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THE COUPLED EQUATION APPROACH TO THE NUMERICAL SOLUTION OF THE BIHARMONIC EQUATION BY FINITE DIFFERENCES. II*

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1. Introduction. The first part of this paper has appeared in this Journal, 5 (1968), pp. 323–339. There, a scheme for numerically solving the biharmonic equation is divided into inner and outer iterations. Part I (§ 1–§ 3) gives the details for the outer iteration scheme.

In § 4 we prove a general theorem for inner iterations (Theorem 10), and then give details for two possible inner iteration schemes, while in § 5 a Chebyshev scheme for the outer iteration is discussed.

The bibliography is given in Part I.

4. The overall iteration scheme.

DEFINITION 13. If L is a nonsingular square matrix, then S_p and $N_p, p = 1, 2, \cdots$, are said to describe a *linear iteration* for L if $I - N_pL = S_p$. The *iterates* for the equation $LU = K$ are given by

$$U_p = S_pU_{p-1} + N_pK.$$

LEMMA 15. *If S_p and N_p describe a linear iteration for L, then the p-th iterate, U_p, is given by*

$$U_p = T_pU_0 + C_pK,$$

where $I - C_pL = T_p$, and $T_p = S_pS_{p-1} \cdots S_1$. Thus, T_p and C_p describe a linear iteration for L.

Proof. By induction on p.

DEFINITION 14. The *outer iteration* is given by

$$LV_{m+1} + 2ch^{-2}M\overline{U}_m = ch^{-2}A, \tag{4.1}$$

$$LU_{m+1} = c^{-1}h^2V_{m+1} + B, \tag{4.2}$$

$$\overline{U}_{m+1} = \omega U_{m+1} + (1 - \omega)\overline{U}_m, \tag{4.3}$$

where

$$A = h^4F + D_2, \quad B = D_1$$

(see Lemma 4).

DEFINITION 15. Let S_p and N_p describe a linear iteration for the discrete Laplace matrix L (see § 1) and let T_p and C_p be the resulting p-step iteration of

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Lemma 15. We assume $\| T_p \| < 1$, and hence C_p is nonsingular. The *overall iteration* (i) is given by

$$V_{m+1} = T_p V_m + ch^{-2} C_p[-2M\overline{U}_m + A], \tag{4.4}$$

$$U_{m+1} = T_p \overline{U}_m + C_p[c^{-1}h^2 V_{m+1} + B], \tag{4.5}$$

$$\overline{U}_{m+1} = \omega U_{m+1} + (1 - \omega)\overline{U}_m, \tag{4.6}$$

while the *overall iteration* (ii) is given by

$$V_{m+1} = T_p V_m + ch^{-2} C_p[-2M\overline{U}_m + A], \tag{4.7}$$

$$U_{m+1} = T_p U_m + C_p[c^{-1}h^2 V_{m+1} + B], \tag{4.8}$$

$$\overline{U}_{m+1} = \omega U_{m+1} + (1-\omega)\overline{U}_m. \tag{4.9}$$

We have replaced the solution of a Laplace difference equation by a p-step approximation to this solution. Notice that (ii) requires less storage if $\overline{U}_m$ is not retained in the interior, while averaging need be performed near boundary points only.

LEMMA 16. *The iterates* $\overline{U}_m$ *of* (i) *satisfy*

$$\overline{U}_{m+1} = P_1 \overline{U}_m + P_2 \overline{U}_{m-1} + K, \tag{4.10}$$

where

$$P_1 = (1 - \omega)I - 2\omega C_p^2 M + \omega T_p + C_p T_p C_p^{-1},$$

$$P_2 = C_p T_p C_p^{-1}[(\omega - 1)I - \omega T_p],$$

$$K = \omega[C_p^2 A + C_p B - C_p T_p B] = \omega C_p^2[A + LB],$$

while the iterates, $\overline{U}_m$ *of* (ii) *satisfy*

$$\overline{U}_{m+1} = Q_1 \overline{U}_m + Q_2 \overline{U}_{m-1} + Q_3 \overline{U}_{m-2} + K, \tag{4.11}$$

