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On Rayleigh–Ritz ratios of a generalized Laplacian matrix of directed graphs

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Abstract

In [C.W. Wu, Algebraic connectivity of directed graphs, Linear and Multilinear Algebra, in press] a generalization of Fiedler's notion of algebraic connectivity to directed graphs was presented that inherits many properties of the algebraic connectivity of undirected graphs and has applications to the synchronization of coupled dynamical systems with both constant and time-varying coupling. However, it did not inherit nonnegativity and some of its bounds for combinatorial properties such as maximum directed cut and isoperimetric number are less strict that their undirected counterparts. In particular, these bounds do not have the corresponding bounds for undirected graphs as limiting cases. The purpose of this paper is to present a refinement of this algebraic connectivity which preserve nonnegativity for strongly connected graphs and have bounds which contain the undirected graphs as special cases. In particular, we study quantities related to a generalized Laplacian matrix of directed graphs and obtain bounds on combinatorial properties such as diameter, bandwidth, and bisection width for general directed graphs. Finally, we give an application to the synchronization of coupled dynamical systems with constant coupling. In particular, we show that strong enough cooperative coupling will synchronize a network of coupled systems if the underlying directed graph is strongly connected.

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1. Introduction

In [2], Fiedler defined the algebraic connectivity of an undirected graph as the second smallest eigenvalue of its Laplacian matrix. In [1] this is extended to directed graphs by studying Rayleigh–Ritz ratios of the Laplacian matrix. This extension inherits many properties of the undirected case such as super-additivity, and can be applied to graphs with both positive and negative weights. It is also useful in deriving synchronization criteria for arrays of coupled dynamical systems for both constant and time-varying coupling. On the other hand, unlike the undirected case, this algebraic connectivity can be negative, even for strongly connected graphs. Furthermore, relationships between this algebraic connectivity and properties such as bisection width and isoperimetric number do not have the undirected graphs as a limiting case.

In this paper, we refine this generalization of algebraic connectivity to address some of these issues. In particular, we modify the generalization in [1] by studying a generalized Laplacian matrix and considering the strongly connected components of the graph. We will mirror the presentation in [1] to some degree.

We consider finite weighted directed graphs (V, E) without loops, and with a vertex set V, edge set E and adjacency matrix A, where $A_{ij} > 0$ if there is a directed edge from vertex i into vertex j, and 0 otherwise. Without loss of generality, we assume that $A_{ij} \leqslant 1$. An exception to this assumption is when a graph is *unweighted*, defined as the case when A_{ij} are natural numbers, with $A_{ij} = k$ denoting k edges from vertex i to vertex j. The number of vertices and edges will be denoted as $n \geqslant 2$ and m respectively. The indegree and outdegree of vertex k are given by $d_i(k) = \sum_j A_{jk}$ and $d_o(k) = \sum_j A_{kj}$ respectively. The complement of a graph G without multiple edges is defined as the graph $\overline{G}$ with the same vertex set as G and adjacency matrix $\overline{A}$ where $\overline{A}_{ij} = 1 - A_{ij}$ for $i \neq j$.

An undirected graph is equivalent to a directed graph by considering each undirected edge with weight w as two directed edges with weight w and opposite orientation. Similarly, a mixed graph (a graph with both undirected and directed edges) can be considered as a directed graph. We define the symmetric part of a graph by replacing each directed edge with an undirected edge of half the weight. This means that the symmetric part of an undirected graph is itself. If A is the adjacency matrix of G then $\frac{1}{2}(A + A^{\mathrm{T}})$ is the adjacency matrix of the symmetric part of G. We define the *reversal* of a directed graph as the directed graph obtained by reversing the orientation of all the edges. The adjacency matrix of the reversal of G is A^{T}.

A graph is strongly connected if for every ordered pair of vertices (v, w), there exists a directed path from v to w. A graph is weakly connected if its symmetric part is strongly connected, i.e. after ignoring the orientation of the edges, the resulting undirected graph is connected.

For a real symmetric matrix A, let the eigenvalues of A be arranged as:

$$\lambda_1(A) \leqslant \lambda_2(A) \leqslant \cdots \leqslant \lambda_n(A).$$

We will also write $\lambda_1(A)$ and $\lambda_n(A)$ as $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ respectively. For a vector or matrix X, $X \geqslant 0$ and $X > 0$ means that all entries of X are nonnegative or positive respectively.

A directed graph G is a *directed tree* or *arborescence* if the symmetric part of G is a tree and there exists a vertex of G, called the *root* of G, which has directed paths to all remaining vertices of G. A vertex in a directed tree is called a leaf if its outdegree is zero.

2. A generalized Laplacian matrix

The Laplacian matrix of a directed graph is defined as $L = D - A$, where D is the diagonal matrix of vertex outdegrees.[1] Let $e = (1, \ldots, 1)^{\mathrm{T}}$. It is clear that L is a zero row sums matrix with nonpositive off-diagonal elements and thus e is a right eigenvector of L, i.e. $Le = 0$. The Laplacian matrix of the complete graph will be denoted as $L_K = nI - ee^{\mathrm{T}}$. Note that for $x \perp e$, $x^{\mathrm{T}} L_K x = nx^{\mathrm{T}}x$.

A directed graph can be decomposed into strongly connected components and the Laplacian matrix (possibly after reordering of the vertices) can be written in Frobenius normal form [4]:

$$L = \begin{pmatrix} B_1 & B_{12} & \cdots & B_{1k} \\ & B_2 & \cdots & B_{2k} \\ & & \ddots & \vdots \\ & & & B_k \end{pmatrix}, \tag{1}$$

where B_i are square irreducible matrices corresponding to the strongly connected components of G. We will call $k - 1$ the *degree of reducibility* of $L(G)$. Each B_i can be further decomposed as $B_i = L_i + D_i$ where L_i is the zero row sum Laplacian matrix corresponding to the strongly connected components of G and D_i are nonnegative diagonal matrices. We can then write L as $L = L_s + L_r$ where L_s is the block diagonal matrix with L_i as diagonal blocks. In other words L_s and L_r are the Laplacian matrices of subgraphs G_s and G_r where G_s is a disjoint union of the strongly connected components and G_r is the acyclic residual subgraph whose edges are those edges in G which go from one strongly connected component to another strongly connected component. We define $d_o^s(v)$ and $d_o^r(v)$ as the outdegree of vertex v in subgraphs G_s and G_r respectively. We define d_i^s and d_i^r similarly. Note that $d_o(v) = d_o^s(v) + d_o^r(v)$ and $d_i(v) = d_i^s(v) + d_i^r(v)$.

