[1] Weyl, H., "Almost Periodic Invariant Vector Sets in a Metric Vector Space," *Am. J. Math.*, **71**, 178–205 (1949).

[2] Neumann, J. v., "Almost Periodic Functions in a Group I," *Trans. A.M.S.*, **36**, 445–492 (1934).

[3] Bochner, S., and Neumann, J. v., "Almost Periodic Functions in a Group II," *Ibid.*, **37**, 21–50 (1935).

[4] Maak, W., "Moduln fastperiodischer Funktionen," *Abh. Math. Sem. Hamburg Univ.* **16**, 56–71 (1949).

[5] Maak, W., "Abstrakte fastperiodische Funktionen," *Ibid.*, **11**, 367–380 (1936).

ON MEMBRANES AND PLATES

BY G. SZEGÖ

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY

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I. Introduction.—1. We deal with three closely related problems of the Calculus of Variations having some connection with the theory of elastic deformations.

Let D be an arbitrary domain in the plane bounded by a simple curve C. We denote the area-element of D by $d\sigma$. The admissible functions u are defined in D and satisfy certain boundary conditions on C. We define the positive numbers λ_1, λ_2, λ_3 as follows:

$$\lambda_1^2 = \min. \frac{\int_D\int |\operatorname{grad} u|^2\, d\sigma}{\int_D\int u^2\, d\sigma}, \qquad u = 0 \text{ on } C, \tag{a}$$

$$\lambda_2^4 = \min. \frac{\int_D\int (\nabla^2 u)^2\, d\sigma}{\int_D\int u^2\, d\sigma}, \qquad u = \frac{\partial u}{\partial n} = 0 \text{ on } C, \tag{b}$$

$$\lambda_3^2 = \min. \frac{\int_D\int (\nabla^2 u)^2\, d\sigma}{\int_D\int |\operatorname{grad} u|^2\, d\sigma}, \qquad u = \frac{\partial u}{\partial n} = 0 \text{ on } C. \tag{c}$$

Obviously, $\lambda_2^2 \geqq \lambda_1\lambda_3$.

In (a) we allow only continuous functions which have piecewise continuous first derivatives. In (b) and (c) we allow only functions with continuous first derivatives. For the sake of simplicity we assume that C is an analytic curve.

2. Problems (a) and (b) are classical; the quantities λ_1 and λ_2 are the fundamental frequencies of a membrane with fixed boundary and of a clamped plate, respectively. Problem (c) occurs in the study of the buckling of plates.[1] All the three problems have a considerable literature for which we refer to the book of Weinstein.

As to (a) Lord Rayleigh has formulated the following conjecture which was first proved by G. Faber:[2] *For all membranes of given area the circular one has the gravest fundamental tone (lowest fundamental frequency).*

Our purpose is to prove an analogous theorem for the problems (*b*) and (*c*), i.e., for the quantities λ_2 and λ_3. This can be done under a certain hypothesis concerning the functions u for which the minima in (*b*) and (*c*) are attained. The *hypothesis* is the following: *The functions u minimizing problems (b) and (c) are different from zero throughout the domain D, that is, the fundamental vibrations do not produce any nodal lines.*

3. We note the Euler-Lagrange differential equations associated with the minimum problems formulated above:

$$\nabla^2 u + \lambda_1^2 u = 0, \qquad (a')$$

$$\nabla^2\nabla^2 u - \lambda_2^4 u = 0, \qquad (b')$$

$$\nabla^2\nabla^2 u + \lambda_3^2\nabla^2 u = 0. \qquad (c')$$

It is of interest to point out the minima in question for the special case of a circle. Denoting the radius of the circle by a we have

$$\lambda_1 = j/a, \qquad \lambda_2 = k/a, \qquad \lambda_3 = j'/a$$

where j, k, j' denote the smallest positive root of the Bessel functions

$$J_0(x), \qquad J_0(x)I_0'(x) - J_0'(x)I_0(x), \qquad J_0'(x),$$

respectively. We have

$$j = 2.405, \qquad k = 3.19, \qquad j' = 3.832. \qquad (3)$$

Thus the principal result can be expressed as follows: *Let D be an arbitrary domain and let a be the radius of the circle of the same area as D. Then, under the hypothesis formulated above, we have the bounds*

$$\lambda_1 \geqq j/a, \qquad \lambda_2 \geqq k/a, \qquad \lambda_3 \geqq j'/a. \qquad (4)$$

