $\alpha_m^{(k)}$ and $\beta_m^{(k)}$ satisfy a difference equation of the form

$$\lambda_k - A_m(\varepsilon, \delta)\lambda_{k-1} + \varepsilon\delta\lambda_{k-2} = 0, \tag{5.11}$$

for $k = 2, 3, \cdots$. Hence for each k there are two linearly independent solutions which are represented by

$$z_{1m}^k \quad \text{and} \quad z_{2m}^k \quad \text{if } A_m^2 \neq 4\varepsilon\delta$$

or

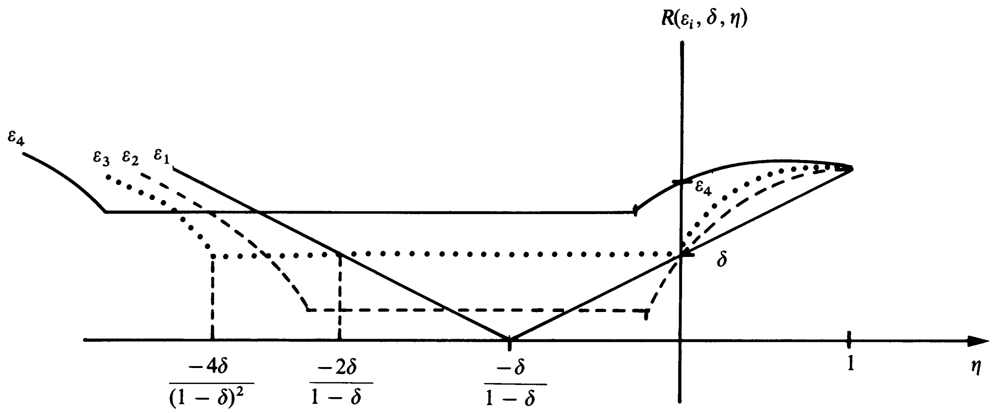
$$A_m^k \quad \text{and} \quad kA_m^{k-1} \quad \text{if } A_m^2 = 4\varepsilon\delta.$$

The results (5.9) and (5.10) follow immediately. This completes the proof.

Before stating and proving our main theorems, we shall investigate the behavior of the function

$$R(\varepsilon, \delta, \eta) = \max\{|z_1|, |z_2|\}.$$

In Fig. 1, δ is considered to be fixed, and curves $R(\varepsilon_i, \delta, \eta)$, $0 = \varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon_4 < 1$, are sketched. The curves are not sketched to scale but show the relationship of one curve to the other for different ε's.



$0 = \varepsilon_1 < \varepsilon_2 < \varepsilon_3 = \delta < \varepsilon_4 < 1$

FIG. 1

The sketch results immediately from the elementary calculus. For a fixed ε we have the following for $0 \leqq \varepsilon, \delta < 1$.

LEMMA 9. $R(\varepsilon, \delta, 1) = 1$.

LEMMA 10. $R(\varepsilon, \delta, 0) = \max(\varepsilon, \delta)$.

LEMMA 11.

(i) *If* $-(\sqrt{\varepsilon} - \sqrt{\delta})^2/((1 - \varepsilon)(1 - \delta)) < \eta \leqq 1$, *then* $R(\varepsilon, \delta, \eta)$ *is monotonically increasing and concave downward* (*i.e.,* $\partial^2 R/\partial \eta^2 \geqq 0$).

(ii) *If* $\eta < -(\sqrt{\varepsilon} + \sqrt{\delta})^2/((1 - \varepsilon)(1 - \delta))$, *then* $R(\varepsilon, \delta, \eta)$ *is monotonically decreasing and concave downward.*

LEMMA 12. *If* $-(\sqrt{\varepsilon} + \sqrt{\delta})^2/((1 - \varepsilon)(1 - \delta)) \leqq \eta \leqq -(\sqrt{\varepsilon} - \sqrt{\delta})^2/((1 - \varepsilon)(1 - \delta))$, *then* $R(\varepsilon, \delta, \eta) = \sqrt{\varepsilon\delta}$.

For the behavior of $R(\varepsilon, \delta, \eta)$ for different ε, we have the following.

LEMMA 13. *If* $0 < \eta < 1$, *then*

$$R(\varepsilon_1, \delta, \eta) < R(\varepsilon_2, \delta, \eta)$$

for every $\varepsilon_1, \varepsilon_2$ *such that* $0 \leqq \varepsilon_1 < \varepsilon_2 < 1$.