where

$$Q_1 = T_p + C_p T_p C_p^{-1} + (1 - \omega)I - 2\omega C_p^2 M,$$

$$Q_2 = (\omega - 1)T_p + (\omega - 1)C_p T_p C_p^{-1} - C_p T_p C_p^{-1} T_p,$$

$$Q_3 = (1 - \omega)C_p T_p C_p^{-1} T_p.$$

Moreover,

$$I - P_1 - P_2 = \omega C_p^2[L^2 + 2M]. \tag{4.12}$$

$$I - Q_1 - Q_2 - Q_3 = \omega C_p^2[L^2 + 2M]. \tag{4.13}$$

Equation (4.10) *may be written in the matrix form*

$$\begin{bmatrix} \overline{U}_{m+1} \\ \overline{U}_m \end{bmatrix} = \begin{bmatrix} P_1 & P_2 \\ I & 0 \end{bmatrix} \begin{bmatrix} \overline{U}_m \\ \overline{U}_{m-1} \end{bmatrix} + \begin{bmatrix} \omega C_p^2 & 0 \\ 0 & \omega C_p^2 \end{bmatrix} \begin{bmatrix} A + LB \\ 0 \end{bmatrix}, \tag{4.14}$$

while (4.11) *may be written in the form*

$$\begin{bmatrix} \overline{U}_{m+1} \\ \overline{U}_m \\ \overline{U}_{m-1} \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 & Q_3 \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} \overline{U}_m \\ \overline{U}_{m-1} \\ \overline{U}_{m-2} \end{bmatrix} + \begin{bmatrix} \omega C_p^2 & 0 & 0 \\ 0 & \omega C_p^2 & 0 \\ 0 & 0 & \omega C_p^2 \end{bmatrix} \begin{bmatrix} A + LB \\ 0 \\ 0 \end{bmatrix}. \tag{4.15}$$

We observe that (4.12) and (4.13) ensure that when $\bar{U}_m$ is convergent to U, we have $[L^2 + 2M]U = A + LB$. U is thus a solution of (1.12).

Proof. Equation (4.10) follows by using (4.4) to eliminate V_{m+1} in (4.5), then using (4.5) for $m - 1$ to eliminate V_m in the resulting equation. Equation (4.6) is used to eliminate U_{m+1} and U_m. Equation (4.11) follows in the same way from (4.7), (4.8) and (4.9).

LEMMA 17. *If*

$$H = \begin{bmatrix} A_1 & A_2 \\ I & 0 \end{bmatrix},$$

where A_1, A_2, I and 0 are square matrices of the same order, then

$$\rho(H) \leqq 2^{-1}[\|A_1\| + (\|A_1\|^2 + 4\|A_2\|)^{1/2}] \leqq \|A_1\| + \|A_2\|^{1/2}. \tag{4.16}$$

Proof. If an eigenvector of H is properly partitioned, we arrive at $A_1U + A_2V = \lambda U$, $U = \lambda V$, where $V \not\equiv 0$. If we eliminate U from the preceding equations, and form the (complex) scalar product with V in the resulting equation, then, (assuming $(V, V) = 1$) $\lambda^2 - \alpha_1\lambda - \alpha_2 = 0$, where $\alpha_1 = (A_1V, V)$ and $\alpha_2 = (A_2V, V)$. Since $|\alpha_1| \leqq \|A_1\|$ and $|\alpha_2| \leqq \|A_2\|$, the result follows from the quadratic formula.

THEOREM 10. *If*

$$H = \begin{bmatrix} P_1 & P_2 \\ I & 0 \end{bmatrix}$$

is the iteration matrix of (4.14), *then*

$$\rho(H) \leqq \gamma_p, \tag{4.17}$$

where

$$\gamma_p = \tau(1 + \tau)^{-1} + e_p + (1 + \tau)^{-1}\|T'_p\| + [\|T_p\|(1 + \tau)^{-1}\|T'_p\|]^{1/2},$$
$$\tau = \tau_h = \rho(L^{-2}M), \quad T'_p = LT_pL^{-1},$$
$$e_p = 3\|T_p\| + 2\|T'_p\| + 2\|T_p\|\|T'_p\| + \|T_p\|^{1/2}$$

and $\omega = (1 + \tau)^{-1}$. *Moreover,*

$$\gamma_p = 1 - \sigma_h h + e_p[1 + O(h^{1/2})] + O(h^2) \quad \text{as} \quad h \to 0. \tag{4.18}$$

Proof. $\rho(H) = \rho(H')$, where

$$H' = \begin{bmatrix} A_1 & A_2 \\ I & 0 \end{bmatrix}$$

and $A_1 = C_p^{-1}P_1C_p$, $A_2 = C_p^{-1}P_2C_p$. If we notice that $C_p^{-1}T_pC_p = T'_p$ and $C_p = (I - T_p)L^{-1}$, then

$$A_1 = (1 - \omega)I - 2\omega L^{-1}ML^{-1} + 2\omega T_pL^{-1}ML^{-1} + 2\omega L^{-1}ML^{-1}T'_p$$
$$- 2\omega T_p(L^{-1}ML^{-1})T'_p + \omega T'_p + T_p,$$
$$A_2 = T_p[(\omega - 1)I - \omega T'_p].$$