The first inequality is the content of the theorem of Rayleigh-Faber, the second and the third are proved in the present paper (under the hypothesis mentioned above). In all the three cases the proof is based on the process of symmetrization as it was in the proof of Faber. However we shall modify Faber's argument at a point which will be essential in dealing with the more difficult second and third problem.

II. Problem (a): Membrane with Fixed Boundary.—1. We denote by u the minimizing function of I (*a*) which is *known* to be different from zero in the interior of D. Hence, we can assume that $0 \leqq u \leqq 1$. We denote the level curve $u = \rho$ by C_ρ, $0 \leqq \rho \leqq 1$, so that $C_0 = C$ and C_1 coincides with the point(s) at which the maximum $u = 1$ is attained. The set C_ρ consists of a finite number of separated curves; for the sake of simplicity we assume that C_ρ is a single Jordan curve.

Let us denote by $A(\rho)$ the area of the domain inside of C_ρ. Thus $A(0) = A$ is the area of the given domain D, $A(1) = 0$. We define the quantity R by the equation $A(\rho) = \pi R^2$ so that R is a decreasing function of ρ; the maximum R_0 of R is the radius of the circle which has the same area as the given domain D.

2. We symmetrize the level curves C_ρ by replacing C_ρ by a circle of radius R about a fixed point, say the origin. The domain D is replaced then by a circular disk $\bar{D}$ of radius R_0. We define on $\bar{D}$ a function $\bar{u}$ by the condition that $\bar{u} = f(\rho)$ on the circle of radius R; the function f will be determined in such a way that the integral in the numerator of I (a) does not change in the transition from D to $\bar{D}$ and from u to $\bar{u}$. That is, if $\overline{d\sigma}$ is the area-element of $\bar{D}$,

$$\int_D\int |\operatorname{grad} u|^2 d\sigma = \int_{\bar{D}}\int |\operatorname{grad} \bar{u}|^2 \overline{d\sigma}. \tag{1}$$

On the other hand, we shall prove that

$$\int_D\int u^2 d\sigma \leqq \int_{\bar{D}}\int \bar{u}^2 \overline{d\sigma}. \tag{2}$$

Also the boundary condition $\bar{u} = 0$ will be satisfied. This yields the assertion immediately.

We observe that this argument differs from that of Faber. He defines the "symmetrized" function $\bar{u}$ by the condition $\bar{u} = \rho$ and shows that this process diminishes the integral in the numerator and leaves the integral in the denominator unchanged.

3. In what follows we use the notation

$$G = |\operatorname{grad} u| = \frac{d\rho}{dn} \tag{3}$$

where $d\rho > 0$ and dn is the piece of the normal of the level curve C_ρ between the level curves C_ρ and $C_{\rho+d\rho}$. The area of the ring-shaped domain between these curves is $A(\rho) - A(\rho + d\rho) = -A'(\rho)\, d\rho$. On the other hand, if ds is the arc-element of C_ρ, we find for the element $d\sigma$ of this area $d\sigma = ds\cdot dn = ds\cdot d\rho/G$ so that

$$-A'(\rho) = |A'(\rho)| = \int_{u=\rho} G^{-1}\, ds. \tag{4}$$

By Schwarz's inequality

$$\int_{u=\rho} G\, ds \int_{u=\rho} G^{-1}\, ds \geqq \left(\int_{u=\rho} ds\right)^2 = (L(\rho))^2 \tag{5}$$

where $L(\rho)$ is the length of C_ρ. Using the isoperimetric inequality $(L(\rho))^2 \geqq 4\pi A(\rho)$ we obtain the following important inequality for the first integral occurring in (5) which we denote also by $P(\rho)$, $\rho > 0$;

$$P(\rho) = \int_{u=\rho} G\, ds \geqq \frac{4\pi A(\rho)}{|A'(\rho)|}. \tag{6}$$