LEMMA 14. *Let* $-(\sqrt{\varepsilon_1} - \sqrt{\delta})^2/((1 - \varepsilon_1)(1 - \delta)) \leqq \eta < 0$.

(i) *If* $0 \leqq \varepsilon_1 < \delta < 1$, *then* $R(\varepsilon_1, \delta, \eta) < R(\varepsilon, \delta, \eta)$ *for every* ε *such that* $0 \leqq \varepsilon < \varepsilon_1$.

(ii) *If* $0 \leqq \delta \leqq \varepsilon_1$, *then* $R(\varepsilon, \delta, \eta) < R(\varepsilon_1, \delta, \eta)$ *for every* ε *such that* $0 \leqq \varepsilon < \varepsilon_1$.

LEMMA 15. *If* $\eta \leqq -(\sqrt{\varepsilon_1} + \sqrt{\delta})^2/((1 - \varepsilon_1)(1 - \delta))$, *then* $R(\varepsilon_1, \delta, \eta) < R(\varepsilon, \delta, \eta)$ *for every* ε *such that* $0 \leqq \varepsilon < \varepsilon_1$.

THEOREM 6. *Let* $u^{(0)}$ *and* $\omega^{(0)}$ *be given such that they satisfy conditions* (5.2) *and* (5.3). *Then the iteration scheme* (5.1) *converges if and only if*

$$R_m(\varepsilon, \delta) \equiv R(\varepsilon, \delta, \eta_m) < 1, \qquad m = 1, 2, \cdots. \tag{5.12}$$

Furthermore, the inequality (5.12) *is satisfied if* $0 \leqq \varepsilon, \delta < 1$, *and*

$$\frac{(1+\delta)+(1-\delta)\eta_m}{-(1+\delta)+(1-\delta)\eta_m} < \varepsilon < 1, \qquad m = 1, 2, \cdots. \tag{5.13}$$

Proof. We shall denote by $\langle\!\langle \cdot \rangle\!\rangle$ the norm over the Hilbert space spanned by the Dirichlet eigenfunctions. Since $u^{(0)}$ and $\omega^{(0)}$ satisfy conditions (5.2) and (5.3),

$$\langle\!\langle u^{(k)} - u \rangle\!\rangle^2 = \langle\!\langle B_k \rangle\!\rangle^2 = \sum_{m=1}^{\infty} |\alpha_m^{(k)}|^2. \tag{5.14}$$

We define

$$M = \{m | A_m^2(\varepsilon, \delta) = 4\varepsilon\delta\}.$$

Notice that M has at most two elements. Then from Lemma 8,

$$\langle\!\langle B_k \rangle\!\rangle^2 = \sum_{\substack{m=1 \\ m \notin M}}^{\infty} |c_{1m} z_{1m}^k + c_{2m} z_{2m}^k|^2 + \sum_{m \in M} |c_{1m} A_m^k + k c_{2m} A_m^{k-1}|^2. \tag{5.15}$$

Now $R_m(\varepsilon, \delta) = \max\{|z_{1m}|, |z_{2m}|\}$.

(i) Let us assume that $R_m(\varepsilon, \delta) < 1$. If $m \notin M$,

$$c_{1m} = \frac{\alpha_m^{(0)} z_{2m} - \alpha_m^{(1)}}{z_{2m} - z_{1m}}, \qquad c_{2m} = \frac{\alpha_m^{(1)} - \alpha_m^{(0)} z_{1m}}{z_{2m} - z_{1m}};$$

and if $m \in M$,

$$c_{1m} = \alpha_m^{(0)}, \qquad c_{2m} = \alpha_m^{(1)} - \alpha_m^{(0)} A_m.$$

