Wavelets and Approximation

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- Digitized Image: Pixel values: $p_I := \frac{1}{|I|} \int_I f(x)\,dx$

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- Deblurring: Given a blurred image $P(f)$ approximate f

Goals of This Talk

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- Future Directions

Univariate Approximation

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- $\|f\|_{L_\infty} := \sup_{x\in\Omega} |f(x)|, \quad p = \infty$

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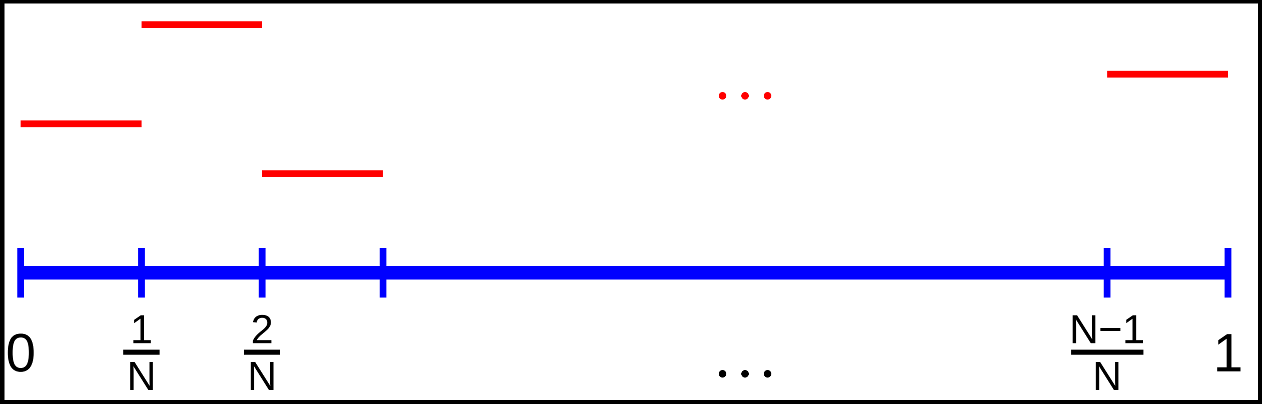
Typical function in $\mathcal{S}_n$



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- Stop when $\mathcal{B}_\epsilon = \emptyset$, $\mathcal{P}_\epsilon := \mathcal{G}_\epsilon$, $N_\epsilon := \#(\mathcal{P}_\epsilon)$