Obviously

$$\iint_D |\text{grad } u|^2 d\sigma = \iint_D G^2 d\sigma = \int_0^1 \int_{u=\rho} G^2 ds \cdot \frac{d\rho}{G} = \int_0^1 P(\rho)d\rho. \quad (7)$$

4. Now we define the function $\bar{u} = f(\rho)$ as follows:

$$f(\rho) = \int_0^\rho (P(t))^{1/2} \left(\frac{|A'(t)|}{4\pi A(t)}\right)^{1/2} dt, \qquad \rho > 0. \quad (8)$$

This integral exists since $P(\rho)$ [see (7)] and $A'(\rho)$ are integrable. Also it tends to zero with ρ so that $\bar{u}$ satisfies the boundary condition. We have $A'(\rho)\, d\rho = 2\pi R\, dR$, hence

$$|\text{grad } \bar{u}|^2 = \left(\frac{d\bar{u}}{dR}\right)^2 = (f'(\rho))^2 \left(\frac{d\rho}{dR}\right)^2 = (f'(\rho))^2 \frac{4\pi^2 R^2}{(A'(\rho))^2} = (f'(\rho))^2 \frac{4\pi A(\rho)}{(A'(\rho))^2} = \frac{P(\rho)}{|A'(\rho)|}. \quad (9)$$

The area between the circles of radii R and $R + dR$ is $2\pi R\, dR = |A'(\rho)|\, d\rho$ so that we find

$$\iint_{\bar{D}} |\text{grad } \bar{u}|^2 d\sigma = \int_0^1 P(\rho)d\rho \quad (10)$$

which proves indeed (1).

On the other hand we conclude from (8), by (6),

$$f(\rho) \geqq \int_0^\rho dt = \rho. \quad (11)$$

Hence

$$\int_0^1 \rho^2 \, |A'(\rho)| \, d\rho \leqq \int_0^1 (f(\rho))^2 \, |A'| (\rho) \, d\rho \quad (12)$$

which proves (2).

This establishes the assertion.

III. Problem (b): Clamped Plate.—1. We denote the minimizing function again by u and *assume* that $0 \leqq u \leqq 1$ holds throughout the domain D. Let $0 < \rho < 1$. We consider the open set $u < \rho$ which consists in general of several simply or multiply connected components K_ρ, K_ρ', ... One of them, say K_ρ, has the curve C ($u = 0$) as part of its boundary. On the rest of the boundary of $u < \rho$ we have $u = \rho$. We denote by $A(\rho)$ the area of the complementary set characterized by the condition $u \geqq \rho$. The function $A(\rho)$ is continuous and monotonically decreasing; $A(0)$ is the area of the given domain D and $A(1) = 0$.

2. We write again $|\text{grad } u| = G$. There is no need for any change in the argument of II 3; in particular II (4) and inequality (5) hold. Here $L(\rho)$ is the total length of the set of curves $u = \rho$. Now we apply

the isoperimetric inequality $(L(\rho))^2 \geqq 4\pi A(\rho)$. Obviously more than this is true: $L(\rho)$ can be replaced by the total length $L_1(\rho)$ of the curves which together with C bound K_ρ. Also $A(\rho)$ can be replaced by the area $A_1(\rho)$ bounded by the curves just mentioned. We have $L(\rho) \geqq L_1(\rho)$, $A(\rho) \leqq A_1(\rho)$.

Thus II (6) holds without any change.

