SIAM J. DISCRETE MATH.
Vol. 7, No. 2, pp. 221-229, May 1994

008

THE LAPLACIAN SPECTRUM OF A GRAPH II*

ROBERT GRONE† AND RUSSELL MERRIS‡

Abstract. Let G be a graph. Denote by $D(G)$ the diagonal matrix of its vertex degrees and by $A(G)$ its adjacency matrix. Then $L(G) = D(G) - A(G)$ is the Laplacian matrix of G. The first section of this paper is devoted to properties of Laplacian integral graphs, those for which the Laplacian spectrum consists entirely of integers. The second section relates the degree sequence and the Laplacian spectrum through majorization. The third section introduces the notion of a d-cluster, using it to bound the multiplicity of d in the spectrum of $L(G)$.

Key words. degree sequence, majorization, Laplacian integral

AMS subject classification. 05C50

1. Laplacian integral graphs. Let $G = (V, E)$ be a graph with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G)$. Denote the degree of vertex v_i by $d(v_i)$. Let $D(G) = \text{diag}\,(d(v_1), d(v_2), \ldots, d(v_n))$ be the diagonal matrix of vertex degrees. The *Laplacian matrix* is $L(G) = D(G) - A(G)$, where $A(G)$ is the (0, 1)-adjacency matrix. It follows from the Geršgorin disc theorem that $L(G)$ is positive semidefinite and from the matrix-tree theorem (or from [3]) that its rank is $n - w(G)$, where $w(G)$ is the number of connected components of G. (More on the Laplacian may be found in [10] or [17].) Denote the spectrum of $L(G)$ by

$$S(G) = (\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$ are the eigenvalues of $L(G)$. If more than one graph is involved, we may write $\lambda_i(G)$ in place of λ_i. The multiplicity of λ as an eigenvalue of $L(G)$ will be denoted $m_G(\lambda)$.

The central theme of this article is the occurrence of integers in $S(G)$. If the spectrum consists entirely of integers, we say G is *Laplacian integral.* The study of graphs whose *adjacency* spectra consist entirely of integers was begun in [11]. Cvetković [4] proved that the set of connected, r-regular, adjacency integral graphs is finite. When $r = 2$, there are three such graphs, namely, C_3, C_4, and C_6. (When $r = 3$, there are 13 such graphs [2], [21].) If G is r-regular, then λ is an eigenvalue of $L(G)$ if and only if $r - \lambda$ is an eigenvalue of $A(G)$. Thus, the theory of Laplacian integral graphs coincides with its adjacency counterpart on regular graphs. Elsewhere, there can be remarkable differences. Consider, for example, the 112 connected graphs on six vertices. Six of them are adjacency integral. Of these six, five are regular: C_6 and its complement, K_6, $K_{3,3}$, and the cocktail party graph. The sixth is the tree obtained by joining the centers of two copies of P_3 with a new edge. As we have observed, the first five of these are also Laplacian integral; the sixth is not. On the other hand, there are a total of 37 connected Laplacian integral graphs on six vertices [18].

One general difference between the two theories concerns complements. Let J_n be the n-by-n matrix, each of whose entries is 1. Then, for any graph G on n vertices,

* Received by the editors November 20, 1991; accepted for publication (in revised form) March 23, 1993.
† Department of Mathematical Sciences, San Diego State University, San Diego, California 92182. The work of this author was supported by National Science Foundation grant DMS9007048.
‡ Department of Mathematics and Computer Science, California State University, Hayward, California 94542. The work of this author was supported by a California State University Research, Scholarship and Creative Activity award and by National Security Agency grant MDA904-90-H-4024.

$L(G) + L(G^c) = nI_n - J_n$. It follows that the eigenvalues of $L(G^c)$ are

$$(1) \qquad \lambda_i(G^c) = n - \lambda_{n-i}(G), \quad 1 \leq i < n, \quad \text{and} \quad 0.$$

In particular, G is Laplacian integral if and only if G^c is Laplacian integral, and $\lambda_1(G) \leq n$, with $m_G(n) = w(G^c) - 1$. Another difference involves trees. While some interesting work has been done on adjacency integral trees [11], [22], a complete description seems unlikely in the near future. On the other hand, T is Laplacian integral if and only if $T = K_{1,n-1}$ [7], [15, Cor. 2].

Let $G = (V, E)$ be a graph with n vertices and m edges. If $e = \{u, v\} \in E$ and $w \notin V$, then e is said to be *subdivided* when it is replaced by $\{u, w\}$ and $\{w, v\}$. Of course, replacing the edge replaces the graph; the new graph has $n + 1$ vertices and $m + 1$ edges.

