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Spectral properties of matrices associated with some directed graphs

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Abstract

We study the spectral properties of certain non-self-adjoint matrices associated with large directed graphs. Asymptotically the eigenvalues converge to certain curves, apart from a finite number that have limits not on these curves.

1. *Introduction*

In an earlier paper we studied the spectral properties of small, possibly random, perturbations of an $n \times n$ Jordan matrix, and proved that most, but not necessarily all, of the eigenvalues lie close to a certain circle as $n \to \infty$. In this paper we consider a similar question when the perturbation is a sparse matrix, whose entries need not be small.

It turns out that this problem is naturally associated with a type of directed graph that is relevant to a variety of problems involving unidirectional flows along one-dimensional channels between several junctions. Since the entity flowing may be a fluid (for example, blood flow through the network of veins and arteries), traffic or electronic data, the ideas here have potentially wide applicability. The results in this paper also provide asymptotic spectral information for a special class of large directed graphs.

The spectral theory of a directed, acyclic graph is not particularly interesting. By labelling the vertices appropriately one may make the adjacency matrix of the graph upper triangular, so that its eigenvalues coincide with its diagonal elements. This correctly suggests that the interesting spectral properties of directed graphs are highly dependent on the structure of its cycles.

Let S be a finite set and let $A_{i,j}$ be real or complex numbers indexed by $i, j \in S$. We make S into a directed graph $(S, \to)$ by putting $i \to j$ if $A_{i,j} \neq 0$. We see that S can be partitioned into disjoint 'junctions' $J_1, J_2, \ldots, J_k$ linked by one-dimensional 'channels' $C_1, C_2, \ldots, C_h$. A natural way of constructing the graph is to draw the junctions as disjoint localized regions, and then join them by directed channels. However, one may also represent the channels by a sequence of horizontal lines, possibly of varying lengths, all directed from left to right. The junctions collectively may then be regarded as quasi-periodic boundary conditions that join the right ends back to the left. In many of the applications the entries of A are non-negative and the spectrum of A may also be investigated by using the Perron–Frobenius theory, (see [**6**]). The theorems in this paper do not depend on this assumption, which is not appropriate in some contexts.

Given the above assumptions, we construct a new graph $(S^{(n)}, \to)$ for every natural number n as follows. We leave the junctions unaltered and replace each channel C_i by a new channel $C_i^{(n)}$ with the same endpoints but with $\#(C_i^{(n)}) = n \times \#(C_i)$. Under suitable assumptions the matrix A induces an associated matrix $A^{(n)}$ whose coefficients are parametrized by pairs of

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points in $S^{(n)}$. Our goal is to investigate the asymptotic behaviour of $\mathrm{Spec}(A^{(n)})$ as $n \to \infty$. Many of the techniques used in our analysis were anticipated by [**10**], which is considered the pioneering work on the subject.

The continuous analogue of our problem replaces each channel by a continuous bounded interval within which the relevant operator is $aD + bI$, where D is first-order differentiation and the constants a and b vary from one interval to another. Instead of junctions one has to specify suitable boundary conditions connecting certain groups of endpoints. In contrast to [**2**], we do not restrict attention to self- or skew-adjoint problems. We finally mention that the wave equation on an undirected graph can be modelled in these terms by replacing each unoriented edge by a pair of oriented edges with the same ends and associating the operators $\pm D$ with the two edges; see [**3**].

Every constant in this paper is independent of the asymptotic parameter n.

2. *Zeros of some analytic functions*

The material in this section is needed for the proofs of many of the later theorems, but does not mention graphs explicitly. The results that we attain are to some extent analogous to calculations in [**1**, **5**, **7**–**9**], but the details are different.

We shall often need to count the number of zeros that an analytic function has in a particular region. The following elementary result enables us to use a change of variable to simplify this task.

LEMMA 2.1. *Let U and V be open subsets of $\mathbf{C}$ and let $g : U \to V$ be conformal* (*that is, analytic with analytic inverse*)*. If $F : V \to \mathbf{C}$ is analytic, then $F \circ g$ has a zero of order m at $u_0 \in U$ if and only if F has a zero of order m at $g(u_0)$.*

We shall be concerned with the limiting behaviour of a particular sequence of sets. Let $\{A_n\}$ be a sequence of subsets in some metric space (X, d) and let A be a closed subset of X. We say that A_n converges to A as $n \to \infty$ if the following two properties are satisfied. (i) For every $z \in A$ there exists a sequence $\{z_n\}$ with $z_n \in A_n$ such that $z_n \to z$ as $n \to \infty$. (ii) For every $z \notin A$ there exists an open set U containing z such that $A_n \cap U = \emptyset$ for all large enough n. We say that A is the limit set of the A_n as $n \to \infty$. Insisting that A be closed ensures that the limit is unique.

We shall make repeated use of a contour that forms the boundary of a sector of an annulus centred at the origin. The contour $C_{r,R,\theta_1,\theta_2}$ is completely determined by four parameters r, R, θ_1, θ_2 where $r < R$ correspond to the two radii of the annulus and $\theta_1 < \theta_2$ correspond to the angular sweep of the sector. More precisely, the contour $C_{r,R,\theta_1,\theta_2} = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ is the concatenation of the following four curves

$$\begin{aligned}
\gamma_1(t) &= te^{i\theta_1}, \quad r \leqslant t \leqslant R, \\
\gamma_2(t) &= Re^{it}, \quad \theta_1 \leqslant t \leqslant \theta_2, \\
\gamma_3(t) &= (r + R - t)e^{i\theta_2}, \quad r \leqslant t \leqslant R, \\
\gamma_4(t) &= re^{i(\theta_1+\theta_2-t)}, \quad \theta_1 \leqslant t \leqslant \theta_2.
\end{aligned}$$

We note that $C_{r,R,\theta_1,\theta_2}$ encloses the open region $\{z \in \mathbf{C} : r < |z| < R, \theta_1 < \arg z < \theta_2\}$.

2.1. *Basic examples*

Before launching into the general analysis that starts with Theorem 2.4, we begin by studying some simpler cases that illustrate the key features of our analysis.

LEMMA 2.2. *Let p_n denote the polynomial*

$$p_n(z) = (z-a)^n(z-b)^n + \alpha(z-a)^n + \beta(z-b)^n + \gamma, \tag{2.1}$$

where $a \neq b$. If α, β and γ are all non-zero, then the roots of p_n converge as $n \to \infty$ to the union of the two circles $|z-a| = 1$ and $|z-b| = 1$.

Proof. Given $\epsilon > 0$, put

$$S_\epsilon = \{z \in \mathbf{C} : |z-a| \geqslant 1+\epsilon\} \cap \{z \in \mathbf{C} : |z-b| \leqslant 1-\epsilon\}.$$

If $z \in S_\epsilon$, then we have

$$\frac{p_n(z)}{(z-a)^n} - \alpha = (z-b)^n + \beta\left(\frac{z-b}{z-a}\right)^n + \frac{\gamma}{(z-a)^n}.$$

Therefore

$$\left|\frac{p_n(z)}{(z-a)^n} - \alpha\right| \leqslant \frac{|\alpha|}{2}$$

for all large enough n, uniformly with respect to $z \in S_\epsilon$. This implies that $p_n(z) \neq 0$. By carrying out similar calculations for the remaining regions, we find that every root of p_n lies in the ϵ-neighbourhood N_ϵ of the union of the two circles, provided n is large enough.

We now show that if z_0 lies in the union of the two circles, then there exists a sequence z_n of complex numbers such that $z_n \to z_0$ and $p_n(z_n) = 0$ for all n. There are essentially two cases to consider, depending on whether z_0 lies on one or both of the circles $|z-a| = 1$ and $|z-b| = 1$.

Firstly, suppose z_0 lies on only one of the circles. For the sake of definiteness we suppose that $|z_0 - a| = 1$ and $|z_0 - b| < 1$. Let U be a small open neighbourhood of z_0 so that $|(z-a)(z-b)| < c < 1$ and $|z-b| < c < 1$ for all z in U. This implies that $q_n(z) = (z-a)^n(z-b)^n + \beta(z-b)^n$ is uniformly exponentially small for z in U as $n \to \infty$. This suggests that inside U the roots of

$$p_n(z) = \alpha(z-a)^n + \gamma + q_n(z)$$

should be close to the roots of $f_n(z) = \alpha(z-a)^n + \gamma$.

Suppose that $z_0 = e^{i\phi_0} + a$. Let $w_n = |\gamma/\alpha|^{1/n}e^{i\phi_n} + a$ be a root of $f_n(z)$ which is closest to z_0. Since $|\gamma/\alpha|^{1/n} \to 1$ and $\phi_n \to \phi_0$, we see that $w_n \to z_0$. Let C_n be the contour $C_{r_n,R_n,\theta_{1,n},\theta_{2,n}} + a$, where $r_n = \left(\frac{1}{2}|\gamma/\alpha|\right)^{1/n}$, $R_n = \left(\frac{3}{2}|\gamma/\alpha|\right)^{1/n}$, $\theta_{1,n} = \phi_n - \pi/2n$ and $\theta_{2,n} = \phi_n + \pi/2n$. The region enclosed by C_n contains w_n.

Let $M = |\gamma|/2$. We show that $z \in C_n$ implies $|f_n(z)| \geqslant M$, where we note the bound is independent of n. Since $C_n = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$, there are four cases to consider. If $z \in \gamma_1$, then $z = \rho e^{i\theta_{1,n}} + a$, where $r_n \leqslant \rho \leqslant R_n$ and so

$$\begin{aligned}
|f_n(z)| &= |\alpha\rho^n e^{i(n\phi_n - \pi/2)} + \gamma| \\
&= |\alpha\rho^n e^{i(n\phi_n - \pi/2)} + \gamma - (\alpha|\gamma/\alpha|e^{in\phi_n} + \gamma)| \\
&= |\alpha||i(-\rho^n) - |\gamma/\alpha|| \\
&\geqslant |\gamma|,
\end{aligned}$$

where the second equality uses $\alpha|\gamma/\alpha|e^{in\phi_n} + \gamma = f_n(w_n) = 0$. The case $z \in \gamma_3$ is similar. If $z \in \gamma_2$, then $z = R_n e^{i\theta} + a$ for some θ and so

$$\begin{aligned}
|f_n(z)| &= \left|\alpha\tfrac{3}{2}|\gamma/\alpha|e^{in\theta} + \gamma\right| \\
&\geqslant \tfrac{3}{2}|\gamma| - |\gamma| \\
&= \tfrac{1}{2}|\gamma|.
\end{aligned}$$

The case $z \in \gamma_4$ is similar. Therefore $|f_n(z)| \geqslant M$ for all $z \in C_n$ and all n.

By taking N large enough we can ensure, for all $n > N$, that C_n is completely contained in U and that $|q_n(z)| < M/2$ for all z in U. Therefore, for $n > N$ and $z \in C_n$ we have

$$|q_n(z)| < M/2 < |f_n(z)|.$$

Hence by Rouche's theorem $f_n(z)$ and $f_n(z) + q_n(z)$ have the same number of roots inside C_n. Therefore $p_n(z)$ has a root z_n inside C_n. Furthermore, $z_n \to z_0$ as $n \to \infty$ because

$$r_n < |z_n - a| < R_n,$$
$$\theta_{1,n} < \arg(z_n - a) < \theta_{2,n}$$

and $r_n, R_n \to 1$ and $\theta_{1,n}, \theta_{2,n} \to \phi_0$.

The remaining cases for when z_0 lies on one of the circles are very similar, but may involve an extra step. For example, if $|z_0 - a| = 1$ and $|z_0 - b| > 1$, then we apply the above analysis instead to the equation $p_n(z)/(z-b)^n = 0$ with

$$f_n(z) = (z-a)^n + \beta \quad \text{and} \quad q_n(z) = \alpha \left(\frac{z-a}{z-b}\right)^n + \gamma \left(\frac{1}{z-b}\right)^n.$$

The final case is when z_0 lies on both circles, that is $|z_0 - a| = 1$ and $|z_0 - b| = 1$. We first choose a sequence $\{z_0^{(m)}\}_m$ such that $z_0^{(m)} \to z_0$ with each $z_0^{(m)}$ belonging to only one of the circles. By the previous case, for each m there exists a sequence $\{z_n^{(m)}\}_n$ such that $p_n(z_n^{(m)}) = 0$ and $z_n^{(m)} \to z_0^{(m)}$. There exist positive integers $M_1 < M_2 < M_3 < \ldots$ so that for each m we have

$$|z_n^{(m)} - z_0^{(m)}| < \frac{1}{2^m} \quad \text{for } n \geqslant M_m.$$

The sequence $\{z_n\}$ defined by putting $z_n = z_n^{(m)}$ when $M_m \leqslant n < M_{m+1}$ satisfies $p_n(z_n) = 0$ and $z_n \to z_0$, as required. □

If one of the coefficients α or β vanishes, then the form of the limit set changes. This is illustrated in Figures 1 and 2 and proved in Lemma 2.3.

Lemma 2.3. *Assume that $\alpha \neq 0$, $\beta = 0$, $\gamma \neq 0$ and $b = -a$, so that*

$$p_n(z) = (z-a)^n (z+a)^n + \alpha (z-a)^n + \gamma.$$

If $0 < a < 1$, then the roots of p_n as $n \to \infty$ converge to the union of two arcs within the circles $|z-a| = 1$ and $|z+a| = 1$ together with an arc in the fourth-order curve $|(z-a)(z+a)| = 1$. The ends of all three arcs are at $\pm i\sqrt{1-a^2}$. However, if $a \geqslant 1$, then the roots of p_n converge to the entire circle $|z+a| = 1$ together with the closed fourth-order curve $|(z-a)(z+a)| = 1$ contained in $\{z \in \mathbf{C} : \Re(z) \geqslant 0\}$.

Proof. First suppose that $0 < a < 1$. Let $a_1 = 1$, $a_2 = \alpha$, $a_3 = \gamma$ and

$$f_1(z) = (z-a)(z+a),$$
$$f_2(z) = (z-a),$$
$$f_3(z) = 1,$$

so that

$$p_n(z) = a_1 f_1(z)^n + a_2 f_2(z)^n + a_3 f_3(z)^n.$$

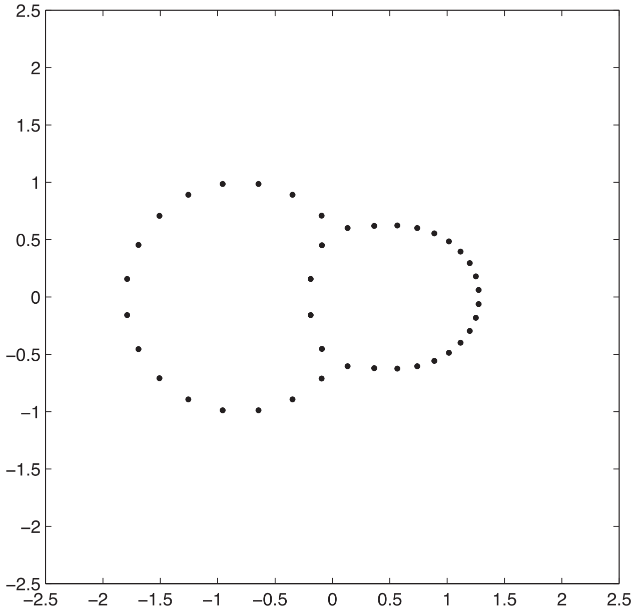

FIGURE 1. *The constant a in $p_n(z)$ of Lemma 2.3 satisfies $0 < a < 1$. The actual parameter values for this example are $a = 0.8$, $n = 20$, $\alpha = \gamma = 1$.*

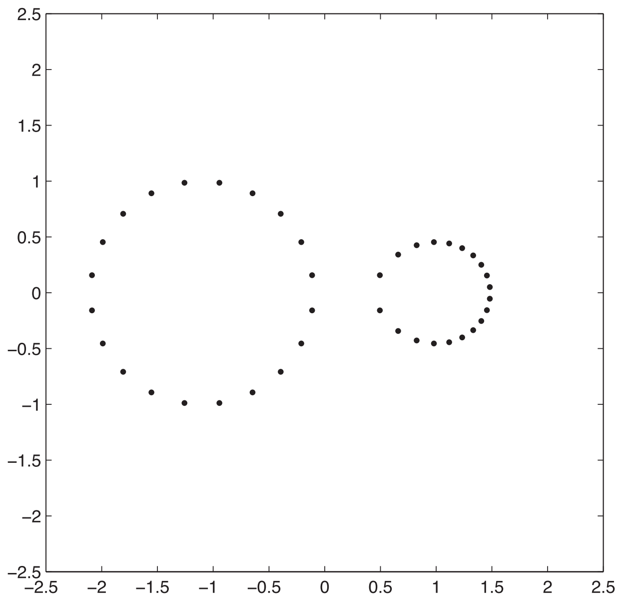

FIGURE 2. *The constant a in $p_n(z)$ of Lemma 2.3 satisfies $a \geqslant 1$. The actual parameter values for this example are $a = 1.1$, $n = 20$, $\alpha = \gamma = 1$.*

