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A Survey of Graph Laplacians

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Let G be a graph on n vertices. Its Laplacian is the n-by-n matrix $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees and $A(G)$ is the (0,1)-adjacency matrix of G. This article surveys recent results on graph Laplacians.

1. INTRODUCTION

This survey is an expository sequel to [32]. Only Theorem 5 has not appeared elsewhere. While some overlap with the previous survey may be necessary to make the present paper self-contained, motivational and historical discussions will not be repeated here.

Let $G = (V, E)$ be a graph with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G) = \{e_1, e_2, \ldots, e_m\}$. For each edge $e_j = \{v_i, v_k\}$, choose one of v_i, v_k to be the positive "end" of e_j and the other to be the negative end. Thus, G is given an "orientation". The vertex-edge **incidence matrix** afforded by an orientation of G is the n-by-m matrix $Q(G) = (q_{ij})$, where $q_{ij} = +1$ if v_i is the positive end of e_j; -1 if it is the negative end; and 0 otherwise. In the space of m-tuples, the kernel of Q is the "cycle space" of G.

It turns out that the **Laplacian Matrix**, $L(G) = Q(G)Q(G)^t$, is independent of the orientation. In fact, $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees and $A(G)$ is the (0,1)-adjacency matrix. On the other hand, $L(G)$ does depend on the numbering of the vertices. In some cases (see the discussion of bandwidth below) it is of interest to investigate optimal numberings of the vertices. For the most part, however, one is interested in $L(G)$ only up to permutation similarity. Indeed, graphs G_1 and G_2 are isomorphic if and only if there is a permutation matrix P such that

$$L(G_2) = P^t L(G_1) P. \tag{1}$$

Of course, two matrices cannot be permutation-similar if they are not similar, and two real symmetric matrices are similar if and only if they have the same eignevalues.

[1]This article was prepared in conjunction with the ICMS workshop on Algebraic Graph Theory at the University of Edinburgh, July 12–16, 1993.

Denote the spectrum of $L(G)$ by

$$S(G) = (\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$. (We may write $\lambda_i(G)$ if more than one graph is under consideration.) Following Fiedler, we define $a(G) = \lambda_{n-1}(G)$, calling it the **algebraic connectivity** of G. Eigenvectors of $L(G)$ affording $a(G)$ were called "characteristic valuations" by Fiedler, but his terminology is rapidly being eclipsed by the more popular term, **Fiedler vector.**

There is an "edge version" of the Laplacian, namely $K(G) = Q(G)^t Q(G)$. Unlike its vertex counterpart, $K(G)$ depends on the orientation for the signs of its off-diagonal elements. Of course, $K(G)$ and $L(G)$ share the same nonzero spectrum.

2. DIAMETER

The eigenvalues of $L(G)$ are natural graph-theoretic invariants, at least from an algebraic perspective. What $S(G)$ means combinatorially is a subject of much investigation with new results appearing almost weekly. For example, let $d(G) = (d_1, d_2, \ldots, d_n)$, where $d_1 \geq d_2 \geq \cdots \geq d_n$ are the degrees of the vertices of G arranged in nonincreasing order (d_i need not be the degree of vertex v_i).

THEOREM 1 [37] *Let G be a graph on n vertices. Then*

$$\text{diameter } (G) \leq 2\left\lceil \frac{d_1 + a(G)}{4a(G)} \right\rceil \ln(n-1),$$

where $\lceil\ \rceil$ *is the "ceiling" function.*

For $G \neq K_n$, define $t(G) = [\lambda_1(G) + a(G)]/[\lambda_1(G) - a(G)]$.

THEOREM 2 [7] *If* $G \neq K_n$, *then*

$$\text{diameter}(G) \leq \left\lfloor \frac{\cosh^{-1}(n-1)}{\cosh^{-1}(t(G))} \right\rfloor + 1,$$

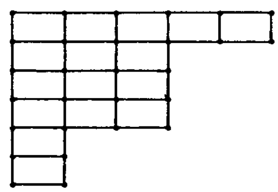
where $\lfloor\ \rfloor$ *is the "floor" function.*

THEOREM 3 [11] *If* $G \neq K_n$, *then*

$$\text{diameter}(G) \leq \left\lfloor \frac{\ln(n-1)}{\ln(t(G))} \right\rfloor + 1$$

3. MAXIMAL GRAPHS

Of course, $d(G) = (d_1, d_2, \ldots, d_n)$ is a partition of $2m$. Partitions are frequently studied with the help of Ferrers diagrams. The diagram, e.g., for $(c) = (5,3,3,3,1,1)$ is shown in Figure 1. (Isolated vertices are not explicitly represented in Ferrers diagrams.)



