SIAM J. NUMER. ANAL.
Vol. 5, No. 2, June, 1968
Printed in U.S.A.

THE COUPLED EQUATION APPROACH TO THE NUMERICAL SOLUTION OF THE BIHARMONIC EQUATION BY FINITE DIFFERENCES. I*

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Introduction. The boundary problem $\Delta\Delta u = f$ in a rectangle R, where u and the normal derivative, $\partial u/\partial n$, are known on the boundary of R, may be reduced to the study of the system $c\Delta u = v$, $\Delta v = cf$ with the same boundary conditions. A reduction to a system of difference equations by the usual techniques (for details see Theorem 3) leads to a pair of discrete Poisson equations in which, however, v is not known a priori on the boundary. This difficulty has been overcome in various ways by several workers with the aid of schemes involving an "inner" and "outer iteration" (see, e.g., Esch [1], Pearson [6], Peebles [7]).

In §1, we discuss the outer iteration scheme (see Definition 4). This scheme involves solution of two discrete Poisson equations at each step, m. Boundary conditions are given at each stage, but the conditions on v vary with m. If h is the mesh size, then the spectral radius of this outer iteration scheme is shown to be given by $2(h\sigma_h)^{-1}$, where σ_h is the solution of a certain discrete minimum problem which tends to a positive number

$$\sigma = \min \left\{ \frac{\iint (\Delta u)^2 \, dx \, dy}{\int (\partial u/\partial n)^2 \, ds} \right\}$$

as $h \to 0$. This shows that the iteration scheme is divergent. Convergence may be attained, however, with the aid of an averaging step, "relaxation" (see Definition 6). The spectral radius of the averaged outer iteration is found to be $\lesssim 1 - \sigma h$ as $h \to 0$ (see Lemma 4).

In §2 the existence of σ is proved. A careful discussion of the question $\sigma_h \to \sigma$ is given.

In §3, a brief discussion is given of the behavior of σ under symmetrization (see Pólya and Szegö [8]). This yields a practical estimate for the size of σ.

Since each step of the outer iteration involves the solution of Poisson

* Received by the editors November 1, 1966, and in final revised form December 18, 1967.

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difference equations, we are at liberty to solve these equations (approximately) using various direct or iterative techniques. In Part II (to appear), a particular iterative technique which has been used successfully in numerical applications (see Peebles [7]) is studied, and estimates are obtained for its rate of convergence. The general question of the relation between so-called inner and outer iterations has been discussed in Ortega and Rheinboldt [5].

It is to be observed that while an extension of these results to domains with sides parallel to the axes is probably within reach of the present techniques, the inclusion of boundaries whose sides are slanted at an arbitrary angle leads to serious complications in setting up the difference equations.

1. The basic equations and the outer iteration scheme.

DEFINITION 1. $R = \{(x, y): 0 < x < a \text{ and } 0 < y < b\}$, where $a = Jh$, $b = Kh$, $h > 0$, and J and K are positive integers; $P = (x, y)$; B is the boundary of R; $\bar{R} = R \cup B$; $C = (0, 0) \cup (a, 0) \cup (a, b) \cup (0, b)$; $C^m(R)$ = functions m times continuously differentiable in R; $C^m(\bar{R})$ = functions m times continuously differentiable in $\bar{R}$; $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$; $\partial/\partial n$ = derivative in direction of the exterior normal along $B - C$.

THEOREM 1. *Let $f \in C^1(\bar{R})$, $g \in C^2(\bar{R})$. Then there exists a unique function $u \in C^1(\bar{R}) \cap C^4(R)$ such that*

$$\Delta\Delta u(P) = f(P), \qquad P \in R, \tag{1.1}$$

$$u(P) = g(P), \qquad P \in B, \tag{1.2}$$

$$\frac{\partial u}{\partial x}(P) = \frac{\partial g}{\partial x}(P), \qquad \frac{\partial u}{\partial y}(P) = \frac{\partial g}{\partial y}(P), \qquad P \in B. \tag{1.3}$$

Hence if $g_1(P) \equiv \partial g(P)/\partial n$, we have

$$\frac{\partial u}{\partial n}(P) = g_1(P), \qquad P \in B - C. \tag{1.3'}$$

Proof. See Friedrichs [3].

DEFINITION 2. $M_h = \{(x, y): x = jh, y = kh, \text{ where } j \text{ and } k \text{ are integers}\}$; $R_h = R \cap M_h$; $B_h = B \cap M_h$; $\bar{R}_h = \bar{R} \cap M_h$. If $P \in B_h - C$, then P^- is its closest neighbor in R_h and P^+ is the reflection of P^- in the side of B containing P. $R_h^+ = \bar{R}_h \cup \{P^+ : P \in B_h - C\}$. If S is a set then $E(S)$ is the collection of real-valued functions on S. $U \in \mathring{E}(\bar{R}_h)$ if $U \in E(\bar{R}_h)$ and $U(P) = 0$ for $P \in B_h$. If $S \subset M_h$ and $U \in E(S)$, then whenever these expressions are defined,

$$U_x(x, y) = h^{-1}[U(x + h, y) - U(x, y)],$$

$$U_{\bar{x}}(x, y) = h^{-1}[U(x, y) - U(x - h, y)],$$

$$U_y(x, y) = h^{-1}[U(x, y + h) - U(x, y)],$$
$$U_{\bar{y}}(x, y) = h^{-1}[U(x, y) - U(x, y - h)],$$
$$\Delta_h U(x, y) = U_{x\bar{x}}(x, y) + U_{y\bar{y}}(x, y).$$

THEOREM 2. *Let* $F \in E(R_h)$, $G \in E(B_h)$ *and* $G_1 \in E(B_h - C)$. *Then there exists a unique* $U \in E(R_h^+)$ *such that*

$$\Delta_h\Delta_h U(P) = F(P), \qquad P \in R_h, \tag{1.4}$$
$$U(P) = G(P), \qquad P \in B_h, \tag{1.5}$$
$$(2h)^{-1}[U(P^+) - U(P^-)] = G_1(P), \qquad P \in B_h - C. \tag{1.6}$$

Moreover, if u *is the function of Theorem* 1 *and* F, G, G_1 *are the restrictions of* f, g *and* g_1 *to* R_h, B_h *and* $B_h - C$, *respectively, then* $u - U = O(h^{3/2})$ *as* $h \to 0$ *provided that* $u \in C^6(\bar{R})$.

Proof. See Zlámal [11]. Also see Stetter [9].