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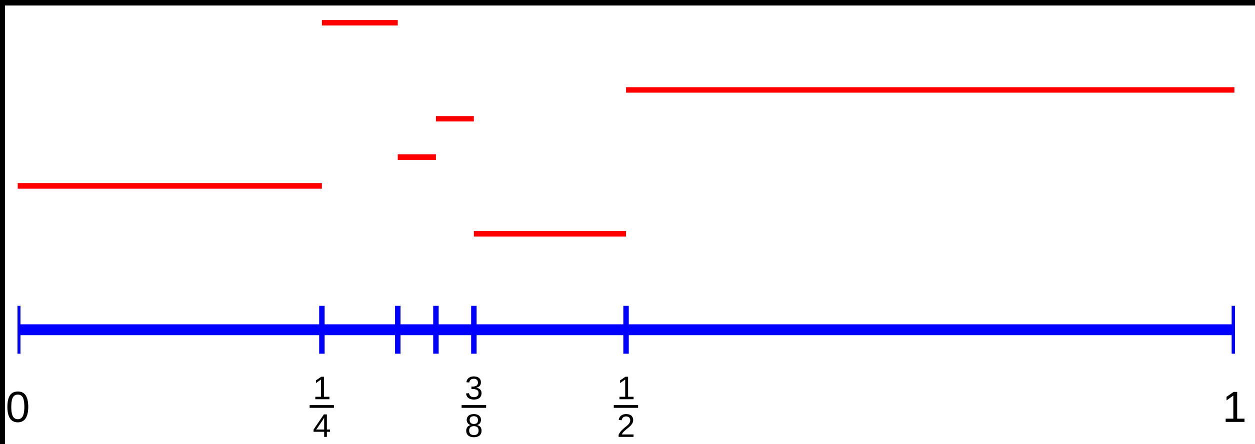
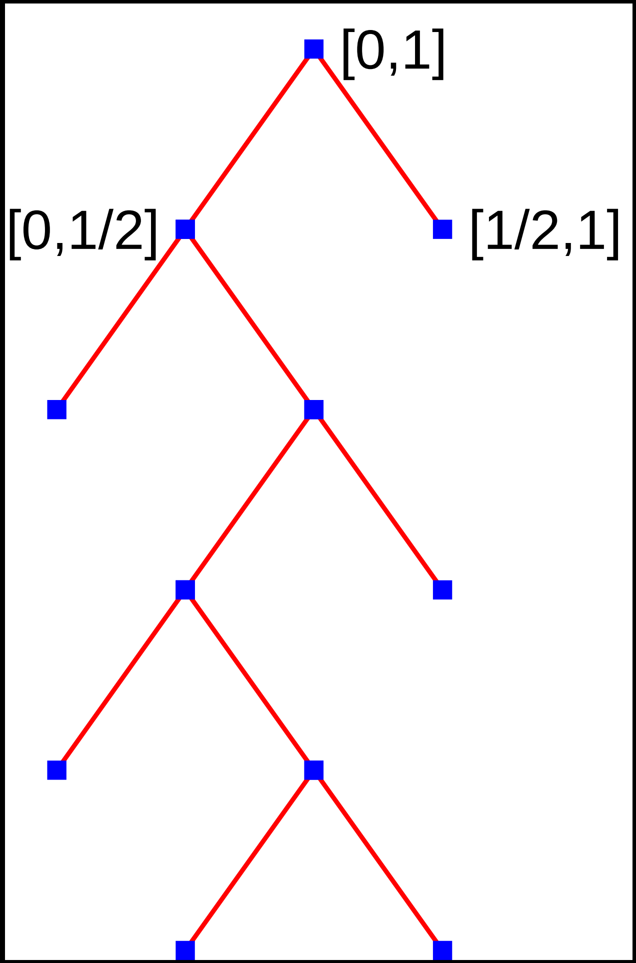
Adaptively generated partition

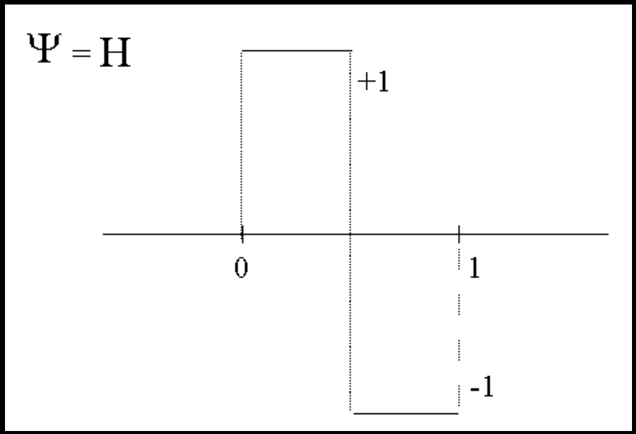


Tree associated to adaptive partition



Wavelets: Haar Wavelet

$$H(x) := \begin{cases} -1, & x \in [0, 1/2) \\ +1, & x \in [1/2, 1] \end{cases},$$