[1] There exists other definitions of Laplacian matrices of directed (or mixed) graphs, see for instance [3].

By Gershgorin's circle criterion, all eigenvalues of the Laplacian matrix L have nonnegative real parts. From Perron–Frobenius theory [5], there exists a nonnegative nonzero vector $w \geqslant 0$ such that $w^{\mathrm{T}}L = 0$. If the graph is strongly connected, then 0 is a simple eigenvalue of L and there exists a positive vector $w > 0$ such that $w^{\mathrm{T}}L = 0$. The special structure of Laplacian matrices allows us to further characterize when the vector w is positive and when 0 is a simple eigenvalue.

Lemma 1. *For a graph G with Laplacian matrix L, there exists a positive vector $w > 0$ such that $w^{\mathrm{T}}L = 0$ if and only if G is a disjoint union of strongly connected graphs.*

Proof. If G is a disjoint union of strongly connected graphs, each such subgraph H_i has a positive vector w_i such that $w_i^{\mathrm{T}}L(H_i) = 0$. Concatenating these vectors will form a positive vector w such that $w^{\mathrm{T}}L(G) = 0$. If G is not a disjoint union of strongly connected subgraphs, then $L(G)$ written in Frobenius normal form (Eq. (1)) satisfies:

(1) B_1 is irreducible,
(2) one of $B_{12}, \ldots, B_{1k}$ is not equal to the zero matrix.

Since L has zero row sums this means that $(B_1)_{ii} \geqslant \sum_{i \neq j} |(B_1)_{ij}|$. $B_{1l} \neq 0$ for some l means that for at least one i, $(B_1)_{ii} > \sum_{i \neq j} |(B_1)_{ij}|$. By [6], B_1 is nonsingular, and so any vector w such that $w^{\mathrm{T}}L = 0$ must be of the form $(0 \;\; w_2)^{\mathrm{T}}$. □

Corollary 1. *If the indegree of each vertex is equal to its outdegree, then the graph is weakly connected if and only if it is strongly connected.*

Proof. The hypothesis on the graph implies that its Laplacian matrix L has zero column sums, i.e. $e^{\mathrm{T}}L = 0$. By Lemma 1 this implies that the graph consists of disjoint strongly connected components. Thus if the graph is weakly connected, it is also strongly connected. □

Lemma 2. *The following two statements are equivalent*:

(1) $\mathrm{Re}(\lambda) > 0$ *for all eigenvalues λ of L not corresponding to the eigenvector e.*
(2) *The reversal of G has a spanning directed tree.*

Proof. Suppose that the reversal of G has a spanning directed tree. Let r be the root of this tree. Since all vertices of G has a directed path to r, in the Frobenius normal form (Eq. (1)), r must be in the connected component corresponding to B_k, the diagonal block in the lower right corner. Furthermore, for each $i \neq k$, one of the matrices B_{il} must be nonzero. As in the proof of Lemma 1, B_i is nonsingular for $i \neq k$. So the zero eigenvalue of L must come from B_k. Since B_k is irreducible, the zero eigenvalue of L is simple. Furthermore, Gershgorin's circle criterion implies that all nonzero eigenvalues of L have positive real parts.

There exists a spanning directed tree in the reversal of G if and only if for any pair of vertices v and w, there exists a vertex z such that there is a directed path from v to z and a directed path from w to z [7]. If the reversal of G does not have a spanning directed tree, then there exist a pair of vertices v and w such that for all vertices z, there is either no directed paths from v to z or no directed paths from w to z. Let $R(v)$ and $R(w)$ be the set of vertices reachable from v and w respectively. Let $H(v)$ and $H(w)$ be the subgraphs of G corresponding to $R(v)$ and $R(w)$ respectively. Expressing the Laplacian matrix of $H(v)$ in Frobenius normal form, let $B(v)$ be the square irreducible matrix in the lower right corner. We define $B(w)$ similarly. Note that $B(w)$ and $B(v)$ are zero row sums singular matrices. By the construction, it is easy to see that $B(v) = B_i$ and $B(w) = B_j$ in the Frobenius normal form (Eq. (1)) of the Laplacian matrix of G for some i, j. Since the eigenvalues of L are the eigenvalues of B_i's, this means that the zero eigenvalue of L is not simple. Furthermore, the kernel of L contains 2 or more linearly independent eigenvectors. □

In [8] Lemma 2 is generalized by showing that for reducible Laplacian matrices, the multiplicity of the zero eigenvalue is equal to the minimum number of directed trees needed to span the reversal of the graph.

2.1. *Strongly connected graphs*

For w such that $w^{\mathrm{T}}L = 0$, let $W = \mathrm{diag}(w)$, i.e. W is a diagonal matrix whose diagonal entries are the entries of w. The components of the vector w is denoted as $w(v)$ for $v \in V$. For a strongly connected graph, we define a *generalized* Laplacian matrix $\tilde{L}$ which depends on w as $\tilde{L}(w) = WL$ which has both zero row sums and zero column sums. We define the following quantities:

Definition 1. For a strongly directed graph G with Laplacian matrix L, let

$$\tilde{a}(G) = \min_{x \perp e, \|x\|=1} x^{\mathrm{T}} W L x = \lambda_2 \left(\frac{1}{2}(\tilde{L} + \tilde{L}^{\mathrm{T}}) \right),$$

$$\tilde{b}(G) = \frac{\max_{x \perp e, \|x\|=1} x^{\mathrm{T}} W L x}{\min_v w(v)} = \frac{\lambda_{\max} \left(\frac{1}{2}(\tilde{L} + \tilde{L}^{\mathrm{T}}) \right)}{\min_v w(v)} \geqslant \tilde{a}(G),$$

where w is the unique vector such that $w^{\mathrm{T}}L = 0$ and $\max_v w(v) = 1$.