THEOREM 3. *Let* $F \in E(R_h)$, $G \in E(B_h)$ *and* $G_1 \in E(B_h - C)$. *Then, if* $c \neq 0$, *there exists a unique* $U \in E(R_h^+)$ *and a unique* $V \in E(\bar{R}_h - C)$ *such that*

$$c\Delta_h U(P) = V(P), \qquad P \in \bar{R}_h - C, \tag{1.7}$$
$$\Delta_h V(P) = cF(P), \qquad P \in R_h, \tag{1.7$'$}$$
$$U(P) = G(P), \qquad P \in B_h, \tag{1.8}$$
$$(2h)^{-1}[U(P^+) - U(P^-)] = G_1(P), \qquad P \in B_h - C. \tag{1.9}$$

Moreover, U *is the function of Theorem* 2.

DEFINITION 3. If P and Q belong to R_h, P *precedes* Q if (i) P and Q have the same ordinate and P lies to the left of Q or (ii) the ordinate of P is below that of Q. If $U \in E(R_h)$, $E(\bar{R}_h)$ or $E(R_h^+)$, we associate with it a $(J - 1) \times (K - 1)$-dimensional vector (also called U) whose components are the function values at the points $P \in R_h$ ordered in accordance with the above rule of precedence.

LEMMA 1. *Let* U, V *and* F *be as in Theorem* 3. *Then for the associated vectors,*

$$cLU = h^2V + cD_1, \tag{1.10}$$
$$LV + 2ch^{-2}MU = ch^2F + ch^{-2}D_2, \tag{1.11}$$
$$L^2U + 2MU = D, \qquad D = LD_1 + D_2 + h^4F, \tag{1.12}$$

where D_1 *is a known vector arising from the values of* G, *and* D_2 *is a known vector which is a linear function of* h *whose coefficients arise from values of* G *and* G_1.

$$L = \begin{bmatrix} L_1 & I & & & 0 \\ I & L_2 & I & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot \\ 0 & & & \cdot & \cdot & I \\ & & & & I & L_{K-1} \end{bmatrix}, \qquad M = \begin{bmatrix} M_1 & & & 0 \\ & M_2 & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & \cdot \\ & & & M_{K-1} \end{bmatrix},$$

where L_k and M_k are $(J-1) \times (J-1)$ matrices given by

$$L_k = \begin{bmatrix} -4 & 1 & & & 0 \\ 1 & -4 & 1 & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & 1 \\ 0 & & & & \cdot & -4 \\ & & & & 1 & -4 \end{bmatrix}, \qquad k = 1, 2, \cdots, K-1,$$

$$M_1 = M_{K-1} = \begin{bmatrix} 2 & & & \\ & 1 & 0 & \\ & & \ddots & \\ & 0 & 1 & \\ & & & 2 \end{bmatrix}, \qquad M_k = \begin{bmatrix} 1 & & & \\ & 0 & 0 & \\ & & \ddots & \\ & 0 & 0 & \\ & & & 1 \end{bmatrix},$$

$$k = 2, 3, \cdots, K-2.$$

Proof. To obtain (1.10) write down (1.7) at each $P \in R_h$ in the order of Definition 3; then use (1.8) to eliminate $U(P)$ for $P \in B_h$; finally, rearrange terms and multiply by h^2. To obtain (1.11) write down (1.7′) at each $P \in R_h$ in the order of Definition 3; use (1.7) to eliminate $V(P)$ at each $P \in B_h - C$; then use (1.9) to eliminate $U(P^+)$ at each $P \in B_h - C$; having done this use (1.8) to eliminate $U(P)$ at each $P \in B_h$; finally, rearrange terms and multiply by h^2. Equation (1.12) follows from (1.10) and (1.11).

Definition 4 (Basic iteration).

(a) Let $U_m \in E(\bar{R}_h)$ satisfy $U_m(P) = G(P), P \in B_h$.

(b) Extend U_m to $E(R_h^+)$ by the formula (1.9).

(c) Let $V_{m+1} \in E(\bar{R}_h - C)$ be defined by requiring $V_{m+1}(P) = c\Delta_h U_m(P), P \in B_h - C$; $\Delta_h V_{m+1}(P) = cF(P), P \in R_h$.

(d) Let $U_{m+1} \in E(\bar{R}_h)$ be defined by $U_{m+1}(P) = G(P)$, $P \in B_h$; $c\Delta_h U_{m+1}(P) = V_{m+1}(P), P \in R_h$.

LEMMA 2. *In vector notation we have, for the iterates of Definition* 4,

$$LV_{m+1} + 2ch^{-2}MU_m = ch^2F + ch^{-2}D_2, \tag{1.13}$$

$$cLU_{m+1} = h^2V_{m+1} + cD_1, \tag{1.14}$$

$$L^2U_{m+1} + 2MU_m = D, \tag{1.15}$$

and hence,

$$U_{m+1} = HU_m + L^{-2}D, \qquad H = -2L^{-2}M. \tag{1.16}$$

Proof. An argument similar to that in Lemma 1 establishes (1.13), (1.14) and (1.15). It is well known (see, e.g., Forsythe and Wasow [2]) that $-L$ is positive definite.

DEFINITION 5 (The spectral radius). If A is a square matrix, $\rho(A)$ is the maximum modulus of the eigenvalues of A.

THEOREM 4. *If* $\tau_h = \rho(L^{-2}M)$, *then* $\tau_h = (h\sigma_h)^{-1}$, *where* $\sigma_h \to \sigma$ *as* $h \to 0$, $0 < \sigma < \infty$. *Thus* $\rho(H) \sim 2(\sigma h)^{-1}$ *as* $h \to 0$.