Example 1. Denote by G_n the graph obtained from K_{n-1}, $n > 2$, by subdividing one edge; then G_n is Laplacian integral. This is because the complement of G_n is a graph with two connected components, one isomorphic to K_2 and the other to $K_{1,n-3}$. The same result does not hold for the adjacency matrix; $A(G_5)$ has three irrational eigenvalues. If two edges of K_3 are subdivided, the result is C_5, which is neither adjacency nor Laplacian integral.

If every edge of G is subdivided, the resulting graph $s(G)$ has $n + m$ vertices and $2m$ edges. It is called the *subdivision* of G. (*Note.* $S(G)$ is an n-tuple; $s(G)$ is a graph.)

THEOREM 1. *Let G be a connected, r-regular, Laplacian integral graph on n vertices. Then $s(G)$ is Laplacian integral if and only if $G = K_n$.*

Proof. The result is trivial for $r < 2$. If $r = 2$, then (as we have seen above) $G = C_3$, C_4, or C_6, Of these, $s(C_3) = C_6$ is Laplacian integral, whereas $s(C_4) = C_8$ and $s(C_6) = C_{12}$ are not. Thus, we may assume that $r \geq 3$ (so $n \geq 4$). Let m denote the number of edges in G. It was shown in [13] (see [17]) that

$$\det(xI_{m+n} - L(s(G))) = (-1)^n(x-2)^{m-n} \det(x(r+2-x)I_n - L(G)).$$

Therefore, α is an eigenvalue of $L(s(G))$ if and only if $\alpha = 2$ or $\alpha(r + 2 - \alpha)$ is an eigenvalue of $L(G)$. If $G = K_n$, then $r = n - 1$, and the eigenvalues of $L(s(G))$ satisfy $m_{s(G)}(0) = 1$, $m_{s(G)}(1) = n - 1$, $m_{s(G)}(2) = n(n-3)/2$, $m_{s(G)}(n) = n - 1$, and $m_{s(G)}(n + 1) = 1$.

Conversely, the eigenvalues of $L(G)$ are all of the form $\lambda = \alpha(r + 2 - \alpha)$. Since $r + 1$ is the minimum value taken by this product when both factors are constrained to be positive integers, we deduce that $\lambda_{n-1}(G) \geq r + 1$. Thus, the trace of $L(G)$ is at least $(n-1)(r+1)$. However, trace $L(G) = rn$. Combining these, we conclude that $r \geq n - 1$. $\square$

It is proved in [11, Cor. 6] that the line graph of a regular adjacency integral graph is adjacency integral. Since the line graph of a regular graph is regular, this result carries over to the Laplacian case. So, for example, the Petersen graph is Laplacian integral since it is the complement of the line graph of K_5. Recall that a graph is (r, s)-*semiregular* if it is bipartite with a bipartition (V_1, V_2) in which each vertex of V_1 has degree r and each vertex of V_2 has degree s. The next result was proved by Mohar [17, Thm. 3.9].

PROPOSITION 1. *Let G be a connected, (r, s)-semiregular, Laplacian integral graph. Then its line graph is Laplacian integral.*

COROLLARY 1. *The line graph of the subdivision of K_n is Laplacian integral.*

Ultimately, we would like to "explain" all the integral graphs (whether Laplacian or adjacency). Harary and Schwenk [11] identified various families of adjacency integral graphs. Inevitably, they found graphs that belong to none of them. One such graph is the line graph of the subdivision of K_4. It is clear from Corollary 1 that this graph does, in fact, belong to a natural family. However, because the characteristic polynomial of

$A(s(K_4))$ is $x^2(x^2 - 2)^3(x^2 - 6)$, this fact is not immediately evident from the adjacency perspective.

Let G and H be graphs on disjoint sets of vertices. Their *union* $G + H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H)$. The *join* of G and H may be defined by $G \vee H = (G^c + H^c)^c$. It is the graph obtained from $G + H$ by adding new edges from each vertex of G to every vertex of H. Clearly, the union and join of Laplacian integral graphs are Laplacian integral.

The *product* of graphs G and H is the graph $G \times H$, whose vertex set is the Cartesian product $V(G) \times V(H)$. Suppose that $v_1, v_2 \in V(G)$ and $u_1, u_2 \in V(H)$. Then (v_1, u_1) and (v_2, u_2) are adjacent in $G \times H$ if and only if one of the following conditions is satisfied: (i) $v_1 = v_2$ and $\{u_1, u_2\} \in E(H)$ or (ii) $\{v_1, v_2\} \in E(G)$ and $u_1 = u_2$. For example, the line graph of $K_{p,q}$ is $K_p \times K_q$. If G and H are Laplacian integral graphs, then $G \times H$ is Laplacian integral [17, Thm. 3.5]. Let $G^2 = G \times G$. The subgraph of G^2 induced on $W = \{(v_i, v_j): i < j\} \subset V(G^2)$ is called $G^{[2]}$ [8]. Since both $(x^2 - 7x + 8)^2$ and $(x^2 - 10x + 20)$ are factors of the characteristic polynomial of $L(C_6^{[2]})$, $G^{[2]}$ does not generally propagate Laplacian integrality. Still, this construction is the source of numerous Laplacian integral graphs.