3. Now we introduce the notation, $\rho > 0$,

$$\int_{u=\rho} (\nabla^2 u)^2 \frac{ds}{G} = Q(\rho) \tag{1}$$

so that

$$\iint (\nabla^2 u)^2 \, d\sigma = \int_0^1 \int_{u=\rho} (\nabla^2 u)^2 \frac{ds}{G} \, d\rho = \int_0^1 Q(\rho) d\rho. \tag{2}$$

To be sure, $Q(\rho)$ has no meaning for $\rho = 0$ but the integral (2) is finite. By Schwarz's inequality:

$$\int_{u=\rho} (\nabla^2 u)^2 \frac{ds}{G} \int_{u=\rho} \frac{ds}{G} \geqq \left(\int_{u=\rho} \nabla^2 u \frac{ds}{G} \right)^2 \tag{3}$$

or

$$(Q(\rho))^{1/2} \mid A'(\rho) \mid^{1/2} \geqq \int_{u=\rho} \nabla^2 u \frac{ds}{G}. \tag{4}$$

The integral of the function on the left-hand side of (4) exists in $0 \leqq \rho \leqq 1$ since $Q(\rho)$ and $A'(\rho)$ are integrable. Integrating we obtain

$$\int_0^\rho (Q(t))^{1/2} \mid A'(t) \mid^{1/2} dt \geqq \int_0^\rho \int_{u=t} \nabla^2 u \frac{ds}{G} \, dt =$$
$$\iint_{0 \leqq u \leqq \rho} \nabla^2 u \, d\sigma = - \int_{u=0,\ u=\rho} \frac{\partial u}{\partial n} ds = - \int_{u=\rho} \frac{\partial u}{\partial n} ds =$$
$$\int_{u=\rho} G \, ds = P(\rho) \geqq \frac{4\pi A(\rho)}{\mid A'(\rho) \mid}. \tag{5}$$

The integration indicated by the condition $u = \rho$ has to be extended over a set of curves $u = \rho$ and the normal is directed in each case into the interior of the domain $0 < u < \rho$.

4. We define now the function $f(\rho)$ in the following way. Let

$$g(\rho) = \frac{\mid A'(\rho) \mid}{4\pi A(\rho)} \int_0^\rho (Q(t))^{1/2} \mid A'(t) \mid^{1/2} dt, \ \rho > 0, \tag{6$_1$}$$

and

$$f(\rho) = \textstyle\int_0^\rho g(\rho) d\rho, \qquad f(0) = 0. \tag{6$_2$}$$

The integral in (6_2) exists since ($0 < \rho_0 < \rho$)

$$\int_{\rho_0}^{\rho} -A'(\rho)d\rho \int_0^{\rho} (Q(t))^{1/2} (-A'(t))^{1/2}\, dt$$
$$= A(\rho_0) \int_0^{\rho_0} (Q(t))^{1/2} (-A'(t))^{1/2}\, dt - A(\rho) \int_0^{\rho} (Q(t))^{1/2} (-A'(t))^{1/2}\, dt$$
$$+ \int_{\rho_0}^{\rho} A(\rho)(Q(\rho))^{1/2} (-A'(\rho))^{1/2}\, d\rho.$$

In view of (5) we have $g(\rho) \geqq 1$, $f'(\rho) \geqq 1$ so that we obtain the important inequality

$$f(\rho) \geqq \rho, \qquad 0 \leqq \rho \leqq 1. \tag{7}$$

5. We determine $R = R(\rho)$ as in I, $A(0) = \pi R_0^2$ being the area of D. We consider the function $\bar{u} = \bar{u}(R) = f(\rho)$ defined on the circle $\bar{D}$ of radius R_0. This function satisfies the boundary conditions

$$\left.\begin{aligned} &(\bar{u})_{R = R_0} = f(0) = 0, \\ &\left(\frac{\partial \bar{u}}{\partial n}\right)_{R = R_0} = \lim_{R \to R_0} {}^{u}{}'(R) = \lim_{\rho \to 0} f'(\rho) \frac{2\pi R}{|A'(\rho)|} = 0, \end{aligned}\right\} \tag{8}$$

in view of (6_1).