Thus, we have

$$\|A_1\| \leqq \|(1-\omega)I - 2\omega L^{-1}ML^{-1}\|$$
$$+ 2\omega\|L^{-1}ML^{-1}\|(\|T_p\| + \|T'_p\| + \|T_p\|\,\|T'_p\|) + \omega\|T'_p\| + \|T_p\|,$$
$$\|A_2\| \leqq \|T_p\|\,\|(1-\omega)I + \omega T'_p\|.$$

On the other hand $L^{-1}ML^{-1}$ is symmetric and $L^{-2}M = L^{-1}[L^{-1}ML^{-1}]L$ so $\|L^{-1}ML^{-1}\| = \rho(L^{-2}M) = \tau$ while

$$\|(1-\omega)I - 2\omega L^{-1}ML^{-1}\| = \rho((1-\omega)I - 2\omega L^{-2}M) \leqq \tau(1+\tau)^{-1}.$$

Hence, we obtain with the aid of Lemma 17,

$$\rho(H) \leqq \tau(1+\tau)^{-1}(1 + 2\|T_p\| + 2\|T'_p\| + 2\|T_p\|\,\|T'_p\|)$$
$$+ (1-\tau)^{-1}\|T'_p\| + \|T_p\|$$
$$+ (\|T_p\|\,\|\tau(1+\tau)^{-1}I + (1+\tau)^{-1}T'_p\|)^{1/2},$$

since $\omega = (1+\tau)^{-1}$. If we observe that $\tau(1+\tau)^{-1} \leqq 1$, it is easy to verify (4.17). Equation (4.18) follows from a direct examination of γ_p, combined with Theorem 4, and the relation $\omega = (1+\tau)^{-1}$.

An analysis of (4.15) similar to that we have made for (4.14) may be made with the aid of Vieta's formula for the roots of a cubic polynomial. We omit a detailed discussion.

DEFINITION 16. If $r > 0$, then

$$k = (\log r)(\log \rho)^{-1},$$

where $\rho = \rho(H)$ and H is given in Theorem 1, is said to be the *number of outer iterations required to reduce the initial error by a relative amount r.*

For a discussion of the motivation for this definition, see Varga [10, p. 61].

DEFINITION 17. If p is the number of steps used in the matrix H of Theorem 10 (H depends on p, see Lemma 16) and k is given in Definition 16, them $m = 2pk$ is said to be the *total number of iterations required to reduce the initial error by a relative amount r.*

DEFINITION 18. Let $k' = (\log r)(\log \gamma_p)^{-1}$. Evidently, $k' \geqq k$ (see Theorem 10).

COROLLARY 2. *Let $S_p = S$ be the Jacobi matrix for L, (and hence $T_p = S^p$, see* [10]), *let $\beta = \rho(S)$, let $\alpha_h: 0 < \alpha_h < \sigma_h$, and let $p_0 = (\log \beta)^{-1}(2 \log \alpha_h h)$. Then*

$$e_{p_0} = h\alpha_h + O(h^2) \tag{4.19}$$

and hence,

$$\gamma_{p_0} = 1 - (\sigma_h - \alpha_h)h + O(h^{3/2}). \tag{4.20}$$

It follows that, as $h \to 0$,

$$\begin{aligned} 2p_0k' &= 2(\log \beta \log \gamma_{p_0})^{-1}(2\log \alpha_h h)(\log r) \\ &= 16h^{-3}\mu^{-1}(\sigma_h - \alpha_h)^{-1}(-\log \alpha_h h)(-\log r)[1 + O(h^{1/2})], \end{aligned} \tag{4.21}$$

where $\mu = \pi^2(a^{-2} + b^{-2})$.

