Linear Algebra and its Applications 436 (2012) 99–111

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

On weighted directed graphs

R.B. Bapat [a], D. Kalita [b,1], S. Pati [b,*]

[a] Stat-Math Unit, Indian Statistical Institute Delhi, 7-SJSS Marg, New Delhi 110 016, India
[b] Department of Mathematics, Indian Institute of Technology Guwahati, Assam 781 039, India

ARTICLE INFO

Article history:
Received 6 January 2011
Accepted 16 June 2011
Available online 18 July 2011

Submitted by S. Kirkland

AMS classification:
05C50
05C05
15A18

Keywords:
Laplacian matrix
Mixed graph
Weighted directed graph
3-Colored digraph
Edge singularity

ABSTRACT

The study of a mixed graph and its Laplacian matrix have gained quite a bit of interest among the researchers. Mixed graphs are very important for the study of graph theory as they provide a setup where one can have directed and undirected edges in the graph. In this article we present a more general structure, namely the weighted directed graphs and supply appropriate generalizations of several existing results for mixed graphs related to singularity of the corresponding Laplacian matrix. We also prove many new combinatorial results relating the Laplacian matrix and the graph structure.

1. Introduction

All our graphs are simple. All our directed graphs have simple **underlying** undirected graphs. We denote the set of vertices and the set of edges of a graph G by $V(G)$ and $E(G)$, respectively. A *mixed graph* is a graph with some directed and some undirected edges. Several researchers have studied different aspects like coloring, balancing, matrix-tree type theorem, the spectral properties of the Laplacian matrices of mixed graphs, see for example [1,2,13,14,11,3,8,9] and the references therein. Our aim in this article is to present and study a more general class of graphs of which the set of mixed graphs is a particular subset.

Let G be a mixed graph on vertices $1, \ldots, n$. We write $ij \in E(G)$ to mean the existence of an undirected edge between the vertices i and j. We write $(i, j) \in E(G)$ to mean the existence of the

* Corresponding author.
E-mail addresses: rbb@isid.ac.in (R.B. Bapat), k.debajit@iitg.ernet.in (D. Kalita), pati@iitg.ernet.in (S. Pati).
1 The financial assistance for this author was provided by CSIR, India, through a SRF.

0024-3795/$ - see front matter
doi:10.1016/j.laa.2011.06.035

directed edge from the vertex i to the vertex j. The *adjacency matrix* $A(G) = [a_{ij}]$ of G is the matrix with

$$a_{ij} = \begin{cases} 1 & \text{if } ij \in E(G), \\ -1 & \text{if } (i,j) \in E(G) \text{ or } (j,i) \in E(G), \\ 0 & \text{else,} \end{cases}$$

where the word 'or' is used in the exclusive sense. The *degree* d_i of a vertex i in G is the number of edges (both directed and undirected) incident with i. Let $D(G)$ be the diagonal matrix with d_i as the ith diagonal entry. Bapat, Grossman and Kulkarni [1] introduced the *Laplacian matrix* $L(G) = D(G) + A(G)$ of a mixed graph G and supplied a generalization of the Matrix Tree Theorem for mixed graphs. They also discussed the singularity of a mixed graph. Zhang and Li [13], and Zhang and Luo [14] presented some upper bounds for the spectral radius and the second smallest eigenvalue of the Laplacian matrix of a mixed graph. Fan studied the Laplacian spectral integral variations of mixed graphs occurring in one place in [5], largest eigenvalue and least eigenvalue of unicyclic mixed graphs in [6,7], the eigenvectors corresponding to the smallest Laplacian eigenvalue of a mixed graph containing exactly one nonsingular cycle in [8]. Tan and Fan [12] introduced the notion of edge singularity for mixed graphs and proved some inequalities between the edge singularity and smallest Laplacian eigenvalue of a mixed graph. Fan et al. [9] studied the first eigenvalue and the first eigenvectors of nonsingular unicyclic mixed graphs.

Notice that the adjacency (Laplacian) matrix of a mixed graph is indifferent about the orientation of a directed edge; it only differentiates between the directed and undirected edges. Hence their eigenvectors are not expected to give us any information relating the orientations of edges. The adjacency (resp. Laplacian) matrix of a mixed graph may be viewed as the adjacency (resp. Laplacian) matrix of a directed graph, where some of the edges have weight 1 and remaining have weight -1. Motivated by this we focus on the edge weighted directed graphs, where the weights are some complex numbers of absolute value 1 with nonnegative imaginary part, that is, weights are chosen from the upper half part of the unit circle on the complex plane. Henceforth, we shall understand that **weights** are complex numbers of unit modulus with nonnegative imaginary part, unless otherwise specified.

Let G be a weighted directed graph on vertices $1, \ldots, n$. (Recall that the underlying undirected graph of G is simple.) Let us denote the weight of the edge (i,j) by w_{ij}. Let $\overline{\mathrm{w}}_{ij}$ denote the complex conjugate of w_{ij}. The *adjacency matrix* $A(G) = [a_{ij}]$ of G is a matrix with

$$a_{ij} = \begin{cases} \mathrm{w}_{ij} & \text{if } (i,j) \in E(G), \\ \overline{\mathrm{w}}_{ji} & \text{if } (j,i) \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Note that choosing the weights only from the 'upper half part of the unit circle' is not really a restriction: if G has an edge (i,j) with a weight $x + y\mathrm{i}$, then we may replace (i,j) by (j,i) with the weight $x - y\mathrm{i}$, where $\mathrm{i} = \sqrt{-1}$.

The *degree* d_i of a vertex i in a weighted directed graph G is the number of edges incident with i. It may be viewed as the sum of the absolute values of the weights of the edges incident with the vertex i.

Definition 1. Let G be a weighted directed graph. We define the *Laplacian matrix* $L(G)$ of G as the matrix $D(G) - A(G)$, where $D(G)$ is the diagonal matrix with d_i as the ith diagonal entry.

1. Notice that if weight of each edge in G is 1, then our definition of $L(G)$ coincides with the usual Laplacian matrix of an unweighted undirected graph. This has motivated us to use $D(G) - A(G)$ rather than $D(G) + A(G)$ for the Laplacian matrix.