We will prove that as $n \to \infty$ the roots of p_n converge to the union of the following subsets of $\mathbf{C}$, which we denote by L.

$$\begin{aligned} L_{1,2} &= \{z \in \mathbf{C} : |f_1(z)| = |f_2(z)| > |f_3(z)|\} = \{z \in \mathbf{C} : |z+a| = 1,\ \Re(z) < 0\},\\ L_{1,3} &= \{z \in \mathbf{C} : |f_1(z)| = |f_3(z)| > |f_2(z)|\} = \{z \in \mathbf{C} : |(z-a)(z+a)| = 1,\ \Re(z) > 0\},\\ L_{2,3} &= \{z \in \mathbf{C} : |f_2(z)| = |f_3(z)| > |f_1(z)|\} = \{z \in \mathbf{C} : |z-a| = 1,\ \Re(z) < 0\},\\ L_{\mathrm{end}} &= \{\pm i\sqrt{1-a^2},\},\end{aligned}$$

where L_{end} is the set of endpoints of the arcs $L_{1,2}$, $L_{1,3}$ and $L_{2,3}$. Let L_ϵ be the ϵ-neighbourhood of L. We will show that $\mathbf{C} \setminus L_\epsilon$ is contained in the union of finitely many sets of the form

$$B_{r,\delta} = \{z \in \mathbf{C} : |f_r(z)(1-\delta)| > \max\{|f_s(z)| : s \neq r\}\}, \quad \text{where } \delta \in (0,1).$$

Since L_ϵ is bounded and since $f_1(z)$ dominates both $f_2(z)$ and $f_3(z)$ in absolute value as $|z| \to \infty$, there exists $R > 0$ and $1 > \delta_1 > 0$ such that $L_\epsilon \subset B(0,R)$ and such that $z \notin B(0,R)$ implies that

$$|f_1(z)(1-\delta_1)| > \max\{|f_2(z)|, |f_3(z)|\}.$$

Therefore $\mathbf{C} \setminus B(0,R) \subseteq B_{1,\delta_1}$. Let $K = \bar{B}(0,R) \setminus L_\epsilon$. For each $z \in K$ there is an $r_z \in \{1,2,3\}$ and $1 > \delta_z > 0$ such that $z \in B_{r_z,\delta_z}$. Since each B_{r_z,δ_z} is open and K is compact, we can cover K with finitely many such B_{r_z,δ_z}. In fact, if we take δ small enough, then we conclude that

$$\mathbf{C} \setminus L_\epsilon \subseteq \bigcup_{r=1}^{3} B_{r,\delta}.$$

Inside the set $B_{r,\delta}$ the term $a_r f_r(z)^n$ dominates the remaining terms of $p_n(z)$ so that for large enough n we have $p_n(z) \neq 0$ for all $z \in B_{r,\delta}$. More precisely, if $z \in B_{r,\delta}$, then $f_r(z) \neq 0$ and

$$\begin{aligned} \left|\frac{p_n(z)}{a_r f_r(z)^n}\right| &= \left|1 + \sum_{s \neq r} (a_s/a_r) \left(\frac{f_s(z)}{f_r(z)}\right)^n\right| \\ &\geqslant 1 - \sum_{s\neq r} |a_s/a_r|(1-\delta)^n \\ &> 0 \end{aligned}$$

for n large enough, where we note that the bound is uniform for $z \in B_{r,\delta}$. Consequently $p_n(z) \neq 0$ for all $z \in B_{r,\delta}$ for n large enough. Since $\mathbf{C} \setminus L_\epsilon$ is contained in the union of finitely many such $B_{r,\delta}$, it follows that all the zeros of $p_n(z)$ are in L_ϵ for n large enough, as required.

We now show that if z_0 is in L, then there is a sequence $\{z_n\}$ such that $p_n(z_n) = 0$ and $z_n \to z_0$. There are a number of cases to consider, but the proofs are similar. We prove the case $z_0 \in L_{1,3}$ in a manner that emphasizes the general technique. Since $z_0 \in L_{1,3}$, we have

$$|f_1(z_0)| = |f_3(z_0)| > |f_2(z_0)|.$$

By continuity there is an open neighbourhood U of z_0 such that $|f_2(z)/f_3(z)| < c < 1$ for all $z \in U$. We want to solve $p_n(z) = 0$ for z close to z_0, that is

$$a_1 f_1(z)^n + a_2 f_2(z)^n + a_3 f_3(z)^n = 0.$$

After dividing both sides by a_1 and $f_3(z)^n$ this is equivalent to solving

$$\left(\frac{f_1(z)}{f_3(z)}\right)^n + \frac{a_3}{a_1} + \frac{a_2}{a_1}\left(\frac{f_2(z)}{f_3(z)}\right)^n = 0.$$

Putting $f(z) = f_1(z)/f_3(z)$, $a = a_3/a_1$ and $g_n(z) = (a_2/a_1)(f_2(z)/f_3(z))^n$, we want to solve

$$f(z)^n + a + g_n(z) = 0 \tag{2.2}$$

for z near z_0. We note that $f'(z_0) \neq 0$, $f(z_0) = e^{i\theta_0}$ for some θ_0 and $g_n(z)$ is uniformly exponentially small in U because $|g_n(z)| \leqslant |a_2/a_1| c^n$, where $c \in (0,1)$.

Since $f'(z_0) \neq 0$, the inverse mapping theorem implies that there exist open sets V and W such that $z_0 \in V$, $f(z_0) \in W$ and $f : V \to W$ is conformal with analytic inverse $g : W \to V$. By reducing V, if necessary, we may assume that $V \subseteq U$.

Under the change of variable $w = f(z)$ equation (2.2) becomes

$$w^n + a + \tilde{g_n}(w) = 0,$$

where $\tilde{g_n}(w) = g_n(g(w))$. We want to solve this equation for w near $f(z_0)$. Since $\tilde{g_n}(w)$ is uniformly exponentially small in W as $n \to \infty$ we expect the solutions to be close to those of

$$w^n + a = 0 \tag{2.3}$$

by Rouche's theorem. Let $w_n = |a|^{1/n} e^{i\phi_n}$ be a solution of (2.3) with argument ϕ_n closest to θ_0. This ensures $w_n \to f(z_0)$ as $n \to \infty$. Let C_n be the contour $C_{r_n, R_n, \theta_{1,n}, \theta_{2,n}}$ with $r_n = \left(\frac{1}{2}|a|\right)^{1/n}$, $R_n = \left(\frac{3}{2}|a|\right)^{1/n}$, $\theta_{1,n} = \phi_n - \pi/2n$ and $\theta_{2,n} = \phi_n + \pi/2n$. We note that w_n lies in the interior of C_n. Routine estimates show that $w \in C_n$ implies that

$$|w^n + a| \geqslant |a|/2$$

for all n. For n large enough we have C_n and its interior completely contained in W. Since $\tilde{g_n}(w) \to 0$ uniformly as $n \to \infty$ for $w \in W$, for n large enough we have

$$|\tilde{g_n}(w)| < |a|/2 \leqslant |w^n + a|$$

for all $w \in C_n$. Therefore by Rouche's theorem $w^n + a + \tilde{g_n}(w) = 0$ has a solution u_n inside C_n. Let $z_n = g(u_n)$. The sequence $\{z_n\}$ satisfies $p_n(z_n) = 0$ and $z_n \to z_0$, as required.

The case $a \geqslant 1$ is handled very similarly, with the only difference being that now the set L is the union of

$$\begin{aligned}
L_{1,2} &= \{z \in \mathbf{C} : |f_1(z)| = |f_2(z)| > |f_3(z)|\} = \{z \in \mathbf{C} : |z + a| = 1,\ \Re(z) < 0\},\\
L_{1,3} &= \{z \in \mathbf{C} : |f_1(z)| = |f_3(z)| > |f_2(z)|\} = \{z \in \mathbf{C} : |(z - a)(z + a)| = 1,\ \Re(z) > 0\},\\
L_{2,3} &= \{z \in \mathbf{C} : |f_2(z)| = |f_3(z)| > |f_1(z)|\} = \emptyset,
\end{aligned}$$

along with the point 0 in the case $a = 1$. The proof now continues as before. □

2.2. *The general case*

The correct context for understanding Lemmas 2.2 and 2.3 involves functions of the type

$$F_n(z) = \sum_{r=1}^{m} a_r(z) f_r(z)^n, \tag{2.4}$$

where $\{f_r\}_{r=1}^m$ and $\{a_r\}_{r=1}^m$ are non-zero analytic functions on an open connected set R, and $m \geqslant 2$. We impose the following hypotheses on the f_r.

(1) There exist a compact set $K \subseteq R$ and constant $c \in (0,1)$ such that $z \in R \setminus K$ implies that

$$|f_r(z)/f_1(z)| < c \quad \text{if } r \neq 1.$$

(2) For each $r \neq s$ the function f_r/f_s is not constant.

(3) For all choices of the three distinct integers r, s, t the set

$$\{z \in R : |f_r(z)| = |f_s(z)| = |f_t(z)|\}$$

is finite.

The set

$$L = \bigcup_{r=1}^{m} \{z \in R : |f_r(z)| = \max\{|f_s(z)| : s \neq r\}\}$$

is the union of what might be called the anti-Stokes lines of the function F_n. We shall show that the zeros of F_n converge to L as n increases, except for a discrete set of zeros that converge to certain zeros of the a_r.

Recall that a function $f : (u, v) \to \mathbf{C}$ is said to be real-analytic if it is C^∞ and if for all a in (u, v) there is a corresponding $\delta_a > 0$ such that

$$f(x + a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} x^n$$

is an absolutely convergent series for $|x| < \delta_a$. Note that f is real-analytic if and only if $\Re f$ and $\Im f$ are real-analytic, and that f is real-analytic if and only if f is the restriction of a complex analytic function $\tilde{f} : U \to \mathbf{C}$ to (u, v), where U is some complex neighbourhood of (u, v). Sums, products, quotients and compositions of real-analytic functions are real-analytic, and the square root of a strictly positive real-analytic function is real-analytic because the logarithm has an analytic branch defined on $\mathbf{C} \setminus [-\infty, 0]$.

We begin by studying the structure of L. The following subsets of R will play an important role. We put $A = A_1 \cup A_2 \cup A_3$, where

$$\begin{aligned} A_1 &= \{z \in R : (f_r/f_s)'(z) = 0 \text{ or some distinct } r, s\}, \\ A_2 &= \{z \in R : a_r(z) = 0 \text{ for some integer } r\}, \\ A_3 &= \{z \in R : |f_r(z)| = |f_s(z)| = |f_t(z)| \text{ for distinct } r, s, t\}. \end{aligned}$$

Theorem 2.4. *The set*

$$L = \bigcup_{r=1}^{m} \{z \in R : |f_r(z)| = \max\{|f_s(z)| : s \neq r\}\}$$

is the disjoint union of sets D and C_j, where $j = 1, \ldots, N$, with the following properties. We denote by D a finite subset of $\mathbf{C}$. Each C_j is a real-analytic curve, that is, the range of a real-analytic one-to-one function $\gamma_j : (u_j, v_j) \to \mathbf{C}$ whose derivative does not vanish anywhere. These curves satisfy the following properties:

(i) *γ_j is contained in*

$$\{z \in R : |f_r(z)| = |f_s(z)| > |f_t(z)|\}$$

for distinct integers r, s (dependent on j) and all $t \neq r, s$. If $z \in \gamma_j$ then $(f_r/f_s)'(z) \neq 0$;

(ii) *γ_j may be completed to a closed curve, or its ends lie in $D \cap A$;*

(iii) *$\gamma_j \cap (A_2 \cup A_3) = \emptyset$ for $j = 1, \ldots, N$.*

Proof. We begin by showing that the result is true for each set

$$L_r = \{z \in R : |f_r(z)| = \max\{|f_s(z)| : s \neq r\}\}.$$

The first step is to show that for each z_0 in L_r there is a corresponding open neighbourhood U_{z_0} such that $L_r \cap U_{z_0} \setminus \{z_0\}$ is contained in the range of finitely many analytic arcs each of which has range contained in L_r. There are a number of cases to consider.

If $z_0 \notin A_3$, then there exists a unique s not equal to r such that

$$|f_r(z_0)| = |f_s(z_0)| > |f_t(z_0)|$$

for all $t \neq r, s$. By continuity there exist an open disc centred at z_0, say B_{z_0}, and constant $c > 0$, such that $z \in B_{z_0}$ implies

$$|f_r(z)| > c > |f_t(z)| \quad \text{for all } t \neq r, s$$

and $f_s(z) \neq 0$. Consequently the set $L_r \cap B_{z_0}$ consists of precisely those points z in B_{z_0} such that $|f_r(z)| = |f_s(z)|$, or equivalently $|f_r(z)/f_s(z)| = 1$. Define $f : B_{z_0} \to \mathbf{C}$ by $f(z) = f_r(z)/f_s(z)$. Since $|f(z_0)| = 1$, there exists $\theta_0 \in \mathbf{R}$ such that $f(z_0) = e^{i\theta_0}$. There are now two cases to consider depending on whether $f'(z_0)$ is zero or non-zero.

If $f'(z_0) \neq 0$, then by the inverse mapping theorem there exist open neighbourhoods U of z_0 and V of $f(z_0)$ so that $f : U \to V$ has an analytic inverse $g : V \to U$. We may assume that V is a small disc of radius $r < 1/2$ centred at $f(z_0)$ and that $U \subseteq B_{z_0}$. There exist $u, v \in \mathbf{R}$ satisfying $u < v < u + 2\pi$ such that

$$V \cap S^1 = \{e^{it} : t \in (u, v)\},$$

where S^1 denotes the unit circle centred at 0. Define $\gamma : (u, v) \to \mathbf{C}$ by $\gamma(t) = g(e^{it})$. We note that γ is a real-analytic arc because it is the restriction of $g(e^{iz})$ to (u, v). Since

$$L_r \cap U = \{\gamma(t) : t \in (u, v)\},$$

it is sufficient to define U_{z_0} to be U.

If $f'(z_0) = 0$, then the analysis is more difficult. Suppose that

$$f(z) = a_0 + a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \ldots,$$

where $m \geqslant 2$ and $a_m \neq 0$. Obviously $a_0 = f(z_0) = e^{i\theta_0}$. We show that $f(z)$ essentially behaves as $z \mapsto z^m$ for z close to z_0. We now make this precise. Let

$$k(z) = a_m + a_{m+1}(z - z_0) + a_{m+2}(z - z_0)^2 + \ldots.$$

Since $k(z_0) \neq 0$ we see that in a small enough open neighbourhood of z_0, say U_{z_0}, we can take an analytic branch of $k(z)^{1/m}$. Let $r(z)$ be such a branch. Now

$$\begin{aligned} f(z) &= a_0 + (z - z_0)^m(a_m + a_{m+1}(z - z_0) + a_{m+2}(z - z_0)^2 + \ldots) \\ &= a_0 + (z - z_0)^m k(z) \\ &= a_0 + [(z - z_0)r(z)]^m \\ &= a_0 + h(z)^m, \end{aligned}$$

where $h(z) = (z - z_0)r(z)$ satisfies $h(z_0) = 0$ and $h'(z_0) = r(z_0) \neq 0$. Since $h'(z_0) \neq 0$ we may shrink U_{z_0} so that $U_{z_0} \subseteq B_{z_0}$ and $h : U_{z_0} \to U_0$ is conformal, where $0 \in U_0 = h(U_{z_0})$. Choose $\epsilon < 1/2$ so that $B(0, \epsilon) \subseteq U_0$, and shrink U_{z_0} again so that $U_{z_0} = h^{-1}(B(0, \epsilon))$. This enables us to factorize $f(z)$ as the composition of the following three surjective maps

$$U_{z_0} \xrightarrow{z \mapsto h(z)} B(0, \epsilon) \xrightarrow{z \mapsto z^m} B(0, \epsilon^m) \xrightarrow{z \mapsto z + a_0} B(a_0, \epsilon^m).$$

Let $s : B(0, \epsilon^m) \setminus [0, -a_0] \to B(0, \epsilon)$ be an analytic branch of $z \mapsto z^{1/m}$, where the branch cut $[0, -a_0]$ is the line connecting 0 to $-a_0$. The m analytic branches of $z \mapsto z^{1/m}$ on $B(0, \epsilon^m) \setminus [0, -a_0]$ are then given by

$$s_k(z) = e^{(2\pi i/m)k} s(z) \quad \text{for } k = 1, \ldots, m.$$

Recalling that $a_0 = e^{i\theta_0}$, there exist $\theta_-, \theta_+ \in \mathbf{R}$ such that $\theta_- < \theta_0 < \theta_+ < \theta_- + 2\pi$ and

$$B(a_0, \epsilon^m) \cap S^1 = \{e^{it} : t \in (\theta_-, \theta_+)\}.$$

For $k = 1, \ldots, m$ define $\alpha_k : (\theta_-, \theta_0) \to \mathbf{C}$ by

$$\alpha_k(t) = h^{-1}(s_k(e^{it} - a_0)).$$

Similarly, for $k = 1, \ldots, m$ define $\beta_k : (\theta_0, \theta_+) \to \mathbf{C}$ by

$$\beta_k(t) = h^{-1}(s_k(e^{it} - a_0)).$$

We note that α_k and β_k are real-analytic curves that satisfy

$$L_r \cap U_{z_0} \setminus \{z_0\} = \bigcup_{k=1}^{m} \{\alpha_k(t) : t \in (\theta_-, \theta_0)\} \cup \{\beta_k(t) : t \in (\theta_0, \theta_+)\},$$

as required.