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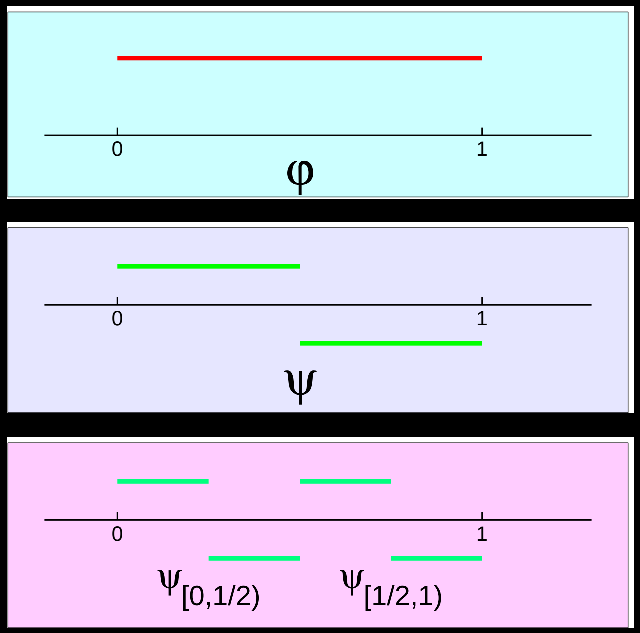
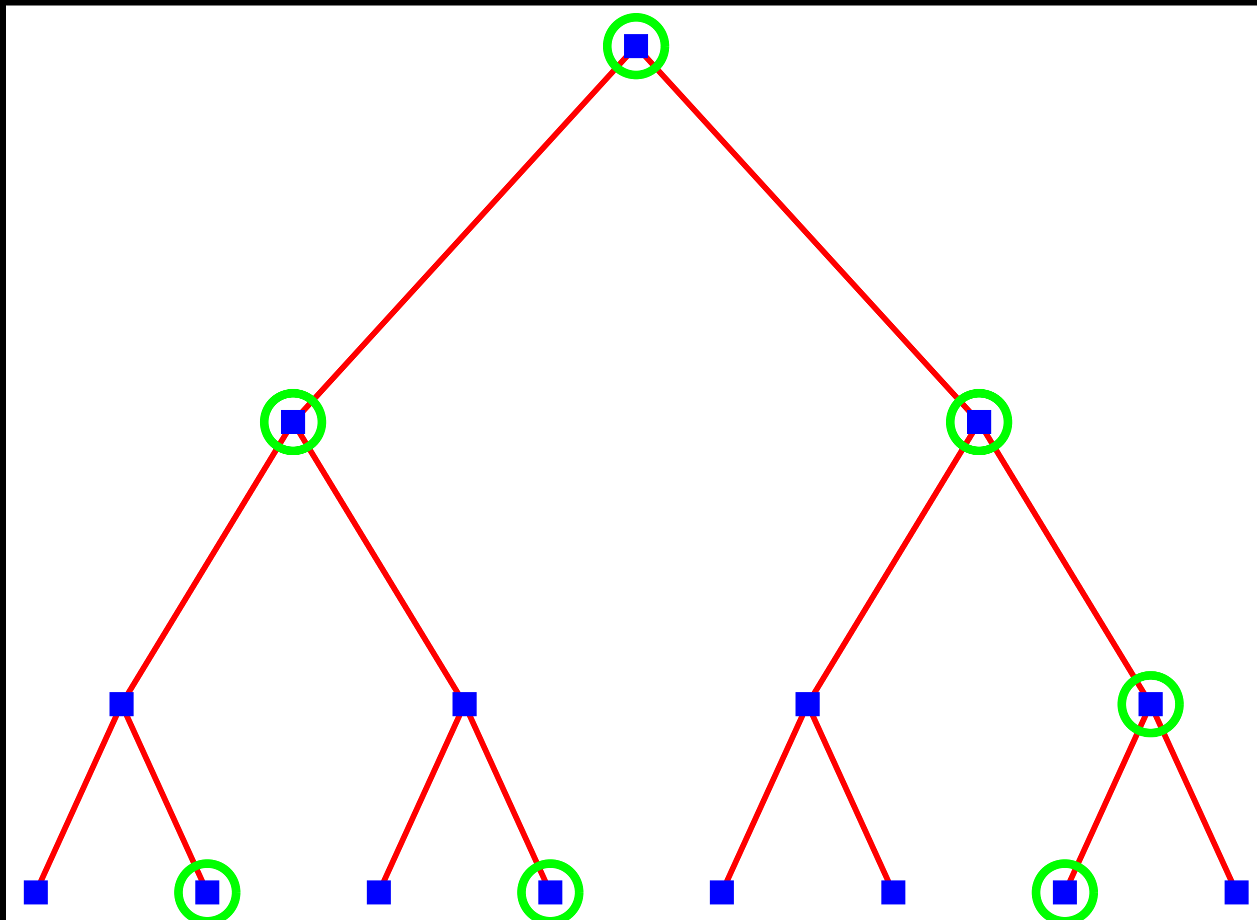
- $\mathcal{D}_+ := \{I \in \mathcal{D} : |I| \leq 1\}$
- $\{\chi_{[0,1]}\} \cup \{H_I\}_{I \in \mathcal{D}_+}$ is a complete orthonormal system in $L_2[0,1]$