Proof. The eigenvalues of $L^{-2}M$ satisfy $Mu = \lambda L^2 u$. Since M is non-negative definite and $-L$ is positive definite we see that (for the real inner product $(\cdot, \cdot)$),

$$\tau_h = \max\left\{\frac{(MU, U)}{(LU, LU)} : U \neq 0\right\} = \left[\min\left\{\frac{(LU, LU)}{(MU, U)} : MU \neq 0\right\}\right]^{-1}. \tag{1.17}$$

However,

$$(LU, LU) = h^4 \sum_{P \in R_h} (\Delta_h U(P))^2,$$

$$(MU, U) = h^2 \sum_{P \in B_h} (\delta_n U(P))^2,$$

where the function U appearing on the right belongs to $\mathring{E}(\bar{R}_h)$, and $\delta_n U(P) = h^{-1}[U(P) - U(P^-)]$, $P \in B_h - C$; $\delta_n U(P) = 0$, $P \in C$. Thus, $\tau_h = (h\sigma_h)^{-1}$, where

$$\sigma_h = \min\left\{\frac{h^2 \sum (\Delta_h U(P))^2}{h \sum (\delta_n U(P))^2} : U \in \mathring{E}(\bar{R}_h), \delta_n U \not\equiv 0\right\}. \tag{1.18}$$

In §2 we show that $\sigma_h \to \sigma$, where σ is the finite positive solution of the corresponding continuous minimum problem:

$$\sigma = \min\left\{\frac{\iint_R (\Delta u)^2\, dx\, dy}{\int_B (\partial u/\partial n)^2\, ds} : u \in \mathring{H}_2, \frac{\partial u}{\partial n} \not\equiv 0\right\}. \tag{1.19}$$

LEMMA 3. *Let A be a square matrix with eigenvalues $\alpha_j = \mu_j + i\nu_j$, μ_j, ν_j real, where $a \leqq \mu_j \leqq b$ $(0 < a < b)$, and $|\nu_j| \leqq d$. Then, the matrix $\bar{H} = I - \omega A$ for $\omega = 2(a+b)^{-1}$ satisfies $\rho(\bar{H}) \leqq (b - a + 2d)(b+a)^{-1}$.*

Proof. If λ_j are the eigenvalues of $\bar{H}$, then $\lambda_j = 1 - \omega\alpha_j$ and

$$\begin{aligned} |\lambda_j| &\leqq |1 - \omega\mu_i| + |\omega\nu_i| \\ &\leqq \max\{|1 - \omega x| : a \leqq x \leqq b\} + \omega d \\ &\leqq (b-a)(b+a)^{-1} + 2(a+b)^{-1} d. \end{aligned}$$

(See Forsythe and Wasow [2, p. 225].)

DEFINITION 6 (Modified iteration).

(a) Let $U_0 \in E(\bar{R}_h)$, $U_0(P) = G(P)$, $P \in B_h$ be given. Then $U_1 \in E(\bar{R}_h)$ is given by the iteration of Definition 4. $\bar{U}_0 = U_0$, $\bar{U}_1 = \omega U_1 + (1-\omega)\bar{U}_0$.

(b) If $U_0, U_1, \cdots, U_m$, and $\bar{U}_0, \bar{U}_1, \cdots, \bar{U}_{m-1}$ in $E(\bar{R}_h)$ (all equal to $G(P)$ for $P \in B_h$) have been calculated and extended to $E(R_h^+)$ by (1.9), define $\bar{U}_m = \omega U_m + (1-\omega)\bar{U}_{m-1}$.

(c) $V_{m+1}(P) = c\Delta_h \bar{U}_m(P), P \in B_h - C; \Delta_h V_{m+1}(P) = cF(P), P \in R_h$.

(d) $U_{m+1}(P) = G(P), P \in B_h; c\Delta_h U_{m+1}(P) = V_{m+1}(P), P \in R_h$.

Observe that the values $\bar{U}_m$ need only be retained at points near the boundary from step to step since they enter only in (c). This means, however, that the sequence U_m must be analyzed for convergence to a solution of (1.4).

LEMMA 4. *In vector notation the iteration of Definition 6 becomes*

$$LV_{m+1} + 2ch^{-2}M\bar{U}_m = ch^2F + ch^{-2}D_2, \tag{1.20}$$

$$cLU_{m+1} = h^2V_{m+1} + cD_1, \tag{1.21}$$

$$\bar{U}_{m+1} = \omega U_{m+1} + (1-\omega)\bar{U}_m, \tag{1.22}$$

$$\bar{U}_{m+1} = \bar{H}\bar{U}_m + \omega L^{-2}D, \quad \bar{H} = \omega H + (1-\omega)I, \tag{1.23}$$

where $H = -2L^{-2}M$,

$$U_{m+1} = H\bar{U}_m + L^{-2}D. \tag{1.24}$$

Moreover, for $\omega = (1+\tau_h)^{-1}$,

$$\rho(\bar{H}) \leqq \tau_h(1+\tau_h)^{-1} \sim 1 - \sigma h \quad \text{as} \quad h \to 0. \tag{1.25}$$

Proof. Equations (1.20)–(1.24) follow in the same manner as Lemma 2. Now $\bar{H} = I - \omega A$, where $A = I + 2L^{-2}M$ is a matrix whose eigenvalues α_i satisfy $1 \leqq \alpha_i \leqq 1 + 2\tau_h$. Using Lemma 3 we find for $\omega = (1+\tau_h)^{-1}$, that $\rho(\bar{H}) \leqq \tau_h(1+\tau_h)^{-1}$. The asymptotic relation follows from Theorem 4.

LEMMA 5. *Let $E_m = U_m - U$, where U_m is the vector representative of the iterate of Definition 6 and U is the solution to* (1.12). *Then*

$$E_{m+1} = (\bar{H})^m HE_0. \tag{1.26}$$

Proof. Let $\bar{E}_m = \bar{U}_m - U$. Since $U = HU + L^{-2}D$, (1.24) shows that $E_{m+1} = H\bar{E}_m = H(\bar{H})^m\bar{E}_0 = H(\bar{H})^m E_0 = (\bar{H})^m HE_0$.

Equation (1.26) shows that except perhaps for a distortion by the factor H (which may be large since $\rho(H)$ is large as $h \to 0$), the iterates U_m converge as rapidly as the $\bar{U}_m$. Storage requirements for a computer would seem to indicate a preference for calculating U_m without retention of $\bar{U}_m$. Finally, it should be noted that by varying ω from step to step a Chebyshev scheme may be used to accelerate convergence of the present scheme.

2. $\sigma_h \to \sigma$.