(c) = (5,3,3,3,1,1)

FIGURE 1.

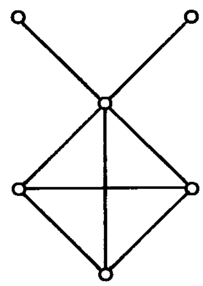


FIGURE 2.

The partition $(a) = (a_1, a_2, \ldots, a_n)$ is **graphic** if there exists a (simple) graph whose degree sequence is (a). The partition (c) is graphic; it is the degree sequence of the graph in Figure 2.

The largest square array of boxes that will fit in the upper left-hand corner of a Ferrers diagram is called its "Durfee square". The size of the Durfee square of partition (a), call it $f(a)$, is the cardinality, $o\{i:\ a_i \geq i\}$. So, $f(a)$ is the length of the main diagonal in the Ferrers diagram for (a). In Figure 1, $f(c) = 3$. The **conjugate** of (a) is the partition $(a^*) = (a^*_1, a^*_2, \ldots)$, whose ith part is $a_i^* = o\{j : a_j \geq i\}$. So, $(c^*) = (6,4,4,1,1)$. Since the diagram for (a^*) is just the transpose of the diagram for (a), $f(a) = f(a^*)$. While the following condition for a sequence to be graphic has been attributed to Hasselbarth [52], it seems to have been discovered earlier by Ruch and Gutman [50]: The partition (a) of $2m$ is graphic if and only if

$$\sum_{i=1}^{k}(a_i + 1) \leq \sum_{i=1}^{k} a_i^*, \quad 1 \leq k \leq f(a). \tag{2}$$

The inequalities in (2) are reminiscent of majorization. If $(a) = (a_1, a_2, \ldots, a_s)$ and $(b) = (b_1, b_2, \ldots, b_t)$, then (b) **majorizes** (a) if

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i, \quad 1 \leq k \leq \min\{s, t\}$$

and

$$\sum_{i=1}^{s} a_i = \sum_{i=1}^{t} b_i.$$

If (b) is a graphic partition that majorizes (a), it follows from (2) that (a) is a graphic partition. We will say that a graph is **maximal** if its degree sequence is majorized by no other graphic partition. So, G is mximal if and only if $d(G)$ satisfies

$$d_i + 1 = d_i^*, 1 \leq i \leq f(d(G)). \tag{3}$$

called "maximally branched" in [50], these graphs were introduced in an attempt to quantify the chemically important but vaguely defined concept of branching. Apart from isolated vertices, maximal graphs are characterized by their Laplacian spectra.

THEOREM 4 [34] *Let G be a graph with no isolated vertices. Then G is maximal if and only if*

$$d^*(G) = (\lambda_1(G), \lambda_2(G), \ldots, \lambda_{n-1}(G)),$$

that is, if and only if the conjugate of its degree sequence is identical to its nonzero Laplacian spectrum.

It is known that both $S(G)$ and $d^*(G)$ majorize $d(G)$. It was conjectured in [21] that $d^*(G)$ majorizes $S(G)$. If the conjecture is true, then (Theorem 4) equality holds if and only if the largest connected component of G is maximal and the remaining components are isolated vertices. Theorem 4 also affords an infinite family of "Laplacian integral" graphs.

4. IMMANANTAL POLYNOMIALS

Recall that G_1 and G_2 are isomorphic if and only if $L(G_1)$ and $L(G_2)$ are permutation-similar. It is not so much the similarity invariants of $L(G)$ that are important, but the *permutation* -similarity invariants.