Haar Basis



Wavelet tree



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- This is linear approximation because X_n is a linear space

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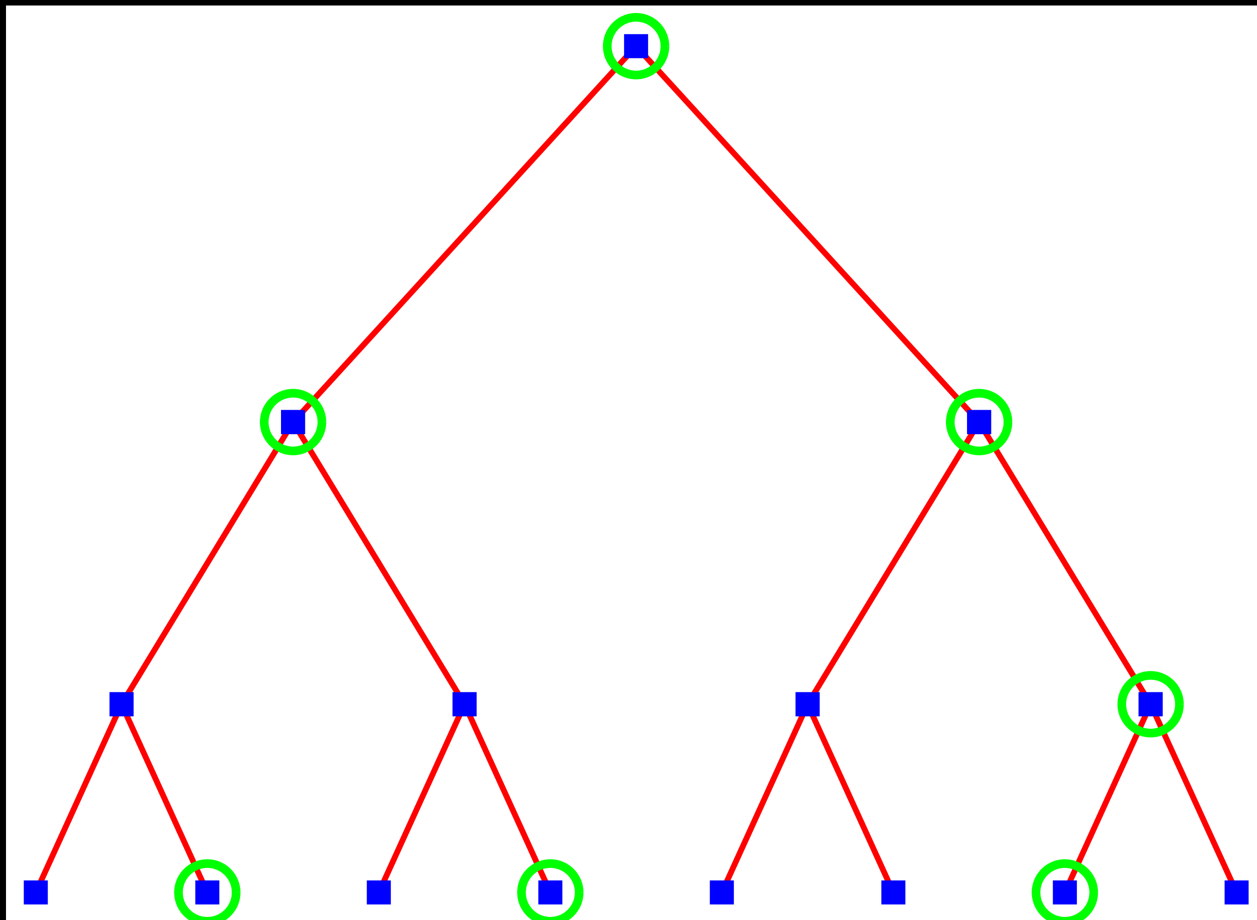
n-term approximation