Now we consider the cases when $z_0 \in A_3$. Since $z_0 \in L_r$, it follows that f_r and at least two other functions, say f_{s_i} for $i = 1, \ldots, k$, attain the maximum absolute value, that is

$$|f_r(z_0)| = |f_{s_1}(z_0)| = \ldots = |f_{s_k}(z_0)| = \max\{|f_s(z_0)| : s \neq r\}.$$

There are two cases to consider depending on whether or not $f_r(z_0)$ is 0. We first suppose that $f_r(z_0) \neq 0$. By continuity there exist $\epsilon, c > 0$ so that if $z \in B(z_0, \epsilon)$, then

$$|f_r(z)| > c > |f_t(z)|,$$
$$|f_{s_i}(z)| > c > |f_t(z)| \quad \text{for } i = 1, \ldots, k$$

for $t \neq r, s_1, \ldots, s_k$. If necessary, reduce ϵ so that $B(z_0, \epsilon) \cap A_3 = \{z_0\}$. For $i = 1, \ldots, k$ there exists an open set $U_i \subseteq B(z_0, \epsilon)$, with $z_0 \in U_i$, such that the set

$$\{z \in U_i \setminus \{z_0\} : |f_r(z)| = |f_{s_i}(z)|\}$$

can be parametrized by finitely many analytic arcs, say $\gamma_1^{(i)}, \ldots, \gamma_{n_i}^{(i)}$. Let $\gamma : (u, v) \to \mathbf{C}$ be one of these curves so that $|f_r(\gamma(x))| = |f_{s_i}(\gamma(x))|$ for all $x \in (u, v)$. We show that either $\gamma \subseteq L_r$ or $\gamma \cap L_r = \emptyset$. Choose $t \neq r, s_i$ and suppose that there exist $x_1, x_2 \in (u, v)$ such that

$$|f_t(\gamma(x_1))| < |f_r(\gamma(x_1))|$$

and

$$|f_t(\gamma(x_2))| > |f_r(\gamma(x_2))|.$$

By the intermediate value theorem there is a point x_3 between x_1 and x_2 such that

$$|f_t(\gamma(x_3))| = |f_r(\gamma(x_3))|$$

and so $\gamma(x_3) \in A_3$. However, by construction $\gamma \cap A_3 = \emptyset$ and so this is impossible. Hence we either have

$$|f_t(z)| < |f_r(z)| \quad \text{for all } z \in \gamma$$

or we have

$$|f_t(z)| > |f_r(z)| \quad \text{for all } z \in \gamma.$$

If there is a $t \neq r, s_i$ such that the latter case holds, then $\gamma \cap L_r = \emptyset$, otherwise $\gamma \subseteq L_r$. Let

$$U_{z_0} = \bigcap_{i=1}^{k} U_i$$

and

$$\mathcal{C} = \bigcup_{i=1}^{k} \{\gamma_j^{(i)} \subseteq L_r : j = 1, \ldots, n_i\}.$$

It is clear that

$$L_r \cap U_{z_0} \setminus \{z_0\} \subseteq \bigcup_{\gamma \in \mathcal{C}} \gamma \subseteq L_r,$$

as required.

The remaining case is when $f_r(z_0) = 0$. This implies that all of the functions f_s vanish at z_0 and so one of these functions has a zero of lowest order, say of order k. For $s = 1, \ldots, m$ put $g_s(z) = f_s(z)/(z - z_0)^k$. Not all the $g_s(z)$ vanish at z_0. If $|g_r(z_0)| = \max\{|g_s(z_0)| : s \neq r\}$, then $g_r(z_0) \neq 0$ and so we apply one of the previous cases to the g_s since for all z we have

$$|f_r(z)| = \max\{|f_s(z)| : s \neq r\} \text{if and only if } |g_r(z)| = \max\{|g_s(z)| : s \neq r\}.$$

If equality does not hold, then there is an open neighbourhood U_{z_0} of z_0 such that $L_r \cap U_{z_0} = \{z_0\}$.

We have now shown that for each z in L_r there is an open set U_z containing z such that $L_r \cap U_z \setminus \{z\}$ is contained in the union of finitely many analytic arcs, each of which is contained in L_r. If this z is not on one of these analytic arcs, then $z \in A$. Since L_r is compact, it can be covered by finitely many such open sets, say $U_{z_1}, \ldots, U_{z_n}$, and consequently L_r is the union of finitely many analytic arcs, say $\gamma_1, \ldots, \gamma_n$ and a finite subset of A. We note that each γ_i has non-vanishing gradient since each is of the form $(f_r/f_s)^{-1}(e^{it})$ for some analytic branch of $(f_r/f_s)^{-1}$. However, as it stands, the γ_i might not be disjoint from themselves and this must now be rectified.

Now, suppose that the γ_i are not disjoint. Without loss of generality we may suppose that $\gamma_1 : (u_1, v_1) \to \mathbf{C}$ and $\gamma_2 : (u_2, v_2) \to \mathbf{C}$ are not disjoint, that is, there exist $t_1 \in (u_1, v_1)$ and $t_2 \in (u_2, v_2)$ such that $\gamma_1(t_1) = \gamma_2(t_2) = w_0$ for some $w_0 \in \mathbf{C}$. We know that $\gamma_1(t) = (f_r/f_{s_1})^{-1}(e^{it})$ for some s_1 and that $\gamma_2(t) = (f_r/f_{s_2})^{-1}(e^{it})$ for some s_2. If $s_1 \neq s_2$, then $w_0 \in A_3$, which contradicts $\gamma_i \cap A_3 = \emptyset$ for all i. Therefore $s_1 = s_2 = s$. Now

$$e^{it_1} = (f_r/f_s)(w_0) = e^{it_2}$$

and so $t_1 = t_2 + 2\pi n$ for some integer n. Since e^{it} is 2π periodic, we can reparametrize the range of γ_2 by

$$\tilde{\gamma_2} : (\tilde{u_2}, \tilde{v_2}) \to \mathbf{C}, \quad \text{where } \tilde{\gamma_2}(t) = (f_r/f_s)^{-1}(e^{it}),$$

where $\tilde{u_2} = u_2 + 2\pi n$, $\tilde{v_2} = v_2 + 2\pi n$ and where $(f_r/f_s)^{-1}$ is the same analytic branch used in the definition of γ_2. This ensures that $\gamma_1(t_1) = \tilde{\gamma_2}(t_1) = w_0$. Let $f = f_r/f_s$ and note that $f(w_0) = e^{it_1}$. Since $f'(w_0) \neq 0$, there exist open neighbourhoods U of w_0 and V of e^{it_1} such that $f : U \to V$ has analytic inverse $g : V \to U$. If $\delta > 0$ is small enough, then for all t in $(t_1 - \delta, t_1 + \delta)$ we have

$$\gamma_1(t) = g(e^{it}) = \tilde{\gamma_2}(t).$$

Since γ_1 and $\tilde{\gamma_2}$ are analytic and agree on a small interval, we deduce that they must also agree on all of $(u_1, v_1) \cap (\tilde{u_2}, \tilde{v_2})$ by the principle of isolated zeros. Hence we can replace γ_1 and $\tilde{\gamma_2}$ by the single analytic arc

$$\tau : (\min\{u_1, \tilde{u_2}\}, \max\{v_1, \tilde{v_2}\}) \longrightarrow \mathbf{C}, \quad \text{where } \tau(t) = \begin{cases} \gamma_1(t) & \text{if } t \in (u_1, v_1), \\ \tilde{\gamma_2}(t) & \text{otherwise.} \end{cases}$$

If we continue the above process of replacing two overlapping arcs by a single analytic arc, then after finitely many repetitions we will arrive at finitely many disjoint analytic arcs, say $\tau_1, \ldots, \tau_m$, such that $\bigcup_i \tau_i = \bigcup_i \gamma_i$. It is clear that for each τ_i there is a corresponding s_i such that the range of τ_i is contained in

$$\{z \in R : |f_r(z)| = |f_{s_i}(z)| > |f_t(z)|\}$$

for all $t \neq r, s_i$.

There is no guarantee that each $\tau_i : (u_i, v_i) \to \mathbf{C}$ is one-one, but this situation can be easily rectified. Suppose that $\tau_i : (u_i, v_i) \to \mathbf{C}$ is not one-one. Then there exist $a, b \in (u_i, v_i)$ with $a < b$ such that $\tau_i(a) = \tau_i(b)$ and $\tau_i|_{(a,b)}$ is one-one. Since

$$e^{ia} = (f_r/f_{s_i})(\tau(a)) = (f_r/f_{s_i})(\tau(b)) = e^{ib},$$

we conclude that $b-a=2\pi n$ for some integer n. Let $\tilde{\tau}_i:(u_i,v_i)\to\mathbf{C}$ be the unique $2\pi n$ periodic function such that $\tilde{\tau}_i|_{[a,b)}=\tau_i|_{[a,b)}$. If $\delta>0$ is small enough, then for $|t|<\delta$ we have

$$\tau(a+t)=\tau(b+t).$$

This implies $\tilde{\tau}_i$ is analytic and so $\tau_i(t)=\tilde{\tau}_i(t)$ for all $t\in(u_i,v_i)$. Therefore we may replace τ_i by the one-one function $\tau_i|_{(a,b)}$ and its endpoint $\tau_i(a)$. Therefore, we can assume that all the τ_i are one-one.

We conclude that for each $r=1,\ldots,m$ the set L_r is the disjoint union of a finite subset D_r of $\mathbf{C}$ and one-one analytic curves $\gamma_1^{(r)},\ldots,\gamma_{n_r}^{(r)}$ that satisfy Properties 1 and 2 in the statement of the theorem. It is possible, for $r\neq s$, that $\gamma_i^{(r)}$ is not disjoint from $\gamma_j^{(s)}$ for some i and j. It follows by Property 1 that $\gamma_j^{(s)}\subseteq L_r$. Hence $\gamma_j^{(s)}$ is redundant and can be deleted. We continue this process until the remaining curves are disjoint. We conclude that $L=\bigcup_r L_r$ is the disjoint union of a finite subset D of $\mathbf{C}$ and one-one real-analytic curves, say $\gamma_1,\ldots,\gamma_n$, that satisfy Properties 1 and 2.

Finally, we want each γ_i to also satisfy Property 3, that is $\gamma_i\cap(A_2\cup A_3)=\emptyset$. We first note that $L\cap(A_2\cup A_3)$ is finite because L is compact and $A_2\cup A_3$ is both closed and discrete. If $\gamma_i:(u_i,v_i)\to\mathbf{C}$ meets $A_2\cup A_3$, that is $\gamma_i\cap(A_2\cup A_3)\neq\emptyset$, then there exist finitely many points, say $a_1,\ldots,a_m$, such that

$$u_i<a_1<a_2<\ldots<a_m<v_i$$

and $\gamma_i(t)\in(A_2\cup A_3)$ if and only if $t\in\{a_1,\ldots,a_m\}$. We now just replace γ_i by

$$\gamma_i|_{(u_i,a_1)},\gamma_i|_{(a_1,a_2)},\ldots,\gamma_i|_{(a_m,v_i)}$$

and enlarge D by the points $\gamma_i(a_k)$ for all k. After doing this for each γ_i, we obtain a finite subset D of $\mathbf{C}$ and a finite collection of analytic curves that satisfy all the required properties. □

It should be mentioned that the set L may contain isolated points. By the above proof, if $z_0\in L$ is isolated, then $f_r(z_0)=0$ for $r=1,\ldots,m$. An important consequence is that $F_n(z_0)=0$ for all n.

We now show that each arc $\gamma:(u,v)\to\mathbf{C}$ in Lemma 2.4 can be parametrized by arc length with the arc length parametrization also being real-analytic. We begin by showing that the length of γ is finite, that is

$$l=\int_u^v|\gamma'(t)|\,\mathrm{d}t<\infty.$$

This is achieved by showing that $|\gamma'(t)|=\mathrm{O}((t-u)^a)$ as $t\to u$ for some $a>-1$, and likewise at the opposite endpoint v. The problematic case is when for t near u we have

$$\gamma(t)=h^{-1}(s(e^{it}-e^{iu})),$$

where h is conformal and s is a branch of $z\mapsto z^{1/m}$. Differentiating it suffices to show that

$$|s'(e^{it}-e^{iu})|=\mathrm{O}((t-u)^a)$$

for some $a>-1$ as $t\to u$. Now

$$\begin{aligned}|s'(e^{it}-e^{iu})|&=\frac{1}{m}|e^{i(t-u)}-1|^{-(m-1)/m}\\&\leqslant c_0(t-u)^{-(m-1)/m}\end{aligned}$$

for some constant $c_0 > 0$ as $t \to u$, as required. Hence $l < \infty$, as required. This implies the length function $s : (u, v) \to (0, l)$ where

$$s(t) = \int_u^t |\gamma'(x)| \, \mathrm{d}x$$

is well defined. Since

$$|\gamma'(x)| = (\gamma'(x)\overline{\gamma'(x)})^{1/2}$$

and $\gamma'(x)$ does not vanish, we conclude that $|\gamma'(x)|$ is real-analytic. Therefore s is real-analytic. Since $s'(x) > 0$ for all $x \in (u, v)$, we conclude that s^{-1} is real-analytic by the inverse mapping theorem. Therefore the arc length parametrization $\tilde{\gamma} : (0, l) \to \mathbf{C}$ of γ defined by $\tilde{\gamma}(t) = \gamma(s^{-1}(t))$ is real-analytic.

Our next theorem analyses the distribution of the zeros of F_n along L. The proof identifies the exceptional points and the densities in the statement.

THEOREM 2.5. *There exist a finite subset D of L, real-analytic curves $\gamma_r : (0, l_r) \to \mathbf{C}$ for $r = 1, \ldots, N$ parametrized by arc length with non-vanishing gradients and bounded real-analytic densities $\rho_r : (0, l_r) \to (0, \infty)$ with the following properties.*

(i) *The set L is the disjoint union of D and the ranges of all the γ_r.*

(ii) *For every choice of r and every closed subinterval $[\alpha, \beta]$ of $(0, l_r)$ we have*

$$\lim_{\epsilon \to 0} N_\epsilon = \int_\alpha^\beta \rho_r(s) \, \mathrm{d}s,$$

where

$$N_\epsilon = \lim_{n \to \infty} \frac{1}{n} |\{z \in R : F_n(z) = 0\} \cap S_\epsilon|$$

and

$$S_\epsilon = \{z \in \mathbf{C} : \mathrm{dist}(z, \gamma_r([\alpha, \beta])) < \epsilon\}.$$

Proof. Let $\gamma : (0, l) \to \mathbf{C}$ be one of the analytic curves in L. It has range contained in

$$\{z \in R : |f_r(z)| = |f_s(z)| > |f_t(z)|\}$$

for some integers $r \neq s$ and all $t \neq r, s$. Without loss of generality we may suppose that γ is the arc length parametrization of $\tau : (u, v) \to \mathbf{C}$, where τ is of the form $(f_r/f_s)^{-1}(e^{it})$, as in Theorem 2.4. By taking a sufficiently small open neighbourhood U of $\gamma([\alpha, \beta])$, we can ensure that $z \in U$ implies that $a_r(z) \neq 0$, $f_s(z) \neq 0$,

$$\left| \frac{a_k(z)}{a_r(z)} \right| \leqslant M \quad \text{and} \quad \left| \frac{f_k(z)}{f_s(z)} \right| < 1 - \delta$$

for some $\delta \in (0, 1)$, $M > 0$, and all $k \neq r, s$. Since $a_r(z) f_s(z)^n \neq 0$ for all $z \in U$, the solutions (counted with multiplicity) of $F_n(z) = 0$ in U are precisely the same as those of $F_n(z)/(a_r(z) f_s(z)^n) = 0$. Now

$$\frac{F_n(z)}{a_r(z) f_s(z)^n} = \left(\frac{f_r(z)}{f_s(z)} \right)^n + \frac{a_s(z)}{a_r(z)} + \sum_{k \neq r,s} \frac{a_k(z)}{a_r(z)} \left(\frac{f_k(z)}{f_s(z)} \right)^n$$

and so with

$$f(z) = f_r(z)/f_s(z), \quad a(z) = a_s(z)/a_r(z) \quad \text{and} \quad g_n(z) = \sum_{k \neq r,s} (a_k(z)/a_r(z))(f_k(z)/f_s(z))^n$$

we are interested in solving the equation

$$f(z)^n + a(z) + g_n(z) = 0$$

inside U. We note that $g_n(z)$ is uniformly exponentially small in U as $n \to \infty$.