There is a one-to-one correspondence between the partitions of n and the irreducible characters of the symmetric group S_n. The **immanant** corresponding to the character χ of S_n is defined by

$$d_\chi(A) = \sum_{p \in S_n} \chi(p) \prod_{i=1}^{n} a_{ip}(i),$$

where $A = (a_{ij})$ is any n-by-n matrix. (If $\chi = \varepsilon$ is the alternating character, then d_χ is the determinant. If $\chi \equiv 1$ is the principal character, then d_χ is the permanent.) Because χ is a conjugacy class function, $d_\chi(P^tAP) = d_\chi(A)$, for all χ, all A, and every n-by-n permutation matrix P. In other words, for every character χ of S_n, $d_\chi(A)$ is a permutation-similarity invariant of A .

It follows from these remarks that two graphs on n vertices, G_1 and G_2 are isomorphic only if they share a complete set of **immanantal polynomials,** i.e., only if

$$d_\chi(xI - L(G_1)) = d_\chi(xI - L(G_2)),$$

for every character χ of S_n. On the other hand, even taken all together, the immanantal polynomials are not enough to distinguish nonisomorphic graphs.

THEOREM 5 *Suppose k is a fixed but arbitrary positive integer. Let* $n = 17 \cdot \lceil \log_2(k) \rceil + 1$, *where* $\lceil\ \rceil$ *is the ceiling function. Then there exists a "coimmanantal" family of k trees on n vertices. That is, there exist nonisomorphic trees* $T_1, T_2, \ldots, T_k$ *on n vertices such that, for every character* χ *of* S_n,

$$d_\chi(xI - L(T_1)) = d_\chi(xI - L(T_2)) = \cdots = d_\chi(xI - L(T_k)).$$

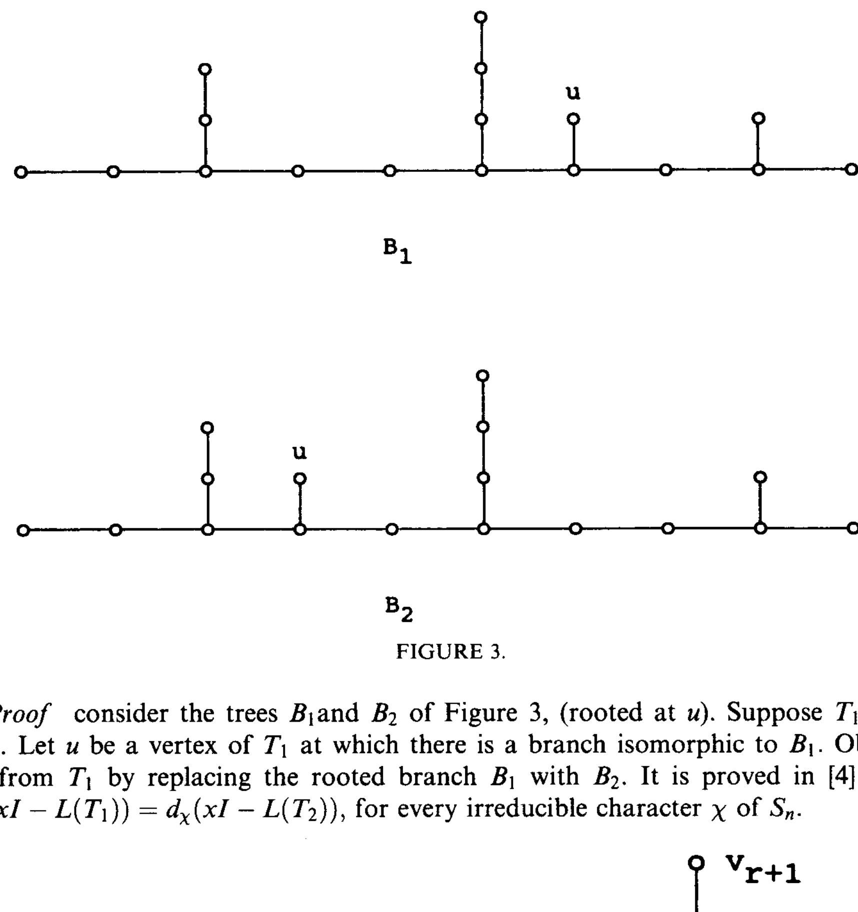


FIGURE 3.