2. If weights of the edges in G are ± 1, then (viewing the edges of weight 1 as directed and the edges of weight -1 as undirected) our definition of $L(G)$ coincides with the Laplacian matrix as defined in [1].
3. If the weight of the edges in G are -1, then our definition of $L(G)$ coincides with the well studied signless Laplacian (see for example, Cvetkovic et al. [4]) of an undirected graph G.

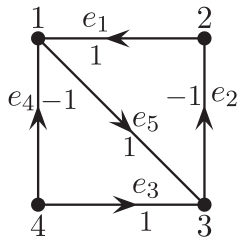
Note that with this setup, the Laplacian matrix of a weighted directed graph is positive semidefinite. The justification is as follows. The *vertex edge incidence matrix* $M = M(G) = [m_{i,e}]$ of G is a matrix with rows labeled by the vertices and columns labeled by the edges in G satisfying

$$m_{i,e} = \begin{cases} 1 & \text{if } e = (i,j) \text{ for some vertex } j, \\ -\overline{\mathrm{w}}_{ji} & \text{if } e = (j,i) \text{ for some vertex } j, \\ 0 & \text{otherwise.} \end{cases}$$

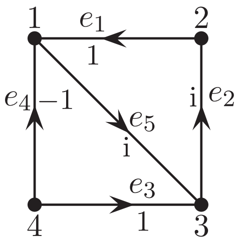
Since $L(G) = MM^*$, it is a positive semidefinite matrix. In particular, if the edges of the weighted directed graph G have weights ± 1, then $M(G)$ coincides with the vertex edge incidence matrix of a mixed graph.

Example 1. Consider the weighted directed graphs G and H shown below. Weights of the edges are written beside them. Their vertex edge incidence matrices are also supplied. Observe that in the graph G if we view the edges having weight 1 as directed and the edges having weight -1 as undirected, then $M(G)$ is the same as the vertex edge incidence matrix of a mixed graph.

G



H



$$M(G) = \begin{bmatrix} -1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \qquad M(H) = \begin{bmatrix} -1 & 0 & 0 & 1 & 1 \\ 1 & \mathrm{i} & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & \mathrm{i} \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Let G be a weighted directed graph on vertices $1, \ldots, n$. Note that for any $x \in \mathbb{C}^n$, and for any edge $e = (i,j)$, we have $(M^*x)_e = x_i - \mathrm{w}_{ij}x_j$. It follows that

$$x^*L(G)x = (M^*x)^*(M^*x) = \sum_{(i,j)\in E(G)} |x_i - \mathrm{w}_{ij}x_j|^2. \tag{1}$$

A i_1-i_k-*walk* W in a weighted directed graph G is a sequence $W : i_1, i_2, \ldots, i_k$ of vertices such that, for $1 \leq p \leq k-1$, we have either $(i_p, i_{p+1}) \in E(G)$ or $(i_{p+1}, i_p) \in E(G)$. If $e = (i_p, i_{p+1}) \in E(G)$, then we say e is *directed along* the walk, otherwise we say e is *directed opposite* to the walk. We call $\mathrm{w}_W = a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{k-1}i_k}$ the weight of the walk W, where a_{ij} are entries of $A(G)$.

We call a connected weighted directed graph *singular* (resp. *nonsingular*) if its corresponding Laplacian matrix is singular (resp. nonsingular).

The organization of the article is as follows. In Section 2, we give several characterizations of singularity for the more general class of weighted directed graphs. This provides a better combinatorial insight. Many results in this section generalize the known results in the literature. We introduce the concept of D-similarity here. We provide a characterization of the connected weighted directed graphs which are D-similar to mixed graphs, which is new of its kind.

In Section 3, we introduce and study the edge singularity for weighted directed graphs and provide the appropriate generalizations of some of the known results for mixed graphs. The problem of characterizing mixed graphs with a fixed edge singularity has never been addressed. We provide a combinatorial characterization of connected weighted directed graphs having a fixed edge singularity.

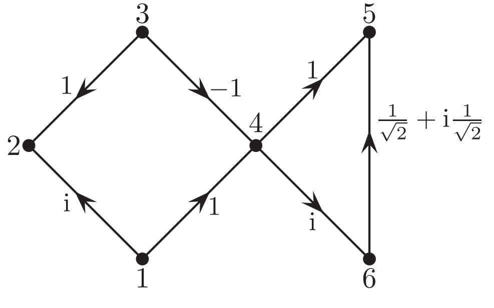
In Section 4, we consider the directed graphs with edges having colors red, blue, or green. We assign each red edge the weight 1, each blue edge the weight -1 and each green edge the weight i. We call this graph a 3-colored digraph. Note that this class is a very small subclass of the weighted directed graphs and is still a larger class than the mixed graphs. We provide some more characterizations of singularity for connected 3-colored digraphs in this section.

The study of spectral properties of the Laplacian and the adjacency matrices of a weighted directed graph is not included in this document.

2. *D*-similarity and singularity in weighted directed graphs

It was first observed in [1], that unlike the Laplacian matrix of an undirected graph, the Laplacian matrix of a mixed graph is sometimes nonsingular. Several characterizations of singularity for mixed graphs were provided in [1]. It is natural to ask for similar characterization of singularity for the weighted directed graphs. In this section, we provide some characterizations of the singular weighted directed graphs.

Example 2. Consider the weighted directed graph G shown below. Observe that $W_1 : 1, 4, 5, 6$ and $W_2 : 1, 4, 6$ are two different 1-6-walks in G with the weights $\mathrm{w}_{W_1} = \frac{1}{\sqrt{2}} - \mathrm{i}\frac{1}{\sqrt{2}}$ and $\mathrm{w}_{W_2} = \mathrm{i}$, respectively. Clearly $\mathrm{w}_{W_1} \neq \mathrm{w}_{W_2}$.



In view of Example 2 above, a natural question is the following: Does there exist a weighted directed graph G such that each u-v-walk in G has the same weight, for each fixed $u, v \in V(G)$?

The answer to this question is in the affirmative, for example, we consider a weighted directed graph G with all the edges having weight 1. Note that such a weighted directed graph is always singular. So it is natural to ask the following question : Does there exist a nonsingular weighted directed graph H such that each u-v-walk in H has the same weight, for each fixed $u, v \in V(H)$?