Take $q := \min_{m \notin M} |z_{2m} - z_{1m}| \neq 0$ and

$$C_m = \max\{|\alpha_m^{(0)}|, |\beta_m^{(0)}|\} \geqq |\alpha_m^{(1)}|.$$

Then

$$|c_{1m}| \leqq \frac{2}{q} C_m, \quad |c_{2m}| \leqq \frac{2}{q} C_m, \qquad m \notin M,$$

and

$$|c_{1m}| \leqq C_m, \quad |c_{2m}| \leqq 2C_m, \qquad m \notin M.$$

It now follows that

$$\begin{aligned} \langle\!\langle B_k \rangle\!\rangle^2 &\leqq \frac{16}{q^2} \sum_{\substack{m=1 \\ m \notin M}}^{\infty} C_m^2 R_m^{2k} + \sum_{m \in M} (1+2k)^2 C_m^2 R_m^{2k-2} \\ &\leqq \frac{16}{q^2} \left\{ \sum_{\substack{m=1 \\ m \notin M}}^{N} C_m^2 R_m^{2k} + \sum_{\substack{m=N+1 \\ m \notin M}}^{\infty} C_m^2 \right\} + \sum_{m \in M} (1+2k)^2 C_m^2 R_m^{2k-2}. \end{aligned} \tag{5.16}$$

By Lemma 8, $\sum_{m=1}^{\infty} |\alpha_m^{(0)}|^2$ and $\sum_{m=1}^{\infty} |\beta_m^{(0)}|^2$ converge, hence $\sum_{m=1}^{\infty} C_m^2$ converges. Hence from (5.16),

$$\lim_{k \to \infty} \langle\!\langle B_k \rangle\!\rangle^2 \leqq \frac{16}{q^2} \sum_{m=N+1}^{\infty} C_m^2$$

for every N. It now follows that

$$\lim_{k\to\infty} \langle\langle u^{(k)} - u \rangle\rangle = \lim_{k\to\infty} \langle\langle B_k \rangle\rangle = 0.$$

(ii) To prove the necessity, let us suppose that for some μ, $R_\mu(\varepsilon, \delta) \geqq 1$. To fix ideas let $\mu \notin M$ and $|z_{1\mu}| > |z_{2\mu}|$. Then from (5.15),

$$\langle\langle B_k \rangle\rangle^2 \geqq |c_{1\mu} z_{1\mu}^k + c_{2\mu} z_{2\mu}^k|^2 \geqq |z_{1\mu}|^{2k} \left| |c_{1\mu}| - |c_{2\mu}| \left| \frac{z_{2\mu}}{z_{1\mu}} \right|^k \right|^2$$

$$\geqq \left| |c_{1\mu}| - |c_{2\mu}| \left| \frac{z_{2\mu}}{z_{1\mu}} \right|^k \right|^2 .$$

This implies

$$\lim_{k\to\infty} \langle\langle B_k \rangle\rangle \geqq |c_{1\mu}|^2 .$$

But $c_{1\mu}$ is, subject to the conditions of (5.2), essentially arbitrary. This is a contradiction to the assumption that the iteration scheme converges for any given $u^{(0)}$ and $\omega^{(0)}$ which satisfy conditions (5.2) and (5.3). This ends the proof of (5.12).

(iii) From Lemmas 9, 10 and 11, we have

$$R_m(\varepsilon, \delta) < 1 \quad \text{for } -\frac{(\sqrt{\varepsilon} + \sqrt{\delta})^2}{(1-\varepsilon)(1-\delta)} < \eta_m < 1, \qquad 0 \leqq \varepsilon, \delta < 1.$$

If $\eta_m < -(\sqrt{\varepsilon} + \sqrt{\delta})^2/((1-\varepsilon)(1-\delta))$, then $R_m(\varepsilon, \delta) < 1$ if and only if

$$R_m(\varepsilon, \delta) = \tfrac{1}{2}\{-A_m(\varepsilon, \delta) + \sqrt{A_m^2(\varepsilon, \delta) - 4\varepsilon\delta}\} < 1.$$

Notice that $A_m(\varepsilon, \delta) < 0$ in this region. Hence by a simple calculation,

$$\varepsilon > \frac{(1+\delta) + (1-\delta)\eta_m}{-(1+\delta) + (1-\delta)\eta_m}.$$

This completes the proof.

We are now faced with determining, if possible, an optimal ε, say $\varepsilon_0(\delta)$, for each δ. The rate of convergence of the iterative scheme clearly depends upon $R_m(\varepsilon, \delta) \equiv R(\varepsilon, \delta, \eta_m)$. In fact the convergence factor for the iterative scheme is ρ, where

$$\rho(\varepsilon, \delta) = \sup_m R_m(\varepsilon, \delta).$$

Now if the Dirichlet eigenvalue problem for the domain G has infinitely many eigenvalues, then $\rho(\varepsilon, \delta) \geqq 1$ for every ε. This is evident from Fig. 1, where $\eta_m = 1 - cv_m^{-1}$. In this sense there is no optimal ε.