2.2. *General directed graphs*

Consider the Laplacian matrix expressed in Frobenius normal form (Eq. (1)). Let w_i be the positive vector such that $w_i^{\mathrm{T}} L_i = 0$ and $\max_v w_i(v) = 1$ and define:

$$w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_k \end{pmatrix}.$$

As before, we define $W = \mathrm{diag}(w)$ and $\tilde{L} = WL$. We can now define $\tilde{a}(G)$ and $\tilde{b}(G)$:

Definition 2. For a directed graph with Laplacian matrix L in Frobenius normal form (Eq. (1)), $\tilde{a}(G)$ and $\tilde{b}(G)$ are defined as:

$$\tilde{a}(G) = \min_{x \perp e, \|x\|=1} x^{\mathrm{T}} W L x \leqslant \frac{\max_{x \perp e, \|x\|=1} x^{\mathrm{T}} W L x}{\min_{v,i} w_i(v)} = \tilde{b}(G).$$

It is easy to show that WL has zero row sums and zero column sums if and only if G is a disjoint union of strongly connected graphs. By Courant–Fischer min–max theorem, we have:

$$\lambda_1\left(\frac{1}{2}(\tilde{L} + \tilde{L}^{\mathrm{T}})\right) \leqslant \tilde{a}(G) \leqslant \lambda_2\left(\frac{1}{2}(\tilde{L} + \tilde{L}^{\mathrm{T}})\right),$$

$$\frac{\lambda_{n-1}\left(\frac{1}{2}(\tilde{L} + \tilde{L}^{\mathrm{T}})\right)}{\min_{v,i} w_i(v)} \leqslant \tilde{b}(G) \leqslant \frac{\lambda_{\max}\left(\frac{1}{2}(\tilde{L} + \tilde{L}^{\mathrm{T}})\right)}{\min_{v,i} w_i(v)}.$$

As in [1], $\tilde{a}$ and $\tilde{b}$ can be written as

$$\tilde{a}(G) = \lambda_{\min}\left(\frac{1}{2} Q^{\mathrm{T}}(WL + L^{\mathrm{T}}W)Q\right),$$

$$\tilde{b}(G) = \frac{\lambda_{\max}\left(\frac{1}{2} Q^{\mathrm{T}}(WL + L^{\mathrm{T}}W)Q\right)}{\min_{i,v} w_i(v)},$$

where Q is an n by $n - 1$ matrix whose columns form an orthonormal basis of $e^{\perp}$.

The quantity $\tilde{a}(G)$ can be considered as another extension of the algebraic connectivity to directed graphs and $\tilde{a}(G)$ and $\tilde{b}(G)$ coincide with the definitions of $a(G)$ and $b(G)$ in [1] for graphs where the indegree of each vertex in a strongly connected component subgraph is equal to its outdegree, i.e. each L_i in the Frobenius normal form has zero column sums. This is due to the fact that in this case $w_i = e$ for each strongly connected component and thus $W = I$.

3. Properties of $\tilde{a}(G)$ and $\tilde{b}(G)$

Lemma 3. *If G is strongly connected, then $\tilde{a}(G) \leqslant \mathrm{Re}(\lambda)$ for all eigenvalues λ of $L(G)$ not belonging to the eigenvector e.*

Proof. The proof will be given in Section 5. $\square$

Lemma 4. *If G is strongly connected, then*

$$0 \leqslant \tilde{a}(G) \leqslant \tilde{b}(G) \leqslant \frac{2\max_v\{w(v)d_o(v)\}}{\min_v w(v)},$$

where w is a vector such that $w^{\mathrm{T}}L = 0$.

Proof. $\frac{1}{2}(\tilde{L} + \tilde{L}^{\mathrm{T}})$ is a symmetric matrix with zero row sums, nonnegative diagonal elements and nonpositive off-diagonal elements. Its diagonal elements are $\frac{w(v)d_o(v)}{\min_v w(v)}$. By Gershgorin's circle criterion, the eigenvalues of $\tilde{L}$ satisfy

$$0 \leqslant \lambda \leqslant \frac{2\max_v\{w(v)d_o(v)\}}{\min_v w(v)}. \qquad \square$$

It is easy to see that if G is a disjoint union of 2 or more strongly connected graphs $H_1, \ldots, H_k$, then $\tilde{a}(G) = 0$ and $\tilde{b}(G) = \max_i \tilde{b}(H_i)$.

Lemma 5. *If G is strongly connected, then*

$$\tilde{a}(G) = n \min_{x \notin \mathrm{span}(e)} \frac{x^{\mathrm{T}}WLx}{x^{\mathrm{T}}L_K x},$$
$$\tilde{b}(G) = n \max_{x \notin \mathrm{span}(e)} \frac{x^{\mathrm{T}}WLx}{x^{\mathrm{T}}L_K x}.$$

Proof. Decompose $x \notin \mathrm{span}(e)$ as $x = \alpha e + y$, where $y \perp e$. Since $e^{\mathrm{T}}WL = WLe = e^{\mathrm{T}}L_K = L_K e = 0$, the proof is then complete by noting that

$$\frac{x^{\mathrm{T}}WLx}{x^{\mathrm{T}}L_K x} = \frac{y^{\mathrm{T}}WLy}{y^{\mathrm{T}}L_K y} = \frac{y^{\mathrm{T}}WLy}{ny^{\mathrm{T}}y}. \qquad \square$$

Definition 3. Let S_1 and S_2 be subsets of vertices. Define

$$e(S_1, S_2) = \sum_{v_1 \in S_1, v_2 \in S_2} A_{v_1,v_2}$$

which is the sum of the weights of edges which start in S_1 and end in S_2. In general, $e(S_1, S_2) \neq e(S_2, S_1)$. For the decomposition $L = L_s + L_r$ in the Frobenius normal form, we also define

$$e_s(S_1, S_2) = \sum_{v_1 \in S_1, v_2 \in S_2} A^s_{v_1,v_2},$$
$$e_r(S_1, S_2) = \sum_{v_1 \in S_1, v_2 \in S_2} A^r_{v_1,v_2},$$

where A^s and A^r are $-L_s$ and $-L_r$ respectively with the diagonal elements set to zero.

Note that $e(S_1, S_2) = e_s(S_1, S_2) + e_r(S_1, S_2)$ and since $A_{ij} \leqslant 1$, $e(S_1, S_2) \leqslant |S_1||S_2|$. The following two lemmas are simple to prove but are quite useful to relate combinatorial properties of G with $\tilde{a}(G)$ and $\tilde{b}(G)$.