Let G be a connected weighted directed graph on vertices $1, \ldots, n$. Assume that weight of any 1-i-walk is the same. By $\mathbf{n}$ we denote the vector of size n defined by $\mathbf{n}_1 = 1$ and $\mathbf{n}_i =$ conjugate of the weight of a 1-i-walk which is the same as the weight of a i-1-walk. The following result answers the previous question in the negative.

Lemma 1. *Let G be a connected weighted directed graph on vertices* $1, \dots, n$. *Then* $L(G)$ *is singular if and only if the weight of any* 1-*i*-*walk is the same. Furthermore, when* $L(G)$ *is singular,* 0 *is a simple eigenvalue with an eigenvector* $\mathbf{n}$.

Proof. Suppose that $L(G)$ is singular. Let $x \neq 0$ be a null vector. Then from (1), we have $x_u = \mathrm{w}_{uv}x_v$, whenever (u, v) is an edge. Note that, if $x_u = 0$, then for each neighbor w of u, we have $x_w = 0$. As G is connected, this implies that $x = 0$. Hence the eigenvalue 0 has multiplicity one. Let W be any 1-i-walk. Using (1), we have $x_1 = \mathrm{w}_W x_i$. Hence each 1-i-walk has the same weight and $x = x_1\mathbf{n}$.

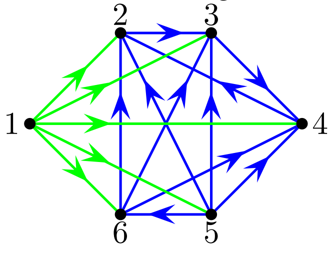
Conversely, suppose that the weight of any 1-i-walk is the same. Note that if $(i, j) \in E(G)$, then $\mathbf{n}_j = \overline{\mathrm{w}}_{ij}\mathbf{n}_i$. Using (1), we have

$$\mathbf{n}^*L(G)\mathbf{n} = \sum_{(i,j)\in E(G)} |\mathbf{n}_i - \mathrm{w}_{ij}\mathbf{n}_j|^2 = 0.$$

Therefore $\|M^*\mathbf{n}\|^2 = 0$ and $L(G)\mathbf{n} = MM^*\mathbf{n} = 0$. So $L(G)$ is singular. □

It follows that the class of singular connected weighted directed graphs is same as the class of connected weighted directed graphs G having the property that each u-v-walk in G has the same weight, for each fixed $u, v \in V(G)$.

Example 3. The graph in the following picture is a weighted directed graph. Here the blue colored edges have a weight -1 which can be given any direction and the green colored edges have a weight i. Note that the graph is nonsingular and the smallest eigenvalue has multiplicity 5.



Let G be a weighted directed graph on vertices $1, \dots, n$ and D be a diagonal matrix with $|d_{ii}| = 1$, for each i. Then $D^*L(G)D$ is the Laplacian matrix of another weighted directed graph which we denote by ${}^DG$. If $(i, j) \in E(G)$ has a weight w_{ij}, then it has the weight $\overline{d}_{ii}\mathrm{w}_{ij}d_{jj}$ in ${}^DG$. Let H and G be two weighted directed graphs on vertices $1, \dots, n$. We say *H is D-similar to G*, if there exists a diagonal matrix D (with $|d_{ii}| = 1$, for each i) such that $H = {}^DG$.

The following result tells that a singular connected weighted directed graph is nothing but an unweighted undirected graph allowing D-similarity.

Lemma 2. *Let G be a connected weighted directed graph on vertices* $1, \dots, n$. *Then* $L(G)$ *is singular if and only if G is D-similar to an unweighted undirected graph.*

Proof. Suppose that $L(G)$ is singular. By Lemma 1, the vector $\mathbf{n}$ is well defined. Take D to be the diagonal matrix with $d_{ii} = \mathbf{n}_i$, for each i. We have $(D^*L(G)D)_{ij} = \overline{\mathbf{n}}_i\, l_{ij}\, \mathbf{n}_j$. If $(i, j) \in E(G)$, then $l_{ij} = -\mathrm{w}_{ij} = -\mathbf{n}_i/\mathbf{n}_j$, so that $\overline{\mathbf{n}}_i\, l_{ij}\, \mathbf{n}_j = -1$. If $(j, i) \in E(G)$, then $l_{ij} = -\overline{\mathrm{w}}_{ji} = -\overline{\mathbf{n}}_j/\overline{\mathbf{n}}_i$, so that $\overline{\mathbf{n}}_i\, l_{ij}\, \mathbf{n}_j = -1$. Furthermore $l_{ii} = d_i$, so that $\overline{\mathbf{n}}_i\, l_{ii}\, \mathbf{n}_i = d_i$. The converse is trivial. □

Remark 3. Notice that when G is a singular mixed graph, the vector $\mathbf{n}$ is the vector with entries 1 or -1. Hence in this case the diagonal matrix D in Lemma 2 is nothing but a signature matrix.

The following result is a characterization of singularity of weighted directed cycles. It will be used to give another characterization of a nonsingular weighted directed graph.

Lemma 4. *Let C be a weighted directed graph on vertices $1, \dots, n$ whose underlying undirected graph is a cycle. Then C is singular if and only if $\mathrm{w}_C = 1$.*

Proof. If C is singular then by Lemma 1, we have $1 = \mathrm{w}_C$. Conversely let $\mathrm{w}_C = 1$ and W_1 be a 1-i-path, $i \neq 1$. Let W_2 be the other 1-i-path. Denote by W_3 the i-1 path obtained by tracing back W_2. Then $1 = \mathrm{w}_C = \mathrm{w}_{W_1}\mathrm{w}_{W_3}$, which implies that $\mathrm{w}_{W_1} = 1/\mathrm{w}_{W_3} = \mathrm{w}_{W_2}$. Hence by Lemma 1, C is singular. □

Throughout this article a cycle C in a weighted directed graph is said to be *nonsingular* if its weight $\mathrm{w}_C \neq 1$. Otherwise we call it a *singular* cycle.