Proof consider the trees $B_1$and B_2 of Figure 3, (rooted at u). Suppose T_1 is a tree. Let u be a vertex of T_1 at which there is a branch isomorphic to B_1. Obtain T_2 from T_1 by replacing the rooted branch B_1 with B_2. It is proved in [4] that $d_\chi(xI - L(T_1)) = d_\chi(xI - L(T_2))$, for every irreducible character χ of S_n.

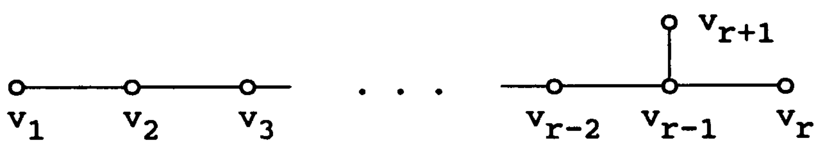


FIGURE 4.

Starting with the "skeleton" illustrated in Figure 4 attach rooted branches, isomorphic either to B_1or to B_2, at each of the vertices $v_1, v_2, \ldots, v_r$. Any one of the 2^r different trees obtained in this way has $17r + 1$ vertices. Moreover, from our remarks above, all 2^r of them share a complete set of immanantal polynomials. choosing $r = \lceil \log_2 k \rceil$ completes the proof. ■

If $n = 17\lceil \log_2(k) \rceil + 1$ then $k \sim 2^{(n-1)/17.}$ In particular, the size of these coimmanantal families grows exponentially with n.

5. UNIMODULAR CONGRUENCE

We have been treating (1) as a similarity condition, but it might just as well be viewed in terms of unimodular congruence. Recall that an n-by-n integer matrix U is *unimodular* if det $U = \pm 1$; the unimodular matrices are those integer matrices with integer inverses. Two matrices A and B are *unimodularly congruent* if there is a unimodular matrix U such that $B = U^tAU$.

Unimodular congruence for the edge version, $K(G) = Q(G)^tQ(G)$, is not very interesting. If G_1 and G_2 are connected, then $K(G_1)$ and $K(G_2)$ are unimodularly congruent if and only if they have the same numbers of vertices and edges [35]. On the other hand, if G_1 is 3-connected, then $L(G_1)$ and $L(G_2)$ are unimodularly congruent if and only if G_1 and G_2 are isomorphic [59].

There is no easy way to tell if two arbitrary integer matrices are unimodularly congruent. But, it is not known whether there is an easy way to tell if two Laplacian matrices are unimodularly congruent. Suppose we partition $L(G)$ in the form

$$L(G) = \begin{pmatrix} P & Q \\ Q^t & R \end{pmatrix}, \tag{4}$$

where P is k-by-k. If we denote by sum(P) the sum of the entries in the matrix P, then the equation, sum(P) + sum(Q) = 0, is a simple consequence of the fact that every row (and every column) of $L(G)$ sums to 0. So, sum(P) = –sum(Q) = –sum(Q^t) = sum(R). This common sum is the cardinality, $o(E_X)$, of the "edge cut" or "**coboundary**"

$$E_X = \{\{u, v\} \in E(G) : u \in X \text{ and } v \notin X\}, \tag{5}$$

of the set $X = \{v_1, v_2, \ldots, v_k\}$.

The minimum of $o(E_Y)$, over all nontrivial subsets Y of $V(G)$, is the edge connectivity of G. The maximum is the cardinality version of the max-cut of G, the subject of Section 8. In statistical physics, $o(E_Y)$ is associated with the energy of phase transitions ([10] and [60]).

From a number-theoretic point of view, $o(E_Y)$ is a value of the integer quadratic form,

$$uL(G)u^t = \sum_{\substack{i<j \\ \{v_i, v_j\} \in E}} (u_i - u_j)^2, \tag{6}$$

where $u = (u_1, u_2, \ldots, u_n)$; if $u_1 = \cdots = u_k = 1$ and $u_{k+1} = \cdots = u_n = 0$, then $uL(G)u^t = \text{sum}(P) = |\text{sum}(Q)| = o(E_X)$. Let $M(G)$ be the multiset $\{uL(G)u^t : u$ is a (0,1)-vector$\}$.