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Comparison of these different types of approximation

- Approximation classes: $\alpha > 0$ define $\mathcal{A}^\alpha(L_p,\ linear\ splines)$ as the set of all $f \in L_p[0,1]$ such that

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- $\mathcal{A}^s_q(L_p)$ finer scaling: same approximation order s

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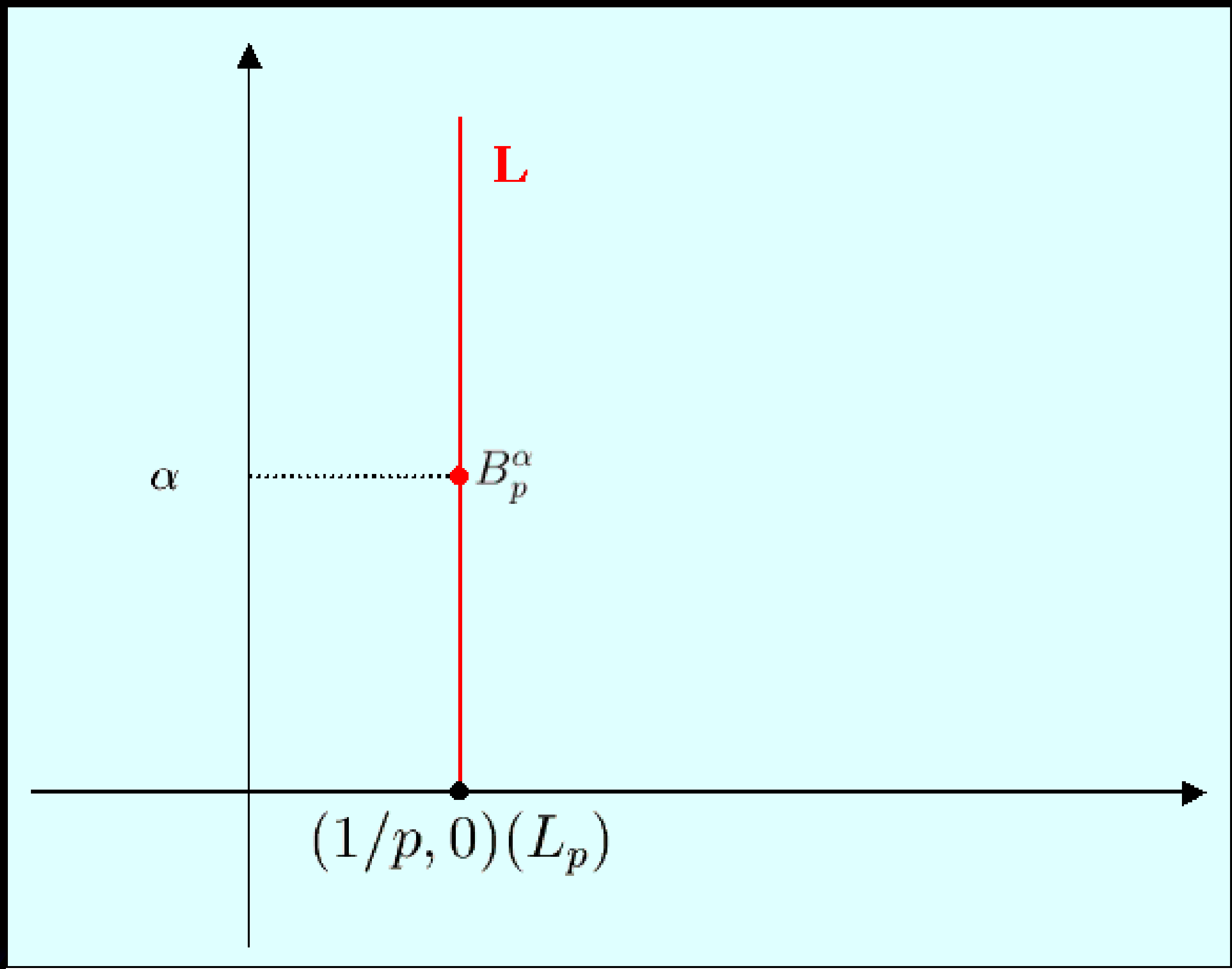
Linear Approximation: $\mathcal{A}^s_\infty(L_p)$ Besov space of smoothness s