2. Majorization and the degree sequence. Recall that a nonincreasing sequence $(d) = (d_1, d_2, \ldots, d_n)$ of positive integers is said to be *graphic* if there exists a (simple) graph having degree sequence (d). The theory of graphic sequences has a rich tradition nicely summarized in [19].[1] Another perspective on the discussion of §1 would be to approach nonincreasing sequences $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of nonnegative real numbers in a similar way; i.e., what are necessary and/or sufficient conditions for (λ) to be the spectrum of the Laplacian matrix of a (simple) graph, and what special conditions arise if we require (λ) to be an integer sequence? One condition follows immediately from a theorem of Schur. It involves the property of majorization. Suppose that (a) and (b) are finite nonincreasing sequences of real numbers. Then (a) is said to *majorize* (b) if

$$a_1 \geq b_1,$$

$$a_1 + a_2 \geq b_1 + b_2,$$

$$a_1 + a_2 + a_3 \geq b_1 + b_2 + b_3,$$

and so on, with equality at the end; that is, the sum of all the a's is equal to the sum of all the b's.

In the introductory paragraph, we defined $D(G)$ to be the diagonal matrix of vertex degrees. We abuse that language now by letting $D(G)$ also denote the *sequence* of vertex degrees in nonincreasing order, i.e., $D(G) = (d_1, d_2, \ldots, d_n)$. (We do not assume that $d_i = d(v_i)$.) Since the spectrum of any symmetric, positive semidefinite matrix majorizes its main diagonal [20], [14, p. 218], the following is immediate from the definitions.

PROPOSITION 2. *Let G be a graph. Then $S(G)$ majorizes $D(G)$.*

The most immediate consequence of Proposition 2 is the inequality $\lambda_1 \geq d_1$. In fact, this inequality is subject to some improvement, as we now show.

THEOREM 2. *Suppose that $G = (V, E)$ is a connected a graph with $n > 2$ vertices. If $S = \{u_1, u_2, \ldots, u_k\} \subset V$, let $G[S] = (S, E[S])$ be the subgraph of G induced on S. Suppose that $E[S]$ consists of r pairwise disjoint edges. Then*

$$\lambda_1 + \cdots + \lambda_k \geq d(u_1) + \cdots + d(u_k) + k - r.$$

(Recall that $d(u_i)$ need not equal d_i.)

[1] Unfortunately, [19] contains several annoying misprints.

The inequality in Theorem 2 suggests it might be useful to define

$$\Sigma_k(G) = \max\left\{\sum_{i=1}^{k} d(u_i)\colon \{u_1, \ldots, u_k\} \text{ independent in } G\right\}.$$

Curiously enough, the analogous quantity

$$\sigma_k(G) = \min\left\{\sum_{i=1}^{k} d(u_i)\colon \{u_1, \ldots, u_k\} \text{ independent in } G\right\}$$

is useful in the study of Hamiltonian cycles [9].

COROLLARY 2. *If G has an edge, then*

$$\lambda_1 \geq d_1 + 1. \tag{2}$$

Proof. If $d_1 = 1$, then every connected component of G is either an isolated vertex or a copy of K_2. In this case, $\lambda_1 = 2$ and $d_1 = 1$. If $d_1 > 1$, let w be a vertex of G of degree d_1. Let C be a connected component of G containing w. Then we may apply Theorem 2 to the subgraph C with $k = 1$ and $u_1 = w$. Since $L(C)$ is a direct summand of $L(G)$, $\lambda_1(G) \geq \lambda_1(C)$. □

Corollary 2 improves the earlier result [7, Thm. 3.7] $\lambda_1 \geq d_1 + d_1/(n-1)$. (It can be shown that inequality in (2) is strict whenever $d_1 < n - 1$.)

COROLLARY 3. *Let G be a connected graph on $n > 2$ vertices. Then $\lambda_1 + \lambda_2 \geq d_1 + d_2 + 1$. If there are two nonadjacent vertices in G having degrees d_1 and d_2, then $\lambda_1 + \lambda_2 \geq d_1 + d_2 + 2$.*

Proof. The proof is immediate from Theorem 2. Either $r = 1$ or $r = 0$. □

Example 2. Let G be the graph shown in Fig. 1. Then $S(G) = (5.1, 3.8, 1.5, 1, 0.6, 0)$ and $\lambda_1 + \lambda_2 < d_1 + d_2 + 2$.

