Note to My Paper "On Membranes and Plates,"

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Then p is congruent modulo 12 to 1, 5, 7, 11. Therefore, the 4 cases are actually possible, and we get

$$h = \begin{cases} {}^1/_{12}\cdot(p-1) & p \equiv 1 \mod 12 \\ {}^1/_{12}\cdot(p-5)+1 & p \equiv 5 \mod 12 \\ {}^1/_{12}\cdot(p-7)+1 & p \equiv 7 \mod 12 \\ {}^1/_{12}\cdot(p-11)+2 & p \equiv 11 \mod 12. \end{cases}$$

If we remark that the number of supersingular invariants for $p = 3$ is 1, we see that we can express these results by the Eichler formula stated in the introduction.

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[1] M. Eichler, "Ueber die Idealklassenzahl total definiter Quaternionenalgebren," *Math. Z.*, **43**, 102–109, 1938.

[2] M. Deuring, "Die Typen der Multiplikatorenringe elliptischer Funktionenkörper," *Abhandl. Math. Sem. Hans. Univ.*, **14**, 197–272, 1941.

[3] *Ibid.*, pp. 253–255.

[4] H. Hasse, "Existenz separabler zyklischer unverzweigter Erweiterungskörper von Primzahlgrade p über elliptischen Funktionenkörpern der Characteristik p," *J. reine u. angew. Math.*, **172**, 77–85, 1934.

NOTE TO MY PAPER "ON MEMBRANES AND PLATES,"

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1. We follow the notation of the paper quoted in the title:[1] D is a given simply connected domain, C its boundary curve; u is the minimizing function of the quotient

$$\frac{\int_D\int (\nabla^2 u)^2 d\sigma}{\int_D\int u^2 d\sigma}, \qquad u = \frac{\partial u}{\partial n} = 0 \text{ on } C; \tag{1}$$

we *assume* that $0 \leqq u \leqq 1$ in D (the part $u \leqq 1$ of the assumption is, of course, immaterial); C_ρ is the set of curves $u = \rho$, $0 \leqq \rho \leqq 1$; $A(\rho) = \pi R^2$ is the area of the domain $\rho \leqq u \leqq 1$;

$$G = |\text{grad } u|, \qquad \int_{C_\rho} G ds = P(\rho), \qquad \int_{C_\rho} G^{-1}(\nabla^2 u)^2 ds = Q(\rho).$$

2. In a recent communication Dr. P. R. Beesack, of McMaster University, has called my attention to the fact that the "symmetrized" function $\bar{u} = f(\rho)$, introduced by the formulas (6_1) and (6_2) of the paper quoted, is *not* eligible for the circular plate $\bar{D}$ of radius R_0, $A(0) = \pi R_0^2$. This is due to the convergence of $A(\rho)$ to 0 as $\rho \to 1$. The argument must be modified as follows.

3. We conclude as in III (4) of the previous paper that

$$[Q(\rho)]^{1/2}|A'(\rho)|^{1/2} \geqq - \int_{C_\rho} G^{-1}\nabla^2 u \, ds;$$

hence

$$\int_\rho^1 [Q(t)]^{1/2}|A'(t)|^{1/2}\,dt \geqq -\int_\rho^1 \int_{C_t} G^{-1}\nabla^2 u\,ds\cdot dt =$$

$$-\int_{\rho\leqq u\leqq 1}\int \nabla^2 u\,d\sigma = \int_{C_\rho} \frac{\partial u}{\partial n}\,ds = P(\rho) \geqq \frac{4\pi A(\rho)}{|A'(\rho)|}; \quad (2)$$

here the normal is directed into the *interior* of the domain $\rho \leqq u \leqq 1$, that is, in the direction of the increasing values of u, along $u = \rho$. The contribution of C_t tends to zero as $t \to 1$; indeed, if $u = 1$ holds at certain isolated points, we have grad $u = 0$, and, in addition, the curve shrinks to a point; should $u = 1$ hold along certain curves, grad $u = 0$ must be used.[2]

4. We define the constant ρ_0, $0 < \rho_0 < 1$, by the condition

$$\int_0^{\rho_0} [Q(t)]^{1/2}|A'(t)|^{1/2}\,dt = \int_{\rho_0}^1 [Q(t)]^{1/2}|A'(t)|^{1/2}\,dt, \quad (3)$$

and now introduce the continuous function

$$g(\rho) = \begin{cases} \dfrac{|A'(\rho)|}{4\pi A(\rho)}\displaystyle\int_0^\rho [Q(t)]^{1/2}|A'(t)|^{1/2}\,dt \text{ for } 0 \leqq \rho \leqq \rho_0, \\ \dfrac{|A'(\rho)|}{4\pi A(\rho)}\displaystyle\int_\rho^1 [Q(t)]^{1/2}|A'(t)|^{1/2}\,dt \text{ for } \rho_0 \leqq \rho \leqq 1. \end{cases} \quad (4)$$