Remark 5. Notice that if we consider mixed graphs, then a cycle C is singular if and only if $\mathrm{w}_C = 1$, that is there are an even number of undirected edges (viewing the edges of weight -1 as undirected) on the cycle. That is the cycle is nonsingular if and only if it has an odd number undirected edges. So the previous lemma generalizes Lemma 1 of [1].

The following result gives another characterization of singularity of a connected weighted directed graph.

Lemma 6. *Let G be a connected weighted directed graph on vertices $1, \dots, n$. Then $L(G)$ is singular if and only if there exist a partition $V(G) = V_1 \cup \cdots \cup V_k$ such that the following conditions are satisfied.*

(i) There are distinct complex numbers w_i of unit modulus associated to each V_i, for $i = 1, \dots, k$.
(ii) Any edge between V_i and V_j, $i < j$ is either directed from V_i to V_j with a weight $\mathrm{w}_i\overline{\mathrm{w}}_j \neq 1$ or is directed from V_j to V_i with a weight $\overline{\mathrm{w}}_i\mathrm{w}_j \neq 1$.
(iii) Each edge within V_i has a weight 1, for $i = 1 \dots, k$.

Proof. Suppose that $L(G)$ is singular. By Lemma 1, 0 is a simple eigenvalue and $\mathbf{n}$ is a null vector of $L(G)$. Let $V_i = \{j \in V(G) : \mathbf{n}_j = \mathbf{n}_i\}$. Let $u \in V_i$, $v \in V_j$, $i < j$ such that (u, v) is an edge. If $\mathrm{w}_{uv} = 1$, then $\mathbf{n}_u = \mathbf{n}_v$, which is not possible. Since $\mathbf{n}_u = \mathbf{n}_i$ and $\mathbf{n}_v = \mathbf{n}_j$, we must have $\mathrm{w}_{uv} = \mathbf{n}_i\overline{\mathbf{n}}_j \neq 1$, by Lemma 1 and the definition of $\mathbf{n}$. Similarly, if (v, u) is an edge, then we must have $\mathrm{w}_{vu} = \mathbf{n}_j\overline{\mathbf{n}}_i \neq 1$. So to each V_i we associate the complex number $\mathrm{w}_i = \mathbf{n}_i$. By the definition of $\mathbf{n}$, it is easy to see that edges within V_i have weights 1.

Conversely, suppose that $V(G) = V_1 \cup \cdots \cup V_k$, and (i)–(iii) are satisfied. Let D be the diagonal matrix with the diagonal entries $d_{uu} = \mathrm{w}_i$, if $u \in V_i$. Note that $(D^*L(G)D)_{uv} = \overline{d}_{uu}l_{uv}d_{vv}$. If $(u, v) \in E(G)$ has a weight 1, then (as the edges of weight 1 appear only inside a V_i) both $u, v \in V_i$, for some i, where $1 \leq i \leq k$. In that case $d_{uu} = d_{vv}$ and $l_{uv} = -1$, so that $\overline{d}_{uu}l_{uv}d_{vv} = -1$. If $(u, v) \in E(G)$ has a weight other than 1, then $u \in V_i$, $v \in V_j$, for some $i \neq j$. In that case we have $\mathrm{w}_{uv} = \mathrm{w}_i\overline{\mathrm{w}}_j$, by (ii). Thus $\overline{d}_{uu}l_{uv}d_{vv} = \overline{\mathrm{w}}_i(-\mathrm{w}_i\overline{\mathrm{w}}_j)\mathrm{w}_j = -1$. Furthermore, $\overline{d}_{uu}l_{uu}d_{uu} = l_{uu}$. Noting that $D^*L(G)D$ is Hermitian, we see that $D^*L(G)D$ is the Laplacian matrix of the underlying unweighted undirected graph of G. Hence $L(G)$ is singular, by Lemma 2. □

Remark 7. Notice that if we have mixed graph in Lemma 6, then we have only two types of weights. Hence a connected mixed graph is singular if and only if there exist a partition $V(G) = V_1 \cup V_2$ such that edges inside V_i have weights 1 and edges between V_1 and V_2 have weights -1.

The following theorem which is a summary of the previous discussions is a generalization of Theorem 4 of [1].

Theorem 8. *Let G be a connected weighted directed graph. Then the following are equivalent.*

(a) G is singular.
(b) There is a diagonal matrix D with diagonal entries of unit modulus such that $D^*L(G)D$ is the Laplacian matrix of the underlying unweighted undirected graph.
(c) Each cycle C in G has weight $\mathrm{w}_C = 1$.

(d) There exist a partition $V(G) = V_1 \cup \cdots \cup V_k$ such that the following conditions are satisfied.
(i) There are distinct complex numbers w_i of unit modulus associated to each V_i, for $i = 1, \ldots, k$.
(ii) Any edge between V_i and V_j, $i < j$ is either directed from V_i to V_j with a weight $\mathrm{w}_i\overline{\mathrm{w}}_j \neq 1$ or is directed from V_j to V_i with a weight $\overline{\mathrm{w}}_i\mathrm{w}_j \neq 1$.
(iii) Each edge within V_i has a weight 1, for $i = 1 \ldots, k$.

Proof. (a) $\Leftrightarrow$ (b) Follows from Lemma 2.

(b) $\Leftrightarrow$(c) Suppose that there is a diagonal matrix D with diagonal entries of unit modulus such that $D^*L(G)D$ is the Laplacian matrix of the underlying unweighted undirected graph. Let ${}^DG$ be the underlying unweighted undirected graph of G. If $(i,j) \in E(G)$ has a weight w_{ij}, then it has the weight $\overline{d}_{ii}\mathrm{w}_{ij}d_{jj}$ in ${}^DG$. So the weight of a cycle C in G remains the same in ${}^DG$. Note that each cycle in ${}^DG$ has weight 1. Hence the result holds.