Define the argument function $\theta : (0, l) \to \mathbf{R}$ by $\theta(t) = -i \log(f(\gamma(t))$ using an analytic branch of $\log f \circ \gamma$. For definiteness, we may put $\theta(t) = s^{-1}(t)$, where $s : (u, v) \to (0, l)$ is the length function

$$s(t) = \int_u^t |\tau'(x)| \, \mathrm{d}x.$$

This uses the fact that $\gamma(t) = \tau(s^{-1}(t))$. We note that θ is real-analytic and can be extended continuously to the closed interval $[0, l]$. Also, define the density function $\rho : (0, l) \to \mathbf{R}$ by $\rho(t) = \theta'(t)/2\pi$. We show that ρ is bounded by showing that $(s^{-1})'(t)$ is bounded. It suffices to check that there exists a constant $C > 0$ such that $|s'(t)| \geqslant C$ for all $t \in (u, v)$. Since

$$|s'(t)| = |\tau'(t)| = |(f^{-1})'(e^{it})|,$$

where f^{-1} is an analytic branch of a local inverse of f, it suffices to check that $f'(z)$ is bounded for $z \in \tau$ (or equivalently γ.); but if one observes the construction of τ, this is clearly the case. Hence ρ is bounded. Also ρ is positive because $(s^{-1})'(t)$ is non-zero and s is increasing.

The idea of the proof is to now cover $\gamma([\alpha, \beta])$ with finitely many sets with disjoint interiors such that in each set we can count the number of solutions of $F_n(z) = 0$. The boundaries of these sets contribute $\mathrm{O}(1)$ zeros as $n \to \infty$ and so their contribution can be ignored. This will be achieved over a number of steps.

Let $z_0 \in \gamma([\alpha, \beta])$ and let $w_0 = f(z_0)$. Since $f'(z_0) \neq 0$, there are open neighbourhoods V of z_0 and W of w_0 such that $f : V \to W$ is conformal with analytic inverse $g : W \to V$. We may assume that V is sufficiently small so that $V \subseteq U$ and $z \in V$ implies

$$|a(z) - a(z_0)| < \frac{|a(z_0)|}{8}.$$

Since $|w_0| = 1$, there exists $\phi_0 \in \mathbf{R}$ such that $w_0 = e^{i\phi_0}$. Let C_{w_0} be the contour $C_{r,R,\theta_1,\theta_2}$, where $r = 1 - \delta$, $R = 1 + \delta$, $\theta_1 = \phi_0 - \delta$ and $\theta_2 = \phi_0 + \delta$, where $\delta > 0$ is small enough so that C_{w_0} and its interior is contained in W. Let A_{w_0} be the union of C_{w_0} and its interior, so that A_{w_0} is a sector of an annulus with angular sweep $\theta_2 - \theta_1$. Finally let $N_{z_0} = g(A_{w_0})$. We show that

$$\#\{z \in N_{z_0}^\circ : F_n(z) = 0\} = \frac{n(\theta_2 - \theta_1)}{2\pi} + \mathrm{O}(1)$$

as $n \to \infty$. Let $\tilde{F_n}(w) = F_n(g(w))$ for $w \in W$. Since $g : W \to V$ is conformal, Lemma 2.1 implies that

$$\#\{z \in N_{z_0}^\circ : F_n(z) = 0\} = \#\{w \in A_{w_0}^\circ : \tilde{F_n}(w) = 0\}.$$

The solutions of $\tilde{F_n}(w) = 0$ counted with multiplicity are precisely the same as those of

$$w^n + \tilde{a}(w) + \tilde{g_n}(w) = 0,$$

where $\tilde{a}(w) = a(g(w))$ and $\tilde{g_n}(w) = g_n(g(w))$. We use Rouche's theorem to show that each such solution in $A_{w_0}^\circ$ is close to a root of

$$w^n + \tilde{a}(w_0) = 0,$$

which has roots given by

$$w_j = |\tilde{a}(w_0)|^{1/n} e^{is_j} \quad \text{for } j = 1, \ldots, n,$$

where $s_{j+1} - s_j = 2\pi/n$. For $j = 1, \ldots, n$ define $C^{j,n}$ to be the contour $C_{r_n,R_n,\theta_{1,j},\theta_{2,j}}$, where $r_n = \left(\frac{1}{2}|\tilde{a}(w_0)|\right)^{1/n}$, $R_n = \left(\frac{3}{2}|\tilde{a}(w_0)|\right)^{1/n}$, $\theta_{1,j} = s_j - \pi/2n$ and $\theta_{2,j} = s_j + \pi/2n$. We note that the $C^{j,n}$ approach the unit circle uniformly as $n \to \infty$ and are evenly distributed around the unit circle by an angle of $2\pi/n$.

We show that if $C^{j,n} \subseteq A_{w_0}$, then $\tilde{F}_n(w) = 0$ has precisely one solution inside $C^{j,n}$. Routine estimates show that if $w \in C^{j,n}$, then $|w^n + \tilde{a}(w_0)| > |\tilde{a}(w_0)|/2$, where we note that the lower bound is independent of n. Also, since $\tilde{g_n}(w) \to 0$ uniformly as $n \to \infty$, there exists $N > 0$ such that for $n > N$ we have $|\tilde{g_n}(w)| < |\tilde{a}(w_0)|/8$ for all $w \in W$. Therefore for $w \in C^{j,n}$ and $n > N$ we have

$$\begin{aligned}|w^n + \tilde{a}(w) + \tilde{g_n}(w)| &\geqslant |w^n + \tilde{a}(w_0)| - |\tilde{a}(w_0) - \tilde{a}(w)| - |\tilde{g_n}(w)| \\ &\geqslant \frac{|\tilde{a}(w_0)|}{2} - \frac{|\tilde{a}(w_0)|}{8} - \frac{|\tilde{a}(w_0)|}{8} \\ &= \frac{|\tilde{a}(w_0)|}{4}\end{aligned}$$

and

$$\begin{aligned}|(w^n + \tilde{a}(w) + \tilde{g_n}(w)) - (w^n + \tilde{a}(w_0))| &\leqslant |\tilde{a}(w) - \tilde{a}(w_0)| + |\tilde{g_n}(w)| \\ &< \frac{|\tilde{a}(w_0)|}{8} + \frac{|\tilde{a}(w_0)|}{8} \\ &= \frac{|\tilde{a}(w_0)|}{4}.\end{aligned}$$

Therefore by Rouche's theorem $w^n + \tilde{a}(w) + \tilde{g_n}(w) = 0$ and $w^n + \tilde{a}(w_0) = 0$ have the same number of solutions inside $C^{j,n}$, and so $\tilde{F}_n(w) = 0$ has precisely one solution counted with multiplicity inside $C^{j,n}$.

Let $D^{j,n}$ be the open region enclosed by $C^{j,n}$. Routine estimates show that if $w \in A_{w_0} \setminus \bigcup_j D^{j,n}$, then $|w^n + \tilde{a}(w_0)| \geqslant |\tilde{a}(w_0)|/2$. Consequently for $w \in A_{w_0} \setminus \bigcup_j D^{j,n}$ and $n > N$, we have

$$|w^n + \tilde{a}(w) + \tilde{g_n}(w)| > \frac{|\tilde{a}(w_0)|}{4}$$

and so $\tilde{F}_n(w) = 0$ has no solutions outside the $C^{j,n}$. Therefore

$$\begin{aligned}\#\{z \in N_{z_0}^{\circ} : F_n(z) = 0\} &= \#\{w \in A_{w_0}^{\circ} : \tilde{F}_n(w) = 0\} \\ &= \#\{j : C^{j,n} \subseteq A_{w_0}\} + \mathrm{O}(1) \\ &= \frac{n(\theta_2 - \theta_1)}{2\pi} + \mathrm{O}(1)\end{aligned}$$

as $n \to \infty$. We also note that the boundary of N_{z_0} has $\mathrm{O}(1)$ (in fact, at most 2) solutions as $n \to \infty$. Our choice of A_{w_0} (and hence N_{z_0}) was quite arbitrary and it will be useful to impose further conditions. There exists $t_0 \in [\alpha, \beta]$ such that $z_0 = \gamma(t_0)$. We may choose A_{w_0} so that there exist a, b satisfying $0 < a < t_0 < b < l$ so that

$$S^1 \cap A_{w_0} = \{e^{i\theta(t)} : t \in [a, b]\}$$

and

$$\gamma \cap N_{z_0} = \gamma([a, b]).$$

This uses the fact that $f(\gamma(t)) = e^{i\theta(t)}$ by definition. These sets satisfy the relation

$$f(\gamma \cap N_{z_0}) = S^1 \cap A_{w_0}.$$

Furthermore we can impose the condition that $\theta(b) - \theta(a) < \pi/2$.

For convenience, let $A_\delta(\theta_1, \theta_2)$ denote the closed region enclosed by the contour $C_{1-\delta,1+\delta,\theta_1,\theta_2}$. We show that there is a $\delta > 0$ small enough and a partition of $[\alpha, \beta]$ of the form

$$t_0 < t_1 = \alpha < t_2 < t_3 < \ldots < t_m = \beta < t_{m+1}$$

so that f^{-1} has an analytic branch g_i in a neighbourhood of $A_i = A_\delta(\theta(t_i), \theta(t_{i+1}))$ so that the sets $N_i = g_i(A_i)$ for $i = 0, \ldots, m$ have the following properties.

(i) The interior of $\bigcup_i N_i$ is an open set containing $\gamma([\alpha,\beta])$. The interiors of each N_i are disjoint.

(ii) For $i = 0,\ldots,m$ we have $S^1 \cap A_i = \{e^{i\theta(t)} : t \in [t_i, t_{i+1}]\}$ and $\gamma \cap N_i = \gamma([t_i, t_{i+1}])$.

(iii) For $i = 0,\ldots,m$, we have

$$\#\{z \in N_i^\circ : F_n(z) = 0\} = \frac{n(\theta(t_{i+1}) - \theta(t_i))}{2\pi} + \mathrm{O}(1)$$

as $n \to \infty$. The boundary of N_i has $\mathrm{O}(1)$ solutions as $n \to \infty$.

Let $z \in \gamma([\alpha,\beta])$. By the previous step there exist subsets of $\mathbf{C}$, denoted by N_z and A_z, and real numbers $a_z < b_z$, such that $z \in N_z^\circ$, $A_z = A_{\delta_z}(\theta(a_z), \theta(b_z))$ for some $\delta_z > 0$, such that we have

$$S^1 \cap A_z = \{e^{i\theta(t)} : t \in [a_z, b_z]\}, \quad \gamma \cap N_z = \gamma([a_z, b_z])$$

and $\theta(b_z) - \theta(a_z) < \pi/2$. There are also open subsets of $\mathbf{C}$, say U_z and V_z, such that $N_z \subseteq U_z$, $A_z \subseteq V_z$ and $f : U_z \to V_z$ has analytic inverse $g_z : V_z \to U_z$. Since $\gamma([\alpha,\beta])$ is compact, there exist finitely many points, say $z_1,\ldots,z_M$, such that

$$\gamma([\alpha,\beta]) \subseteq \bigcup_{i=1}^{M} N_{z_i}^\circ.$$

Let P be the set formed by removing from

$$\{a_{z_i}, b_{z_i} : i = 1,\ldots,M\} \cup \{\alpha,\beta\}$$

all elements less than α, except for the largest such one, and similarly, all elements greater than β, except for the smallest such one. The set P can be labelled in increasing order so that

$$t_0 < t_1 = \alpha < t_2 < t_3 < \ldots < t_m = \beta < t_{m+1}.$$

Initially let δ be the minimum of the δ_{z_i}. For $i = 0,\ldots,m$ there exists z_k (depending on i) such that $[t_i, t_{i+1}] \subseteq [a_{z_k}, b_{z_k}]$. Letting $g_i = g_{z_k}$ we put $A_{i,\delta} = A_\delta(\theta(t_i), \theta(t_{i+1}))$ and $N_{i,\delta} = g_i(A_{i,\delta})$.

It now suffices to show that by reducing δ, if necessary, we can ensure that the interiors of the $N_{i,\delta}$ are disjoint. Suppose there exists $i < j$ such that $N_{i,\delta}^\circ \cap N_{j,\delta}^\circ \neq \emptyset$. This implies $\theta(t_{i+1}) < \theta(t_j)$ since if $\theta(t_{i+1}) = \theta(t_j)$, then we would have $A_{i,\delta}^\circ \cap A_{j,\delta}^\circ = \emptyset$ (because $\theta(t_{j+1}) - \theta(t_i) < \pi$) and so $N_{i,\delta}^\circ \cap N_{j,\delta}^\circ = \emptyset$, which is a contradiction. Thus $[t_i, t_{i+1}]$ and $[t_j, t_{j+1}]$ are disjoint and hence $\gamma([t_i, t_{i+1}])$ and $\gamma([t_j, t_{j+1}])$ are disjoint. Consequently there exist disjoint open sets U_i and U_j such that $\gamma([t_i, t_{i+1}]) \subseteq U_i$ and $\gamma([t_j, t_{j+1}]) \subseteq U_j$. Now reduce δ so that $g_i(A_{i,\delta}) \subseteq U_i$ and $g_j(A_{j,\delta}) \subseteq U_j$. This implies $N_{i,\delta} \cap N_{j,\delta} = \emptyset$. We continue this process of reducing δ until the interiors of the $N_{i,\delta}$ are disjoint. We then put $A_i = A_{i,\delta} = A_\delta(\theta(t_i), \theta(t_{i+1}))$ and $N_i = g_i(A_i)$ and it is clear that these sets satisfy the required properties.