Let us define $\varepsilon_0(\delta)$ as follows:

$$\varepsilon_0(\delta) = \min_{\varepsilon_1} \{\varepsilon_1 | \hat\rho(\varepsilon_1, \delta) \leqq \hat\rho(\varepsilon, \delta) \text{ for } 0 \leqq \varepsilon, \varepsilon_1 < 1\}, \tag{5.17}$$

where

$$\hat\rho(\varepsilon, \delta) = \sup_{m \ni \eta_m \leqq 0} R_m(\varepsilon, \delta).$$

Such a choice of $\varepsilon_0(\delta)$ is always possible if $\eta_1 \leqq 0$. This, from Fig. 1, indicates a speed-up of the rate of convergence for $\eta_m \leqq 0$ and a slowing of the rate of convergence for $\eta_m > 0$.

THEOREM 7. *Let δ be chosen such that $0 \leqq \delta < 1$. Then the optimal choice of ε, $\varepsilon_0(\delta)$, is*

$$\varepsilon_0(\delta) = \frac{2\delta + (1-\delta)\eta_1}{-2 + (1-\delta)\eta_1} \tag{5.18}$$

and

$$\hat{\rho}(\varepsilon_0(\delta), \delta) = \max\{\delta, \varepsilon_0(\delta)\}. \tag{5.19}$$

Proof. Clearly from Fig. 1, for fixed ε, δ, the function $R(\varepsilon, \delta, \eta)$ decreases for $\eta < (\sqrt{\varepsilon} + \sqrt{\delta})^2/((1-\varepsilon)(1-\delta))$, remains constant $(= \sqrt{\varepsilon\delta})$ for

$$-\frac{(\sqrt{\varepsilon} + \sqrt{\delta})^2}{(1-\varepsilon)(1-\delta)} \leqq \eta \leqq -\frac{(\sqrt{\varepsilon} - \sqrt{\delta})^2}{(1-\varepsilon)(1-\delta)},$$

and increases for

$$-\frac{(\sqrt{\varepsilon} - \sqrt{\delta})^2}{(1-\varepsilon)(1-\delta)} < \eta \leqq 0.$$

(i) If $\eta_1 \geqq -4\delta/(1-\delta)^2$, then

$$R(\varepsilon, \delta, 0) = \delta \quad \text{for } \varepsilon \leqq \delta.$$

There are at most two values of $\varepsilon \leqq \delta$ for which $R(\varepsilon, \delta, \eta_1) = \delta$. One is clearly $\varepsilon = \delta$ (i.e., $A_1^2(\varepsilon, \delta) \leqq 4\varepsilon\delta$). The other is obtained by assuming $A_1^2(\varepsilon, \delta) > 4\varepsilon\delta$. We choose the second as optimal and Lemma 15 shows that this is the smallest ε. That is,

$$\begin{gathered} R(\varepsilon_0(\delta), \delta, 0) = R(\varepsilon_0(\delta), \delta, \eta_1) = \delta, \\ A_1^2(\varepsilon, \delta) > 4\varepsilon\delta. \end{gathered} \tag{5.20}$$

(ii) If $\eta_1 < -4\delta/(1-\delta)^2$, then form Lemmas 14 and 15, $\hat{\rho}(\varepsilon, \delta) > \delta$. Hence the optimal choice is

$$R(\varepsilon_0(\delta), \delta, 0) = R(\varepsilon_0(\delta), \delta, \eta_1) = \varepsilon_0(\delta) > \delta. \tag{5.21}$$

Both (5.20) and (5.21) lead by a simple calculation to the equality of (5.18). Clearly,

$$\hat{\rho}(\varepsilon_0(\delta), \delta) = \begin{cases} \delta & \text{if } \eta_1 \geqq -4\delta/(1-\delta)^2, \\ \varepsilon_0(\delta) & \text{otherwise}. \end{cases}$$

This completes the proof.