Lemma 6. *Let S_1 and S_2 be two nontrivial disjoint subsets of vertices of a graph G (i.e. $0 < |S_1|$, $0 < |S_2|$ and $S_1 \cap S_2 = \emptyset$) and $\overline{S}_i = V \backslash S_i$. Then*

$$\tilde{a}(G) \leqslant \frac{|S_2|^2 e(S_1, \overline{S}_1) + |S_1||S_2|(e(S_1, S_2) + e(S_2, S_1)) + |S_1|^2 e(S_2, \overline{S}_2)}{|S_1||S_2|^2 + |S_1|^2|S_2|}$$
$$\leqslant \tilde{b}(G).$$

Furthermore,

$$\tilde{a}(G) \leqslant \frac{e(S_1, \overline{S}_1)}{|S_1|} + \frac{e(S_2, \overline{S}_2)}{|S_2|},$$
$$\tilde{b}(G) \geqslant \frac{e(S_1, S_2)}{|S_1|} + \frac{e(S_2, S_1)}{|S_2|}.$$

Proof. Let x be a vector such that $x_v = |S_2|$ if $v \in S_1$, $x_v = -|S_1|$ if $v \in S_2$ and $x_v = 0$ otherwise. Then $x \perp e$ and $x^{\mathrm{T}}x = |S_1||S_2|^2 + |S_1|^2|S_2|$.

$$x^{\mathrm{T}}WDx = |S_2|^2 \sum_{v \in S_1} w(v)d_o(v) + |S_1|^2 \sum_{v \in S_2} w(v)d_o(v),$$
$$x^{\mathrm{T}}WAx = |S_2|^2 \sum_{v \in S_1} w(v)e(v, S_1) - |S_1||S_2| \sum_{v \in S_1} w(v)e(v, S_2)$$
$$- |S_1||S_2| \sum_{v \in S_2} w(v)e(v, S_1) + |S_1|^2 \sum_{v \in S_2} w(v)e(v, S_2).$$

Since $e(v, S) + e(v, \overline{S}) = e(v, V) = d_o(v)$, this implies that

$$x^{\mathrm{T}}WLx = x^{\mathrm{T}}WDx - x^{\mathrm{T}}WAx$$
$$= |S_2|^2 \sum_{v \in S_1} w(v)e(v, \overline{S}_1) + |S_1|^2 \sum_{v \in S_2} w(v)e(v, \overline{S}_2)$$
$$+ |S_1||S_2| \left(\sum_{v \in S_1} w(v)e(v, S_2) + \sum_{v \in S_2} w(v)e(v, S_1) \right). \quad (2)$$

Since $\tilde{a}(G) \leqslant \frac{x^{\mathrm{T}}WLx}{x^{\mathrm{T}}x} \leqslant \tilde{b}(G)$, the first set of inequalities follows, noting that $\min_v w(v) \leqslant w(v) \leqslant 1$ in the definitions of $\tilde{a}(G)$ and $\tilde{b}(G)$. The last 2 inequalities follow from the fact that $e(S_1, \overline{S}_1) \geqslant e(S_1, S_2)$ and $e(S_2, \overline{S}_2) \geqslant e(S_2, S_1)$. $\square$

Lemma 7. *Let S be a nontrivial subset of vertices of a graph G (i.e. $0 < |S| < n$) and $\overline{S} = V\backslash S$. Then*

$$\tilde{a}(G) \leqslant \frac{e(S,\overline{S})}{|S|} + \frac{e(\overline{S},S)}{n-|S|} \leqslant \tilde{b}(G),$$

$$\tilde{a}(G) \leqslant \frac{n}{|S|(n-|S|)}\min(e_s(S,\overline{S}), e_s(\overline{S},S)) + \frac{e_r(S,\overline{S})}{|S|} + \frac{e_r(\overline{S},S)}{n-|S|},$$

$$\tilde{b}(G) \geqslant \frac{n}{|S|(n-|S|)}\max(e_s(S,\overline{S}), e_s(\overline{S},S)) + \frac{e_r(S,\overline{S})}{|S|} + \frac{e_r(\overline{S},S)}{n-|S|}.$$

Proof. By choosing $S_1 = S$ and $S_2 = \overline{S}$ in Lemma 6, the first set of inequalities follows. Furthermore, Eq. (2) becomes:

$$\begin{aligned} x^{\mathrm{T}}WLx &= (n-|S|)^2\sum_{v\in S} w(v)e(v,\overline{S}) + |S|^2\sum_{v\in\overline{S}} w(v)e(v,S) \\ &\quad + |S|(n-|S|)\left(\sum_{v\in S} w(v)e(v,\overline{S}) + \sum_{v\in\overline{S}} w(v)e(v,S)\right) \\ &= n(n-|S|)\sum_{v\in S} w(v)e(v,\overline{S}) + n|S|\sum_{v\in\overline{S}} w(v)e(v,S). \end{aligned} \tag{3}$$

Note that

$$\sum_{v\in S} w(v)e_s(v,\overline{S}) + \sum_{v\in S} w(v)e_s(v,S) = \sum_{v\in S} w(v)d_o^s(v). \tag{4}$$

Since WL_s has zero row sums and zero column sums, this implies that $w(v)d_o^s(v) = \sum_{u\in V} w(u)e_s(u,v)$. Therefore

$$\begin{aligned} \sum_{v\in S} w(v)d_o^s(v) &= \sum_{u\in V} w(u)\sum_{v\in S} e_s(u,v) = \sum_{u\in V} w(u)e_s(u,S) \\ &= \sum_{u\in S} w(u)e_s(u,S) + \sum_{u\in\overline{S}} w(u)e_s(u,S). \end{aligned} \tag{5}$$

Combining Eqs. (4) and (5) we get:

$$\sum_{v\in S} w(v)e_s(v,\overline{S}) = \sum_{v\in\overline{S}} w(v)e_s(v,S).$$

Combining this with Eq. (3) and the fact that $e(v,S) = e_s(v,S) + e_r(v,S)$ we get the desired result. □

Corollary 2. *Let S be a nontrivial subset of vertices of G. Then*

$$\frac{|S|(n-|S|)}{n}\tilde{a}(G) \leqslant \max(e(S,\overline{S}), e(\overline{S},S)), \tag{6}$$

$$\frac{|S|(n-|S|)}{n}\tilde{b}(G) \geqslant \min(e(S,\overline{S}), e(\overline{S},S)). \tag{7}$$

Proof. Since

$$\frac{e_r(S,\overline{S})}{|S|}+\frac{e_r(\overline{S},S)}{n-|S|}=\frac{n}{|S|(n-|S|)}e_r(S,\overline{S})+\frac{e_r(\overline{S},S)-e_r(S,\overline{S})}{n-|S|},$$

we get by Lemma 7

$$\tilde{a}(G)\leqslant\frac{n}{|S|(n-|S|)}e(S,\overline{S})+\frac{e_r(\overline{S},S)-e_r(S,\overline{S})}{n-|S|}$$

Similarly,

$$\tilde{a}(G)\leqslant\frac{n}{|S|(n-|S|)}e(\overline{S},S)+\frac{e_r(S,\overline{S})-e_r(\overline{S},S)}{|S|}$$

Since one of the terms $\frac{e_r(\overline{S},S)-e_r(S,\overline{S})}{n-|S|}$ and $\frac{e_r(S,\overline{S})-e_r(\overline{S},S)}{|S|}$ is nonpositive, we get Eq. (6). Eq. (7) follows from a similar reasoning. □

Corollary 3. *Let S be a nontrivial subset of vertices of a graph G. If G is a disjoint union of strongly connected graphs, then*

$$\frac{|S|(n-|S|)}{n}\tilde{a}(G)\leqslant\min(e(S,\overline{S}),e(\overline{S},S)),$$
$$\frac{|S|(n-|S|)}{n}\tilde{b}(G)\geqslant\max(e(S,\overline{S}),e(\overline{S},S)).$$

Proof. Follows from Lemma 7 and the fact that $L_r=0$ in this case. □

Lemma 8. *Let v, u be nonadjacent vertices of a graph G, i.e. $A_{vu}=A_{uv}=0$. Then*

$$\tilde{a}(G)\leqslant\frac{1}{2}(d_o(v)+d_o(u))\leqslant\tilde{b}(G).$$

In particular, if G has two vertices with zero outdegrees, then $\tilde{a}(G)\leqslant 0$.