CONJECTURE 1. *Let G be a graph with $m \geq 1$ edges. Then $S(G)$ majorizes the sequence $(d_1 + 1, d_2, d_3, \ldots, d_{n-1}, d_n - 1)$.*

Proof of Theorem 2. Order the m edges in E arbitrarily. For each edge $e = \{v, w\}$, designate one of v, w to be the "positive end" of e and the other to be the "negative end." Thus, G is given a fixed but arbitrary *orientation.* If $e \in E$ and $v \in V$, define $s(v, e) = +1$ if v is the positive end of e, -1 if it is the negative end, and 0 otherwise. The vertex-edge *incidence matrix* afforded by the orientation is the n-by-m matrix $Q = Q(G)$ whose (v, e)-entry is $s(v, e)$. It turns out that $L(G) = QQ^t$, independently of the orientation. The matrix $K(G) = Q^tQ$ depends on the orientation for the signs of its off-diagonal entries. In any event, $K(G)$ and $L(G)$ share the same nonzero eigenvalues.

Suppose that x is some real m-tuple. It is convenient to think of its components as indexed by E, so the "vth" component" of Qx is

$$\sum_{e \in E} s(v, e)x_e.$$

Therefore,

$$x^tK(G)x = \sum_{v \in V}\left[\sum_{e \in E} s(v, e)x_e\right]^2. \tag{3}$$

Suppose that $E[S] = \{e_1, e_2, \ldots, e_r\}$. Without loss of generality, we may assume $e_i = \{u_i, u_{k-i+1}\}$ and $d(u_i) \leq d(u_{k-i+1})$, $1 \leq i \leq r$. Choose an orientation of $E(G)$ so that u_i is the positive end of each of the $d(u_i)$ edges incident with it, $1 \leq i \leq k - r$. In addition, we may prescribe that u_i is the positive end of each of the $d(u_i) - 1 > 0$ edges,

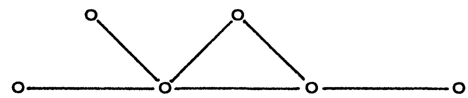

FIG. 1

other than e_{k-i+1}, incident with it, $k - r < i \le k$. For each $u \in S$, define a real m-tuple $x(u)$ as follows: $x(u)_e$, the coordinate of $x(u)$ corresponding to the edge $e \in E$, is 1 if u is the positive end of e, and 0 otherwise. Then $\{x(u)\colon u \in S\}$ is an orthogonal set of vectors. Moreover, $\|x(u_i)\|^2 = d(u_i)$ if $i \le k - r$ and $d(u_i) - 1$ if $i > k - r$. Let $y(u) = x(u)/\|x(u)\|$, $u \in S$. Then, for $1 \le i \le k - r$,

$$\sum_{e\in E} s(v, e)y(u_i)_e = \begin{cases} d(u_i)^{1/2} & \text{if } v = u_i, \\ -d(u_i)^{-1/2} & \text{if } \{v, u_i\} \in E, \\ 0 & \text{otherwise,} \end{cases}$$

and, for $i > k - r$,

$$\sum_{e\in E} s(v, e)y(u_i)_e = \begin{cases} (d(u_i) - 1)^{1/2} & \text{if } v = u_i, \\ -(d(u_i) - 1)^{-1/2} & \text{if } \{v, u_i\} \in E\backslash E[S], \\ 0 & \text{otherwise.} \end{cases}$$

So, from (3),

$$y(u_i)^t K(G) y(u_i) = \begin{cases} d(u_i) + 1, & i \le k - r, \\ d(u_i), & i > k - r. \end{cases}$$

Since $\{y(u_i)\colon 1 \le i \le k\}$ is an orthonormal set of vectors,

$$\sum_{i=1}^{k} \lambda_i \ge \sum_{i=1}^{k} y(u_i)^t K(G) y(u_i)$$
$$= \sum_{i=1}^{k} d(u_i) + k - r. \qquad \square$$

Nonincreasing integer sequences are frequently pictured by means of so-called Ferrers–Sylvester diagrams. The diagram for $(d) = (5, 5, 5, 4, 4, 4, 3)$ is pictured on the left in Fig. 2. Its *transpose* is the diagram pictured on the right corresponding to the conjugate partition $(d)^t = (7, 7, 7, 6, 3)$. In general, the *conjugate* of a nonincreasing integer sequence

$$(a) = (a_1, a_2, \ldots, a_p)$$

is

$$(a)^t = (a_1^t, a_2^t, \ldots, a_q^t),$$

where $q = a_1$ and a_i^t is the number of elements in the set $\{j\colon a_j \ge i\}$.