Note that $R(\varepsilon, \delta, \eta) \equiv R(\delta, \varepsilon, \eta)$. Hence given an ε, one may choose an optimal δ, $\delta_0(\varepsilon)$, in exactly the same manner. We are still free to choose δ in Theorem 7. Let us choose an optimal δ. That is, setting $\varepsilon_{\text{opt}} = \varepsilon_0(\delta_{\text{opt}})$, δ_{opt} is that δ for which

$$\hat{\rho}(\varepsilon_{\text{opt}}, \delta_{\text{opt}}) = \min_{0 \leqq \delta < 1} \hat{\rho}(\varepsilon_0(\delta), \delta). \tag{5.22}$$

THEOREM 8.

$$\varepsilon_{\text{opt}} = \delta_{\text{opt}}, \tag{5.23}$$

$$\delta_{\text{opt}} = 1 + \frac{-2 + 2\sqrt{1 - \eta_1}}{\eta_1} = \frac{\sqrt{c} - \sqrt{v_1}}{\sqrt{c} + \sqrt{v_1}}, \tag{5.24}$$

or

$$\eta_1 = \frac{-4\delta_{\text{opt}}}{(1 - \delta_{\text{opt}})^2}.$$

The proof follows simply from Theorem 7 and (5.22).

From the above discussion of the averaging scheme, the question arises: "Why average at all if a c can be chosen such that $0 < c < 2v_1$?" This we shall attempt to answer below.

Let $u^{(0)}$ and $\omega^{(0)}$ be given. Then let

$$B_k = \sum_{m=1}^{\infty} \alpha_m^{(k)} D_m, \qquad k = 0, 1, \cdots,$$

denote the biharmonic function obtained from (5.1) with $\varepsilon = \delta = 0$ and $c = c_1$.

Now let

$$\hat{B}_k = \sum_{m=1}^{\infty} \hat{\alpha}_m^{(k)} D_m, \qquad k = 0, 1, \cdots,$$

denote the biharmonic function obtained from (5.1) with

$$\varepsilon = \delta = \frac{\sqrt{\hat{c}} - \sqrt{v_1}}{\sqrt{\hat{c}} + \sqrt{v_1}} \quad \text{and} \quad c = \hat{c}.$$

Set

$$B_k^N = \sum_{m=1}^{N} \alpha_m^{(k)} D_m \quad \text{and} \quad \hat{B}_k^N = \sum_{m=1}^{N} \hat{\alpha}_m^{(k)} D_m.$$

THEOREM 9. *For every N such that*

$$\alpha_N^{(1)} \neq 0, \quad v_N > c_1 \quad \textit{and} \quad v_N < \left(\frac{2v_N - c_1}{c_1}\right)^2 v_1,$$

there exist positive constants σ, L and $\hat{c}$, independent of k, such that for $0 < \sigma < 1$,

$$\frac{\langle\!\langle \hat{B}_k^N \rangle\!\rangle}{\langle\!\langle B_k^N \rangle\!\rangle} < L\sigma^k, \qquad k = 0, 1, 2, \cdots. \tag{5.25}$$

Furthermore,

$$v_N \leqq \hat{c} < \left(\frac{2v_N - c_1}{c_1}\right)^2 v_1.$$

Proof. From the hypotheses on v_N,

$$v_N < \left(\frac{2v_N - c_1}{c_1}\right)^2 v_1.$$

Choose $\hat{c}$ such that

$$\nu_N \leqq \hat{c} < \left(\frac{2\nu_N - c_1}{c_1}\right)^2 \nu_1 .$$

Hence there is a σ, $0 < \sigma < 1$, such that

$$\frac{\sqrt{\hat{c}} - \sqrt{\nu_1}}{\sqrt{\hat{c}} + \sqrt{\nu_1}} = \sigma(1 - c_1\nu_N^{-1}) < (1 - c_1\nu_N^{-1}). \tag{5.26}$$

From (4.5) and (4.6), we have

$$B_k^N = \sum_{n=1}^{N} (1 - c\nu_m^{-1})^{k-1}\alpha_m^{(1)} D_m,$$

hence,

$$\langle\!\langle B_k^N \rangle\!\rangle \geqq |1 - c_1\nu_N^{-1}|^{k-1}|\alpha_N^{(1)}| .$$

Since $\hat{c} > \nu_N$ or $1 - \hat{c}\nu_N^{-1} < 0$ we have from (5.16), (5.22) and (5.24) that there exists an $L_1 > 0$ such that

$$\langle\!\langle \hat{B}_k^N \rangle\!\rangle < L_1\delta^k .$$

Taking $L = (L_1\delta)/(\sigma|\alpha_N^{(1)}|)$, we obtain (5.25). This completes the proof.