Conversely suppose that each cycle C in G has weight $\mathrm{w}_C = 1$. Let T be a weighted directed spanning tree of G. Take D to be the diagonal matrix with $d_{11} = 1$ and $d_{ii} =$ weight of the i-1-path in T, for $i \neq 1$. Consider the graph ${}^DG$ whose Laplacian matrix is $D^*L(G)D$. Note that $(D^*L(G)D)_{ij} = \overline{d}_{ii}l_{ij}d_{jj}$. If $(i,j) \in E(T)$ has weight w_{ij}, then $d_{jj} = \overline{\mathrm{w}}_{ij}d_{ii}$. In that case, weight of the edge (i,j) is 1 in ${}^DG$. If $(i,j) \in E(G) - E(T)$, then G contains a cycle, say C formed by the edge (i,j) and the unique i-j-path, say P in T. Since weight of an edge $e \in E(T)$ changes to the weight 1 in ${}^DG$, we see that weight of the path P is 1 in ${}^DG$. Note that the weight of a cycle in G remains the same in ${}^DG$. Hence weight of the edge (i,j) in ${}^DG$ must be 1, as $\mathrm{w}_C = 1$. Thus ${}^DG$ is the underlying unweighted undirected graph of G.

(b)$\Leftrightarrow$(d). Follows from Lemma 6. $\square$

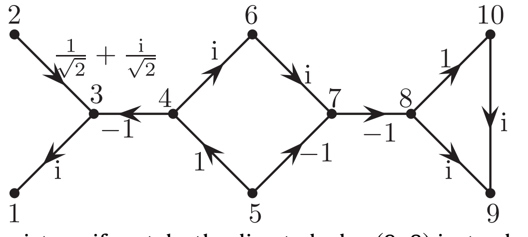
The following result is an immediate consequence.

Corollary 9. *Let G be a connected weighted directed graph. Then G is nonsingular if and only if it contains a nonsingular cycle. In particular, a weighted directed tree is always singular.*

Example 4. The graph in the following picture is a weighted directed graph. Note that there are two cycles and both of them have weight 1. Hence the graph is singular. Indeed one can check that

$$\mathbf{n} = \left[1,\ -\frac{1}{\sqrt{2}} + \frac{\mathrm{i}}{\sqrt{2}},\ \mathrm{i},\ -\mathrm{i},\ -\mathrm{i},\ -1,\ \mathrm{i},\ -\mathrm{i},\ -1,\ -\mathrm{i}\right]$$

is a null vector.



Observe that in the above picture, if we take the directed edge (9, 8) instead of (8, 9), then the weight of the cycle [8, 10, 9, 8] becomes −1. Hence by Corollary 9, the graph is nonsingular.

Note that by Lemma 2, a connected weighted directed graph is singular if and only if it is D-similar to an unweighted undirected graph. The following is a natural question: which connected weighted directed graphs are D-similar to mixed graphs? The following result characterizes those graphs.

Theorem 10. *Let G be a connected weighted directed graph. Then G is D-similar to a mixed graph if and only if G does not contain a cycle of non-real weight.*

Proof. Suppose that G does not contain a cycle of non-real weight. Then each of the cycle contained in G has a weight ± 1, as the weights of the edges have absolute value 1. Let T be a weighted directed spanning tree of G. By Corollary 9, T is singular. By Lemma 2, there is a diagonal matrix D, such that ${}^{D}T$ is an unweighted undirected tree. Consider the graph ${}^{D}G$ for this D. So the weight of an edge $(i,j) \in E(T)$ changes to 1 in ${}^{D}G$. If $(i,j) \in E(G) - E(T)$, then G contains a cycle, say C formed by the edge (i,j) and the unique i-j-path, say P in T. Since weight of an edge $e \in E(T)$ changes to 1 in ${}^{D}G$, we see that the weight of the path P is 1 in ${}^{D}G$. Note that the weight of a cycle in G remains the same in ${}^{D}G$. Thus the weight of the edge (i,j) in ${}^{D}G$ is either 1 or -1, as $w_C = \pm 1$. Hence the graph ${}^{D}G$ is a mixed graph.

Conversely, suppose that G is D-similar to a mixed graph H. So $L(H) = D^{*}L(G)D$ and $H={}^{D}G$. As the weight of a cycle is the same in both G and ${}^{D}G$, we see that the weights of the cycles are real. □

3. Edge singularity of weighted directed graphs

The edge singularity of a mixed graph was studied in [12]. We continue the same study in the context of weighted directed graphs. The *edge singularity* $\varepsilon s(G)$ of a weighted directed graph is the minimum number of edges whose removal results a weighted directed graph containing no nonsingular cycles or cycles with weight different from 1 (by Lemma 4). That is, all components of the resulting graph are singular.

The following result is very fundamental in nature and it relates the edge singularity with connectivity.

Lemma 11. *Let G be a connected weighted directed graph on vertices $1, \ldots, n$. Let F be a set of $\varepsilon s(G)$ edges in G such that $G - F$ does not contain a cycle of weight different from 1. Then $G - F$ is connected.*

Proof. If G is singular, then the result holds obviously. Suppose that G is nonsingular and $G - F$ is disconnected. Let $G_1, G_2, \ldots G_r$, $(r \geqslant 2)$ be the components of $G - F$. As the graph G is connected, we can choose $r - 1$ edges $e_1, e_2, \ldots e_{r-1}$ from F such that the graph

$$H := G_1 \cup G_2 \cup \ldots G_r + \{e_1, e_2, \ldots e_{r-1}\}$$

is connected. So each edge $e_1, \ldots, e_{r-1}$ must be a bridge in H. By Corollary 9, as G_i's do not contain nonsingular cycles, we see that H does not contain a nonsingular cycle. Thus H is singular, by Corollary 9. Hence $\varepsilon s(G) \leqslant |F| - (r-1) < |F|$, a contradiction. □

The following result generalizes Theorem 2.1 of [12] obtained by Tan and Fan for mixed graphs.

Lemma 12. *Let G be a connected weighted directed graph on vertices $1, \ldots, n$ with m edges. Then $0 \leqslant \varepsilon s(G) \leqslant m - n + 1$. In particular, $\varepsilon s(G) = m - n + 1$ if and only if all the cycles contained in G are nonsingular.*

Proof. Clearly, $\varepsilon s(G) \geqslant 0$. Let T be a spanning tree of G. By Corollary 9, T is singular. Thus removal of the $m - n + 1$ edges which are not in T from the graph G makes the resulting graph singular. Hence $\varepsilon s(G) \leqslant m - n + 1$.