PROPOSITION 3. *Let* (d) *be a graphic sequence. Then* $(d)^t$ *majorizes* (d).

Proof. Suppose that $d_i \ge i$, $1 \le i \le k$, and $d_{k+1} < k + 1$. (In Fig. 2, $k = 4$.) Define $r_i = d_i + 1$, $1 \le i \le k$ and $r_i = d_i$, $k < i \le n$. Ryser's necessary and sufficient condition for (d) to be graphic is that $(r)^t$ majorize (r) (see [1, §11.5]). Note that $r_i^t = d_i^t$, $1 \le i \le k$

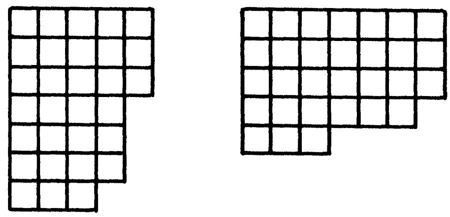

FIG. 2. *Ferrers–Sylvester diagrams.*

and $r_i^t = d_i^t + c_i$, $i > k$, where the c_i are nonnegative integers *that add up to k*. So, for $1 \le j \le k$,

$$\sum_{i=1}^{j} d_i^t = \sum_{i=1}^{j} r_i^t \ge \sum_{i=1}^{j} r_i = \sum_{i=1}^{j} (d_i + 1) = j + \sum_{i=1}^{j} d_i > \sum_{i=1}^{j} d_i.$$

For $j > k$, let $x_j = c_1 + c_2 + \cdots + c_{j-k}$. Then

$$x_j + \sum_{i=1}^{j} d_i^t = \sum_{i=1}^{j} r_i^t \ge \sum_{i=1}^{j} r_i = k + \sum_{i=1}^{j} d_i.$$

So, since $x_j \le k$,

$$\sum_{i=1}^{j} d_i^t \ge (k - x_j) + \sum_{i=1}^{j} d_i \ge \sum_{i=1}^{j} d_i. \qquad \square$$

Propositions 2 and 3 raise the natural question of whether $S(G)$ and $D(G)^t$ are majorization comparable. (An infinite family of "maximal" graphs for which $D(G)^t = S(G)$ is discussed in [16].)

CONJECTURE 2. *Let G be a connected graph. Then $D(G)^t$ majorizes $S(G)$.*

If Conjecture 2 is true, then

$$\lambda_{n-1} \ge d_{n-1}^t; \tag{4}$$

i.e., the number of vertices of G of degree $n-1$ is bounded above by λ_{n-1}, the "algebraic connectivity" of G [7]. To verify (4), suppose that G is a graph with k vertices of degree $n-1$. If $k = n$, then $G = K_n$ and $\lambda_{n-1} = n$. Otherwise, G^c has at least $k+1$ components, the largest of which has at most $n-k$ vertices. So, $\lambda_1(G^c) \le n - k$. Thus, $\lambda_{n-1}(G) = n - \lambda_1(G^c) \ge k$.

If $D(G)^t$ majorizes $S(G)$ for connected graphs, then $D(G)^t$ majorizes $S(G)$ for all graphs as can easily be seen, e.g., by the condition of Hardy, Littlewood, and Polya [14, p. 22]. So, we may as well restrict our attention to connected graphs, where the number of vertices of G having degree at least 1 is $d_1^t = n$. The inequality $n \ge \lambda_1$ is clear from (1). Indeed, if G is a connected r-regular graph, then $D(G) = (r, r, \ldots, r)$ and $D(G)^t = (n, n, \ldots, n)$ majorizes $S(G)$.

A second step toward proving Conjecture 2 would be to show

$$d_1^t + d_2^t \ge \lambda_1 + \lambda_2. \tag{5}$$

We are not able to establish (5) in all cases. However, some partial results along these lines will emerge from the following result.

THEOREM 3. *Let G be a connected graph and suppose that w is a cut vertex of G. If the largest component of $G - w$ contains r vertices, then $r + 1 \ge \lambda_2(G)$.*

Proof. Denote by $L(w)$ the $(n-1)$-by-$(n-1)$ principal submatrix of $L(G)$ obtained by striking out the row and column corresponding to w. Let α be the largest eigenvalue of $L(w)$. By the Cauchy interlacing inequalities, $\lambda_1 \ge \alpha \ge \lambda_2$.