Hence for N sufficiently large, the truncated error term $\hat{B}_k^N$ obtained by averaging, converges faster than B_k^N as $k \to \infty$ if the coefficient $\alpha_N^{(1)}$ of the Nth eigenfunction is not zero. If $\langle\!\langle B_0 - B_0^N \rangle\!\rangle$ is "small," then the rate of convergence of $\hat{B}_k$ is (for computational purposes) effectively that of B_k^N, which has a finite rate of convergence.

6. Lower bounds for ν_1. As we have seen in the preceding paragraphs, a method for computing the smallest eigenvalue of the Dirichlet eigenvalue problem must be obtained. If this is not feasible or possible, a lower bound would be sufficient, as is clear from Theorems 6, 7 and 8.

We shall consider certain dual inequalities as defined by Fichera. For this and more general dual problems, see Fichera [7] and Kuttler and Sigillito [11].

LEMMA 16. *Let v be an arbitrary real harmonic function in G such that $v \in L_2(\Gamma)$. Then there exists a positive constant α such that*

$$\left(\iint_G |v|^2 \, d\xi \, d\eta\right)^{1/2} \leqq \alpha\left(\int_\Gamma |v|^2 \, ds\right)^{1/2} . \tag{6.1}$$

The dual inequality to (6.1) is the inequality (6.2) in the lemma below.

LEMMA 17. *Let α be the constant in Lemma 16. Then for every function $u \in C^2(G) \cap C^1(\bar{G})$ such that $u = 0$ on Γ, we have*

$$\left(\int_\Gamma \left|\frac{\partial u}{\partial n}\right|^2 ds\right)^{1/2} \leqq \alpha\left(\iint_G |\Delta u|^2 \, d\xi \, d\eta\right)^{1/2} . \tag{6.2}$$

Now from the duality principle of Fichera [7],

$$\alpha^{-2} \leqq \nu_1 = \inf_{\substack{\Delta v(z)=0\\ z\in G}} \left(\int_\Gamma v^2\, ds\right) \Big/ \left(\iint_G v^2\, d\xi\, d\eta\right) \tag{6.3}$$
$$= \inf_{\substack{u(z)=0\\ z\in\Gamma}} \left(\iint_G (\Delta u)^2\, d\xi\, d\eta\right) \Big/ \left(\int_\Gamma \left(\frac{\partial u}{\partial n}\right)^2 ds\right).$$

It is of course clear that if a function u_1 exists such that the infimum in (6.3) is attained, then u_1 is biharmonic and u_1 is also a solution of (3.1). Moreover, $v = \Delta u_1$.

Clearly if an α is obtained such that α satisfies (6.1), then α^{-2} is a lower bound for the first eigenvalue of the Dirichlet eigenvalue problem.

There have appeared in the literature several results which give lower bounds for ν_1. See for example [3], [5], [9], [10] and [12].

7. The finite difference approach. In this section we show how the coupled system (4.1) for the continuous problem may be approximated by a coupled system of difference equations. In particular, if the domain G is rectangular and if we take $c = 2h^{-1}$, where h is the step size, then the corresponding difference equations are exactly those used by Greenspan and Schultz [8], and Smith [13]. We also show that our choices of optimal ε and δ correspond to those of Smith [13] for an iteration of the first or second kind and to those of Ehrlich [5] for a combined iteration of the first and second kind.

Let us superimpose a square grid over the domain G with mesh size h. Then G_h shall denote those grid points in G, and Γ_h shall denote those grid points on Γ. If $(x, y) \in G_h$, then we define the discrete Laplace operator, Δ_h, by

(7.1)
$$\Delta_h u(x, y) = \frac{u(x+h, y) + u(x-h, y) + u(x, y+h) + u(x, y-h) - 4u(x, y)}{h^2}.$$