Proof. For integer p, we find since T_p is symmetric and commutes with L, that $\|T_p\| = \|T'_p\| = \rho(T_p) = \beta^p$. It follows that

$$e_p = 5\beta^p + 2\beta^{2p} + (\beta^p)^{1/2}. \tag{4.22}$$

For arbitrary real p we define e_p as in (4.22). γ_p may be defined for all real p in a similar manner. Now $\beta^{p_0} = (\alpha_h h)^2$ so that

$$e_{p_0} = h\alpha_h + 5h^2\alpha_h^2 + 2h^4\alpha_h^4,$$

and (4.19) follows. Applying (4.19) to (4.18), (4.20) follows. It is well known, that $\beta = 1 - 4^{-1}\mu h^2 + O(h^4)$ and hence as $h \to 0$, $-\log\beta = 4^{-1}\mu h^2 + O(h^4)$. On the other hand $-\log\gamma_{p_0} = (\sigma_h - \alpha_h) + O(h^{3/2})$. A short calculation yields (4.21).

DEFINITION 19. Let $S_j = [I - \alpha_{j-1}(-L)]$, $j = 1, 2, \cdots, p$, where the α_j^{-1} are the roots of the polynomial

$$Q_p(x) = C_p[(c + d - 2x)(d - c)^{-1}]/C_p[y_0].$$

In this formula $C_p(y) = \cos(p\cos^{-1} y)$ is the Chebyshev polynomial of degree p, and $y_0 = (d + c)(d - c)^{-1}$. c is the smallest eigenvalue of $-L$, and d is its largest eigenvalue.

For a discussion of this method for solving equations with a positive definite matrix the reader is referred to Forsythe and Wasow [2], where the method is called "Richardson's Method". It is sometimes called the "Extrapolated Jacobi Method."

COROLLARY 3. *Let S_p and y_0 be as in Definition 19, let $\lambda = y_0 - (y_0^2 - 1)^{1/2}$, suppose that $\alpha_h: 0 < \alpha_h < \sigma_h$ and let*

$$p_0 = (\log\lambda)^{-1}(2\log 2^{-1/2}h\alpha_h).$$

Then, we may find $\bar{e}_{p_0}$ and $\bar{\gamma}_{p_0}$ such that

$$e_{p_0} \leqq \bar{e}_{p_0} = h\alpha_h + O(h^2) \tag{4.23}$$

and

$$\gamma_p \leqq \bar{\gamma}_{p_0} = 1 - (\sigma_h - \alpha_h)h + O(h^{3/2}). \tag{4.24}$$

It follows that, as $h \to 0$,

$$\begin{aligned} 2p_0\bar{k} &\equiv 2p_0(\log r)(\log\bar{\gamma}_{p_0})^{-1} \\ &= 2(\log\lambda\log\bar{\gamma}_{p_0})^{-1}(2\log 2^{-1/2}h\alpha_h)(\log r) \\ &= 2^{5/2}\mu^{-1/2}h^{-2}(\sigma_h - \alpha_h)^{-1}(-\log 2^{-1/2}h\alpha_h)(-\log r)[1 + O(h^{1/2})]. \end{aligned} \tag{4.25}$$

Proof. Let $\rho_p = \rho(T_p)$. For integer p, it follows again since T_p is symmetric and commutes with L that $\|T_p\| = \|T'_p\| = \rho_p$. In this case we have

$$e_p = 5\rho_p + 2\rho_p^2 + \rho_p^{1/2}.$$

Using the definition of T_p we find

$$T_p = [I - \alpha_{p-1}(-L)][I - \alpha_{p-2}(-L)]\cdots[I - \alpha_0(-L)] = Q_p(-L).$$

From Forsythe and Wasow [2, p. 228] we see that $\|T_p\| \leqq 2\lambda^p$. Thus

$$e_p \leqq 10\lambda^p + 8\lambda^{2p} + 2^{1/2}\lambda^{p/2}.$$

We let $\bar{e}_p$ be the right-hand side of this expression. $\bar{\gamma}_p$ is defined by replacing e_p by $\bar{e}_p$ in (4.18). Evidently, $e_p \leqq \bar{e}_p$ and $\gamma_p \leqq \bar{\gamma}_p$. Since

$$2\lambda^{p_0} = (h\alpha_h)^2,$$

we obtain (4.23). Equation (4.24) follows from the definition of $\bar{\gamma}_{p_0}$ and (4.23). A calculation similar to that of [2, p. 230] yields

$$c = 4(\sin^2 2^{-1}a^{-1}\pi h + \sin^2 2^{-1}b^{-1}\pi h),$$

$$d = 4[\sin^2 (1 - a^{-1}h)2^{-1}\pi + \sin^2 (1 - b^{-1}h)2^{-1}\pi].$$

This furnishes the result

$$y_0 = 1 + 4^{-1}\mu h^2 + O(h^4),$$

and hence

$$\lambda = y_0 - (y_0^2 - 1)^{1/2} = 1 - 2^{-1/2}\mu^{1/2}h + O(h^2).$$

This, together with (4.24), yields (4.25).