If $C_1, C_2, \ldots, C_k$ are the connected components of $G - w$, let S_i be the union of $\{w\}$ and the vertices of C_i, $1 \le i \le k$. Let $G_i = G[S_i]$ be the subgraph of G induced by S_i and write $L_i = L(G_i)$, $1 \le i \le k$. Then $L(w)$ is the direct sum of $L_i(w)$, $1 \le i \le k$, where $L_i(w)$ is the principal submatrix of L_i obtained by striking out the row and column corresponding to w. It follows that α is the largest eigenvalue of $L_i(w)$ for some i. By another application of the Cauchy inequalities, we conclude that $\alpha \le \lambda_1(G_i)$ for some i. By (1) and the hypotheses, $r + 1 \ge \lambda_1(G_i)$ for all i. □

A pendant *neighbor* is a vertex adjacent to a vertex of degree 1.

COROLLARY 4. *Let G be a connected graph with $n > 2$ vertices. Suppose that w is a pendant neighbor of G adjacent to k pendant vertices. Then $n - k \ge \lambda_2$.*

Proof. If $G = K_{1,n-1}$, then $n - k = 1 = \lambda_2$. Otherwise, the largest component of $G - w$ contains at most $n - k - 1$ vertices, so the result is immediate from Theorem 3. □

We now return to (5). Since $n = d_1^t \ge \lambda_1$, it would suffice to show that $d_2^t \ge \lambda_2$. Now d_2^t is the number of vertices of G having degree at least 2. That is, $d_2^t = n - p$, where p is the number of pendant vertices. However, as we now see, it is not generally true, even for trees, that $n - p \ge \lambda_2$.

Example 3. Let T be the tree on six vertices obtained by joining the centers of two copies of P_3 by a new edge. Then $D(T) = (3, 3, 1, 1, 1, 1)$ and $D(T)^t = (6, 2, 2)$. To one decimal place, $S(T) = (4.6, 3, 1, 1, 0.4, 0)$. (Recall that T is the only adjacency integral graph on six vertices that is not Laplacian integral.) Here $n = 6$, $p = 4$, and $n - p = 2 < 3 = \lambda_2(T)$.

COROLLARY 5. *Let G be a connected, Laplacian integral graph with n vertices, p of which are pendants. Then $n - p \ge \lambda_2$.*

Proof. Suppose that G has a total of q pendant neighbors altogether. It is proved in [10, Thm. 3.11] that the number of eigenvalues of $L(G)$, multiplicities included, lying in the open interval $[0, 1)$ is at least q. Since we are assuming that G is connected, $m_G(0) = 1$. Thus, $q > 1$ contradicts the hypothesis that G is Laplacian integral. We conclude that all p pendants of G share the same neighbor, so the result follows from Corollary 4. □

3. Eigenvalues and graph structure. In a natural way, the majorization questions of §2 have led to the relationship between the Laplacian spectrum and graph structure. We proceed to develop more results along these lines, beginning with a relative of Corollary 5.

PROPOSITION 4. *Let G be a graph with $n > 2$ vertices, p of which are pendants. If $\lambda_1 = n$, then all p of the pendants are adjacent to the same neighbor w, $d(w) = n - 1$, and $\lambda_2 \le n - p$. (In particular, if T is a tree, then $\lambda_1(T) = n$ if and only if $T = K_{1,n-1}$.)*

Proof. From (1), $\lambda_1(G) = n$ if and only if G^c is disconnected. If G had two distinct pendant neighbors, then G^c would be connected. So, there is a unique pendant neighbor w. We conclude from Corollary 4 that $\lambda_2 \le n - p$. If $d(w)$ were less than $n - 1$ then, again, G^c would be connected. □

PROPOSITION 5. *Let G be a connected graph. Let $P = \{v_1, v_2, \ldots, v_k\}$ be a set of pendant vertices of G, all of which are adjacent to the same neighbor w. Suppose that $\lambda \neq 1$. If (the multiplicity) $m_G(\lambda) > 1$, then $m_{G-P}(\lambda) > 0$. (In particular, $\lambda \le n - k$.) Moreover, if $m_{G-P}(\lambda) > 1$, then $m_G(\lambda) > 0$.*

Proof. Let $w = v_{k+1}$. If λ is a multiple eigenvalue of $L(G)$, it has an eigenvector x whose $(k + 1)$st component is 0. Then $L(G)x = \lambda x$ forces $x_1 = x_2 = \cdots = x_k = 0$. It follows that $y = (0, x_{k+2}, \ldots, x_n)$ is an eigenvector of $L(G - P)$, affording λ. Because $G - P$ has $n - k$ vertices, $\lambda \le n - k$. To obtain the final assertion, let w be the first vertex of $G - P$. If λ is a multiple eigenvalue of $L(G - P)$, then it is afforded by an eigenvector

y whose first component is 0. Let x be the vector obtained from y be inserting k zeros at the beginning. Then x is an eigenvector of $L(G)$, affording λ. □