THEOREM 6 [60] *If $L(G_1)$ and $L(G_2)$ are unimodularly congruent, then the multisets $M(G_1)$ and $M(G_2)$ are identical.*

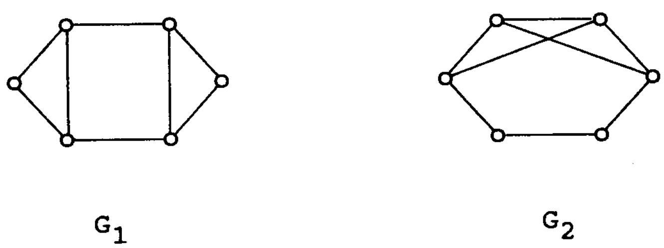


FIGURE 5.

Unfortunately, the converse of Theorem 6 is false. If G_1 and G_2 are the two graphs in Figure 5, then $M(G_1)$ and $M(G_2)$ are identical, but $L(G_1)$ and $L(G_2)$ are not even unimodularly equivalent, much less unimodularly congruent [60].

Some interesting work on unimodular equivalence of the unoriented vertex-edge incidence matrix is reported in [22]–[23].

6. OPTIMAL NUMBERINGS

It is natrual to identify vertex numberings of G with one-to-one functions $f : V(G) \to \{1, 2, \ldots, n\}$. For each such "numbering", define

$$b_f(G) = \underset{u,v \in E(G)}{\text{maximum}} \; | f(u) - f(v) | .$$

If $L(G)$ is the Laplacian matrix corresponding to the vertex numbering f, then the nonzero entries in row i of $L(G)$ lie in columns j satisfying

$$i - b_f(G) \le j \le i + b_f(G).$$

For computational purposes, it is frequently useful to choose a numbering f for which $b_f(G)$ is as small as possible. This leads to the *bandwidth* of G, namely,

$$bw(G) = \min_f b_f(G).$$

If, for example, G is connected, then $L(G)$ can be brought to tridiagonal form by a permutation similarity if and only if $bw(G) \le 1$, if and only if G is the path P_n.

THEOREM 7 ([29],[42]) *Let G be a graph with at least one edge. Then*

$$1 + bw(G) \ge \left(\frac{a(G)}{2\lambda_1(G) - a(G)} \right) n,$$

with equality, e.g., for $G = K_n$.

These notions can be generalized in several ways. For example, one could assign a "Weight" to each edge of G (see Section 8). In the definition of $b_f(G)$, one might replace the maximum with a sum or, even more generally, with a "p-sum" [28]. Another interesting idea is to generalize vertex numberings: If H is another graph, define

$$bw_H(G) = \min_f \max_{E(G)} d_H(f(v_i), f(v_j)), \tag{7}$$

where the minimum is over all one-to-one functions $f : V(G) \to V(H)$, the maximum is over all $\{v_i, v_j\} \in E(G)$, and $d_H(f(v_i), f(v_j))$ is the length of a shortest path in H from $f(v_i)$ to $f(v_j)$. If $H = P_n$, then $bw_H(G) = bw(G)$. More generally, H is called a "host graph" [6]. (The dual of $bw_H(G)$, namely

$$\text{sep}(G, H) = \max_f \min_{E(G)} d_H(f(v_i), f(v_j)), \tag{8}$$

has been called the "separation of G into H" [36].)

Suppose $G = (V, E)$ is a graph. Let X be a subset of V of cardinality $o(X)$. The **boundary** of X is

$$B_X = \{v \in X : \{u, v\} \in E \text{ for some } u \notin X\}$$

(Some authors refer to $B_{V/X}$ as the boundary of X.) The next result is an immediate consequence of the definitions.

THEOREM 8 ([6]) *Let G be a graph. Then for any host graph H,*

$$bw_H(G) \geq \max_k \min_{o(X)=k} o(B_X).$$

Roughly speaking, a graph has good "expansion" if $o(B_{V/X})$ is large relative to $o(X)$, for every sufficiently small set, X, of vertices. Exploitation of the connection between good expansion and large $a(G)$ has led to the construction of so called "Ramanujan graphs", a family of graphs with good expansion. (See [30], [31], [39], [44], [51] and the references in [32].)