Since the numbers $2p_0k'$ and $2p_0\bar{k}$ represent, roughly the overall number of iterations required for reducing the initial error by a relative amount r, we would like to minimize these quantities with respect to α_h which is still at our disposal. Thus, we must study the minimization of the function $(-\log C\alpha)(\sigma_h - \alpha)^{-1}$, $C > 0$, in the interval $0 < \alpha < \sigma_h$, where $C = h$ in the first instance and $C = 2^{-1/2}h$ in the second.

LEMMA 18. *Let* $C > 0$, $A > 0$, *and* $CA < 1$. *Then, the function* $\phi(\alpha) = -(A - \alpha)^{-1} \log C\alpha$ *has a unique minimum in the interval* $0 < \alpha < A$ *which is attained at the value* $\alpha = \alpha_0$. *Here* $\alpha_0 = eC^{-1}x_0$ *and* x_0 *is the unique solution of the transcendental equation*

$$-x \log x = e^{-1}CA.$$

Moreover, $\phi(\alpha_0) = \alpha_0^{-1}$. *e is the base of natural logarithms.*

Proof. By calculus.

LEMMA 19. *Let* $f(x) = -x \log x$, $0 \leqq x \leqq e^{-1}$ *and let* $g(y)$, $0 \leqq y \leqq e^{-1}$, *be the function inverse to* $f(x)$ *in this interval. Then*

$$g(y) \sim y(-\log y)^{-1} \quad as \quad y \to 0.$$

Proof. Since $-g(y) \log g(y) = y$, we need only show that $\log g(y) \sim \log y$ as $y \to 0$. Moreover, the same formula shows that $y^{-1}g(y) \to 0$ as $y \to 0$, since $g(y)$ evidently approaches zero. But we also know that $g'(y) = -[\log g(y) + 1]^{-1} = g(y)[g(y) - y]^{-1}$; thus if we apply L'Hôpital's rule to the ratio $\log g(y) [\log y]^{-1}$ we obtain $yg'(y)[g(y)]^{-1} = -y[g(y) - y]^{-1}$. This ratio tends to 1 since $y^{-1}g(y) \to 0$.

COROLLARY 2'. *If* α_h *is chosen to be* α_0 *as in Lemma* 18 *with* $A = \sigma_h$ *and* $C = h$, *then as* $h \to 0$,

$$2p_0k \sim 16h^{-3}\mu^{-1}\sigma^{-1}[-\log (e^{-1}\sigma h)](-\log r). \tag{4.26}$$

Proof. $-(\log \alpha_h h)(\sigma_h - \alpha_h)^{-1} = \phi(\alpha_h) = \alpha_0^{-1}$, for $\alpha_h = \alpha_0$. But

$$\begin{aligned}\alpha_0 &= eC^{-1}g(e^{-1}CA)\\ &\sim (eC^{-1})(e^{-1}CA)[-\log(e^{-1}CA)]^{-1}\\ &= \sigma_h[-\log(e^{-1}h\sigma_h)]^{-1}\\ &\sim \sigma[-\log(e^{-1}\sigma h)]^{-1}.\end{aligned}$$

COROLLARY 3′. *If α_h is chosen to be α_0 as in Lemma* 18 *with $A = \sigma_h$ and*

$$C = 2^{-1/2}h,$$

then

$$2p_0\bar{k} \sim 2^{5/2}\mu^{-1/2}h^{-2}\sigma^{-1}[-\log(e^{-1}2^{-1/2}h\sigma)](-\log r). \tag{4.27}$$

If $S_p = S$ is chosen to be the successive over-relaxation matrix for L, we see that $T_p = S^p$. Thus, if we desire to estimate e_p in Theorem 10, for large values of p we have available the estimate (see Varga [10, p. 65 and p. 114])

$$\|T_p\| \sim \nu p\bar{\lambda}^{p-1} \quad \text{as} \quad p \to \infty,$$

and since $T'_p = LT_pL^{-1}$ we also have

$$\|T'_p\| \sim \nu' p\bar{\lambda}^{p-1} \quad \text{as} \quad p \to \infty.$$

Here $\bar{\lambda}$ is the spectral radius of S, the S-O-R matrix for L. In order to carry through the program of Corollary 2 for this iteration we are faced with two problems. First, the estimates we have are valid only for large p, whereas the estimate (4.22) holds for all p. In the second place, although the behavior of $\bar{\lambda}$ is well known as $h \to 0$ we must also determine the growth of ν and ν' as $h \to 0$ in order to obtain a result comparable to (4.19). Further work is needed to complete the study of this method.