By definition S_ϵ is the ϵ-neighbourhood of $\gamma([\alpha,\beta])$. There exists $c > 0$ such that for each $i = 1,\ldots,m-1$ we have

$$f(N_i \cap S_\epsilon) \supseteq A_c(\theta(t_i), \theta(t_{i+1})).$$

Therefore as $n \to \infty$, we have

$$\#\{z \in N_i^\circ \cap S_\epsilon : F_n(z) = 0\} = \frac{n(\theta(t_{i+1}) - \theta(t_i))}{2\pi} + \mathrm{O}(1).$$

Also, for each $\epsilon > 0$ small enough there is a corresponding $c_\epsilon > 0$ such that $c_\epsilon \to 0$ as $\epsilon \to 0$ and such that

$$f(N_0 \cap S_\epsilon) \subseteq A_c(\theta(t_1) - c_\epsilon, \theta(t_1))$$

and

$$f(N_m \cap S_\epsilon)) \subseteq A_c(\theta(t_m), \theta(t_m) + c_\epsilon).$$

Therefore as $n \to \infty$ we have for $i = 0, m$

$$K_{i,\epsilon} := \#\{z \in N_i^\circ \cap S_\epsilon : F_n(z) = 0\} \leqslant \frac{nc_\epsilon}{2\pi} + \mathrm{O}(1).$$

Consequently, as $n \to \infty$, we have

$$\begin{aligned}|\{z \in R : F_n(z) = 0\} \cap S_\epsilon| &= \sum_{i=0}^{m} \#\{z \in N_i^\circ \cap S_\epsilon : F_n(z) = 0\} + \mathrm{O}(1) \\ &= K_{0,\epsilon} + K_{m,\epsilon} + \sum_{i=1}^{m-1} \frac{n(\theta(t_{i+1}) - \theta(t_i))}{2\pi} + \mathrm{O}(1) \\ &= \frac{n(\theta(\beta) - \theta(\alpha))}{2\pi} + K_{0,\epsilon} + K_{m,\epsilon} + \mathrm{O}(1).\end{aligned}$$

Hence

$$\frac{\theta(\beta) - \theta(\alpha)}{2\pi} \leqslant N_\epsilon \leqslant \frac{\theta(\beta) - \theta(\alpha)}{2\pi} + \frac{c_\epsilon}{\pi},$$

and so

$$\lim_{\epsilon \to 0} N_\epsilon = \frac{\theta(\beta) - \theta(\alpha)}{2\pi}.$$

Finally,

$$\int_\alpha^\beta \rho(s)\, \mathrm{d}s = \int_\alpha^\beta \frac{\theta'(s)}{2\pi}\, \mathrm{d}s = \frac{\theta(\beta) - \theta(\alpha)}{2\pi},$$

and so

$$\lim_{\epsilon \to 0} N_\epsilon = \int_\alpha^\beta \rho(s)\, \mathrm{d}s,$$

as required. □

The above result is sharp in the sense that one cannot let $\alpha = 0$ or $\beta = l_r$. To see this consider the function

$$F_n(z) = f_1(z)^n + f_2(z)^n + f_3(z)^n,$$

where $f_1(z) = z^2(z-1)$, $f_2(z) = z - 1$ and $f_3(z) = (z-1)^2$. Since $|f_1(1)| = |f_2(1)| = |f_3(1)| = 0$ we note that 1 does not lie on any of the analytic arcs. The problem is that $F_n(z) = 0$ has n solutions counted with multiplicity at 1. If z is close to 1 and satisfies $|z| = 1$, then we have

$$|f_1(z)| = |f_2(z)| > |f_3(z)|$$

and so there is an analytic arc $\gamma : (0, l) \to \mathbf{C}$ with endpoint 1. Without loss of generality suppose that $\gamma(0+) = 1$. If we choose $\beta > 0$ small enough, then we get

$$\int_0^\beta \rho(s)\, \mathrm{d}s = \frac{1}{2\pi}(\theta(\beta) - \theta(0)) < \frac{1}{2}.$$

This makes use of the fact that θ can be extended continuously to $[0, l]$. For each $\epsilon > 0$ the ϵ-neighbourhood S_ϵ of $\gamma((0, \beta])$ contains 1 and so $N_\epsilon \geqslant 1$. Consequently we have

$$\liminf_{\epsilon \to 0} N_\epsilon > \int_0^\beta \rho(s)\, \mathrm{d}s.$$

Corollary 2.6. *The limit set of* $\{z \in R : F_n(z) = 0\}$ *as* $n \to \infty$ *is equal to the union of* L *and the discrete set*

$$Z := \bigcup_{r=1}^{m} \{z \in R : a_r(z) = 0,\ |f_r(z)| > \max\{|f_s(z)| : s \neq r\}\}.$$

Proof. Letting $A_n := \{z \in R : F_n(z) = 0\}$ and $A := L \cup Z$, the task is to show that $A_n \to A$ as $n \to \infty$. There are a number of cases to consider.

Suppose $z_0 \in L$. If z_0 lies on an analytic arc, or is an endpoint of an analytic arc, then by Theorem 2.5 there exists a sequence $z_n \in A_n$ such that $z_n \to z_0$ as $n \to \infty$. The other possibility is that z_0 is an isolated point of L, and we recall that this can only occur if $f_r(z_0) = 0$ for all r. In this case $z_0 \in A_n$ for all n.

Suppose that $z_0 \in Z$. There exists $r \in \{1, \ldots, m\}$ such that $a_r(z_0) = 0$ and $|f_r(z_0)| > |f_s(z_0)|$ for $s \neq r$. Suppose this root has multiplicity m_0. Applying Rouche's theorem to $F_n(z)/f_r(z)^n$, we see that F_n has m_0 zeros in any disc $B(z_0, \delta)$, provided δ is small enough and n large enough. Therefore there exists a sequence $z_n \in A_n$ converging to z_0 as $n \to \infty$.

Finally, if $z_0 \notin L \cup Z$, then there exists $r \in \{1, \ldots, m\}$ such that $a_r(z_0) \neq 0$ and $|f_r(z_0)| > |f_s(z_0)|$ for $s \neq r$. Let U be a sufficiently small open neighbourhood of z_0 such that $z \in U$ implies that

$$|f_s(z)/f_r(z)| < c < 1 \quad \text{for } s \neq r,$$
$$|a_r(z)| > M > 0.$$

By considering $F_n(z)/f_r(z)^n$ we see that $A_n \cap U = \emptyset$ for all large enough n. □

The convergence of A_n to A in the above corollary is locally uniform in the following sense. If K is a compact subset of $\mathbf{C}$ and $N(A, \epsilon)$ an ϵ-neighbourhood of A, then for all large enough n we have $A_n \cap K \subseteq N(A, \epsilon) \cap K$. This is seen by covering $K \setminus N(A, \epsilon)$ by finitely many open sets $U_1, \ldots, U_k$ such that $A_n \cap U_i = \emptyset$ for all large enough n.

Example 1. Given $c > 0$, the solutions of the equation $(z^2 - 1)^n = c$ are given explicitly by

$$z_r = \pm\{1 + c^{1/n} e^{2\pi i r/n}\}^{1/2},$$

where $1 \leqslant r \leqslant n$. Putting $f(z) = z^2 - 1$, the solutions converge as $n \to \infty$ to $L = \{z : |f(z)| = 1\}$. As in Theorem 2.4 one may parametrize L by $\gamma(u) = \pm\sqrt{1 + e^{iu}}$, where $-\pi < u < \pi$. For this initial parametrization one has $f(\gamma(u)) = e^{iu}$, $\theta(u) = u$, $\rho(u) = 1/(2\pi)$ and $\int_{-\pi}^{\pi} \rho(u)\, du = 1$. If instead one parametrizes the curve γ by arc length using the formula

$$\frac{ds}{du} = |\gamma'(u)| = |1 + e^{iu}|^{-1/2},$$

then one sees that the density of the zeros vanishes at $z = 0$ because $s'(u) \to \infty$ as $u \to \pm\pi$.

2.3. *An important degenerate case*

If each f_r is a power of z, then the above general analysis does not hold. In this case the f_r fail to satisfy hypothesis 3, since for each r we have $|f_r(z)| = 1$ for each z on the unit circle $|z| = 1$. Understanding this degenerate case will be important to us later on, and so we have the following result.

Lemma 2.7. *Let p be a polynomial in two variables of the form*

$$p(z, w) = a_m(z)w^m + a_{m-1}(z)w^{m-1} + \ldots + a_1(z)w + a_0(z).$$

Let $\epsilon > 0$. As $n \to \infty$ the solutions of $p(z, z^n) = 0$ that satisfy $|z| < 1 - \epsilon$ converge to the solutions of $a_0(z) = 0$, satisfying $|z| < 1 - \epsilon$, while the solutions of $p(z, z^n) = 0$ that satisfy $|z| > 1 + \epsilon$ converge to the solutions of $a_m(z) = 0$, satisfying $|z| > 1 + \epsilon$.

Assuming that neither $a_0(z) = 0$ nor $a_m(z) = 0$ has a solution of modulus 1, the remaining solutions of $p(z, z^n) = 0$ are asymptotically uniformly distributed around the circle $|z| = 1$ in *the sense that for all α and β such that $\beta - \alpha \leqslant 2\pi$ and for all $\delta > 0$ we have*

$$\frac{\#\{z \in \mathbf{C} : p(z, z^n) = 0, 1-\delta < |z| < 1+\delta, \alpha < \arg z < \beta\}}{\#\{z \in \mathbf{C} : p(z, z^n) = 0\}} \longrightarrow \frac{\beta - \alpha}{2\pi} \quad \text{as } n \to \infty,$$

where we count the roots with multiplicity.

Proof. We note that $p(z, z^n)$ has $nm + d$ roots counted with multiplicity, where d is the degree of $a_m(z)$. We begin by proving the last statement. Fix an arbitrary point z_0 of modulus one. We first prove that the result is true in a neighbourhood of z_0. The equation $p(z_0, w) = 0$ has exactly m solutions counted with multiplicity because $a_m(z_0) \neq 0$, and each solution is non-zero because $p(z_0, 0) = a_0(z_0) \neq 0$. Let $w_1, \dots, w_k$ be the distinct solutions and let $m_1, \dots, m_k$ be the corresponding multiplicities so that $m_1 + \ldots + m_k = m$. The nm solutions of $p(z_0, z^n) = 0$ are precisely the nth roots of the w_i, that is

$$\{z \in \mathbf{C} : p(z_0, z^n) = 0\} = \bigcup_{i=1}^{k} \{z \in \mathbf{C} : z^n = w_i\}.$$

Let z_i be an nth root of w_i. We show that the solution z_i of $p(z_0, z^n) = 0$ has multiplicity m_i. Substituting z^n for w and z_i^n for w_i in

$$p(z_0, w) = a_{m_i}(w - w_i)^{m_i} + a_{m_i+1}(w - w_i)^{m_i+1} + \ldots \quad (a_{m_i} \neq 0)$$

yields

$$\begin{aligned} p(z_0, z^n) &= a_{m_i}(z^n - z_i^n)^{m_i} + a_{m_i+1}(z^n - z_i^n)^{m_i+1} + \ldots \\ &= (z - z_i)^{m_i}(z^{n-1} + z^{n-2}z_i + \ldots + z_i^{n-1})^{m_i}(a_{m_i} + a_{m_i+1}(z^n - z_i^n) + \ldots) \\ &= (z - z_i)^{m_i} g(z), \end{aligned}$$

where $g(z)$ is analytic and satisfies $g(z_i) \neq 0$, and so the root z_i has multiplicity m_i.

Let $C_1, \dots, C_k$ be contours of the form $C_{r,R,\theta_1,\theta_2}$ such that the regions enclosed by the C_i are disjoint and such that w_i lies inside the region enclosed by C_i. The preimage of C_i under the map $z \mapsto z^n$ consists of n disjoint contours of the form $C_{r,R,\theta_1,\theta_2}$ which are equally spaced around the unit circle in the sense that they may be labelled $C_i^{1,n}, \dots, C_i^{n,n}$ so that

$$C_i^{j,n} = \zeta^{j-1} C_i^{1,n} \quad \text{for } j = 1, \dots, n, \text{ where } \zeta = e^{i2\pi/n},$$

where multiplication by ζ effectively rotates the contour by an angle of $2\pi/n$. As $n \to \infty$ the radii of the preimage contours converge to 1 and their angular sweep converge to 0. We may label the nth roots of w_i as $z_i^{1,n}, \dots, z_i^{n,n}$ so that $z_i^{j,n}$ is inside the contour $C_i^{j,n}$.

Let $C = \bigcup_{i=1}^{k} C_i$. If $w \in C$, then $p(z_0, w) \neq 0$ since the only zeros of this polynomial are the w_i. Since C is compact, there is a constant $M_1 > 0$ such that $|p(z_0, w)| > M_1$ for all $w \in C$. Note that if $z \in C_i^{j,n}$, then z^n is on C and so $|p(z_0, z^n)| > M_1$. Let

$$M_2 = \max\{|w^k| : w \in C, 0 \leqslant k \leqslant m\} > 0.$$

The continuity of the polynomials $a_i(z)$ imply that there exists an open neighbourhood of z_0, say A, such that

$$|a_i(z) - a_i(z_0)| \leqslant \frac{M_1}{2(m+1)M_2} \quad \text{for } 0 \leqslant i \leqslant m \text{ and } z \in A.$$

By reducing A, if necessary, we may assume that

$$A = \{z \in \mathbf{C} : s < |z| < S,\ \phi_1 < \arg z < \phi_2\}.$$

Inside the contour $C_i^{j,n}$ the equation $p(z_0, z^n) = 0$ has precisely m_i solutions, and these are all at the point $z_i^{j,n}$. If the contour $C_i^{j,n}$ lies completely inside the region A and if $z \in C_i^{j,n}$, then we have

$$\begin{aligned}|p(z_0, z^n) - p(z, z^n)| &\leqslant |a_m(z_0) - a_m(z)||z^{nm}| + \ldots + |a_1(z_0) - a_1(z)||z^n| + |a_0(z_0) - a_0(z)| \\ &\leqslant (m+1)\frac{M_1}{2(m+1)M_2}M_2 \\ &< M_1 \\ &\leqslant |p(z_0, z^n)|.\end{aligned}$$

Therefore by Rouche's theorem the equation $p(z, z^n) = 0$ has precisely m_i solutions inside the contour $C_i^{j,n}$. Since the angle between successive contours $C_i^{j,n}$ is $2\pi/n$, we conclude

$$\#\{j : C_i^{j,n} \subset A\} = \frac{n(\phi_2 - \phi_1)}{2\pi} + \mathrm{O}(1)$$

as $n \to \infty$, where the error term O(1) is actually $\leqslant 2$ for all large enough n. Consequently the number of solutions of $p(z, z^n) = 0$ that lie in A is at least

$$\sum_{i=1}^{k} m_i \times \#\{j : C_i^{j,n} \subset A\} = \frac{nm(\phi_2 - \phi_1)}{2\pi} + \mathrm{O}(1).$$

Since the unit circle $\{z : |z| = 1\}$ is compact, we may cover it with finitely many sets of the form A, say $A_1, \ldots, A_N$, such that in each A_i we have the above lower bound on the number of solutions of $p(z, z^n) = 0$ in A_i. After trimming these sets and relabelling we may assume, for $i = 1, \ldots, N$, that

$$A_i = \{z \in \mathbf{C} : 1 - \delta_0 < |z| < 1 + \delta_0,\ \phi_{i-1} < \arg z < \phi_i\},$$

where $\delta_0 < \delta$ and $\phi_0 < \phi_1 < \ldots < \phi_N$ with $\phi_N - \phi_0 = 2\pi$. For n large enough we conclude that the number of solutions of $p(z, z^n) = 0$ in the annulus $\{z : 1 - \delta_0 < |z| < 1 + \delta_0\}$ is at least

$$\sum_{i=1}^{N} \frac{nm(\phi_i - \phi_{i-1})}{2\pi} + \mathrm{O}(1) = nm + \mathrm{O}(1).$$

Since $p(z, z^n) = 0$ has only $nm + d$ solutions, we conclude that all but O(1) of these solutions lie inside the contours $C_i^{j,n}$. Therefore, the number of solutions of $p(z, z^n) = 0$ in a region T can be counted with increasing accuracy as $n \to \infty$ by identifying those contours $C_i^{j,n}$ which are contained in T and recalling that the region enclosed by $C_i^{j,n}$ contains exactly m_i solutions. Therefore

$$\begin{aligned}\frac{\#\{z \in \mathbf{C} : p(z, z^n) = 0, 1 - \delta < |z| < 1 + \delta, \alpha < \arg z < \beta\}}{\#\{z \in \mathbf{C} : p(z, z^n) = 0\}} &= \frac{\sum_{i=1}^{k} nm_i(\beta - \alpha)/2\pi + \mathrm{O}(1)}{nm + d} \\ &= \frac{nm(\beta - \alpha)/2\pi + \mathrm{O}(1)}{nm + d} \\ &\longrightarrow \frac{\beta - \alpha}{2\pi}\end{aligned}$$

as $n \to \infty$, as required.

We now return to the first statement. Let $B = \{z \in \mathbf{C} : |z| < 1 - \epsilon\}$ and let $b_n(z) = p(z, z^n) - a_0(z)$ so that $p(z, z^n) = b_n(z) + a_0(z)$. Let $z_1, \ldots, z_k$ be the roots of $a_0(z)$ in B. Without loss of generality we may suppose that ϵ is small enough so that all the roots of $a_0(z)$ with modulus less than 1 are in B. For $i = 1, \ldots, k$ let D_i be a disc of radius δ and centre z_i, and let S_i be its circular boundary. For δ small enough the D_i are disjoint and are subsets of B. There exists $M > 0$ such that $|a_0(z)| > M$ for all z in $B \setminus \bigcup_i D_i$. Since $b_n(z) \to 0$ uniformly on B as $n \to \infty$, we have for large enough n that

$$|a_0(z)| > M/2 > |b_n(z)|$$

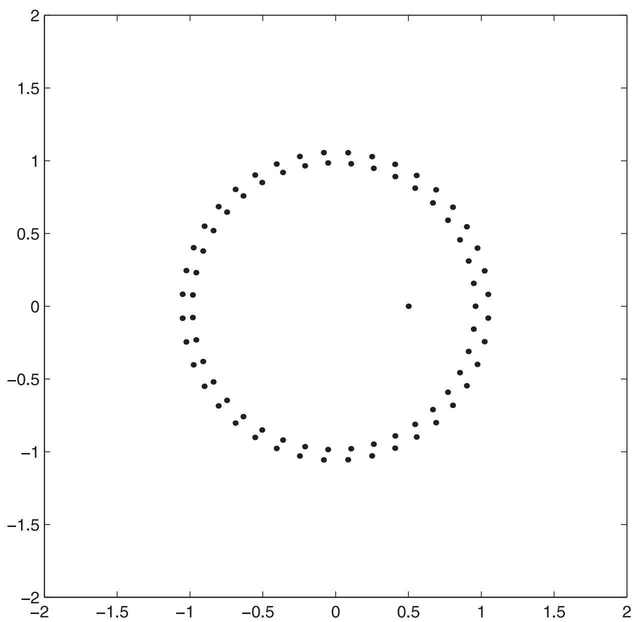

FIGURE 3. *Solutions of* $p(z, z^n) = 0$ *for* $n = 40$ *in Example* 2.

for all z in $B \setminus \bigcup_i D_i$. Consequently, the only solutions of $p(z, z^n) = 0$ in B are in $\cup_i D_i$. Furthermore, Rouche's theorem applied to the contour S_i implies that each D_i contains a solution of $p(z, z^n) = 0$. Since δ is arbitrary, the result follows. The proof of the second statement is similar. □

Only minor modifications are needed if a_0 or a_m has a zero of modulus 1. By the above proof, one can show that if S is an arc of the unit circle on which neither a_0 nor a_m vanishes, then the solutions of $p(z, z^n) = 0$ close to S are uniformly distributed along S.