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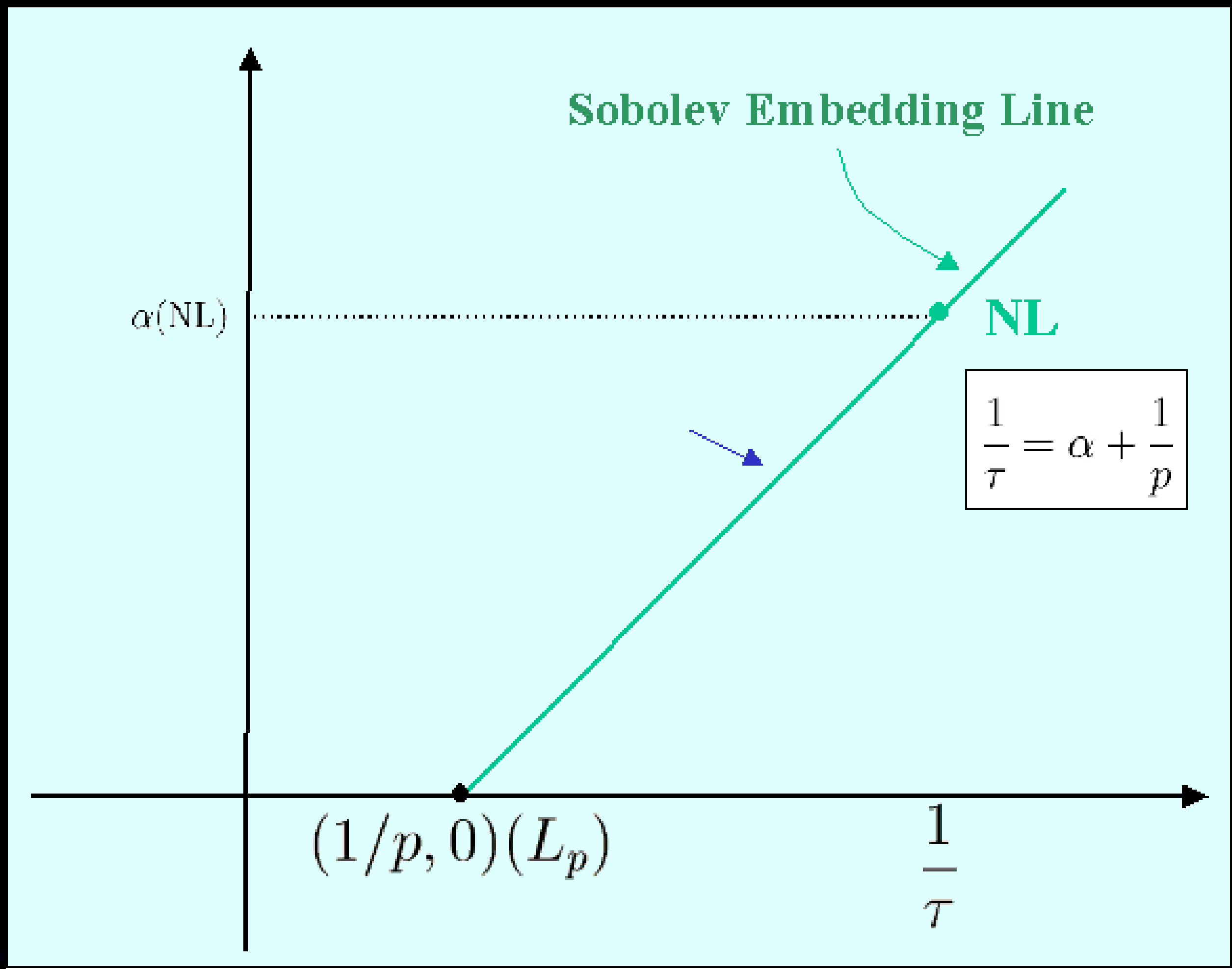