PROPOSITION 6. *Let G be a graph with n vertices and $k \geq 1$ spanning trees. If λ is a positive integer eigenvalue of $L(G)$, then $\lambda \mid nk$. If G is Laplacian integral, then $\lambda^t \mid nk$, where $t = m_G(\lambda)$, the multiplicity of λ as an eigenvalue of $L(G)$.*

Proof. Let $p(x)$ be the characteristic polynomial of $L(G)$. Since G is connected, $L(G)$ has rank $n - 1$, so we may factor $p(x)$ as $xf(x)$. (The polynomial $f(x)$ has been called the "T-polynomial" of G [5], [12].) As $f(0)$ is the coefficient of x in $p(x)$, it is the sum of the determinants of the $(n-1)$-by-$(n-1)$ principal submatrices of $L(G)$. By the matrix-tree theorem, each of these n determinants has the value k. Thus, $f(0) = nk$. Since $f(x)$ is a monic polynomial with integer coefficients, $f(\lambda) = 0$ if and only if $\lambda \mid f(0)$. On the other hand, $f(0)$ is the $(n-1)$th elementary symmetric function of the eigenvalues of $L(G)$, i.e., $f(0)$ is the product of the nonzero eigenvalues of $L(G)$. So, if G is Laplacian integral, $\lambda^t \mid f(0)$. □

DEFINITION. *Let G be a graph. A* cluster *of G is an independent set of two or more vertices of G, each of which has the same set of neighbors. (The set of neighbors of vertex v is $\{u \in V: \{u, v\} \in E\}$.) The* degree *of a cluster is the cardinality of its shared set of neighbors, i.e., the common degree of each vertex in the cluster. A d-*cluster *is a cluster of degree d. The number of vertices in a d-cluster is its* order. *A collection of two or more d-clusters is* independent *if the sets of vertices comprising the d-clusters are pairwise disjoint. (The neighbor sets of independent d-clusters need not be disjoint.)*

The next result extends the work of Faria on the "star degree" of a graph [6].

THEOREM 4. *Let G be a graph with k independent d-clusters of orders $r_1, r_2, \ldots, r_k$. Then $m_G(d) \geq r_1 + r_2 + \cdots + r_k - k$.*

Example 4. The graphs G_1 and G_2 in Fig. 3 are the smallest pair of nonisomorphic, connected, Laplacian integral, Laplacian cospectral graphs. They share the spectrum $S(G_1) = S(G_2) = (7, 6, 6, 4, 4, 3, 0)$. Both pictures are drawn so that the top row of vertices is a 4-cluster: G_1 contains a 4-cluster of order 2, while G_2 contains one of order 3. With $d = 4$ and $k = 1$, Theorem 4 asserts that $m_G(4) \geq 1$ for $G = G_1$ and $m_G(4) \geq 2$ for $G = G_2$; the bound is sharp for G_2 but not for G_1. On the other hand, an examination of the spectrum shows there is no point in looking for a 5-cluster in either graph.

Proof of Theorem 4. The independent d-clusters correspond to k nonoverlapping principal submatrices of $L(G)$. Each submatrix is d-times an r_i-by-r_i identity matrix, $i = 1, 2, \ldots, k$. Suppose that one of these principal submatrices includes rows and columns numbered s and t in $L(G)$, $s < t$. (That is, suppose that v_s and v_t belong to the same d-cluster in G.) Let x be the column vector with $x_i = 1$ if $i = s$, -1 if $i = t$, and 0, otherwise. Because v_s and v_t belong to the same d-cluster, they have the same neighbors. Hence, $L(G)x = dx$. A d-cluster of order r_i affords $r_i - 1$ linearly independent eigenvectors of this type, and eigenvectors of this type arising from independent clusters are linearly independent. □

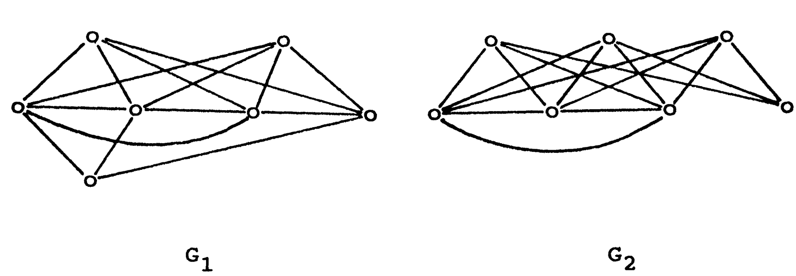

FIG. 3

COROLLARY 6. *Let G be a graph with an r-clique, $r \geq 2$. Suppose that every vertex of the clique has the same set of neighbors outside the clique. Let the degree of each vertex of the clique be d, so $d - r + 1$ is the number of vertices not belonging to the clique but adjacent to every member of the clique. Then $m_G(d + 1) \geq (r - 1)$.*

Proof. The clique becomes an $(n - d - 1)$-cluster of G^c of order r. □

Example 5. Let G be the graph G_2 of Fig. 3. The three vertices of G of degree 5 are a 3-clique satisfying the hypotheses of Corollary 6. Hence $m_G(6) \geq 2$, and, again, we find that G_2 affords a sharp bound.