EXAMPLE 2. Consider the polynomial p given by

$$p(z, w) = w^2 + a_1(z)w + a_0(z),$$

where $a_1(z) = -z^2 - z + 9/2$ and $a_0(z) = z^3 - z^2/2 - 4z + 2$. Letting $z = e^{i\theta}$ and solving for w gives two distinct solutions

$$f_+(e^{i\theta}) = e^{i\theta} - 1/2,$$
$$f_-(e^{i\theta}) = e^{2i\theta} - 4$$

for all $\theta \in [0, 2\pi]$. Since $1/2 \leqslant |f_+(e^{i\theta})| \leqslant 3/2$ and $3 \leqslant |f_-(e^{i\theta})| \leqslant 5$, we should anticipate that for large n the roots of $p(z, z^n) = 0$ will form two distinct rings of solutions, both close to the unit circle, along with an extra zero close to $1/2$. This is illustrated in Figure 3.

EXAMPLE 3. Consider the polynomial p given by

$$p(z, w) = w^2 - 4w - 8z + 3.$$

Letting $z = e^{i\theta}$ and solving for w gives

$$w = 2 \pm \sqrt{8e^{i\theta} + 1}.$$

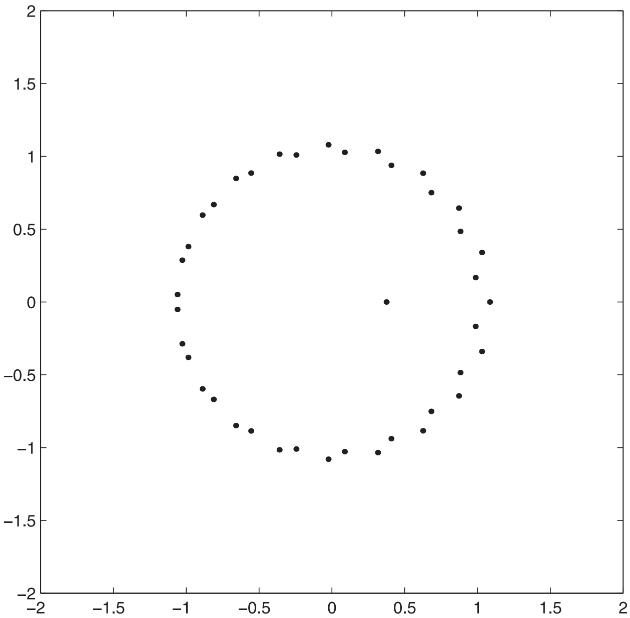

FIGURE 4. *Solutions of* $p(z, z^n) = 0$ *for* $n = 20$ *in Example* 3.

For each θ let $w_1(\theta)$ and $w_2(\theta)$ denote the two roots. In the vicinity of $e^{i\theta}$ there are two (possibly overlapping) families of solutions to $p(z, z^n) = 0$, and these solutions are close to those of $z^n = w_1(\theta)$ and $z^n = w_2(\theta)$. If $|w_1(\theta)| \neq |w_2(\theta)|$ (for exapmle, if $\theta = 0$), then we should expect two distinct families of solutions near $e^{i\theta}$. Futhermore, if $|w_1(\theta)| = |w_2(\theta)|$ (for exapmle, if $\theta = \pi$), then we should expect the families to coincide. This is clearly seen in Figure 4 for $n = 20$.

3. *Spectrum of a directed graph*

In this section we introduce some relevant notions from graph theory and use them to study the adjacency matrix of a directed graph. As well as providing a simple introduction to the general theory, this case has some features of its own that are of interest.

Let $\mathcal{G}$ denote the class of finite, directed graphs $(S, \to)$ that are irreducible in the sense that for every $u, v \in S$ there exists a path $u = s_1 \to s_2 \to \cdots \to s_n = v$. The irreducibility assumption implies that the in-degree and out-degree of every vertex of S is at least 1.

The adjacency matrix A of $(S, \to) \in \mathcal{G}$ is defined by

$$A_{i,j} = \begin{cases} 1 & \text{if } i \to j, \\ 0 & \text{otherwise.} \end{cases}$$

The spectrum of $(S, \to)$ is by definition the spectrum of A.

We partition S into two sets C and J, where C consists of all vertices $v \in S$ that have total degree 2, and where $J := S \setminus C$. If $\#(S) > 1$, then for each $a \in C$ there exist $b, c \neq a$ such that $b \to a \to c$. If $\#(J) = 0$, then $(S, \to)$ is a cycle and

$$\mathrm{Spec}(A) = \{e^{2\pi ir/n} : r = 1, 2, \ldots, n\},$$

where $n = \#(S)$. We henceforth assume that neither C nor J is empty, and set about exploring the structure of these two sets.

The set C may be partitioned into h disjoint subsets $C_1, \dots, C_h$, where the vertices in C_i may be labelled $\{v_{i,1}, \dots, v_{i,d_i}\}$ so that

$$b_i \longrightarrow v_{i,1} \longrightarrow v_{i,2} \longrightarrow \dots \longrightarrow v_{i,d_i} \longrightarrow e_i,$$

where b_i, e_i are in J. The set C_i is called a *channel* and is said to have length d_i and endpoints b_i, e_i.

The structure of J may be quite complex, but every $x \in J$ has total degree at least 3. We may write J as the disjoint union of subsets $\{J_i\}_{i=1}^k$ that are maximally connected with respect to the *undirected* graph structure inherited from $(S, \to)$. It follows that there are no directed edges joining different subsets J_i, which we call junctions. Irreducibility implies that one can pass from any junction to another, but only via intermediate channels. In some cases one may only be able to pass between different points of the same junction via external channels.

We would like to explore the spectrum of $(S, \to)$ as the length of each channel is increased by a factor of n. This involves studying the spectrum of the graph $(S^{(n)}, \to)$ obtained from $(S, \to)$ by replacing each channel

$$b_i \longrightarrow C_i \longrightarrow e_i$$

by the lengthened channel

$$b_i \longrightarrow C_i^{(n)} \longrightarrow e_i,$$

where $\#(C_i^{(n)}) = n \times \#(C_i)$. We denote the corresponding adjacency matrix by $A^{(n)}$.

The key to our analysis is to define a new graph $(\tilde{S}, \to)$ by collapsing each channel C_i in $(S, \to)$ to a single point p_i. In more detail, we put

$$\tilde{S} = J \cup \{p_i : 1 \leqslant i \leqslant h\}$$

and define the edges of the new graph as follows. If $i, j \in J$, then $i \to j$ in $\tilde{S}$ if and only if $i \to j$ in S. Also, we include the edges $b_i \to p_i$ and $p_i \to e_i$ for each channel C_i. The adjacency matrix of $(\tilde{S}, \to)$ is denoted by $\tilde{A}$. Importantly, $(\tilde{S}, \to)$ and $\tilde{A}$ do not depend on n or the lengths d_i of the original channels. With this in mind, the next result simplifies the task of computing the characteristic polynomial of $A^{(n)}$.

THEOREM 3.1. *The eigenvalues of $A^{(n)}$ coincide with the roots of the polynomial pencil $D(z) - \tilde{A}$ on $\tilde{S}$, where $D(z)$ is the diagonal matrix with entries*

$$D_{i,j}(z) = \begin{cases} z & \text{if } i = j \in J, \\ z^{nd_r} & \text{if } i = j = p_r \text{ for some } r \in \{1, \dots, h\}, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed

$$\det(zI - A^{(n)}) = \det(D(z) - \tilde{A}) \tag{3.1}$$

for all $z \in \mathbf{C}$.

Proof. It suffices to prove the following more general result, which will also be of use later on. Let M be a matrix of the form

$$M = \begin{pmatrix} C_1 & & & & E_1 \\ & C_2 & & & E_2 \\ & & \ddots & & \vdots \\ & & & C_h & E_h \\ B_1 & B_2 & \cdots & B_h & J \end{pmatrix},$$

where C_k is the $n_k \times n_k$ matrix

$$\begin{pmatrix} \alpha_k & -1 & & \\ & \alpha_k & \ddots & \\ & & \ddots & -1 \\ & & & \alpha_k \end{pmatrix}$$

with α_k on the diagonal, -1 directly above the diagonal and 0 elsewhere; J is an arbitrary $m \times m$ matrix; B_k is an $m \times n_k$ matrix with 0 everywhere except possibly for entry $(i_k, 1)$, where it has the value β_k; and E_k is an $n_k \times m$ matrix with 0 everywhere except possibly for entry (n_k, j_k), where it has the value ϵ_k. It is clear that the vertices of the graph $(S^{(n)}, \to)$ can be labelled so that the corresponding matrix $zI - A^{(n)}$ has the above form.

On the set of matrices of the above form, we may define a map $M \mapsto \tilde{M}$ by defining the corresponding matrix $\tilde{M}$ by

$$\tilde{M} = \begin{pmatrix} \alpha_1^{n_1} & & & & E_1' \\ & \alpha_2^{n_2} & & & E_2' \\ & & \ddots & & \vdots \\ & & & \alpha_h^{n_h} & E_h' \\ B_1' & B_2' & \cdots & B_h' & J \end{pmatrix},$$

where B_k' is the $m \times 1$ column vector with value β_k in position i_k and 0 elsewhere; E_k' is the $1 \times m$ row vectors with value ϵ_k in position j_k and 0 elsewhere; and where the 1×1 matrix $(a_k^{n_k})$ replaces the matrix C_k. It is clear that the vertices of the graph $(\tilde{S}, \to)$ can be labelled so that the corresponding matrix $D(z) - \tilde{A}$ has the above form.

It now suffices to show that $\det(M) = \det(\tilde{M})$. We prove the result by induction on h, which is the number of matrices C_k along the diagonal of M. The result clearly holds for $h = 0$ and for any size matrix J. For the inductive case, let $N = n_1 + \ldots + n_h$ and let K be the matrix

$$K = \begin{pmatrix} C_2 & & & E_2 \\ & \ddots & & \vdots \\ & & C_h & E_h \\ B_2 & \cdots & B_h & J \end{pmatrix}.$$

Expanding the determinant of M down the first column gives

$$\begin{aligned} \det(M) &= \alpha_1 \det(M_{1,1}) + (-1)^{N+i_1+1}\beta_1 \det(M_{N+i_1,1}) \\ &= \alpha_1^{n_1} \det(K) + (-1)^{N+i_1+1}\beta_1(-1)^{n_1-1}(-1)^{N-n_1+j_1+1}\epsilon_1 \det(K_{N-n_1+i_1,N-n_1+j_1}) \\ &= \alpha_1^{n_1} \det(K) + (-1)^{i_1+j_1+1}\beta_1\epsilon_1 \det(K_{N-n_1+i_1,N-n_1+j_1}), \end{aligned}$$

where $\det(M_{1,1})$ is evaluated by expanding successive minors down the first column and $\det(M_{N+i_1,1})$ is evaluated by expanding successive minors across the first row. Similarly, we have

$$\begin{aligned} \det(\tilde{M}) &= \alpha_1^{n_1} \det(\tilde{K}) + (-1)^{h+i_1+1}\beta_1(-1)^{h-1+j_1+1}\epsilon_1 \det(\tilde{K}_{h-1+i_1,h-1+j_1}) \\ &= \alpha_1^{n_1} \det(\tilde{K}) + (-1)^{i_1+j_1+1}\beta_1\epsilon_1 \det(\tilde{K}_{h-1+i_1,h-1+j_1}). \end{aligned}$$

By the induction hypothesis

$$\det(K) = \det(\tilde{K}),$$
$$\det(K_{N-n_1+i_1,N-n_1+j_1}) = \det(\tilde{K}_{h-1+i_1,h-1+j_1})$$

and so $\det(M) = \det(\tilde{M})$, as required. □

COROLLARY 3.2. *There exists a polynomial p of the form*

$$p(z,w) = a_c(z)w^c + a_{c-1}(z)w^{c-1} + \ldots + a_1(z)w + a_0(z)$$

such that

$$\det(zI - A^{(n)}) = p(z, z^n). \tag{3.2}$$

Moreover $c = \#(C)$. All of the polynomials a_r have degree at most $\#(J)$ and a_c has degree equal to $\#(J)$.

Proof. Theorem 3.1 implies $\det(z - A^{(n)}) = \det(D(z) - \tilde{A})$, and after the change of variable $w = z^n$ we note that

$$\det(D(z) - \tilde{A}) = \begin{vmatrix} w^{d_1} & & & & E'_1 \\ & w^{d_2} & & & E'_2 \\ & & \ddots & & \vdots \\ & & & w^{d_h} & E'_h \\ B'_1 & B'_2 & \ldots & B'_h & zI - J \end{vmatrix}.$$

Using the permutation definition of the determinant we see that the highest-order power of w corresponds to the term in the determinant that includes all of the terms w^{d_r}, and the power of w is then $d_1 + \ldots + d_h = c$. The term in the determinant arising from the identity permutation shows that $a_c(z)$ has degree equal to $\#(J)$. It is clear that this is the maximum degree of the polynomials $a_r(z)$. □

By combining Lemma 2.7 with Corollary 3.2 we obtain the key result of this section.

THEOREM 3.3. *For every $\epsilon > 0$ the spectrum of $(S^{(n)}, \to)$ is almost entirely confined to an annulus of the form $\{z : 1 - \epsilon < |z| < 1 + \epsilon\}$ for n sufficiently large. These eigenvalues are asymptotically uniformly distributed around the unit circle, except near at most finitely many points. There may however be a small number of eigenvalues far from the unit circle.*

In a later section we will show that the corresponding eigenvectors are concentrated around the set J. In general they are concentrated around a single junction, but this may not happen if the graph has certain symmetries.

4. *The general model*

We now turn to the general model described in section 1. Starting with a general $n \times n$ matrix A we may define an associated directed graph $(S, \to)$ by putting $S = \{1, \ldots, n\}$ and declaring $i \to j$ if and only if $A_{i,j} \neq 0$. The edge $i \to j$ is said to have weight $A_{i,j}$. The added presence of weights necessitates a more general definition of channel and junction, which we now explain.

Suppose that S can be partitioned into disjoint sets C and J that satisfy the following properties. The set C may be further partitioned into disjoint subsets $C_1, \ldots, C_h$ called channels, where the vertices in C_i may be labelled $\{v_{i,1}, \ldots, v_{i,d_i}\}$ so that

$$b_i \longrightarrow v_{i,1} \longrightarrow v_{i,2} \longrightarrow \ldots \longrightarrow v_{i,d_i} \longrightarrow e_i,$$

where b_i, e_i are in J. There exists α_i in $\mathbf{C}$ such that the diagonal entry $A_{v,v}$ equals α_i for all v in C_i. There exists $\beta_i \neq 0$ in $\mathbf{C}$ such that the weight of each edge $v_{i,j} \to v_{i,j+1}$ is β_i. The edges $b_i \to v_{i,1}$ and $v_{i,d_i} \to e_i$ also have weight β_i. If $\alpha_i = 0$, then each vertex in C_i has total degree 2, and if $\alpha_i \neq 0$, then each vertex in C_i has total degree 4, the increase in degree being attributed to the edge of weight α_i that begins and ends at each vertex. In summary, each channel C_i is determined by three parameters, the diagonal weight α_i, the off-diagonal weight β_i and the length d_i. As before, the set J can be partitioned into disjoints sets $J_1, \ldots, J_k$ called junctions that are maximally connected with respect to the undirected graph structure induced by $(S, \to)$.

This definition does not require the graph to be irreducible. However, as before, we shall be concerned only with irreducible graphs.