Let us assume that G is a rectangle with length Mh and width Nh, where M and N are positive integers. Let C be a set containing only the points at the corners of G. Then the discrete analogue of problem (4.1) is

$$\begin{aligned}
\Delta_h u^{(k)}(p) &= \omega^{(k-1)}(p), && p \in G_h,\\
u^{(k)}(p) &= f(p), && p \in \Gamma_h,\\
\Delta_h \omega^{(k)}(p) &= \psi(p), && p \in G_h,\\
\omega^{(k)}(p) &= h^{-2}[f(p_1) + f(p_2) - 4f(p) + 2u^{(k)}(p^-)] &&\\
&\quad + cg(p) + (2h^{-2} - ch^{-1})(f(p) - u^{(k)}(p^-)), && p \in \Gamma_h - C,
\end{aligned} \tag{7.2}$$

where $p_1, p_2 \in \Gamma_h$, $p^- \in G_h$ and $\overline{pp_1} = \overline{pp_2} = \overline{pp^-} = h$.

Notice that if $c = 2h^{-1}$, we obtain exactly the difference equations in [8] and [13].

Let $\hat{u}$ denote the solution vector for (7.2), and let $\hat{u}^{(k)}$ denote the kth iteration vector. Smith [13] has shown that

$$\hat{u} - \hat{u}^{(k)} = -2L^{-2}M(\hat{u} - \hat{u}^{(k-1)}), \tag{7.3}$$

where L and M are matrices arising from the difference equations (7.2). Furthermore, $-2h^{-2}M$ is negative semidefinite; and the spectral radius is given by

$$\rho(-2L^{-2}M) = 2/h\sigma_h. \tag{7.4}$$

The constant σ_h is a solution of the minimum problem

$$\sigma_h = \min_{\substack{u(p)=0 \\ p\in\Gamma_h}} \frac{h^2 \sum_{p\in G_h} (\Delta_h u(p))^2}{h \sum_{p\in\Gamma_h - C} (\delta_h u(p))^2}, \tag{7.5}$$

where $\delta_h u(p) = -h^{-1}u(p^-)$, $\overline{pp^-} = h$ and $p^- \in G_h$.

In the continuous case with $c = 2h^{-1}$ and $1 - 2(hv_1)^{-1} < -1$ we have

$$\langle\!\langle u - u^{(k+1)} \rangle\!\rangle = \langle\!\langle B_{k+1} \rangle\!\rangle \leqq |1 - 2(hv_1)^{-1}| \langle\!\langle B_k \rangle\!\rangle. \tag{7.6}$$

Hence, we should have (setting $v_1 = \sigma$)

$$1 - 2h^{-1}\sigma^{-1} \cong -2h^{-1}\sigma_h^{-1}, \tag{7.7}$$

or

$$\sigma_h \cong \sigma - \frac{h\sigma^2}{h\sigma - 2}. \tag{7.8}$$

Let us write $\bar{\tau} = (h\sigma_h)^{-1}$ and consider an iteration of the first or second kind. Then

$$\hat{\rho}(\varepsilon_0(0), 0) = \varepsilon_0(0) = \delta_0(0) = \hat{\rho}(0, \delta_0(0)). \tag{7.9}$$

This is clear from the proof of Theorem 7. Choose $\varepsilon = 0$. Then from Theorem 7 and (7.7) we have

$$\delta_0(0) = \frac{\eta_1}{\eta_1 - 2} = \frac{1 - 2/hv_1}{-1 - 2/hv_1} \cong \frac{\bar{\tau}}{1 + \bar{\tau}}. \tag{7.10}$$

Compare (7.10) with the results of Smith [13]. See also Ehrlich [5, Eq. (3.21)]. Also from (7.10),

$$1 - \delta_0(0) \cong 1/(1 + \bar{\tau}). \tag{7.11}$$

See [5, Eq. (3.20)].

In the case of a combined iteration of the first and second kind we have

$$\varepsilon_{\text{opt}} = \delta_{\text{opt}} = \frac{\sqrt{2/h} - \sqrt{v_1}}{\sqrt{2/h} + \sqrt{v_1}} \cong \frac{\sqrt{1 + 2\bar{\tau}} - 1}{\sqrt{1 + 2\bar{\tau}} + 1}. \tag{7.12}$$

Compare [5, Eq. (3.15)]. Further we have

$$1 - \varepsilon_{\text{opt}} = \frac{2\sqrt{v_1}}{\sqrt{2/h} + \sqrt{v_1}} \cong \frac{2}{1 + \sqrt{1 + 2\bar{\tau}}}, \tag{7.13}$$

which also corresponds to [5, Eq. (3.13)].

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