DEFINITION 7.

$$\langle u, v\rangle = \iint_R uv\, dx\, dy, \qquad \langle u, v\rangle_B = \int_B uv\, ds,$$
$$\| u \|^2 = \langle u, u\rangle, \qquad \| u \|_B^2 = \langle u, u\rangle_B .$$

DEFINITION 8. $u \in T$ if u is a finite sum of the form

$$u = \sum A_{pq} \sin a^{-1}p\pi x \sin b^{-1}q\pi y. \tag{2.1}$$

LEMMA 6. *If $u \in T$, then*

$$4^{-1}abA_{pq} = \langle u, \sin a^{-1}p\pi x \sin b^{-1}q\pi y\rangle, \tag{2.2}$$
$$\| u \|^2 = 4^{-1}ab \sum A_{pq}^2 , \tag{2.3}$$
$$\| \Delta u \|^2 = 4^{-1}ab \sum (a^{-2}p^2\pi^2 + b^{-2}q^2\pi^2)^2 A_{pq}^2 , \tag{2.4}$$
$$\left\| \frac{\partial u}{\partial x} \right\| \leqq C \| \Delta u \|, \qquad \left\| \frac{\partial u}{\partial x} \right\| \leqq C \| \Delta u \|, \tag{2.5}$$
$$| u(x) | \leqq C \|\Delta u\|, \tag{2.6}$$
$$\left\| \frac{\partial^2 u}{\partial x^2} \right\|^2 + 2\left\| \frac{\partial^2 u}{\partial x \partial y} \right\|^2 + \left\| \frac{\partial^2 u}{\partial y^2} \right\|^2 = \| \Delta u \|^2, \tag{2.7}$$
$$\| u \| \leqq C \| \Delta u \|, \tag{2.8}$$
$$\langle u, \Delta v\rangle = \langle v, \Delta u\rangle, \qquad v \in T. \tag{2.9}$$

The constants $C > 0$ are independent of the number of terms in the sum representation of u.

Proof. Equations (2.2), (2.3), (2.4), (2.7) and (2.9) follow from direct calculation with the Fourier expansion. Equations (2.5) and (2.8) follow from Fourier expansion coupled with elementary estimates. Equation (2.6) follows from the Fourier expansion and the calculation (using Schwarz's inequality and the unit bound for the sine function)

$$| u(x) |^2 \leqq \left[\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} (a^{-2}p^2\pi^2 + b^{-2}q^2\pi^2)^{-2} \right] \cdot \left[\sum_{p=1}^{J} \sum_{q=1}^{K} (a^{-2}p^2\pi^2 + b^{-2}q^2\pi^2)^2 \, | A_{pq} |^2 \right].$$

DEFINITION 9. *Let* $\mathring{H}_2 = \mathring{H}_2(R)$ be the completion of T with respect to the norm $\| u \|_2 = \| \Delta u \|$.

LEMMA 7. $u \in \mathring{H}_2$ *is a function in* $C^0(\bar{R})$ *with* $u = 0$ *on* B *and strong first and second derivatives in the* L_2 *sense.*

Proof. Equation (2.8) shows that convergence in the $\| \cdot \|_2$ sense implies convergence in L_2. Equation (2.6) shows uniform convergence and (2.5), (2.7) imply convergence of the derivatives in the L_2 sense.

LEMMA 8. *If* $u \in \mathring{H}_2$, *then, with respect to* $L_2(R)$ *convergence,*

$$u = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} A_{pq} \sin a^{-1}p\pi x \sin b^{-1}q\pi y \tag{2.10}$$

and

$$\Delta u = -\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} (a^{-2}p^2\pi^2 + b^{-2}q^2\pi^2) A_{pq} \sin a^{-1}p\pi x \sin b^{-1}q\pi y, \tag{2.11}$$

where

$$4^{-1}abA_{pq} = \langle u, \sin a^{-1}p\pi x \sin b^{-1}q\pi y \rangle.$$

Thus

$$\| u \|^2 = 4^{-1}ab \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} A_{pq}^2,$$

$$\| \Delta u \|^2 = 4^{-1}ab \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} (a^{-2}p^2\pi^2 + b^{-2}q^2\pi^2)^2 A_{pq}^2.$$

Properties (2.2)–(2.9) *continue to hold for* u.

Proof. Equation (2.10) follows since $u \in L_2(R)$. On the other hand, $\Delta u \in L_2(R)$ and so

$$\Delta u = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} B_{pq} \sin a^{-1}p\pi x \sin b^{-1}q\pi y,$$

$4^{-1}abB_{pq} = \langle \Delta u, \sin a^{-1}p\pi x \sin b^{-1}q\pi y \rangle$. Letting $u_k \to u$, $v_k \to v$ in $\mathring{H}_2$, where $u_k, v_k \in T$, we see that (2.9) holds for all $u, v \in \mathring{H}_2$. Thus,

$$\begin{aligned} 4^{-1}abB_{pq} &= \langle u, \Delta(\sin a^{-1}p\pi x \sin b^{-1}q\pi y) \rangle \\ &= -(a^{-2}p^2\pi^2 + b^{-2}q^2\pi^2) \langle u, \sin a^{-1}p\pi x \sin b^{-1}q\pi y \rangle \end{aligned}$$

and (2.11) follows. The remaining assertions follow by approximation to functions in $\mathring{H}_2$ by functions of T.

Remark. It is not hard to show that $\mathring{H}_2(R)$ contains all functions with strong second derivatives which vanish on B. However we do not need this result.

LEMMA 9. *If* $u \in \mathring{H}_2$, *then* u *has a generalized normal derivative,*

$\partial u/\partial n \in L_2(B)$; *moreover,*

$$\left\| \frac{\partial u}{\partial n} \right\|_B^2 \leqq C_1 \| u \|_{5/3}^2 \leqq C_2 \| u \|_2^2, \tag{2.12}$$

where

$$\| u \|_\alpha^2 = \pi^4 4^{-1} ab \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} [a^{-2}p^2 + b^{-2}q^2]^\alpha A_{pq}^2$$

and

$$4^{-1} ab A_{pq} = \langle u, \sin a^{-1} p\pi x \sin b^{-1} q\pi y \rangle.$$

Proof. In the expansion (2.10) we consider a partial sum which we call $\bar{u}$. Then

$$I_1 \equiv \int_0^a \left| \frac{\partial \bar{u}}{\partial y}(x, 0) \right|^2 dx = 2^{-1} a \sum_{p=1}^{P} \left[\sum_{q=1}^{Q} b^{-1} q\pi A_{pq} \right]^2,$$

$$I_2 \equiv \int_0^a \left| \frac{\partial \bar{u}}{\partial y}(x, b) \right|^2 dx = 2^{-1} a \sum_{p=1}^{P} \left[\sum_{q=1}^{Q} (-1)^q b^{-1} q\pi A_{pq} \right]^2,$$

$$I_3 \equiv \int_0^b \left| \frac{\partial \bar{u}}{\partial x}(0, y) \right|^2 dy = 2^{-1} b \sum_{q=1}^{Q} \left[\sum_{p=1}^{P} a^{-1} p\pi A_{pq} \right]^2,$$