Proof. Follows from Lemma 6 and choosing $S_1=\{v\}$, $S_2=\{u\}$. □

Let $\Delta_o=\max_{v\in V}d_o(v)$, $\delta_o=\min_{v\in V}d_o(v)$, $\Delta_i=\max_{v\in V}d_i(v)$, and $\delta_i=\min_{v\in V}d_i(v)$.

Lemma 9

$$\tilde{a}(G)\leqslant\frac{n}{n-1}\min_v\big(\max(d_o(v),d_i(v))\big),$$
$$\tilde{b}(G)\geqslant\frac{n}{n-1}\max_v\big(\min(d_o(v),d_i(v))\big).$$

If G is a disjoint union of strongly connected graphs,

$$\tilde{a}(G) \leqslant \frac{n}{n-1}\min(\delta_o, \delta_i) \leqslant \frac{n}{n-1}\max(\Delta_o, \Delta_i) \leqslant \tilde{b}(G).$$

Proof. Follows from Corollaries 2 and 3 and choosing $S = \{v\}$. □

Lemma 10. *If G is strongly connected, then $\tilde{a}(G) > 0$. If G does not contain a spanning directed tree, then $\tilde{a}(G) \leqslant 0$.*

Proof. If G is strongly connected, then $\frac{1}{2}(WL + L^{\mathrm{T}}W)$ is irreducible. Therefore by Perron–Frobenius theory, $\tilde{a}(G) = \lambda_2\big(\frac{1}{2}(WL + L^{\mathrm{T}}W)\big) > 0$.

As in the proof of Lemma 2, if the reversal of G does not contain a spanning directed tree, then there exist a pair of vertices v and w such that for all vertices z, either there are no directed paths from v to z or there are no directed paths from w to z. Let $R(v)$ and $R(w)$ denote the nonempty set of vertices reachable from v and w respectively. The hypothesis implies that $R(v)$ and $R(w)$ are disjoint. Furthermore, by definition $e(R(v), \overline{R(v)}) = e(R(w), \overline{R(w)}) = 0$ and the result follows from Lemma 6 by setting $S_1 = R(v)$ and $S_2 = R(w)$. □

Definition 4. The (weak) edge connectivity $e(G)$ of a graph is defined as the smallest weighted sum among all subsets of edges such that its removal results in a graph which is not weakly connected.

Theorem 1. *$\tilde{a}(G) \leqslant e(G)$. If G is strongly connected, $\frac{2(n-1)}{n}\tilde{a}(G) \leqslant e(G)$.*

Proof. Let S be a weakly connected component of the weakly disconnected graph resulting from removal of a minimal set of edges. Then $e(S, \overline{S}) + e(\overline{S}, S) = e(G)$. By Lemma 7,

$$\tilde{a}(G) \leqslant \frac{e(S, \overline{S})}{|S|} + \frac{e(\overline{S}, S)}{n - |S|} \leqslant e(S, \overline{S}) + e(\overline{S}, S).$$

If G is strongly connected, by Corollary 3

$$2\tilde{a}(G) \leqslant (e(S, \overline{S}) + e(\overline{S}, S))\frac{n}{|S|(n - |S|)} \leqslant (e(S, \overline{S}) + e(\overline{S}, S))\frac{n}{n-1}.$$

□

Theorem 2. *If G is strongly connected, then*

$$\tilde{a}(G) \geqslant \frac{1 - \cos\left(\frac{\pi}{n}\right)}{r}e(G),$$

$$\tilde{a}(G) \geqslant \frac{C_1 e(G)}{2r} - C_2 q,$$

where $r = \frac{\max_v w(v)}{\min_v w(v)}$, $q = \max_v w(v)d_o(v)$, $C_1 = 2\big(\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{2\pi}{n}\right)\big)$ *and* $C_2 = 2\cos\left(\frac{\pi}{n}\right)\big(1 - \cos\left(\frac{\pi}{n}\right)\big)$.

Proof. The proof is similar to [2]. Let $K = I - \frac{1}{2q}(WL + L^{\mathrm{T}}W)$ where $q = \max_v\{w(v)d_o(v)\}$. K is doubly stochastic and by [9]

$$\frac{\tilde{a}(G)}{q} = 1 - \lambda_2(K) \geqslant 2\left(1 - \cos\left(\frac{\pi}{n}\right)\right)\mu,$$
$$1 - \lambda_2(K) \geqslant C_1\mu - C_2,$$

where $\mu = \min_{0<|S|<n}\left\{\sum_{v\in S, w\in\overline{S}} K_{vw}\right\}$. Note that

$$\mu q = \frac{1}{2}\min_{0<|S|<n}\left\{\sum_{v\in S} w(v)e(v,\overline{S}) + \sum_{v\in\overline{S}} w(v)e(v,S)\right\} \geqslant \frac{1}{2r}e(G),$$

and the result follows. □

We can also define the strong edge connectivity as follows:

Definition 5. The strong edge connectivity $e_s(G)$ of a graph is defined as the smallest weighted sum among all subsets of edges such that its removal results in a graph which is not strongly connected.

The following results follow similar proofs as the weakly connected case.

Theorem 3. *If G is strongly connected*, $\frac{n-1}{n}\tilde{a}(G) \leqslant e_s(G)$.

Theorem 4. *If G is strongly connected, then*

$$\tilde{a}(G) \geqslant \frac{2\left(1 - \cos\left(\frac{\pi}{n}\right)\right)}{r}e_s(G),$$
$$\tilde{a}(G) \geqslant \frac{C_1 e_s(G)}{r} - C_2 q,$$

where r, q, C_1 *and* C_2 *are as defined in Theorem 2.*

Definition 6. A directed graph is bipartite if its vertices can be partitioned into two sets V and W such that each edge starts from a vertex in V and ends in a vertex in W. If $|V| = p$ and $|W| = q$, then we use $G_{p,q}$ to denote such a graph.