Note added in proof. Theorem 4 was first proved by Isabel Faria, who communicated it to Merris long before the present paper was contemplated. Unfortunately, Merris filed it away and forgot about it. Fortunately, the result will appear under Faria's name in the article, "Multiplicity of Integer Roots of Polynomials of Graphs," to be published by *Linear Algebra Appl.*

REFERENCES

[1] J. A. BONDY AND U. S. R. MURTY, *Graph Theory With Applications*, North-Holland, New York, 1976.

[2] F. C. BUSSEMAKER AND D. M. CVETKOVIĆ, *There are exactly* 13 *connected, cubic, integral graphs*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz., 544–576 (1976), pp. 43–48.

[3] I. M. CHAKRAVARTI, *On a characterization of irreducibility of a nonnegative matrix*, Linear Algebra Appl., 10 (1975), pp. 103–109.

[4] D. M. CVETKOVIĆ, *Cubic integral graphs*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. 498–541 (1975), pp. 107–113.

[5] E. A. DINIC, A. K. KELMANS, AND M. A. ZAITSEV, *Nonisomorphic trees with the same T-polynomial*, Inform. Process. Lett., 6 (1977), pp. 73–76.

[6] I. Faria, *Permanental roots and the star degree of a graph*, Linear Algebra Appl., 64 (1985), pp. 255–265.

[7] M. FIEDLER, *Algebraic Connectivity of graphs*, Czech. Math. J., 23 (98) (1973), pp. 298–305.

[8] ———, *Irreducibility of compound matrices*, Comment. Math. Univ. Carolin., 20 (1979), pp. 737–743.

[9] R. J. GOULD, *Updating the hamiltonian problem—a survey*, J. Graph Theory, 15 (1991), pp. 121–157.

[10] R. GRONE, R. MERRIS, AND V. S. SUNDER, *The Laplacian spectrum of a graph*, SIAM J. Matrix Anal. Appl., 11 (1990), pp. 218–238.

[11] F. HARARY AND A. J. SCHWENK, *Which graphs have integral spectra?*, in Graphs and Combinatorics, R. A. Bari and F. Harary, eds., Lecture Notes in Math 406, Springer-Verlag, Berlin, 1974.

[12] Y. HATTORI, *Nonisomorphic graphs with the same T-polynomial*, Inform. Process. Lett., 22 (1986), pp. 133–134.

[13] A. K. KEL'MANS, *Properties of the characteristic polynomial of a graph*, Kibernetiky-na službu kommunizmu, v. 4, Energija, Moskva-Leningrad, 1967, pp. 27–41. (In Russian.)

[14] A. W. MARSHALL AND I. OLKIN, *Inequalities: Theory of Majorization and Its Applications*, Academic Press, New York, 1979.

[15] R. MERRIS, *Characteristic vertices of trees*, Linear Multilinear Algebra, 22 (1987), pp. 115–131.

[16] ———, *Degree maximal graphs are Laplacian integral*, Linear Algebra Appl., to appear.

[17] B. MOHAR, *The Laplacian spectrum of graphs*, Graph Theory, Combinatorics, and Applications, Y. Alavi et al., eds., John Wiley, New York, 1991, pp. 871–898.

[18] D. L. POWERS, *Graph Eigenvectors*, unpublished tables.

[19] G. SIERKSMA AND H. HOOGEVEEN, *Seven criteria for integer sequences being graphic*, J. Graph Theory, 15 (1991), pp. 223–231.

[20] I. SCHUR, *Über eine Klasse von Mittelbildungen mit Anwendungen die Determinanten*, Theorie Sitzungsber. Berlin. Math. Gesellschaft, 22 (1923), pp. 9–20.

[21] A. J. SCHWENK, *Exactly thirteen connected cubic graphs have integral spectra*, in Theory and Applications of Graphs, Y. Alavi et al., eds., Springer-Verlag, Berlin, 1978, pp. 516–533.

[22] M. WATANABE AND A. J. SCHWENK, *Integral starlike trees*, J. Austral. Math. Soc. Ser. A, 28 (1979), pp. 120–128.