7. SPECTRAL BISECTION

Let G be a graph and H a second graph. Equations (7) and (8) involve one-to-one functions $f : V(G) \to V(H)$. Other applications involve not one-to-one, but onto functions. The problem is already interesting when $H = K_2$.

If the vertices of G are partitioned into two nonempty subsets, X and $Y = V(G)/X$, then the coboundary $E_X = E_Y$. A *bisection* of G is one of these bipartitions in which $o(X) = o(Y)$ or $o(X) = o(Y) + 1$. The graph bisection problem is to find a bisection for which $o(E_X)$ is as small as possible.

When n is even, there is a one-to-one correspondence between bisections of G and $(+1, -1)$-vectors $u = (u_1, u_2, \ldots, u_n)$ that satisfy

$$u_1 + u_2 + \cdots + u_n = 0. \tag{9}$$

Just set $X = \{v_i : u_i = +1\}$ and $Y = \{v_i : u_i = -1\}$. If $\{v_i, v_j\} \in E = E(G)$, then $(u_i - u_j)^2$ is 4 if v_i and v_j are in different sets of the partition, and 0 if they are in the same set. Thus,

$$o(E_X) = \frac{1}{4} \sum_{\{v_i, v_j\} \in E} (u_i - u_j)^2 = \frac{1}{4} uL(G)u^t \geq na(G)/4, \tag{10}$$

with equality if and only if u is a Fiedler vector. While graph eigenvalues have long been recognized as an important tool in combinatorial optimization (see, e.g., [3], [16], and [42]), the most exciting recent developments involve eigen*vectors*. Motivated by the case of equality in (10), computer scientists have used Fiedler vectors in the design of algorithms for the efficient use of parallel processors. (see [1], [2], [18], [24]–[27], [53], and [58].)

8. WEIGHTED GRA

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8. WEIGHTED GRAPHS AND MAX-CUTS

A **C-edge weighted graph**, G_C ("weighted graph" for short), is a pair consisting of a graph G and a postive real valued function C of its edges. The function C is most conveniently described as an n-by-n symmetric, nonnegative matrix $C = (c_{ij})$ with the property $c_{ij} > 0$, if and only if $\{v_i, v_j\} \in E = E(G)$. In fact, for $e = \{v_i, v_j\} \in E$, we will write $w(e) = c_{ij}$, calling it the *weight* of edge e. (If $w(e) = 1$ for all $e \in E$, then $C = A(G)$, the adjacency matrix.) The max-cut of G_C is

$$MC(G_C) = \max \sum_{e \in E_X} w(e),$$

where the maximum is over all proper subsets X of $V(G)$, and E_X is the coboundary of X.

With r_i denoting the i-th row sum of C, define $L(G_C) = \text{diag}\,(r_1, r_2, \ldots, r_n) - C$. Then the symmetric matrix $L(G_C)$ is uniquely determined by its quadratic form:

$$uL(G_C)u^t = \sum c_{ij}(u_i - u_j)^2,$$

where the sum is over the pairs $i < j$ for which $\{v_i, v_j\} \in E$. Note that each row and each column of $L(G_C)$ sums to zero. Adapting (4) to the weighted case, we partition

$$L(G_C) = \begin{pmatrix} P & Q \\ Q^t & R \end{pmatrix},$$

where P is k-by-k. Observe that the property $\text{sum}(P) = -\text{sum}(Q) = -\text{sum}(Q^t) = \text{sum}(R)$ remains valid. If the rows and columns of the matrix P correspond to a set X of vertices, we will call this common sum the *weight* of its coboundary E_X, i.e.,

$$w(E_X) = \text{sum}(P) = \mid \text{sum}(Q) \mid = \sum_{e \in E_X} w(e). \tag{11}$$

so,

$$MC(G_C) = \max_{X \subset V} w(E_X).$$

Let $\lambda_1(G_C) \geq \lambda_2(G_C) \geq \cdots \geq \lambda_{n-1}(G_C) = a(G_C) \geq 0 = \lambda_n(G_C)$ be the eigenvalues of $L(G_C)$. It follows from the same arguments used to prove (10) that

$$\frac{1}{4} na(G_C) \leq MC(G_C). \tag{12}$$

A lower bound for $MC(G_C)$ in terms of the eigenvalues of C was obtained in [33]. Poljak and Turzík gave a polynomial time algorithm that produces a subset $X \subset V(G)$ satisfying

$$w(E_X) \geq \frac{1}{2} \sum_{e \in E(G)} w(e) + \frac{1}{4} \min \sum_{e \in E(T)} w(e), \tag{13}$$

where the minimum is over the spanning trees T of G [47]. (In the unweighted case, (13) produces $MC(G) \geq m/2 + n/4$, a result of Edwards [17].)