$$I_4 \equiv \int_0^b \left| \frac{\partial \bar{u}}{\partial x}(a, y) \right|^2 dy = 2^{-1} b \sum_{q=1}^{Q} \left[\sum_{p=1}^{P} (-1)^p a^{-1} p\pi A_{pq} \right]^2,$$

$$\left\| \frac{\partial \bar{u}}{\partial n} \right\|_B^2 = I_1 + I_2 + I_3 + I_4,$$

$$I_1 \leqq 2^{-1} ab^{-2} \pi^2 \left[\sum_{p=1}^{P} \sum_{q=1}^{Q} q^{10/3} A_{pq}^2 \right] \left[\sum_{q=1}^{\infty} q^{-4/3} \right]$$

$$\leqq 2\pi^{-2} b^{1/3} \sum_{q=1}^{\infty} q^{-4/3} \| \bar{u} \|_{5/3}^2 .$$

Applying this argument to I_2, I_3, I_4 we find

$$\left\| \frac{\partial \bar{u}}{\partial n} \right\|_B^2 \leqq C_1 \| \bar{u} \|_{5/3}^2 \leqq C_2 \| \bar{u} \|_2^2,$$

the second part followed by elementary estimates. Thus, we have (2.12) for the partial sums. Since the sums $\bar{u}$ converge to the limit u in $\mathring{H}_2$, we see that the sums $\partial \bar{u}/\partial n$ converge in $L_2(B)$. We denote this limit by $\partial u/\partial n$. Since the sequence converges in all three norms, (2.12) continues to hold in the limit.

LEMMA 10. *Let* $\{u^k\}$, $k = 1, 2, \cdots$, *be a sequence in* $\mathring{H}_2$ *such that*

$$\| u^k \|_2 \leqq M$$

for some constant M. Then there exists a subsequence u^{k_p} *such that*

(i) u^{k_p} *converges weakly in* $\mathring{H}_2$;

(ii) u^{k_p} *converges strongly in* $\mathring{H}_{5/3}$ (*the completion of T with respect to* $\| \cdot \|_{5/3}$);

(iii) $\partial u^{k_p}/\partial n$ *converges strongly with respect to* $\| \cdot \|_B$.

Proof. Since $\mathring{H}_2$ is a separable Hilbert space, a subsequence (still called u^k) and $u \in \mathring{H}_2$ may be found such that $u^k \rightarrow u$ weakly in $\mathring{H}_2$. If

$$u = \sum A_{pq} \sin a^{-1} p\pi x \sin b^{-1} q\pi y \quad \text{and} \quad u^k = \sum A^k_{pq} \sin a^{-1} p\pi x \sin b^{-1} q\pi y,$$

it is easy to show that $\lim_{k\to\infty} A^k_{pq} = A_{pq}$. Let $u^k = u_P{}^k + v_P{}^k$, where

$$u_P{}^k = \sum_{p=1}^{P} \sum_{q=1}^{P} A^k_{pq} \sin a^{-1} p\pi x \sin b^{-1} q\pi y.$$

$\| u_P{}^k - u_P{}^l \|_{5/3}$ will be small for fixed P and sufficiently large k and l since A^k_{pq} and A^l_{pq} are close. Moreover,

$$\begin{aligned}
\| v_P{}^k - v_P{}^l \|^2_{5/3} &= \pi^4 4^{-1} ab \sum_{p=P+1}^{\infty} \sum_{q=P+1}^{\infty} (a^{-2}p^2 + b^{-2}q^2)^{5/3} \, | A^k_{pq} - A^l_{pq} |^2 \\
&\leqq \pi^4 4^{-1} ab \sum_{p=P+1}^{\infty} \sum_{q=P+1}^{\infty} (a^{-2}p^2 + b^{-2}q^2)^2 (a^{-2}p^2 + b^{-2}q^2)^{1/3} \\
&\qquad \cdot | A^k_{pq} - A^l_{pq} |^2 \\
&\leqq [\pi^4 4^{-1} ab \sum_{p=P+1}^{\infty} \sum_{q=P+1}^{\infty} (a^{-2}p^2 + b^{-2}q^2)^2 \, | A^k_{pq} - A^k_{pq} |^2] \\
&\qquad \cdot [P^2(a^{-2} + b^{-2})]^{-1/3} \\
&\leqq [P^2(a^{-2} + b^{-2})]^{-1/3} \, \| v_P{}^k - v_P{}^l \|^2_2 \\
&\leqq 2M^2 [P^2(a^{-2} + b^{-2})]^{-1/3}.
\end{aligned}$$

By selecting P large enough $\| v_P{}^k - v_P{}^l \|_{5/3}$ is small. Thus u^k is a strongly convergent sequence in $\mathring{H}_{5/3}$, and hence $\partial u^k/\partial n$ is strongly convergent with respect to $\| \cdot \|_B$.

LEMMA 11. *If H is a separable Hilbert space and* $u^k \rightarrow u$ *weakly in H, then*

$$\| u \|^2 \leqq \liminf_{k\to\infty} \| u^k \|^2.$$

Proof. Let $f_1, f_2, \cdots, f_p, \cdots$ be an orthonormal basis for H. Then

$$u = \sum_{p=1}^{\infty} a_p f_p , \quad u^k = \sum_{p=1}^{\infty} a_p{}^k f_p , \quad a_p{}^k = \langle u^k, f_p \rangle, \quad a_p = \langle u, f_p \rangle.$$

Thus $\lim_{k\to\infty} a_p{}^k = a_p$; hence

$$\sum_{p=1}^{P} a_p{}^2 = \lim_{k\to\infty} \sum_{p=1}^{P} (a_p{}^k)^2 \leqq \liminf_{k\to p} \sum_{p=1}^{\infty} (a_p{}^k)^2.$$

THEOREM 5. *There exists a number* σ, $0 < \sigma < \infty$, *such that*

$$\sigma = \min\left\{\frac{\| \Delta u \|^2}{\| \partial u/\partial n \|_B^2} : u \in \mathring{H}_2, \frac{\partial u}{\partial n} \not\equiv 0\right\}. \tag{2.13}$$