In view of Schwarz's inequality, these integrals exist, since the integrals

$$\int_0^1 Q(t)dt \qquad \text{and} \qquad \int_0^1 |A'(t)|dt = A(0)$$

are finite. As before, we define

$$f(\rho) = \int_0^\rho g(\rho)d\rho. \quad (5)$$

The function $\bar{u} = f(\rho)$ has a continuous derivative for $0 < \rho < 1$, including $\rho = \rho_0$. Now

$$\lim_{R\to R_0} \bar{u} = f(0) = 0, \qquad \lim_{R\to R_0} \frac{d\bar{u}}{dR} = \lim_{\rho\to 0} f'(\rho)\,\frac{2\pi R}{A'(\rho)} =$$

$$-\lim_{\rho\to 0} [4\pi A(\rho)]^{-1/2} \int_0^\rho [Q|(t)^{1/2}|A'(t)|^{1/2}\,dt = 0.$$

Moreover, $\lim_{R\to 0} \bar{u}$ is finite, since

$$\int_\rho^1 [Q(t)]^{1/2}|A'(t)|^{1/2}\,dt \leqq \{\int_\rho^1 Q(t)dt\}^{1/2}\{\int_\rho^1 |A'(t)|dt\}^{1/2}$$

$$\leqq [A(\rho)]^{1/2}\{\int_0^1 Q(t)dt\}^{1/2}, \quad (6)$$

and the integral

$$\int_\rho^1 \frac{|A'(\rho)|}{4\pi A(\rho)}[A(\rho)]^{1/2}\,d\rho = -\int_0 \frac{dA}{4\pi A^{1/2}}$$

is convergent. Finally,

$$\lim_{R\to 0} \frac{d\bar{u}}{dR} = \lim_{R\to 0} f'(\rho)\,\frac{2\pi R}{A'(\rho)} = -\lim_{R\to 0} [4\pi A(\rho)]^{-1/2} \int_\rho^1 [Q(t)]^{1/2}|A'(t)|^{1/2}\,dt.$$

The latter quantity is, in view of the first step in (6), less than

$$[4\pi A(\rho)]^{-1/2}\,[A(\rho)]^{1/2}\left\{\int_\rho^1 Q(t)dt\right\}^{1/2} = (4\pi)^{-1/2}\left\{\int_\rho^1 Q(t)dt\right\}^{1/2} \to 0$$

as $\rho \to 1$. Thus $\bar{u}$ as a function of $Re^{i\theta} = x + iy$ has continuous first derivatives with respect to x and y, including at $R = 0$; indeed,

$$\frac{\partial \bar{u}}{\partial x} = \frac{d\bar{u}}{dR}\frac{x}{R} \to 0 \text{ as } R \to 0; \text{ similarly } \frac{\partial \bar{u}}{\partial y}.$$

Recapitulating, we find that $\bar{u}$ is admissible for the circular plate (it has a discontinuity in the second derivatives).

5. The remaining argument needs only unessential modifications. We have

$$\nabla^2 \bar{u} = \pm \left|A'(\rho)\right|^{-1/2}[Q(\rho)]^{1/2}$$

according as $\rho \lesseqgtr \rho_0$, so that the numerator in (1) does not change when we pass from u to $\bar{u}$, D to $\bar{D}$. In view of (2), we again have $g(\rho) \geqq 1, f(\rho) \geqq \rho$, so that the denominator is increased.

This completes the modified argument. The same reasoning settles the problem of the buckling plate.

6. Finally, it is instructive to discuss briefly the case when D is the circle $r \leqq 1$. Then $\bar{u}(R) = \bar{u}(r)$ coincides with the first eigenfunction $v(hr)$, where

$$v(r) = J_0(r)I_0'(h) - J_0'(h)I_0(r), \qquad h = 3.1962$$

(cf. Pólya-Szegö, *op. cit.*, p. 134). This function is decreasing for $0 \leqq r \leqq h$, $v(0) > 0$, $v(h) = 0$. The relation $\rho = [v(0)]^{-1}v(hr)$ holds.
Now

$$\nabla^2\bar{u} = \frac{h}{r}\frac{d}{dr}(r\,v'(hr)),$$

and, taking the equations $(rJ_0'(r))' + rJ_0 = 0$, $(rI_0'(r))' - rI_0 = 0$ into account, we obtain

$$\frac{1}{r}\frac{d}{dr}(r\,v'(r)) = -J_0(r)I_0'(h) - J_0'(h)I_0(r) = -I_0'(h)I_0(r)\left(\frac{J_0(r)}{I_0(r)} + \frac{J_0(h)}{I_0(h)}\right).$$

Here $-1 < J_0(h)/I_0(h) < 0$. Since $J_0(r)/I_0(r)$ is decreasing for $0 < r < j$ [$j = 2.4048$ is the first positive zero of $J_0(r)$], the latter quantity is negative for $r = 0$ and positive for $r = j < h$. Also $\nabla^2\bar{u}$ changes its sign for a single value $r = r_0$, $0 < r_0 < j$, passing there from negative to positive values. This r_0 corresponds to ρ_0, and in this case $\nabla^2\bar{u}$ vanishes at $\rho = \rho_0$.

[1] Cf. also G. Pólya and G. Szegö, "Isoperimetric Inequalities in Mathematical Physics," *Ann. Math. Studies*, **27**, 235–238, 1951, note F.

[2] Discussing the part $0 \leqq u \leqq \rho$ (cf. *loc. cit.*), we must deal in a similar fashion with the points or lines in the interior of D where $u = 0$.