Suppose that $\varepsilon s(G) = m - n + 1$ and G contains a singular cycle C. Let H be the unicyclic spanning subgraph of G containing the cycle C. By Corollary 9, H is singular. Thus by deleting the $m - n$ edges from G we obtain a singular weighted directed graph. Hence $\varepsilon s(G) \leqslant m - n < m - n + 1$, a contradiction.

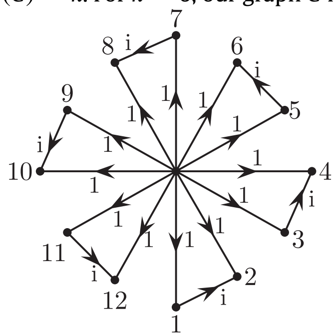
Conversely, suppose that each of the cycles contained in G are nonsingular and $\varepsilon s(G) < m - n + 1$. Let F be a set of $\varepsilon s(G)$ edges in G such that the graph $G - F$ has each component singular. By Lemma 11, $G - F$ is a connected graph and $|E(G - F)| = m - \varepsilon s(G) > n - 1$. Thus $G - F$ contains a cycle, and by the assumption this cycle is nonsingular, a contradiction. Hence the result holds. □

We have two natural questions.

(a) Given a nonnegative integer k, is it possible to find a graph G with $\varepsilon s(G) = k$?
(b) Given n, m and an integer $0 \le k \le m-n+1$, does there exist a graph G with n vertices, m edges for which $\varepsilon s(G) = k$?

The following example will answer the first question.

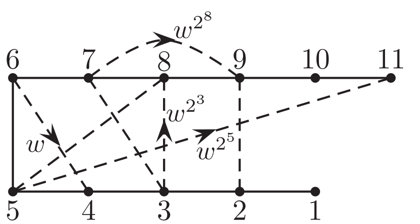
Example 5. Let k be a given nonnegative integer. Consider the weighted directed star H on $2k+1$ vertices with all the edges having a weight 1. Let $1, \ldots, 2k$ be the pendent vertices and v be the vertex of degree $2k$ of the star H. We construct the weighted directed graph G from H by inserting the new directed edges $(j, j+1)$ with an weight i, for $j = 1, 3, \ldots, 2k-1$. Notice that G contains k cycles of length 3 formed by the vertices v, j and $j+1$, for each $j = 1, 3, \ldots, 2k-1$. Let F be any set of edges in G with $|F| < k$. Then $G - F$ contains at least one cycle of the form $[v, j_0, j_0+1, v]$, for some $j_0 \in \{1, 3, \ldots, 2k-1\}$. Hence $\varepsilon s(G) = k$. For $k = 6$, our graph G is shown below.



$G : \varepsilon s(G) = 6$

Let m, n be any given positive integers with $m \leqslant \binom{n}{2}$. Assume that $0 \leqslant k \leqslant m-n+1$. Consider the weighted directed path P_n on n vertices with each edge having a weight 1. Let $w = e^{\mathrm{i}\frac{2\pi}{2^p}}$, where $p \ge k$. Construct a weighted directed graph obtained from P_n by inserting $m-n+1$ new directed edges e_i. To k of these edges assign weights $w^{2^{r_i}}$, where $0 \le r_i \le p-1$ are distinct, $i = 1, \ldots, k$. Assign a weight 1 to the remaining edges. Denote the class of all such graphs by $P(n; m; k)$.

Example 6. Here we give an example of a graph in $P(11; 17; 4)$. It is obtained from the path P_{11} by adding the dotted edges. We choose $p = 9$ and $w = e^{\mathrm{i}\frac{2\pi}{2^9}}$. The undirected edges have weights 1.



In the next result we prove that edge singularity of any graph in $P(n; m; k)$ is k.

Lemma 13. *Let $G \in P(n; m; k)$. Then $\varepsilon s(G) = k$.*

Proof. Let C be a cycle in G which contains l edges of weight different from 1. Then the weight of C is $\mathrm{w}_C = w^K$, where $K = \sum_{i=1}^{l} \pm 2^{r_i}$ and $0 \leqslant r_i \leqslant p-1$ are distinct for $i = 1, \ldots, l$. Since $0 < |K| < 2^p$

and $w = e^{\mathrm{i}\frac{2\pi}{2^p}}$, we see that $w_C = w^K \neq 1$. Thus any cycle in G which contains an edge of weight different from 1 is nonsingular by Lemma 4. We shall use induction to show that $\varepsilon s(G) = k$. For $k = 1$, G contains exactly one edge say, e having a weight $w^{2^{r_1}}$ and e must be contained in a nonsingular cycle of G. Hence $\varepsilon s(G) = 1$. Assume that any graph $H \in P(n; m; k_0)$, $k_0 < k$ has $\varepsilon s(H) = k_0$. Let $G \in P(n; m; k)$, $k > 1$. Let $F = \{e_1, \ldots, e_k\}$ be the set of edges in G such that e_i has a weight $w^{2^{r_i}}$, for $i = 1, \ldots, k$. Since each of the remaining $m - k$ edges in G has an weight 1, $G - F$ does not contain a nonsingular cycle, by Theorem 8. Notice further that, $G - F$ is connected. Thus $\varepsilon s(G) \leqslant k$. If possible, suppose that $\varepsilon s(G) < k$. Let F' be a set of edges in G such that $|F'| = \varepsilon s(G)$ and $G - F'$ does not contain a nonsingular cycle.

Claim. $F' \cap F = \emptyset$. Suppose that our claim is not true. Put $r = |F \cap F'|$. Consider $G - (F \cap F')$. Observe that $\varepsilon s(G - (F \cap F')) \leqslant \varepsilon s(G) - r < k - r$. But, as the graph $G - (F \cap F') \in P(n; m - r; k - r)$, by induction hypothesis, $\varepsilon s(G - (F \cap F')) = k - r$. This is a contradiction. Hence our claim is valid.

Recall that $G - F'$ does not contain a nonsingular cycle. By the observation given in the beginning of the proof we see that each edge $e_i \in F$ must be a bridge in $G - F'$. As $|F| = k$, we see that $G - F' - F$ has at least $k + 1$ components. On the other hand, as the graph $G - F$ is connected and as $|F'| < k$, the graph $G - F - F'$ can have at most k components. This is a contradiction. Hence $\varepsilon s(G) = k$. Our proof is complete. □

Remark 14.