Proof. Let $\sigma = \inf\{\| \Delta u \|^2/\| \partial u/\partial n \|_B^2 : u \in \mathring{H}_2, \partial u/\partial n \not\equiv 0\}$. Taking $u = \sin a^{-1}\pi x \sin b^{-1}\pi y$ we see that $\sigma < \infty$. Select $\sigma_k \to \sigma$ to be a minimizing sequence. If $\sigma_k = \| \Delta u^k \|^2/\| \partial u^k/\partial n \|_B^2$, we may assume $\| \partial u^k/\partial n \|_B^2 = 1$ because $\partial u^k/\partial n \not\equiv 0$ on B; hence, $\sigma_k = \| \Delta u^k \|^2$. Since $\sigma_k \to \sigma$, the condition $\| u^k \|_2 \leqq M$ is satisfied. Thus, according to Lemma 10 a subsequence (still called u^k) may be selected so that $u^k \to u$ weakly in $\mathring{H}_2$ and $\partial u^k/\partial n \to \partial u/\partial n$ strongly with respect to $\| \cdot \|_B$. Evidently, $\| \partial u/\partial n \|_B^2 = 1$, and so according to Lemma 11,

$$\sigma \leqq \| \Delta u \|^2 \leqq \liminf_{k\to\infty} \| \Delta u^k \|^2 = \sigma.$$

This completes the proof of (2.13). If σ were equal to zero, then $\| \Delta u \|^2 = 0$ would imply by (2.12) that $\partial u/\partial n \equiv 0$ on B, and this is contrary to the assumptions.

DEFINITION 10. If U and V belong to $E(\bar{R}_h)$, then

$$[U, V]_h = h^2 \sum_{P\in\bar{R}_h} U(P)V(P), \qquad \| U \|_h^2 = [U, U]_h .$$

If U and V belong to $E(\bar{R}_h)$ or $E(B_h)$, then

$$[U, V]_{Bh} = h \sum_{P\in B_h} U(P)V(P), \qquad \| U \|_{Bh}^2 = [U, U]_{Bh} .$$

DEFINITION 11. If $U \in \mathring{E}(\bar{R}_h)$, then $\delta_n U \in E(B_h)$ is defined by $\delta_n U(P) = h^{-1}[U(P) - U(P^-)]$, $P \in B_h - C$; $\delta_n U(P) = 0$, $P \in C$.

LEMMA 12. *If* $U \in \mathring{E}(\bar{R}_h)$, *then* U *may be extended to be a function of* T *by the formula*

$$U(x, y; h) = \sum_{p=1}^{J-1}\sum_{q=1}^{K-1} a_{pq} \sin a^{-1}p\pi x \sin b^{-1}q\pi y,$$

where

$$4^{-1}aba_{pq} = [U, \sin a^{-1}p\pi x \sin b^{-1}q\pi y]_h .$$

Moreover, if U *and* V *are so extended, we find*

$$[U, V]_h = \langle U, V\rangle \quad and \quad \| U \|_h^2 = \| U \|^2.$$

Proof. Apply the formula

$$[\sin a^{-1}p\pi x \sin b^{-1}q\pi y, \sin a^{-1}j\pi x \sin b^{-1}k\pi y]_h = 4^{-1}ab\delta_{pj}\delta_{qk} ,$$

$$0 < p, j < J, \quad 0 < q, k < K.$$

DEFINITION 12.

$$\sigma_h = \min\left\{\frac{\| \Delta_h U \|_h^2}{\| \delta_n U \|_{Bh}^2} : U \in \mathring{E}(\bar{R}_h), \delta_n U \not\equiv 0\right\}.$$

LEMMA 13. *If* $U \in \mathring{E}(\bar{R}_h)$, *then*

$$\Delta_h U = -\sum_{p=1}^{J-1} \sum_{q=1}^{K-1} (\lambda_p + \mu_q) a_{pq} \sin a^{-1} p\pi x \sin b^{-1} q\pi y, \tag{2.14}$$

where

$$\lambda_p = 2h^{-2}(1 - \cos a^{-1} p\pi h) = 4h^{-2} \sin^2 (2a)^{-1} p\pi h,$$
$$\mu_q = 2h^{-2}(1 - \cos b^{-1} q\pi h) = 4h^{-2} \sin^2 (2b)^{-1} q\pi h.$$

Moreover,

$$\| \delta_n U \|_{Bh}^2 = \| U_x \|_B^2 + \| U_y \|_B^2 \tag{2.15}$$

and

$$\| \delta_n U \|_{Bh} \leqq C_1 \| U \|_{5/3} \leqq C_2 \| \Delta U \|. \tag{2.16}$$

Proof. Equation (2.14) follows easily. Equation (2.15) may be proved as follows:

$$\| \delta_n U \|_{Bh}^2 = I_1 + I_2 + I_3 + I_4,$$
$$I_1 = h \sum_{P \in B_1} U_y^2(x, 0), \qquad I_2 = h \sum_{P \in B_2} U_{\bar{y}}^2(x, b),$$
$$I_3 = h \sum_{P \in B_3} U_x^2(0, y), \qquad I_4 = h \sum_{P \in B_4} U_{\bar{x}}^2(a, y),$$

where, for example, $B_1 = B_h \cap \{(x, 0) : 0 \leqq x \leqq a\}$.

We consider, for instance,

$$U_{\bar{y}}(x, b) = \sum_{p=1}^{J-1} \sum_{q=1}^{K-1} a_{pq} \sin a^{-1} p\pi x \,[\sin q\pi - \sin b^{-1} q\pi(b - h)]h^{-1}$$
$$= \sum_{p=1}^{J-1} \left[\sum_{q=1}^{K-1} (-1)^q a_{pq} h^{-1} \sin b^{-1} q\pi h\right] \sin a^{-1} p\pi x.$$

Evidently, $U_y(x, b) = U_{\bar{y}}(x, b)$, and

$$\int_0^a [U_y(x, b)]^2 \, dx = 2^{-1} a \sum_{p=1}^{J-1} \left[\sum_{q=1}^{K-1} (-1)^q a_{pq} h^{-1} \sin b^{-1} q\pi h\right]^2.$$

On the other hand,

$$h \sum_{P \in B_2} [U_{\bar{y}}(x, b)]^2 = h \sum_{r=1}^{J-1} \left[\sum_{p=1}^{J-1} \sum_{q=1}^{K-1} (-1)^q a_{pq} h^{-1} \sin b^{-1} q\pi h \sin J^{-1} p\pi r\right]^2$$
$$= 2^{-1} a \sum_{p=1}^{J-1} \left[\sum_{q=1}^{K-1} (-1)^q a_{pq} h^{-1} \sin b^{-1} q\pi h\right]^2$$

since

$$h \sum_{r=1}^{J-1} \sin J^{-1}m\pi r \sin J^{-1}p\pi r = 2^{-1}a\delta_{mp} .$$

Equation (2.16) follows from the inequalities $| h^{-1} \sin a^{-1}p\pi h | \leqq a^{-1}p\pi$, $| h^{-1} \sin b^{-1}q\pi h | \leqq b^{-1}q\pi$ and a repetition of the arguments of Lemma 9.