It must be emphasized that there is no canonical way to partition S into sets C and J. We simply choose a partition of S so that the above conditions hold. This is always possible since we may define $C = \emptyset$ and $J = S$, though the lack of channels makes this partition uninteresting.

Suppose that we have partitioned S into sets C and J and have thereby identified the channels $C_1, \ldots, C_h$ in $(S, \to)$. For each positive integer n we construct a new matrix $A^{(n)}$ that has the same graph structure and matrix entries as A, except that the length of each channel is increased by a factor of n. Our aim is to study the asymptotic behaviour of the eigenvalues of $A^{(n)}$ as $n \to \infty$.

Let $B_z^{(n)}$ be the matrix $zI - A^{(n)}$ after we have divided the entries in row i by β_j if i is in C_j. By the proof of Theorem 3.1 we have

$$p_n(z) := \det(zI - A^{(n)}) = k^n \det(B_z^{(n)}) = k^n \sum_s a_s(z) \prod_{i \in s} ((z - \alpha_i)/\beta_i)^{nd_i},$$

where $k = \prod_{i=1}^h \beta_i^{d_i}$, the sum is over all subsets s of $\{1, 2, \ldots, h\}$ and each a_s is a polynomial independent of n and of degree at most $d = \sum_{j=1}^k \#(J_j)$. If $\sigma = \{1, 2, \ldots, h\}$, then a_σ is of degree d. The other a_s may be of lower degree and may vanish identically. The zeros of $p_n(z)$ coincide with those of

$$F_n(z) = \sum_s a_s(z) f_s(z)^n,$$

where the sum is over all subsets s of $\{1, 2, \ldots, h\}$ and

$$f_s(z) = \prod_{i \in s} ((z - \alpha_i)/\beta_i)^{d_i}.$$

The spectral asymptotics are therefore determined by Theorems 2.4 and 2.5, since it is generically true that the f_s satisfy the required hypotheses.

Since there are 2^h subsets of $\{1, 2, \ldots, h\}$, one would expect the number of terms in $p_n(z)$ to grow exponentially as the number of channels is increased. Surprisingly this often does not occur as many of the a_s may vanish identically. We will prove this for the case when each junction consists of a single point. The next result (based on [**4**, Theorem 1.2]) gives a necessary condition for the term a_s to be non-zero in terms of the graph $(S^{(n)}, \to)$ associated with $B_z^{(n)}$. Apart from the weights, we note that $S^{(n)}$ is simply the graph associated with $A^{(n)}$, along with the (possibly extra) edge $i \to i$ for every $i \in S^{(n)}$, since the diagonal entries of $B_z^{(n)}$ are non-zero. In particular, the weight of the edge $v \to v$ for $v \in C_i^{(n)}$ is $(z - \alpha_i)/\beta_i$.

THEOREM 4.1. *Suppose a_s is non-zero for some $s \subseteq \{1, 2, \ldots, h\}$. If we remove every channel $C_i^{(n)}$ for $i \in s$ from the graph $(S^{(n)}, \to)$, then it is possible to cover the remaining subgraph by disjoint cycles so that if C is a channel not in s, then C is completely contained in one of the cycles of length greater than one.*

Proof. It is important to note that by definition the channel $C_i^{(n)}$ does not contain its endpoints b_i, e_i. Now, letting $B := B_z^{(n)}$, we have

$$\det(B_z^{(n)}) = \sum_{\pi} \operatorname{sign}(\pi) \prod_{i \in S^{(n)}} B_{i,\pi(i)},$$

where the sum is over all permutations π of the set $S^{(n)}$. The term $t_\pi = \prod_{i \in S^{(n)}} B_{i,\pi(i)}$ is non-zero if and only if $(i, \pi(i))$ is a directed edge of $(S^{(n)}, \to)$ for every $i \in S^{(n)}$. Now, π may be written as a product of disjoint cyclic permutations, some of which may be of length 1. Therefore, if $t_\pi \neq 0$, π induces a covering $\mathcal{C}$ of the graph $S^{(n)}$ by disjoint cycles. Consider the channel $C_i^{(n)}$. If $\pi(v) = v$ for some $v \in C_i^{(n)}$, then $\pi(w) = w$ for all $w \in C_i^{(n)}$, and so t_π contains the factor $((z - \alpha_i)/\beta_i)^{nd_i}$. In other words, for all $w \in C_i^{(n)}$, the cycle in $\mathcal{C}$ which contains w is the singleton cycle $w \to w$. If $\pi(v) \neq v$ for some $v \in C_i^{(n)}$, then $\pi(w) \neq w$ for all $w \in C_i^{(n)}$, and so the entire channel $C_i^{(n)}$ is contained in a cycle of $\mathcal{C}$ of length greater than one. In this case, the contribution of the channel $C_i^{(n)}$ to t_π is a non-zero constant. Therefore, one obtains

$$a_s(z) \prod_{i \in s} ((z - \alpha_i)/\beta_i)^{nd_i}$$

by summing over all coverings of $S^{(n)}$ by disjoint cycles, such that each point in each channel of s is contained in a singleton cycle, and each channel not in s is completely contained in a cycle of length greater than one. Since a_s is non-zero, such a partition actually exists. □

Corollary 4.2. *Let $(S_{h,k}, \to)$ be a graph consisting of k junctions, each a single point, and h channels. If N is the number of subsets s of $\{1, \ldots, h\}$ such that a_s is non-zero, then $N = \mathrm{O}(h^k)$ as $h \to \infty$.*

Proof. Let G_s denote the graph obtained from $S_{h,k}$ by removing each channel C_i for $i \in s$. If G_s can be covered by disjoint cycles such that each channel C not in s is contained in a single cycle, then for each junction in G_s we either have precisely one in channel and one out channel, or else we have no channels at all. Thus we are now left with a counting problem. The number of subsets s such that d junctions in G_s have precisely one in channel and one out channel with the remaining junctions having no channels is at most $\binom{h}{d}$, since each possibility requires precisely d channels, and we have h from which to choose. Consequently,

$$N \leqslant \binom{h}{0} + \binom{h}{1} + \ldots + \binom{h}{k} = \mathrm{O}(h^k)$$

as $h \to \infty$, as required. □

Despite the above result, there are examples for which a_s is non-zero for all 2^h subsets s of $\{1, \ldots, h\}$. Let S be a graph consisting of one junction and h channels that satisfy the following properties. The junction J is a cycle of length $2h$ with

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow 2h - 1 \longrightarrow 2h \longrightarrow 1.$$

For $i = 1, \ldots, h$ the channel C_i begins at i and ends at $i + 1$ so that

$$i \longrightarrow C_i \longrightarrow i + 1.$$

The length of each channel is not important. If s is a subset of $\{1, \ldots, h\}$, then the only way of covering the graph G_s by disjoint cycles so that each channel C not in s is contained in a single cycle is by using the unique cycle in G_s that contains all the channels. Therefore each a_s is actually a non-zero constant. The spectral asymptotics are particularly simple when none of the a_s vanishes identically.

THEOREM 4.3. *If none of the a_s vanishes identically, then the zeros of F_n as $n \to \infty$ converge to the union of a collection of circles*

$$\Lambda := \bigcup_{i=1}^{h} \{z \in \mathbf{C} : |z - \alpha_i| = |\beta_i|\}$$

and a finite subset of $\mathbf{C}$*. Furthermore, the asymptotic density of the zeros along the circle* $|z - \alpha_i| = |\beta_i|$ *is the constant* $d_i/(2\pi|\beta_i|)$*.*

Proof. By Theorem 2.5 and its corollary, the zeros of F_n as $n \to \infty$ converge to the union of

$$L := \bigcup_{r \subseteq \{1,\ldots,h\}} \{z \in \mathbf{C} : |f_r(z)| = \max\{|f_k(z)| : k \subseteq \{1,\ldots,h\}, k \neq r\}\},$$

and a discrete set of points consisting of certain zeros of the a_s. By the same theorem, the zeros of

$$G_n(z) := \sum_{s \subseteq \{1,\ldots,h\}} f_s(z)^n$$

as $n \to \infty$ also converge to L and with the same asymptotic densities. However,

$$G_n(z) = \prod_{i=1}^{h} \left(1 + \left(\frac{z - \alpha_i}{\beta_i}\right)^{nd_i}\right),$$

and so by a direct calculation we see that the zeros of G_n as $n \to \infty$ converge to the collection of circles Λ, and with the stated asymptotic densities. □

5. *Some examples*

The matrices $A^{(n)}$ considered in this paper may be classified according to the number of channels h and the number of junctions k. In this section we consider some of the phenomena that arise for small values of h and k, our goal being to describe the asymptotic form of $\mathrm{Spec}(A^{(n)})$ as $n \to \infty$. Since each junction has at least one in channel and one out channel by irreducibility, it follows that $k \leqslant h$. Even if we assume for simplicity that each junction consists of a single point and that each channel has the same length n, there are still several graphs for each choice of h and k.

EXAMPLE 4 ($h = k = 1$). We suppose that the graph has one junction containing two points and one channel containing $n - 2$ points. The $n \times n$ matrix is taken to have the form

$$A = \begin{cases} 1 & \text{if } j = i + 1, \\ a & \text{if } i = j = 1, \\ b & \text{if } i = 1 \text{ and } j = n, \\ c & \text{if } j = 1 \text{ and } i = n, \\ d & \text{if } i = j = n, \\ 0 & \text{otherwise} \end{cases}$$

in which the constants a, b, c and d describe boundary conditions at the ends of the ordered interval $\{1, 2, \ldots, n\}$. One may check that the characteristic polynomial of A is

$$p_n(z) = z^{n-2}(z^2 - (a + d)z + ad - bc) - c.$$

The roots of this polynomial converge asymptotically to the unit circle, with the exception of isolated roots that converge to any solution of $z^2 - (a + d)z + ad - bc = 0$ that satisfies $|z| > 1$.

Example 5 ($h = 2,\ k = 1$). We consider the case in which the graph associated to A has only one junction and that junction contains only one point. We also assume that there are $h = 2$ channels, each of which has length n. The matrix A therefore has $2h+1 = 5$ free parameters apart from n. We will see that all the anti-Stokes curves are circles, where we regard straight lines as circles with infinite radii. For $n = 3$ the matrix A is of the form

$$A = \left(\begin{array}{ccc|ccc|c} \alpha_1 & \beta_1 & & & & & \\ & \alpha_1 & \beta_1 & & & & \\ & & \alpha_1 & & & & \beta_1 \\ \hline & & & \alpha_2 & \beta_2 & & \\ & & & & \alpha_2 & \beta_2 & \\ & & & & & \alpha_2 & \beta_2 \\ \hline \beta_1 & & & \beta_2 & & & \gamma \end{array}\right),$$

where we need to assume that β_1 and β_2 are non-zero. For general n the reduced matrix is of the form

$$A(z) = \begin{pmatrix} (z-\alpha_1)^n & & -\beta_1^n \\ & (z-\alpha_2)^n & -\beta_2^n \\ -\beta_1 & -\beta_2 & z-\gamma \end{pmatrix}$$

and the characteristic polynomial is

$$p(z) = (z-\gamma)(z-\alpha_1)^n(z-\alpha_2)^n - \beta_1^{n+1}(z-\alpha_2)^n - \beta_2^{n+1}(z-\alpha_1)^n.$$

This may be simplified further if $\alpha_1 = \alpha_2$ and we assume that this is not the case.

The characteristic polynomial is of the form $\sum_r a_r(z) f_r(z)^n$, where

$$\begin{aligned} f_1(z) &= (z-\alpha_1)(z-\alpha_2), \\ f_2(z) &= \beta_1(z-\alpha_2), \\ f_3(z) &= \beta_2(z-\alpha_1) \end{aligned}$$

and $a_1(z) = z-\gamma$, $a_2(z) = -\beta_1$ and $a_3(z) = -\beta_2$. Each of the functions f_i dominates the others in absolute value in an open set U_i, where U_1 contains all large enough z, $\alpha_1 \in U_2$ and $\alpha_2 \in U_3$. The limit set E of the spectrum of A as $n \to \infty$ is contained in the union of the anti-Stokes lines, with the possible exception of an eigenvalue that converges to γ. Whether or not this eigenvalue exists depends on the parameters of the matrix.

The anti-Stokes lines are defined for $i \neq j$ by

$$K_{i,j} = \{z \in \mathbf{C} : |f_i(z)| = |f_j(z)|\}$$

and are given by the formulae

$$\begin{aligned} K_{1,2} &= \{z \in \mathbf{C} : |z-\alpha_1| = |\beta_1|\}, \\ K_{1,3} &= \{z \in \mathbf{C} : |z-\alpha_2| = |\beta_2|\}, \\ K_{2,3} &= \{z \in \mathbf{C} : |\beta_1|\,|z-\alpha_2| = |\beta_2|\,|z-\alpha_1|\}. \end{aligned}$$

An elementary calculation shows that $K_{2,3}$ is a circle with centre $(\delta\alpha_2 - \alpha_1)/(\delta - 1)$, where $\delta = |\beta_1/\beta_2|^2$. If $\delta = 1$, then $K_{2,3}$ is a straight line. The three circles are part of a coaxial system.

Once one has determined the circles $K_{i,j}$ the regions U_i can be identified by use of the following rules. Each component of $U = \mathbf{C}\backslash(K_{1,2} \cup K_{1,3} \cup K_{2,3})$ is contained in a single set U_i and the unbounded components are contained in U_1. If C is some part of $K_{i,j}$ and U_i is on one side of C, then by applying the maximum principle to $f_i(z)/f_j(z)$ one sees that U_j is on the other side of C. If, however, U_k is on one side of C, where $k \neq i$ and $k \neq j$, then U_k is also on the other side of C and the asymptotic spectrum E does not contain any points in C.

By applying the above rules one can progressively determine which U_i contains each component of U and also eliminate certain curves $C \subseteq K_{i,j}$ as possible parts of E. Figure 5

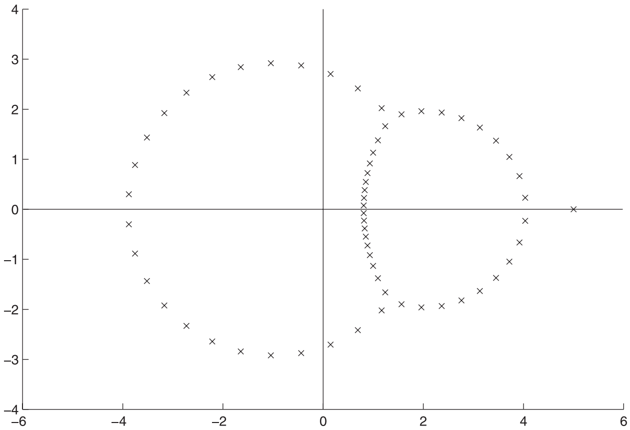

FIGURE 5. *Eigenvalues of a matrix with $h = 2$ and $k = 1$.*

portrays the eigenvalues of the matrix A with $n = 30$, $\alpha = [2, -1]$, $\beta = [2, 3]$ and $\gamma = 5$. The circles involved are $|z - 2| = 2$, $|z + 1| = 3$ and $|z - 4.4| = 3.6$. The arcs removed from each circle are in accordance with the application of the above rules. The eigenvalue close to 5 is given more accurately by $\lambda \sim 5.0000104$.

If the two channels have different lengths, say $c_1 n$ and $c_2 n$, where $n \to \infty$, then only small changes are needed. The anti-Stokes curves $K_{1,2}$ and $K_{1,3}$ are unchanged but $K_{2,3}$ becomes the set of z such that

$$\left|\frac{z - \alpha_1}{\beta_1}\right|^{c_1} = \left|\frac{z - \alpha_2}{\beta_2}\right|^{c_2},$$

which is only a circle if $c_1 = c_2$. The general rules for locating the eigenvalues still apply.