5. The Chebyshev scheme for the outer iteration.

LEMMA 20. *If in the outer iteration of Definition* 14 *we replace ω by ω_{m+1}, then the iterates V_m satisfy*

$$V_{m+1} = V_m + \omega_m(G - KV_m), \qquad m = 1, 2, \cdots, \tag{5.1}$$

where

$$K = I + 2L^{-1}ML^{-1}, \qquad G = ch^{-2}(L^{-1}A - 2L^{-1}ML^{-1}B). \tag{5.2}$$

Moreover, V as defined in Lemma 1 *of* § 1 *satisfies*

$$KV = G. \tag{5.3}$$

Proof. In (4.1) replace m by $m + 1$ and solve for V_{m+2}. Where $\bar{U}_{m+1}$ appears, use (4.3). In the result, use (4.2) to eliminate U_{m+1}. Finally, use (4.1) to eliminate $M\bar{U}_m$. Equation (5.3) follows immediately from (1.10) and (1.11).

We solve for V here since K is a symmetric positive definite matrix.

LEMMA 21. *If $E_m = V_m - V$, then $E_{m+1} = (I - \omega_m K)E_m$, and hence $E_{m+1} = P_m(K)E_1$, where*

$$P_m(x) = \prod_{i=1}^{m} (1 - \omega_i x).$$

Proof. A direct calculation together with (5.3).

LEMMA 22. *The eigenvalues, κ_s, of K satisfy*

$$1 \leqq \kappa_s \leqq 1 + 2\tau_h = 1 + 2(h\sigma_h)^{-1}.$$

Proof. $\rho(L^{-1}ML^{-1}) = \rho(L^{-2}M) = \tau_h = (h\sigma_h)^{-1}$ (see § 1, Theorem 4).

DEFINITION 20. The *average rate of convergence for m iterations of* (5.1) is defined by

$$R[P_m(K)] \equiv -m^{-1} \log \|P_m(K)\|$$

(see Varga [10]).

THEOREM 11. *Let $a = 1$ and $b = 1 + 2\tau_h$, and suppose that the ω_j^{-1}, $j = 1, 2, \cdots, m$, are roots of the polynomial*

$$P_m(x) = C_m[(a + b - 2x)(b - a)^{-1}]/C_m(y_0).$$

In this formula $C_m(y) = \cos(m \cos^{-1} y)$ is the Chebyshev polynomial of degree m, and $y_0 = (b + a)(b - a)^{-1}$. Then,

$$\|P_m(K)\| \leqq 2[y_0 + (y_0^2 - 1)^{1/2}]^{-m}. \tag{5.4}$$

Hence

$$R(P_m(K)) \leqq -m^{-1} \log 2 + \log(y_0 + (y_0^2 - 1)^{1/2}). \tag{5.5}$$

Moreover,

$$\log(y_0 + (y_0^2 - 1)^{1/2}) \sim (2h\sigma)^{1/2} \quad \textit{as} \quad h \to 0. \tag{5.6}$$

Proof. Relations (5.4) and (5.5) follow as in Forsythe and Wasow [2, pp. 228, 229]. Relation (5.6) follows from the same source and the fact that $ab^{-1} \sim \frac{1}{2}h\sigma_h \sim \frac{1}{2}h\sigma$ as $h \to 0$.

Relation (5.5) indicates that for sufficiently large m, a bound for the rate of convergence for the outer iteration with Chebyshev scheme is $\sim (2h\sigma)^{1/2}$, while that for the scheme of § 1 is given by $-\log(1 - \sigma h) \sim \sigma h$. A so-called second order iteration (see Forsythe and Wasow [2]) may be used to obtain the full power of the Chebyshev rate of convergence. This method, however, necessitates additional storage.

Finally, we conjecture on the basis of the results of this section that for a Chebyshev scheme for the outer iteration we should obtain an estimate for the total number of iterations of the form

$$m = 2pk \gtrsim -ah^{-3/2} \log bh\,(-\log r),$$

when Richardson's method is used for the inner iteration.

Note added in proof. The author would like to thank J. F. Kuttler and V. G. Sigillito for pointing out an error in the proof of Theorem 8 (Part I). Although the result is probably true, a correct proof has not yet been found. Corollary 1 which depends on Theorem 8 has been used in calculations to estimate σ. The result is empirically close to an optimal value in applying this method to a biharmonic problem.