(a) Lemma 13 answers the question raised in this section in the affirmative.
(b) In Lemma 13, we only used the fact that the graphs in $P(n; m; k)$ are created from a connected graph. So the statement of the lemma will remain true for graphs in $T(n; m; k)$ which are created from a tree T in a similar way.

The graphs in $P(n; m; k)$ may be viewed as some graphs obtained from a connected undirected graph by adding k edges of weight different from 1. So a natural question is the following: is it true that each connected weighted directed graph G with $\varepsilon s(G) = k$ can be created from a connected unweighted undirected graph by adding k directed edges of weight different from 1?

The answer is in the affirmative as shown below.

Theorem 15. *Let G be a connected weighted directed graph on n vertices with $\varepsilon s(G) = k$. Then there is a diagonal matrix D with diagonal entries of unit modulus such that G is D-similar to a graph H, where H is obtained from a connected unweighted undirected graph by adding k directed edges of weights different from* 1.

Proof. Let F be a set of edges in G such that $|F| = \varepsilon s(G)$ and $G - F$ has each component singular. By Lemma 11, the graph $G - F$ is connected. Let D be the diagonal matrix with ith diagonal entry $d_{ii} = \mathbf{n}_i$, where $\mathbf{n}$ is the null vector of $G - F$. By Lemma 2, $G - F$ is D-similar to the unweighted undirected graph $H_0 := {}^{D}G - {}^{D}F$, where ${}^{D}F$ is the set of edges in ${}^{D}G$ corresponding to F. Note that H_0 is connected as $G - F$ is connected. As $\varepsilon s(G) = \varepsilon s({}^{D}G)$, we see that edges in ${}^{D}F$ must have weights other than 1. Put $H = {}^{D}G$. Then the graph G is D-similar to H which can be obtained from the connected unweighted graph H_0 by adding the k directed edges contained in ${}^{D}F$. □

4. 3-Colored digraphs and their singularity

Let G be a directed graph with edges having colors red, blue, or green. We assign each red edge the weight 1, each blue edge the weight -1 and each green edge the weight i. We call this graph a 3-*colored digraph.* Notice that this notion naturally generalizes the notion of a mixed graph but is much restricted in comparison to the weighted directed graph.

In the Theorem 8 of Section 2, we have given some characterizations of a singular connected weighted directed graph. In this section we supply an additional characterization of singularity of a connected 3-colored digraph. Further information on the structure of a singular connected 3-colored digraph is obtained.

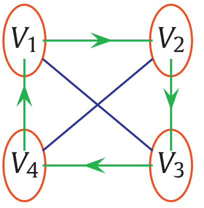


Fig. 1. The structure of a singular 3-colored digraph.

Remark 16. In particular, if a 3-colored digraph G does not contain a green edge, then G is nothing but a mixed graph. In that case an edge with color red corresponds to a directed edge and an edge with color blue corresponds to an undirected edge.

Let G be a 3-colored digraph on vertices $1, \ldots, n$. Note that, if $(i,j) \in E(G)$ has a color red or blue, then $a_{ij} = a_{ji} = 1$ or -1, respectively, where a_{ij} is the ijth entry of the adjacency matrix $A(G)$. So the adjacency (Laplacian) matrix of a 3-colored digraph is indifferent about the orientations of the red and blue colored edges. In view of this fact, we write $ij \in E(G)$ to mean the existence of a red or a blue colored edge between the vertices i and j in G. We write $(i,j) \in E(G)$ to mean the existence of the green colored edge directed from the vertex i to the vertex j in G.

The following theorem characterizes the structure of a singular connected 3-colored digraph. It generalizes the result about the structure of a singular mixed graph obtained by Bapat, Grossman and Kulkarni in [1].

Theorem 17. *Let G be a connected 3-colored digraph on vertices $1, \ldots, n$. Then G is singular if and only if there exist a partition $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$ such that the following conditions are satisfied.*

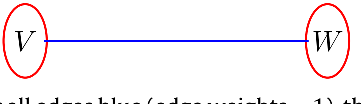
(i) Edges between V_1 and V_3 are blue. Edges between V_2 and V_4 are blue.
(ii) Edges between V_i and V_{i+1} are green and are directed from V_i to V_{i+1}, for each $i \in \mathbb{Z}_4$.
(iii) Edges within V_i are red, $i \in \mathbb{Z}_4$. (See Fig. 1.)

Proof. Suppose that G is singular. By Lemma 1, 0 is a simple eigenvalue and $\mathbf{n}$ is a null vector of $L(G)$. Note that entries of $\mathbf{n}$ are from $\{\pm 1, \pm \mathrm{i}\}$. Let V_1, V_2, V_3, V_4 be the set of those vertices of G which corresponds to the entries $1, -\mathrm{i}, -1$ and i, respectively in $\mathbf{n}$. Let $u \in V_1, v \in V_3$ such that e is an edge in G with u and v as the end vertices. Since $\mathbf{n}_u = 1$ and $\mathbf{n}_v = -1$, we must have the weight $\mathrm{w}_{uv} = -1$, by Lemma 1 and the definition of $\mathbf{n}$. Hence any edge connecting V_1 and V_3 must be blue. Similarly any edge connecting V_2 and V_4 must be blue. Similarly edges connecting V_i and V_{i+1} must be green, directed from V_i to V_{i+1}, for each $i \in \mathbb{Z}_4$. It is easy to see that edges within V_i must be red.

Conversely, suppose that $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$, and (i)–(iii) are satisfied. We associate the complex numbers $1, -\mathrm{i}, -1$ and i to V_1, V_2, V_3 and V_4, respectively. Thus by Theorem 8(d), G is singular. □

Remark 18. (a) Notice that in Theorem 17, some of the V_i's could be empty. For example, taking G an unweighted undirected graph, we have $V(G) = V_1$. Hence the structure of a connected singular 3-colored digraph naturally extends that of the unweighted undirected graph.

(b) Notice further that, as a mixed graph does not have green edges, the components V_2 and V_4 in Theorem 17 are empty. Hence the structure of a connected singular mixed graph is as shown in the following picture.