Corollary 4. *If* $q \geqslant 2$ *for a bipartite directed graph* $G_{p,q}$, *then* $\tilde{a}(G) \leqslant 0$.

Proof. Follows from Lemma 10. □

Theorem 5. *If G is a directed tree with at least two leaves, then* $a(G) \leqslant 0$. *If the reversal of G is a directed tree, then* $\tilde{a}(G) \leqslant \frac{d_i(r)}{n-1}$ *where r is the root of the tree.*

Proof. The proof is the same as in [1]. □

In [10], the eigenvector of L corresponding to $\lambda_2(L)$ is used to construct connected subgraphs of undirected graphs. The following extends this to directed graphs.

Theorem 6. *Let G be a weakly connected graph. Let $y > 0$ be an eigenvector of the Laplacian matrix of the symmetric part of G corresponding to λ_2. For any $\theta \geqslant 0$, the subgraph corresponding to the vertex set*

$$V(\theta) = \{i \in V : y_i \geqslant -\theta\}$$

is weakly connected. Similarly, the subgraph corresponding to

$$V(\theta) = \{i \in V : y_i \leqslant \theta\}$$

is weakly connected.

Proof. Apply Theorem (3,3) in [10] to the symmetric part of G and noting that a graph is weakly connected if and only if its symmetric part is strongly connected. □

3.1. Stationary distribution of Markov chains

The quantity r in Theorems 2 and 4 is related to how different the vector w is from e, which in turn is related to how far L is from having zero column sums. We can get an upper bound on r by using perturbation bounds on the stationary distribution of Markov chains.

First note that for a Laplacian matrix L, $P = I - \alpha L$ is a stochastic matrix for some $\alpha > 0$. For example, we can choose $\alpha = \frac{1}{\max_i L_{ii}}$. Next note that if $\pi^{\mathrm{T}} L = 0$ then $\pi^{\mathrm{T}} P = \pi$. Thus if $\sum_i \pi_i = 1$, then π is the stationary distribution of the Markov chain with transition matrix P. Furthermore, for strongly connected graphs, π is unique and thus $r = \frac{\max_i \pi_i}{\min_i \pi_i}$.

Let $\tilde{P}$ be an irreducible doubly stochastic matrix close to P. In [11] several bounds of the form $\|\pi - e\|_p \leqslant \kappa \|P - \tilde{P}\|_q$ were given for some suitable norms $\|\cdot\|_p$, $\|\cdot\|_q$. These bounds can in turn be used to bound r. For instance, $\|\pi - e\|_\infty \leqslant \delta < 1$ implies $r \leqslant \frac{1+\delta}{1-\delta}$.

4. Combinatorial properties

4.1. Maximum directed cut

Definition 7. The maximum directed cut $md(G)$ is defined as:

$$md(G) = \max_{0<|S|<n} \{e(S, \overline{S})\}.$$

Theorem 7. *If G is a disjoint union of strongly connected graphs*,

$$md(G) \leqslant \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \frac{\tilde{b}(G)}{n}.$$

Proof. Follows from Corollary 3 and the fact that $\frac{|S|(n-|S|)}{n} \leqslant \frac{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil}{n}$. □

4.2. Edge-forwarding index

Consider the definition of edge-forwarding index in [12] applied to directed graphs:

Definition 8. Given a strongly connected unweighted directed graph, a routing is defined as a set of $n(n-1)$ paths $R(u, v)$ between any pair of distinct vertices v, w of G. The load of an edge e, $\pi(G, R, e)$, is defined as the number of paths in the routing R which traverse it. The edge-forwarding index of (G, R) is defined as $\pi(G, R) = \max_{e\in E}(G, R, e)$. The edge-forwarding index of the graph G is defined as $\pi(G) = \min_R \pi(G, R)$.

Theorem 8. *Let G be a strongly connected unweighted directed graph. For* $S \subset V$,

$$\pi(G) \geqslant \max\left(\frac{|S|(n-|S|)}{e(S, \overline{S})}, \frac{|S|(n-|S|)}{e(\overline{S}, S)}\right) \geqslant \frac{n}{\tilde{b}(G)}.$$

Proof. The proof is similar to [13]. Let R be a routing. Each path in R from vertex v in S to vertex w in $\overline{S}$ contains at least one edge in the edge cut of S. Since there are $|S|(n-|S|)$ such paths, $\pi(G) \geqslant \frac{|S|(n-|S|)}{e(S,\overline{S})}$. Similarly, $\pi(G) \geqslant \frac{|S|(n-|S|)}{e(\overline{S},S)}$. The result then follows from Corollary 3. □

4.3. Bisection width

Definition 9. The bisection width is defined as:

$$bw(G) = \min_{|S|=\lfloor \frac{n}{2} \rfloor} \{e(S, \overline{S})\}.$$

A related quantity is

$$\overline{bw}(G) = \max_{|S|=\lfloor \frac{n}{2} \rfloor} \{e(S, \overline{S})\}.$$

It is easy to see that

$$bw(G) + \overline{bw}(\overline{G}) = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil. \tag{8}$$

Theorem 9. *If G is a strongly connected graph*,

$$bw(G) \geqslant \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \frac{\tilde{a}(G)}{n},$$
$$\overline{bw}(G) \leqslant \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \frac{\tilde{b}(G)}{n}.$$

Proof. Follows from Corollary 3 by setting $|S| = \left\lfloor \frac{n}{2} \right\rfloor$. □

4.4. Isoperimetric number and minimum ratio cut

Definition 10. The isoperimetric number $i(G)$ and the minimum ratio cut $rc(G)$ [14] are defined as:

$$i(G) = \min_{0<|S|\leqslant \frac{n}{2}} \left\{ \frac{e(S, \overline{S})}{|S|} \right\},$$
$$rc(G) = \min_{S} \left\{ \frac{e(S, \overline{S})}{|S||\overline{S}|} \right\}.$$

Theorem 10. *If G is strongly connected*, *the isoperimetric number and the minimum ratio cut of a graph satisfy*:

$$i(G) \geqslant \frac{a(G)}{2}, \quad rc(G) \geqslant \frac{\tilde{a}(G)}{n}.$$

Proof. Follows from Corollary 3. □

4.5. Independence number

Definition 11. An independent set of vertices is a set of vertices such that no 2 distinct vertices in the set are adjacent. The independence number of a graph $\alpha(G)$ is the size of the largest independent set of vertices.

The following results extend a theorem in [15] to strongly connected graphs.