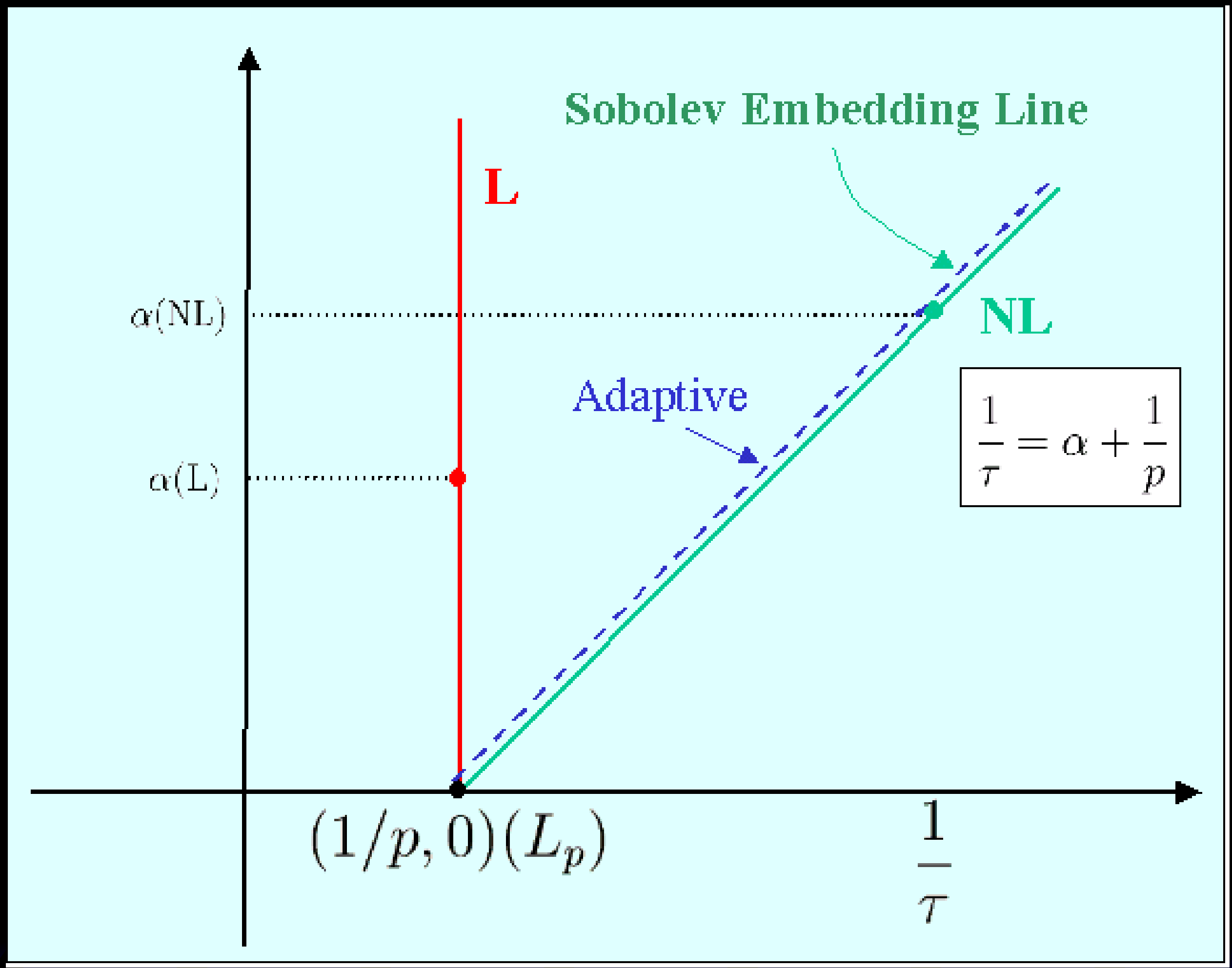
- Petrushev, DeVore-Popov (splines); DeVore-Jawerth-Popov (wavelets)