LEMMA 14. *Let σ be as in Theorem 5. Then,*

$$\limsup \sigma_h \leqq \sigma \quad \textit{as} \quad h \to 0.$$

Proof. Choose a minimizing u as in Theorem 5. ($\sigma = \| \Delta u \|^2$, $\| \partial u/\partial n \|_B^2 = 1$.) Let

$$u = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} A_{pq} \sin a^{-1}p\pi x \sin b^{-1}q\pi y,$$

and let

$$U_h = \sum_{p=1}^{J-1} \sum_{q=1}^{K-1} A_{pq} \sin a^{-1}p\pi x \sin b^{-1}q\pi y.$$

Observe that

$$\sigma_h \| \delta_n U_h \|_{Bh}^2 \leqq \| \Delta_h U_h \|_h^2 \leqq \| \Delta u \|^2 = \sigma,$$

since $\lambda_p \leqq a^{-2}p^2\pi^2$ and $\mu_q \leqq b^{-2}q^2\pi^2$. Using (2.16) we have for some C,

$$\| \delta_n U_h \|_{Bh}^2 = \| U_{hx} \|_B^2 + \| U_{hy} \|_B^2 \leqq C \| \Delta U_h \|^2,$$

and because $U_h \to u$ strongly in $\mathring{H}_2$ it follows that $U_{hx} \to u_1$ and $U_{hy} \to u_2$ strongly in $L_2(B)$. However, since $h^{-1} \sin a^{-1}p\pi h \to a^{-1}p\pi$ and $h^{-1} \sin b^{-1}q\pi h \to b^{-1}q\pi$, we know that $U_{hx} \to \partial u/\partial x$ and $U_{hy} \to \partial u/\partial y$ weakly in $L_2(B)$. Thus $u_1 = \partial u/\partial x$ and $u_2 = \partial u/\partial y$; so as $h \to 0$,

$$\| \delta_n U_h \|_{Bh}^2 \to \left\| \frac{\partial u}{\partial x} \right\|_B^2 + \left\| \frac{\partial u}{\partial y} \right\|_B^2 = \left\| \frac{\partial u}{\partial n} \right\|_B^2 = 1$$

and

$$\limsup \sigma_h \leqq \sigma (\lim \| \delta_n U_h \|_{Bh}^2)^{-1} = \sigma.$$

THEOREM 6. *Let σ be as in Theorem 5 and σ_h be as in Definition 12. Then*

$$\lim \sigma_h = \sigma \quad \textit{as} \quad h \to 0.$$

Proof. Suppose $\liminf \sigma_h = \sigma_0 < \sigma$ as $h \to 0$. Let $\sigma_i = \sigma_{h_i} \to \sigma_0$ as $i \to \infty$, let $U_i = U_{h_i} \in \mathring{E}(\bar{R}_{h_i})$, let $\Delta_i = \Delta_{h_i}$, let $\| \cdot \|_i = \| \cdot \|_{h_i}$ and let $\| \cdot \|_{Bi} = \| \cdot \|_{Bh_i}$. Then, $\sigma_i = \| \Delta_i U_i \|_i^2$ and $\| \delta_n U_i \|_{Bi}^2 = 1$.

Thus, $\| \Delta_i U_i \|_i^2 \leqq M$, and since $2\pi^{-1} \leqq \theta^{-1} \sin \theta \leqq 1$ for $0 \leqq \theta \leqq 2^{-1}\pi$, we see that

$$16\pi^{-4} \| \Delta U_i \|^2 \leqq \| \Delta_i U_i \|_i^2 \leqq \| \Delta U_i \|^2.$$

According to Lemma 5 there is, consequently, a subsequence (still called U_i) such that $U_i \to u_0$ weakly in $\mathring{H}_2$ and strongly in $\mathring{H}_{5/3}$. We may set

$$U_i = \sum_{p=1}^{J_i-1} \sum_{q=1}^{K_i-1} a_{pq}^{i} \sin a^{-1}p\pi x \sin b^{-1}q\pi y$$

and

$$u_0 = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} a_{pq} \sin a^{-1}p\pi x \sin b^{-1}q\pi y.$$

The weak convergence implies that $a_{pq}^{i} \to a_{pq}$ as $i \to \infty$. It follows, since $\lambda_p^{i} \to a^{-2}p^2\pi^2$, $\mu_q^{i} \to b^{-2}q^2\pi^2$ as $i \to \infty$ and

$$\Delta_i U_i = -\sum_{p=1}^{J_i-1} \sum_{q=1}^{K_i-1} (\lambda_p^{i} + \mu_q^{i}) a_{pq}^{i} \sin a^{-1}p\pi x \sin b^{-1}q\pi y,$$

that $\Delta_i U_i \to \Delta u_0$ weakly in $L_2(R)$. According to Lemma 11, we find as $i \to \infty$ that

$$\sigma_0 = \lim \| \Delta_i U_i \|_i^2 = \lim \| \Delta_i U_i \|^2 \geqq \| \Delta u_0 \|^2.$$

From Lemma 13 we conclude that

$$\| U_{ix} \|_B^2 + \| U_{iy} \|_B^2 = \| \delta_n U_i \|_{Bi}^2 \leqq C \| U_i \|_{5/3}^2,$$

so U_{ix} and U_{iy} converge strongly in $L_2(B)$ as $i \to \infty$. But as $i \to \infty$, $a_{pq}^{i} \to a_{pq}$, $h_i^{-1} \sin a^{-1}p\pi h_i \to a^{-1}p\pi$, and $h_i^{-1} \sin b^{-1}q\pi h_i \to b^{-1}q\pi$, so $U_{ix} \to \partial u_0/\partial x$, and $U_{iy} \to \partial u_0/\partial y$ weakly in $L_2(B)$. It follows that $U_{ix} \to \partial u_0/\partial x$ and $U_{iy} \to \partial u_0/\partial y$ strongly in $L_2(B)$. This yields, finally,

$$1 = \lim \| \delta_n U_i \|_{Bi}^2 = \left\| \frac{\partial u_0}{\partial x} \right\|_B^2 + \left\| \frac{\partial u_0}{\partial y} \right\|_B^2 = \left\| \frac{\partial u_0}{\partial n} \right\|_B^2,$$

and so $\sigma \leqq \| \Delta u_0 \|^2 \leqq \sigma_0$, in contradiction to the initial assumption.