Theorem 11. *For a strongly connected graph*, *let the outdegrees and indegrees be ordered as* $d_o(1) \leqslant d_o(2) \leqslant \cdots \leqslant d_o(n)$ *and* $d_i(1) \leqslant d_i(2) \leqslant \cdots \leqslant d_i(n)$ *respectively and define* $\gamma(r) = \frac{1}{r} \max\left(\sum_{j=1}^{r} d_o(j), \sum_{j=1}^{r} d_i(j)\right)$. *If* r_0 *is the smallest integer r such that*

$$\frac{n\gamma(r)}{n-r} > \tilde{b}(G)$$

then $\alpha(G) \leqslant r_0 - 1$.

Proof. Let S be an independent set such that $|S| = r$. Since S is independent, $e(S, \overline{S}) = \sum_{v\in S} d_o(v)$ and $e(\overline{S}, S) = \sum_{v\in S} d_i(v)$ and thus $\max(e(S, \overline{S}), e(\overline{S}, S)) \geqslant r\gamma(r)$. Then by Corollary 3

$$b(G) \geqslant \frac{n\gamma(r)}{n-r}. \qquad \square$$

4.6. Diameter

In [16], an upper bound on the diameter of an undirected graph was proved. A weaker bound was also given for directed graphs where the indegree of each vertex is equal to its outdegree. We extend this latter bound to general directed graphs by using $\tilde{a}(G)$ and $\tilde{b}(G)$.

First note that for a strongly connected graph, there exists a positive vector w in S_a. Let $W = \mathrm{diag}(w)$. Similar to the argument in [16], the diameter d of a strongly connected graph G is the smallest positive integer such that for all $i \neq j$, there exists a polynomial p_m of degree $m \leqslant d$ such that $(p_m(WL)) \neq 0$. The results in [16] can be used to prove the following two results:

Theorem 12. *If G is a strongly connected graph with diameter d and ρ and λ are such that $1 \geqslant \rho > \left\| I - \lambda WL - \frac{1}{n}ee^{\mathrm{T}} \right\|$, then*

$$d \leqslant \left\lceil \frac{\ln(n-1)}{\ln \frac{1}{\rho}} \right\rceil.$$

Proof. The proof is the same as Theorem 6.1 in [16], noting that WL has zero row sums and zero column sums. $\square$

Theorem 13. *If G is a strongly connected graph with diameter d and w is a positive vector such that $w^{\mathrm{T}}L = 0$ and $\max_v w(v) = 1$, then*

$$d \leqslant \left\lfloor \frac{\ln(n-1)}{\ln \frac{\|WL\|}{\sqrt{\|WL\|^2 - \tilde{a}^2}}} \right\rfloor + 1,$$

where $W = \mathrm{diag}(w)$.

Proof. The proof is similar to the proof of Theorem 6.3 in [16]. $\square$

We define a directed graph to be locally connected [17] if the vertices are arranged on a grid and each vertex is adjacent only to vertices in a local neighborhood of fixed radius. The following result extends a result in [17]:

Corollary 5. *If $G(n)$ is a locally connected and strongly connected graph of n vertices, then*

$$\lim_{n\to\infty} \tilde{a}(G(n)) = 0.$$

Proof. If the weights decreases with n such that $\delta_o \to 0$, then clearly $\frac{n}{n-1}\delta_o \geqslant \tilde{a} \to 0$. Otherwise, $\|WL\|$ is bounded away from 0 and the diameter grows faster than $\ln(n)$, and the conclusion follows from Theorem 13. □

In fact any sequence of graphs where the diameter grows as $\omega(\ln(n))$ will have $\tilde{a} \to 0$.

4.7. Linear labelings and bandwidth

Definition 12. For a labeling $\psi : V \to \{1, \ldots, n\}$ the p-discrepancy is defined as

$$\sigma_p(G, \psi) = \left(\sum_{v,w} A_{vw}|\psi(v) - \psi(w)|^p\right)^{\frac{1}{p}}$$

if $p < \infty$ and

$$\sigma_\infty(G, \psi) = \max_{v,w} |\psi(v) - \psi(w)|$$

if $p = \infty$. The min-p-sum of G is defined as:

$$\sigma_p(G) = \min_{\psi} \sigma_p(G, \psi).$$

The cutwidth is defined as

$$c(G, \psi) = \max_{1\leqslant i<n} \left|\left\{(u, v) : A_{uv} \neq 0, \psi(u) \leqslant i < \psi(v)\right\}\right|.$$

$\sigma_\infty(G)$ is also referred as the bandwidth of G. The following results extends the corresponding results in [18,19] to directed graphs, and the proofs are omitted since they are essentially the same as in [18,19].

Theorem 14. *If G is strongly connected*,

$$\tilde{a}(G) \leqslant \frac{3\sigma_1(G)}{n^2 - 1} \leqslant \tilde{b}(G).$$

Proof. The proof is essentially the same as [19], except that for a directed graph, the edge (i, j) is different from (j, i), hence the difference of a factor of two. □

Theorem 15. *If G is strongly connected*,

$$\tilde{a}(G) \leqslant \frac{n}{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil} c(G) \leqslant \tilde{b}(G).$$

Theorem 16. *For a strongly connected graph, let* $\alpha = \left\lceil \frac{\tilde{a}n}{2\tilde{b}-\tilde{a}} \right\rceil$. *If* $\alpha \equiv n \bmod 2$, *then* $\sigma_\infty(G) \geqslant \alpha - 1$, *otherwise* $\sigma_\infty(G) \geqslant \alpha$.

Theorem 17. *For a strongly connected graph, define*

$$\alpha = \left\lceil \frac{\tilde{b}n - \sqrt{(\tilde{b}-\tilde{a})^2 n^2 + 2\tilde{a}\tilde{b} - \tilde{a}^2}}{2\tilde{b} - \tilde{a}} \right\rceil$$

If $\alpha \equiv n \bmod 2$, *then* $\sigma_\infty(G) \geqslant \alpha$, *otherwise* $\sigma_\infty(G) \geqslant \alpha - 1$.

Theorem 18. *For a strongly connected graph, define*

$$\alpha = \left\lceil \frac{\tilde{a}n}{\Delta + \sqrt{\Delta^2 + \tilde{a}^2}} \right\rceil$$

where $\Delta = \min\{\Delta_o, \Delta_i\}$. *If* $\alpha \equiv n \bmod 2$, *then* $\sigma_\infty(G) \geqslant \alpha - 1$, *otherwise* $\sigma_\infty(G) \geqslant \alpha$.

5. Synchronization in networks of coupled dynamical systems

We present in this section an application of $\tilde{a}(G)$ to derive synchronization criteria in a network of coupled dynamical systems.