We have now

$$\nabla^2\bar{u} = \frac{1}{R}\frac{d}{dR}\left(R\frac{d\bar{u}}{dR}\right) = \frac{4\pi}{A'(\rho)}\frac{d}{d\rho}\left(\frac{A(\rho)}{A'(\rho)} f'(\rho)\right) = \frac{(Q(\rho))^{1/2}}{|A'(\rho)|^{1/2}} \tag{9}$$

and we find

$$\iint_{\bar{D}} (\nabla^2\bar{u})^2\, \overline{d\sigma} = \int_0^{R_0} (\nabla^2\bar{u})^2\, 2\pi R\, dR = \int_0^1 \frac{Q(\rho)}{|A'(\rho)|} |A'(\rho)|\, d\rho. \tag{10}$$

Comparing this with (2) we see that this process did not change the integral in the numerator of I (b).

On the other hand, the integral in the denominator will be for the circular domain $\bar{D}$:

$$\int_0^1 (f(\rho))^2 \,|A'(\rho)|\, d\rho \geqq \int_0^1 \rho^2 \,|A'(\rho)|\, d\rho. \tag{11}$$

This completes the proof.

IV. Problem (c): Buckling of a Plate.—1. Dealing with problem (c), the previous argument needs only slight modifications. We define $f(\rho)$ in the same way as in III so that the integral in the numerator of I (c) does not change.

As to the integral in the denominator we have by II (7):

$$\int_D\int |\operatorname{grad} u|^2\, d\sigma = \int_0^1 P(\rho)\, d\rho \tag{1}$$

On the other hand [cf. II (9)]

$$| \text{grad } \bar{u} |^2 = (f'(\rho))^2 \frac{4\pi A(\rho)}{(A'(\rho))^2}, \tag{2}$$

and the area between the circles R and $R + dR$ is $| A'(\rho) | d\rho$ so that

$$\int_D \int | \text{grad } \bar{u} |^2 \overline{d\sigma} = \int_0^1 (f'(\rho))^2 \frac{4\pi A(\rho)}{| A'(\rho) |} d\rho. \tag{3}$$

But

$$f'(\rho) \geqq \frac{| A'(\rho) |}{4\pi A(\rho)} P(\rho) \geqq 1. \tag{4}$$

so that

$$\int_{\bar{D}} \int |\text{grad } \bar{u} |^2 \overline{d\sigma} \geqq \int_0^1 f'(\rho) \frac{4\pi A(\rho)}{| A'(\rho) |} d\rho \geqq \int_0^1 P(\rho) d\rho = \int_D \int | \text{grad } u |^2 d\sigma. \tag{5}$$

This establishes the assertion.

[1] A. Weinstein, "Étude des spectres des équations aux dérivées partielles de la théorie des plaques élastiques," *Mém. Sci. Math.*, vol. **88,** 1937, 62 pp.

[2] Lord Rayleigh, *The Theory of Sound,* 2nd ed., vol. 1, 1894, p. 345; Faber, G., "Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt," *Sitzber. Bayer. Akad.*, 1923, 169–172; independently, a proof was given by Krahn, E., "Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises," *Math. Ann.*, **94,** 97–100 (1924).

SPHERE-GEOMETRICAL UNITARY FIELD THEORY

BY TSURUSABURO TAKASU

TOHOKU UNIVERSITY, SENDAI, JAPAN

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A conformal relativity theory has long been longed for. Now I have arrived at the following results:

1. The Kaluza-Klein space[1,2] is equivalent to the Einstein space $V_4(R^{ij} = {}^1/_2\, g^{ij}R)$ (special) dual-conformal (i.e., N.E.-Laguerre connection) geometrically so that the points in V_4 correspond to the generalized hyperspheres whose developments in the N.E. tangential spaces are hyperspheres of equal radii.

2. The Einstein-Mayer space[3] is equivalent to the Einstein space V_4 (special) Laguerre connection geometrically so that the points in V_4