(c) Observe that if we consider all edges blue (edge weights -1), then $L(G)$ is the signless Laplacian. As we do not have red edges and green edges, we see the following well known result: *the signless Laplacian of a connected undirected graph is singular if and only if the graph is bipartite*.

The following theorem says that a singular 3-colored digraph is D-similar to a 3-colored digraph with all the edges having color red.

Theorem 19. *Let G be a connected 3-colored digraph on vertices $1, \ldots, n$. Then G is singular if and only if there is a diagonal matrix D with diagonal entries from the set $\{\pm 1, \pm \mathrm{i}\}$ such that $D^*L(G)D$ is the Laplacian matrix of the 3-colored digraph ${}^{D}G$ with all the edges having color red.*

Proof. Using Lemma 2 and the information about the entries of D, the proof easily follows. □

Remark 20. Notice that in the case of mixed graphs we do not have green edges. Hence a singular mixed graph has a null vector $\mathbf{n}$ with entries ± 1. In that case the diagonal matrix D in Theorem 19 is nothing but a signature matrix. Thus Theorem 19 is a generalization of Theorem 4 (iii) of [1].

Let $C = [1, \ldots, k, 1]$ be a cycle contained in a 3-colored digraph G on vertices $1, \ldots, n$. Let $n_b(C)$ denote the number of blue edges in C. Let $n_g^+(C)$ and $n_g^-(C)$ denote the number of green edges in C which are directed along the cycle and the number of green edges in C which are directed opposite to the cycle, respectively. The following result is crucial for another characterization of singularity for 3-colored digraphs which is done next.

Lemma 21. *Let G be a 3-colored digraph on vertices $1, \ldots, n$ whose underlying undirected graph is a cycle C. Then G is singular if and only if*

(a) either $n_b(C)$ is even and $n_g^+(C) - n_g^-(C) \equiv 0 \pmod 4$,
(b) or $n_b(C)$ is odd and $n_g^+(C) - n_g^-(C) \equiv 2 \pmod 4$.

Proof. Using Lemma 4, $L(G)$ is singular if and only if

$$1 = \mathrm{w}_C = (-1)^{n_b(C)} \mathrm{i}^{n_g^+(C)} (-\mathrm{i})^{n_g^-(C)} = (-1)^{n_b(C)} \mathrm{i}^{n_g^+(C) - n_g^-(C)},$$

which implies the result. □

Remark 22. Note that Theorem 17, Theorem 19 and Lemma 21 together naturally generalizes Theorem 4 of [1].

The following theorem gives a characterization of connected nonsingular 3-colored digraphs.

Theorem 23. *Let G be a connected 3-colored digraph on vertices $1, \ldots, n$. Then G is nonsingular if and only if G has a cycle $C = [1, \ldots, k, 1]$ satisfying one of the following conditions:*

(a) $n_g^+(C) - n_g^-(C) \equiv 1 \pmod 2$,
(b) $n_b(C)$ is even and $n_g^+(C) - n_g^-(C) \equiv 2 \pmod 4$,
(c) $n_b(C)$ is odd and $n_g^+(C) - n_g^-(C) \equiv 0 \pmod 4$.

Proof. Suppose that G is nonsingular. By Corollary 9, G contains a nonsingular cycle, say C. Hence by Lemma 21, it follows that the cycle C satisfies one of the conditions (a), (b) or (c).

Conversely, suppose that G contains a cycle C satisfying one of the conditions (a), (b) or (c). Then by Lemma 21, the cycle C is nonsingular. Hence G is nonsingular, by Corollary 9. □

Remark 24. Notice that in the case of mixed graphs we do not have green edges. Hence a mixed graph on vertices $1, \ldots, n$ whose underlying undirected graph is a cycle is nonsingular if and only if $n_b(C)$ is odd. Thus in view of remark 16, Theorem 23 is a generalization of Lemma 1 of [1].

Acknowledgements

The authors sincerely thank the referees for carefully reading the manuscript and their suggestions which has improved the manuscript. The authors also thank the editors for some valuable suggestions.

References

[1] R.B. Bapat, J.W. Grossman, D.M. Kulkarni, Generalized matrix tree theorem for mixed graphs, Linear and Multilinear Algebra 46 (1999) 299–312.
[2] R.B. Bapat, J.W. Grossman, D.M. Kulkarni, Edge version of the matrix tree theorem for trees, Linear and Multilinear Algebra 47 (2000) 217–229.
[3] A. Bernath, G. Joret, Well-balanced orientations of mixed graphs, Inform. Process. Lett. 106 (2008) 149–151.
[4] D. Cvetkovic, P. Rowlinson, S.K. Simic, Signless Laplacian of finite graphs, Linear Algebra Appl. 423 (2007) 155–171.
[5] Y.Z. Fan, On spectral integral variations of mixed graphs, Linear Algebra Appl. 374 (2003) 307–316.
[6] Y.Z. Fan, Largest eigenvalue of a unicyclic mixed graph, Appl. Math. J. Chinese Univ. Ser. B 19 (2004) 140–148.
[7] Y.Z. Fan, On the least eigenvalue of a unicyclic mixed graph, Linear and Multilinear Algebra 53 (2005) 97–113.
[8] Y.Z. Fan, On eigenvectors of mixed graphs with exactly one nonsingular cycle, Czechoslovak Math. J. 57 (2007) 1215–1222.
[9] Y.Z. Fan, S.C. Gong, Y. Wang, Y.B. Gao, First eigenvalue and first eigenvectors of a nonsingular unicyclic mixed graph, Discrete Math. 309 (2009) 2479–2487.
[10] R.A Horn, C.R Johnson, Matrix Analysis, Cambridge University Press, 1985.
[11] B. Ries, D. de Werra, On two coloring problems in mixed graphs, European J. Combin. 29 (2008) 712–725.
[12] Y.Y. Tan, Y.Z. Fan, On edge singularity and eigenvectors of mixed graphs, Acta Math. Sinica (English Ser.) 24 (2008) 139–146.
[13] X.D. Zhang, J.S. Li, The Laplacian spectrum of a mixed graph, Linear Algebra Appl. 353 (2002) 11–20.
[14] X.D. Zhang, R. Luo, The Laplacian eigenvalues of mixed graphs, Linear Algebra Appl. 362 (2003) 109–119.