3. Symmetrization and bounds for σ.

THEOREM 7. *Let D be a bounded domain whose boundary is a piecewise analytic, simple closed curve. Then there exists a number $\sigma = \sigma(D)$, $0 < \sigma < \infty$, such that*

$$(3.1) \qquad \sigma(D) = \min \left\{ \frac{\iint_D (\Delta u)^2 \, dx \, dy}{\int_{\partial D} (\partial u/\partial n)^2 \, ds} : u \in \mathring{H}_2(D), \frac{\partial u}{\partial n} \not\equiv 0 \right\}.$$

Proof. The definition of $\mathring{H}_2$ for a general domain may be found in Morrey [4]. A proof of this theorem parallel to that of Theorem 2.1 may be made since the necessary tools and inequalities may be proved without the use of Fourier series (see [4]). We omit the details.

THEOREM 8. *Let* $|D|$ *be the area of* D. *Then, if* D *satisfies the conditions of Theorem* 7, *we have*

$$\sigma(D) \geqq \sigma(B_{R_0}), \tag{3.2}$$

where $B_{R_0} = \{(x, y) : x^2 + y^2 < R_0^2\}$ *and* $\pi R_0^2 = |D|$. *It must be assumed that* u, *the minimizing function of* (3.1), *has no nodal lines.*

Proof. A modification of the arguments in Morrey [4] may be used to establish the regularity of u up to the boundary. The argument of Pólya and Szegö [8, p. 236] for the clamped plate may then be modified by choosing

$$g(\rho) = |A'(\rho)|[A(\rho)]^{-1}\left\{(4\pi)^{-1}\int_0^\rho [Q(t)]^{1/2}[A'(t)]^{1/2}dt + C\right\}, \quad \rho > 0.$$

This leads to a $\bar{u}$ such that

$$\begin{aligned}\left(\frac{\partial \bar{u}}{\partial n}\right)_{R_0} &= \lim_{\rho\to 0}|A'(\rho)|^{-1}f'(\rho)2\pi R = \lim_{\rho\to 0}|A'(\rho)|^{-1}g(\rho)2\pi R \\ &= \lim_{\rho\to 0}\left\{(2\pi R)[4\pi A(\rho)]^{-1}\int_0^\rho Q^{1/2}A'\,dt + [A(\rho)]^{-1}2\pi RC\right\} \\ &= 2CR_0^{-1}.\end{aligned}$$

Thus

$$\int_{\partial B_{R_0}}\left(\frac{\partial \bar{u}}{\partial n}\right)^2 ds = 8\pi C^2 R_0^{-1},$$

so C may be chosen to obtain

$$\int_{\partial D}\left(\frac{\partial u}{\partial n}\right)^2 ds = 8\pi C^2 R_0^{-1}.$$

This leaves $\int_{\partial D}(\partial u/\partial n)^2\,ds$ invariant. It follows from the argument in [8 that $\iint_D (\Delta u)^2\,dx\,dy$ is also invariant so that

$$\sigma(D) = \frac{\iint_D (\Delta u)^2\,dx\,dy}{\int_{\partial D}(\partial u/\partial n)^2\,ds} = \frac{\iint_{B_{R_0}}(\Delta \bar{u})^2\,dx\,dy}{\int_{\partial B_{R_0}}(\partial \bar{u}/\partial n)^2\,ds} \geqq \sigma(B_{R_0}).$$

THEOREM 9.

$$\sigma(B_r) = 2r^{-1}. \tag{3.3}$$

Proof. By the usual perturbation argument involving $u + tv$, where v satisfies $v \equiv 0$ on ∂B_r, we may show that the Lagrange multiplier rule holds in the form $\sigma(B_r) = \min\{\lambda : \Delta\Delta u = 0 \text{ in } B_r,\ \Delta u = \lambda(\partial u/\partial\rho) \text{ on } \partial B_r,\ \partial u/\partial\rho \not\equiv 0,\ u \equiv 0 \text{ on } \partial B_r\}$. ρ is the running radial coordinate from 0 to r. It is well known, however, that a solution of the biharmonic equation in B_r must have the form $u = H_0 + r^{-2}\rho^2 H_1$, where H_0 and H_1 are harmonic in B_r. An application of the boundary conditions leads to $H_0 + H_1 = 0$ on the circle of radius r and $H_0 + H_1 \equiv 0$ in B_r, so that $u = (1 - r^{-2}\rho^2)H$, where H is harmonic. Now $\Delta u = -4r^{-2}H - (4r^{-2}\rho)\partial H/\partial\rho$, whereas $\partial u/\partial\rho = (1 - r^{-2}\rho^2)\partial H/\partial\rho - 2r^{-2}\rho H$. At the boundary, $\Delta u = \lambda(\partial u/\partial\rho)$ leads to $\partial H/\partial\rho = (2^{-1}\lambda - r^{-1})H$. H must have the form $H = 2^{-1}a_0 + \sum (\rho r^{-1})^n(a_n \cos n\theta + b_n \sin n\theta)$. This leads to $r^{-1}na_n = (2^{-1}\lambda - r^{-1})a_n$, and $r^{-1}nb_n = (2^{-1}\lambda - r^{-1})b_n$. We may therefore conclude that the eigenvalues $\lambda = \lambda_n$ have the form $\lambda_n = 2r^{-1}(n + 1)$, so that $\sigma = \lambda_0 = 2r^{-1}$.

COROLLARY 1. *For a rectangle R,*

$$\frac{\pi^2(b^2 + a^2)^2}{4ab(b^3 + a^3)} \geqq \sigma(R) \geqq 2[\pi(ab)^{-1}]^{1/2}.$$

Proof. The lower bound is immediate from Theorems 8 and 9; the upper bound follows by using $u = \sin a^{-1}\pi x \sin b^{-1}\pi y$ in the definition of σ as a minimum.

Acknowledgment. I would like to express my thanks to F. N. Peebles of the University of Tennessee for suggesting this problem and to S. V. Parter of the University of Wisconsin whose written communication suggested and sketched the use of trigonometric series in §2. In fact, the use of trigonometric series made possible a brief self-contained presentation of the existence of σ, while providing the key estimate of Theorem 2.2.

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