The following companion to (12) appeared in [41].

THEOREM 9 *Let G_c be an edge weighted graph. Then*

$$MC(G_C) \leq \frac{1}{4} n\lambda_1(G_C). \tag{14}$$

To give an example, the 60-carbon atom fullerene, C_{60}, is in the shape of a truncated icosahedron. If G is this "soccer ball" graph, then $MC(G) = 78 < 84.27 = 60\lambda_1/4$ [9].

A "correcting vector" is any real n-tuple $u = (u_1, u_2, \ldots, u_n)$ satisfying $u_1 + u_2 + \cdots + u_n = 0$. (Correcting vectors seem first to have been introduced, although not by that name, in [16]) For any correcting vector u, let $f_C(u)$ be the largest eigenvalue of $L(G_C) + \text{diag}\,(u)$. Then the convex function f_C attains a unique minimum at an "optimum correcting vector" u_o. If $\varphi(G_C) = nf_C(u_o)/4$, then

$$MC(G_C) \leq \varphi(G_C), \tag{15}$$

with equality, e.g., for weighted bipartite graphs. While finding a set of vertices X that satisfies $W(E_X) = MC(G_C)$ is an NP-complete problem, $\varphi(G_C)$ can be computed in polynomial time. Moreover, $\varphi(G_C)/MC(G_C) \leq 1.131$ for weighted planar graphs. It remains an open problem whether 1.131 bounds this ratio for all weighted graphs. (See [12]–[14] and [44]–[45].) An analog of (15) for the bisection width appeared in [46]. (Also see [25] and [49].)

The *edge-density* [39] of vertex set X is

$$\rho_C(X) = \sum_{e\in E_x} \frac{w(e)}{o(X)(n - o(X))}.$$

The maximum value of $\rho_C(X)$, as X ranges over the subsets of $V(G)$, is a variation on $MC(G_C)$. The *averaged minimal cut* of G_C is

$$\gamma(G_C) = \min_{0<o(X)<n} \rho_C(X),$$

It was shown in [19] and [39] that

$$\gamma(G_C) \geq a(G_C)/n.$$

(This inequality is consistent with the proposed use of $n/a(G)$ as an alternative to the chemical Wiener Index.) The following companion of (16) was proved in [19]:

$$\gamma(G_C) \leq a(G_C)/2.$$

other bounds can be found in [39].

9. RELATED TOPICS

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9. RELATED TOPICS

Laplacian eigenvalues of random graphs were studied by Juvan and Mohar [29]. Quasi-random graphs, whose nonzero eigenvalues are close to the average vertex degree, were investigated by Chung, *et al.* [8].

Sufficient conditions for the second eigenvalue of the Laplacian differential operator (in a Dirichlet boundary value problem) to have multiplicity at most 2 are given by Friedlander [20].

A necessary condition in terms of Laplacian eigenvalues for a graph to have a Hamiltonian circuit was obtained by Mohar [40].

The Laplacian spectra of infinite graphs has been studied by several authors [38], [43], [54].

While it is a (singular) M-matrix, Styan and Subak-Sharpe have shown that the Moore-Penrose generalized inverse of $L(G_C)$ needn't be nonnegative [55].

A "homogeneous graph" is a graph together with a (vertex-) transitive group of its automorphisms. Laplacians in the context of homogeneous graphs are discussed in [9].

Laplacian matrices of directed graphs date back at least to Tutte's variation of the Matrix-Tree Theorem [56]–[57]. More recent work appeared, e.g., in [5] and [15].

The possibility of extending Laplacian matrices to hypergraphs is discussed in [39].

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