Definition 13. A function $f(y,t)$ is V-uniformly decreasing if $(y-z)^{\mathrm{T}} V(f(y,t) - f(z,t)) \leqslant -c\|y-z\|^2$ for some $c > 0$ and all y, z, t.

Consider the following synchronization result [20–23] for the coupled network of identical dynamical systems with state equations

$$\dot{x} = (f(x_1,t), \ldots, f(x_n,t))^{\mathrm{T}} + (C \otimes D)x, \tag{9}$$

where $x = (x_1, \ldots, x_n)^{\mathrm{T}}$ and C is a zero row sums matrix with nonpositive off-diagonal elements.

Theorem 19. *Let* P *be a matrix and* V *be a symmetric positive definite matrix such that* $f(x,t) + Px$ *is* V*-uniformly decreasing. Then the array in Eq.* (9) *synchronizes in the sense that* $\|x_i - x_j\| \to 0$ *as* $t \to \infty$ *if there exists a symmetric irreducible zero row sums matrix* U *with nonpositive off-diagonal elements such that*

$$(U \otimes V)(C \otimes D - I \otimes P)$$

is negative semidefinite.[2]

[2] A (not necessarily symmetric) real matrix B is positive (negative) semidefinite if $x^{\mathrm{T}}Bx \geqslant 0$ ($\leqslant 0$) for all real vectors x.

Definition 14. Let $\mu(C)$ be the supremum of the set of real numbers μ such that $U(C - \mu I)$ is positive semidefinite for some symmetric zero row sums matrix U with nonpositive off-diagonal elements.

Using Theorem 19 it is easy to show the following [24]:

Theorem 20. *The coupled network*

$$\dot{x} = (f(x_1, t), \ldots, f(x_n, t))^{\mathrm{T}} + (C \otimes D)x \tag{10}$$

synchronizes if $f(y, t) + \alpha Dy$ is V-uniformly decreasing for some symmetric positive definite V, VD is symmetric negative semidefinite and $\mu(C) \geqslant \alpha$.

The matrix C describes the coupling topology between systems whereas the matrix D describes the coupling term between two systems. The term αDy is the amount of feedback needed to stabilize $\dot{y} = f(y, t)$. The array can be considered as coupled via a graph where for $i \neq j$, $C_{ij} \neq 0$ means that there is a term $C_{ij}Dx_j$ in $\dot{x}_i$, i.e. system i is influenced by system j. If we assign a directed edge of weight $-C_{ij}$ from system i to system j, then C is exactly the Laplacian matrix of the underlying graph.[3]

Theorem 20 shows that $\mu(C)$ is a lower bound on the amount of coupling needed to synchronize the array. Next we show that $\tilde{a}$ of the underlying graph provides a lower bound on $\mu(C)$.

Theorem 21. *If $C = L(G)$ and G is strongly connected, then $0 < \tilde{a}(G) \leqslant \mu(C)$.*

Proof. Let w be a positive vector such that $w^{\mathrm{T}}L = 0$ and $\max_v w_v = 1$. Define $U = W - \frac{ww^{\mathrm{T}}}{\sum_v w_v}$ where $W = \mathrm{diag}(w)$. Then U is a symmetric irreducible zero row sums matrix with nonpositive off-diagonal elements. It suffices to show that $B = U(L - \tilde{a}(G)I)$ is positive semidefinite. First note that since $e^{\mathrm{T}}U = Ue = 0$, it follows that $e^{\mathrm{T}}B = Be = 0$. Thus it suffices to show that $\min_{y \perp e} y^{\mathrm{T}}By \geqslant 0$. Since $w^{\mathrm{T}}L = 0$, B can be written as $B = WL - \tilde{a}W + \tilde{a}\frac{ww^{\mathrm{T}}}{\sum_v w_v}$. For $y \perp e$,

$$y^{\mathrm{T}}By = y^{\mathrm{T}}WLy - \tilde{a}y^{\mathrm{T}}Wy + \tilde{a}\frac{(y^{\mathrm{T}}w)^2}{\sum_v w_v} \geqslant \tilde{a}\|y\|^2 - \tilde{a}\|y\|^2 \geqslant 0. \qquad \square$$

[3] From a dynamical systems point of view, it is probably more appropriate to define the edge to go from system j into system i, but the above definition is consistent with the definition of the adjacency matrix.

In fact, the proof of Theorem 21 shows the following stronger result:

$$\tilde{a}(G) \leqslant \min_{x\perp e, x\neq 0} \frac{x^{\mathrm{T}} W L x}{x^{\mathrm{T}}\left(W - \frac{ww^{\mathrm{T}}}{\sum_v w_v}\right)x} \leqslant \mu(L(G)).$$

We are now in a position to prove Lemma 3.

Proof of Lemma 3. Follows from Theorem 21 and the fact that $\mu(C) \leqslant \mathrm{Re}(\lambda)$ for all eigenvalues λ of C not belonging to the eigenvector e [25]. □

In [20] it was shown that if the underlying graph is undirected and connected, sufficiently strong cooperative coupling will synchronize the network. The following result extends this to strongly connected directed graphs.

Corollary 6. *The coupled network*

$$\dot{x} = (f(x_1, t), \ldots, f(x_n, t))^{\mathrm{T}} + \kappa(C \otimes D)x \tag{11}$$

synchronizes if $f(y, t) + Dy$ is V-uniformly decreasing for some symmetric positive definite V, VD is symmetric negative semidefinite, the underlying graph is strongly connected and the scalar κ is large enough.

Proof. Follows from Theorems 20 and 21 and Lemma 10. □

Corollary 6 is still true when the underlying graph is not strongly connected, but its reversal contains a spanning directed tree [26].

Even though we have assumed the weights on the graph to be nonnegative, i.e. L has nonpositive off-diagonal elements, Theorem 21 can also be applied to the case where L has both positive and negative off-diagonal elements provided that a positive vector w exists such that $w^{\mathrm{T}} L = 0$.

6. Conclusions

We propose a refinement $\tilde{a}(G)$ to the generalization of Fiedler's algebraic connectivity to directed graphs described in [1] and study the relationship between $\tilde{a}(G)$ and $\tilde{b}(G)$ and several graph-theoretical properties. We also show that $\tilde{a}(G)$ provides a lower bound on the amount of coupling needed to synchronize a network of coupled dynamical systems. In particular this implies that coupled identical dynamical systems where the underlying graph is strongly connected can be synchronized when the coupling is large enough.

Properties such as independence number and min-p-sum do not depend on the orientation and weight of the edges and it would be interesting to study whether better bounds can be obtained by choosing the orientations and weights of the edges in an intelligent way.

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