EXAMPLE 6 ($h = 3, k = 1$). The analysis in the previous example carries through for larger values of h. In particular for $h = 3$ the reduced matrix has seven free parameters apart from n and is

$$A(z) = \begin{pmatrix} (z-\alpha_1)^n & & & -\beta_1^n \\ & (z-\alpha_2)^n & & -\beta_2^n \\ & & (z-\alpha_3)^n & -\beta_3^n \\ -\beta_1 & -\beta_2 & -\beta_3 & z-\gamma \end{pmatrix}.$$

The characteristic polynomial is

$$\begin{aligned} p(z) = (z-\gamma)(z-\alpha_1)^n(z-\alpha_2)^n(z-\alpha_3)^n - \beta_1^{n+1}(z-\alpha_2)^n(z-\alpha_3)^n \\ - \beta_2^{n+1}(z-\alpha_1)^n(z-\alpha_3)^n - \beta_3^{n+1}(z-\alpha_1)^n(z-\alpha_2)^n. \end{aligned}$$

Assuming that all the α_i are distinct, there are six anti-Stokes lines, namely

$$\begin{aligned} K_{1,2} &= \{z \in \mathbf{C} : |z-\alpha_1| = |\beta_1|\}, \\ K_{1,3} &= \{z \in \mathbf{C} : |z-\alpha_2| = |\beta_2|\}, \\ K_{1,4} &= \{z \in \mathbf{C} : |z-\alpha_3| = |\beta_3|\}, \\ K_{2,3} &= \{z \in \mathbf{C} : |\beta_2|\,|z-\alpha_1| = |\beta_1|\,|z-\alpha_2|\}, \\ K_{2,4} &= \{z \in \mathbf{C} : |\beta_3|\,|z-\alpha_1| = |\beta_1|\,|z-\alpha_3|\}, \\ K_{3,4} &= \{z \in \mathbf{C} : |\beta_2|\,|z-\alpha_3| = |\beta_3|\,|z-\alpha_2|\}. \end{aligned}$$

These are all circles, or straight lines in degenerate cases.

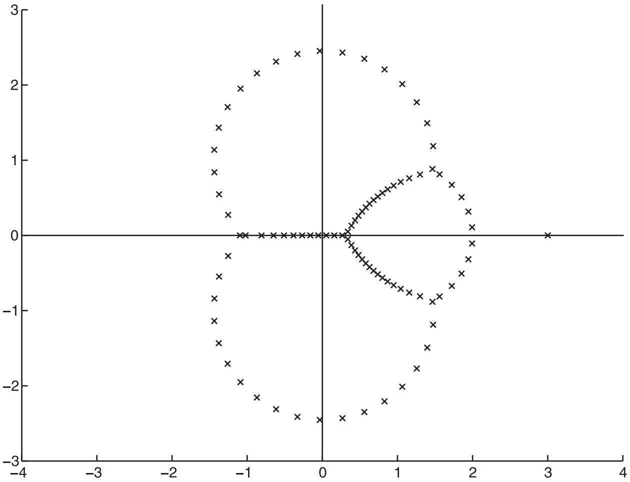

FIGURE 6. *Eigenvalues of a matrix with $h = 3$ and $k = 1$.*

In Figure 6, we put $n = 30$, $\alpha = [1, i, -i]$, $\beta = [1, 3/2, 3/2]$ and $\gamma = 3$. Once again the eigenvalues are close to the parts of the anti-Stokes lines obtained by using the rules given above.

EXAMPLE 7 ($h = k = 2$). We consider a graph with two junctions, each containing one point, and two channels, each containing n points. Each channel goes from one junction to the other, so that the graph has exactly one circuit. The matrix A acts on vectors of length $2n + 2$. The indices $2n + 1$ and $2n + 2$ label the junctions, $\{1, \ldots, n\}$ label the first channel and $\{n + 1, \ldots, 2n\}$ label the second channel. The matrix A is defined by

$$A = \left(\begin{array}{cccc|cccc|cc}
\alpha_1 & \beta_1 & & & & & & & & \\
 & \alpha_1 & \ddots & & & & & & & \\
 & & \ddots & \beta_1 & & & & & & \\
 & & & \alpha_1 & & & & & \beta_1 & \\
\hline
 & & & & \alpha_2 & \beta_2 & & & & \\
 & & & & & \alpha_2 & \ddots & & & \\
 & & & & & & \ddots & \beta_2 & & \\
 & & & & & & & \alpha_2 & \beta_2 & \\
\hline
\beta_1 & & & & & & & & \gamma_1 & \\
 & & & & \beta_2 & & & & & \gamma_2
\end{array}\right)$$

The characteristic polynomial for general n is

$$p(z) = (z - \gamma_1)(z - \gamma_2)(z - \alpha_1)^n (z - \alpha_2)^n - \beta_1^{n+1} \beta_2^{n+1}.$$

The spectrum converges to

$$\{z \in \mathbf{C} : |(z - \alpha_1)(z - \alpha_2)| = |\beta_1 \beta_2|\}, \tag{5.1}$$

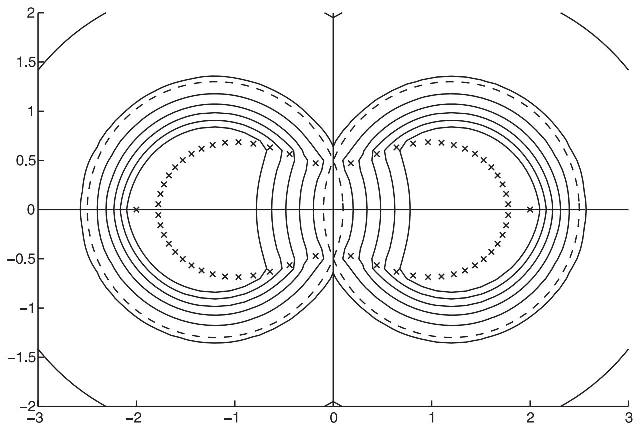

FIGURE 7. *Eigenvalues of a matrix with* $h = 2$, $k = 2$ *and* $n = 30$.

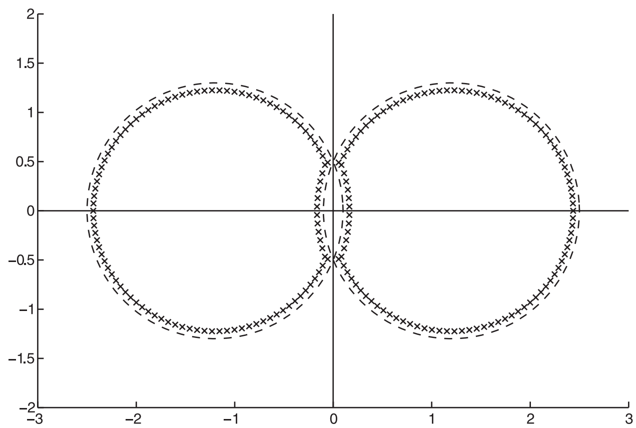

FIGURE 8. *Eigenvalues of a matrix with* $h = 2, k = 1$ *and* $n = 100$.

as $n \to \infty$, apart from the possibility of isolated eigenvalues converging to γ_1 or γ_2. This curve has two points in common with each of the circles $|z - \alpha_i| = |\beta_i|$, provided these circles intersect.

We carried out computations for the case $n = 30$, $\alpha = [-1.2, 1.2]$, $\beta = [1.3, 1.3]$ and $\gamma = [-2, 2]$. Figure 7 plots the eigenvalues as crosses and the circles $|z - \alpha_i| = |\beta_i|$ as dotted curves. The curve (5.1) is simple but non-convex and crosses both circles at $\pm 0.5i$.

The other curves in the figure are the pseudospectral contours of A for $\epsilon = 10^{-m}$, where $m = 0, \ldots, 6$, the outermost one, for $\epsilon = 1$, being only partly visible; see [**6**, **11**] for the definition. One deduces from these contours that the eigenvalues are highly unstable; indeed for $n = 50$ they are not easily computable because of rounding errors. Note that the pseudospectral contours are related much more strongly to the pair of circles than they are to the eigenvalues.

If one links the two junctions together weakly by putting

$$A(2n + 1, 2n + 2) = A(2n + 2, 2n + 1) = 10^{-2},$$

then the spectrum of A changes radically and approximates the union of the two circles more closely as n increases. The case $n = 100$ is plotted in Figure 8.

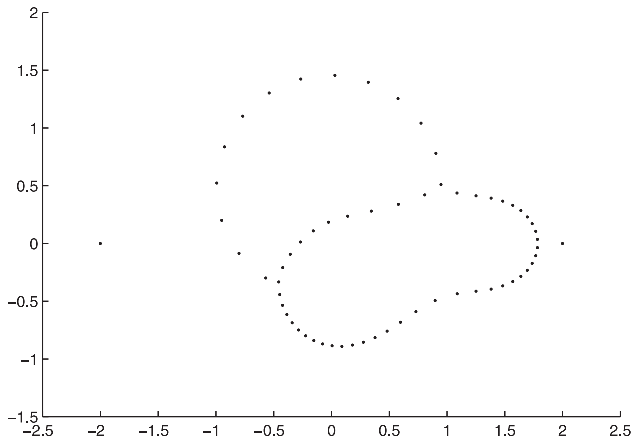

FIGURE 9. *The spectrum of Example* 8.

EXAMPLE 8 ($h = 3, k = 2$). The $(3n+3) \times (3n+3)$ matrix

$$A_{i,j} = \begin{cases} \dfrac{i}{2} & \text{if } 1 \leqslant i = j \leqslant n, \\ -2 & \text{if } i = j = n+1, \\ -\dfrac{i}{2} & \text{if } n+2 \leqslant i = j \leqslant 2n+1, \\ 2 & \text{if } i = j = 2n+2, \\ \dfrac{3}{2} & \text{if } 2n+3 \leqslant i = j \leqslant 3n+2, \\ 0 & \text{if } i = j = 3n+3, \\ 1 & \text{if } i+1 = j, \\ 1 & \text{if } i = 3n+3, j = 1, \\ 1 & \text{if } i = n+1, j = 3n+3, \\ 0 & \text{otherwise} \end{cases}$$

has characteristic polynomial

$$\det(zI - A) = z(z^2 - 4)(z - i/2)^n(z + i/2)^n(z - 3/2)^n - (z - 2)(z + i/2)^n(z - 3/2)^n - 1.$$

Its spectrum is depicted in Figure 9 for $n = 25$. The corresponding graph has two junctions and three channels, each of length n. Since $h = 3$, there are potentially eight terms in (4), but the explicit expression of $\det(zI - A)$ shows that only three are non-zero. These give rise to the three arcs in Figure 9. The eigenvalues near ± 2 are associated with two of the three diagonal entries at the junctions.

6. *Localization of eigenvectors*

In the general model each channel C_r is determined by three parameters, its length d_r, its diagonal entries α_r and its non-zero, off-diagonal entries β_r. We may associate with each channel C_r the circle

$$S_r = \{z \in \mathbf{C} : |z - \alpha_r| = |\beta_r|\}.$$

If $v^{(n)}$ is an eigenvector of $A^{(n)}$ corresponding to the eigenvalue $\lambda^{(n)}$, it will be shown that the coordinates of $v^{(n)}$ corresponding to the channel $C_r^{(n)}$ are influenced by the proximity of $\lambda^{(n)}$

to the circle S_r. For convenience we label the vertices of channel $C_r^{(n)}$ by $\{1,\ldots,nd_r\}$ so that the coordinates of $v^{(n)}$ corresponding to $C_r^{(n)}$ are $v_1^{(n)},\ldots,v_{nd_r}^{(n)}$.

Theorem 6.1. *Let $\{\lambda^{(n)}\}$ be a sequence such that $\lambda^{(n)}$ is an eigenvalue of $A^{(n)}$ and let $v^{(n)}$ denote any eigenvector corresponding to $\lambda^{(n)}$. Then the following statements hold.*

(i) *If there exists $\delta>0$ such that*

$$\operatorname{dist}(\lambda^{(n)},S_r)\geqslant\delta \tag{6.1}$$

for all n, then there exists $c\in(0,1)$ such that for each n either

$$|v_i^{(n)}|\leqslant c^{i-1}|v_1^{(n)}|\quad \text{for } i=1,\ldots,nd_r$$

or

$$|v_{nd_r-i}^{(n)}|\leqslant c^{i}|v_{nd_r}^{(n)}|\quad \text{for } i=0,\ldots,nd_r-1.$$

(ii) *If there exists $C>0$ such that*

$$\operatorname{dist}(\lambda^{(n)},S_r)\leqslant\frac{C}{n} \tag{6.2}$$

for all n, then there exists $d>0$ such that for each n large enough, either

$$d^{-1}\leqslant\left|\frac{v_i^{(n)}}{v_j^{(n)}}\right|\leqslant d\quad \text{for all } i,j\in\{1,\ldots,nd_r\}$$

or

$$v_i^{(n)}=0\quad \text{for } i=1,\ldots,nd_r.$$

Proof. The equation

$$(\lambda^{(n)}I-A^{(n)})v^{(n)}=0$$

implies that

$$(\lambda^{(n)}-\alpha_r)v_i^{(n)}=\beta_r v_{i+1}^{(n)}\quad \text{for } i=1,\ldots,nd_r-1.$$

If $v_1^{(n)}=0$, then $v_i^{(n)}=0$ for $i=1,\ldots,nd_r$. Therefore suppose that $v_1^{(n)}\neq 0$. Condition (6.1) implies that there exists a positive constant $c<1$ such that either $|(\lambda^{(n)}-\alpha_r)/\beta_r|<c$ or $|\beta_r/(\lambda^{(n)}-\alpha_r)|<c$ for each n. If $|(\lambda^{(n)}-\alpha_r)/\beta_r|<c$, then

$$|v_i^{(n)}|\leqslant c^{i-1}|v_1^{(n)}|\quad \text{for } i=1,\ldots,nd_r$$

and if $|\beta_r/(\lambda^{(n)}-\alpha_r)|<c$, then

$$|v_{nd_r-i}^{(n)}|\leqslant c^{i}|v_{nd_r}^{(n)}|\quad \text{for } i=0,\ldots,nd_r-1.$$

This proves (i). To prove (ii) we observe that condition (6.2) implies that there exists $a>0$ such that

$$1-\frac{a}{n}\leqslant\left|\frac{\lambda^{(n)}-\alpha_r}{\beta_r}\right|\leqslant 1+\frac{a}{n},$$

and so

$$\left(1-\frac{a}{n}\right)^{i-1}|v_1^{(n)}|\leqslant|v_i^{(n)}|\leqslant\left(1+\frac{a}{n}\right)^{i-1}|v_1^{(n)}|$$

for $i=1,\ldots,nd_r$. Therefore, if $v_1^{(n)}\neq 0$, then

$$\frac{(1-a/n)^{nd_r}}{(1+a/n)^{nd_r}}\leqslant\left|\frac{v_i^{(n)}}{v_j^{(n)}}\right|\leqslant\frac{(1+a/n)^{nd_r}}{(1-a/n)^{nd_r}}$$

for all $i, j \in \{1, \ldots, nd_r\}$. Since $(1 \pm a/n)^{nd_r} \to e^{\pm ad_r}$ as $n \to \infty$, the result follows. □

If the eigenvalues $\lambda^{(n)}$ are a positive distance away from all the circles S_r, then the corresponding eigenvectors $v^{(n)}$ decrease exponentially along all the channels. This suggests that in this case the eigenvectors will be concentrated around the junctions. We make this precise as follows.

Given n and N, we define the subset $C_{n,N}$ of the graph $(S^{(n)}, \to)$ by

$$C_{n,N} = \bigcup_{r=1}^{h} \{i \in C_r^{(n)} : N \leqslant i \leqslant nd_r - N\}.$$

We say that a sequence of normalized eigenvectors $v^{(n)}$ are localized around the junctions if for all $\epsilon > 0$ there exists N, depending on ϵ, such that $\|v^{(n)}|_{C_{n,N}}\|_2 < \epsilon$ for all n. Note that if

$$n \times \max\{d_r : 1 \leqslant r \leqslant h\} < 2N,$$

then $C_{n,N} = \emptyset$ and so the bound is automatic. The localization condition therefore refers to the asymptotic behaviour of $A^{(n)}$ as $n \to \infty$.

THEOREM 6.2. *Let $\{v^{(n)}\}$ be a sequence of normalized eigenvectors of $A^{(n)}$ with corresponding eigenvalues $\lambda^{(n)}$. If there exists $\delta > 0$ such that*

$$\operatorname{dist}(\lambda^{(n)}, S_r) \geqslant \delta \quad \text{for all } n \text{ and all } r = 1, \ldots, h,$$

then the $v^{(n)}$ are localized around the junctions.

Proof. By Theorem 6.1 there exists a positive constant $c < 1$ such that for $r = 1, \ldots, h$ and all n either

$$|v_i^{(n)}| \leqslant c^{i-1} \quad \text{for } i = 1, \ldots, nd_r$$

or

$$|v_{nd_r - i}^{(n)}| \leqslant c^{i-1} \quad \text{for } i = 0, \ldots, nd_r - 1.$$

Let $\epsilon > 0$ and choose N large enough so that

$$\frac{c^{2(N-1)}}{1 - c^2} < \frac{\epsilon^2}{h}.$$

Now,

$$\|v^{(n)}|_{C_{n,N}}\|_2^2 = \sum_{r=1}^{h} \sum_{i=N}^{nd_r - N} |v_i^{(n)}|^2 \leqslant \sum_{r=1}^{h} \sum_{i=N}^{nd_r - N} (c^2)^{i-1} \leqslant h \frac{c^{2(N-1)}}{1 - c^2} < \epsilon^2$$

for all n, as required. □

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