Approximation class for n-term approximation



Adaptive approximation



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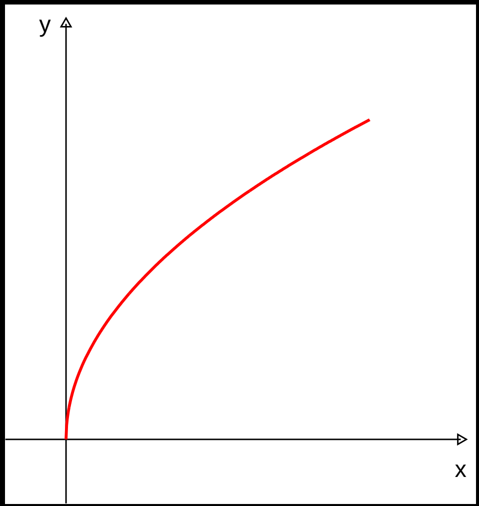
- Tree approximation $f' \in L_p$ for some $p > 1$

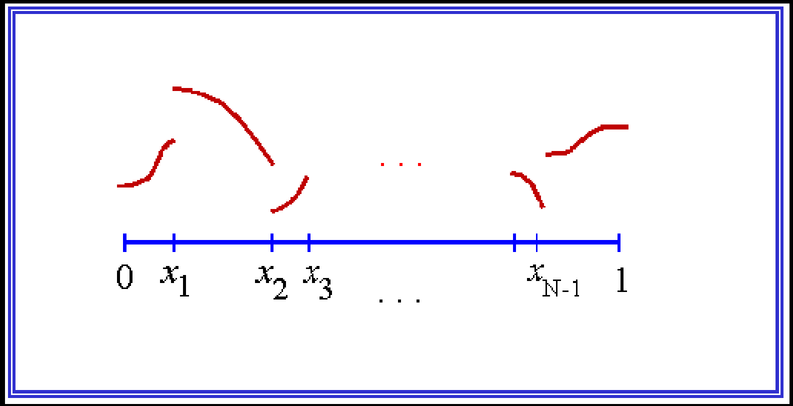
Example: $f(x) = x^\alpha$, $\alpha > -1/p$



$$E_n(f)_p \approx Cn^{-(\alpha+1/p)} \quad \sigma_n(f)_p \leq Cn^{-1}$$

Break points/ wavelets concentrate near singularity at 0

Example: piecewise smooth



$$E_n(f)_p \geq Cn^{-1/p} \quad \sigma_n(f)_p \leq Cn^{-1}$$

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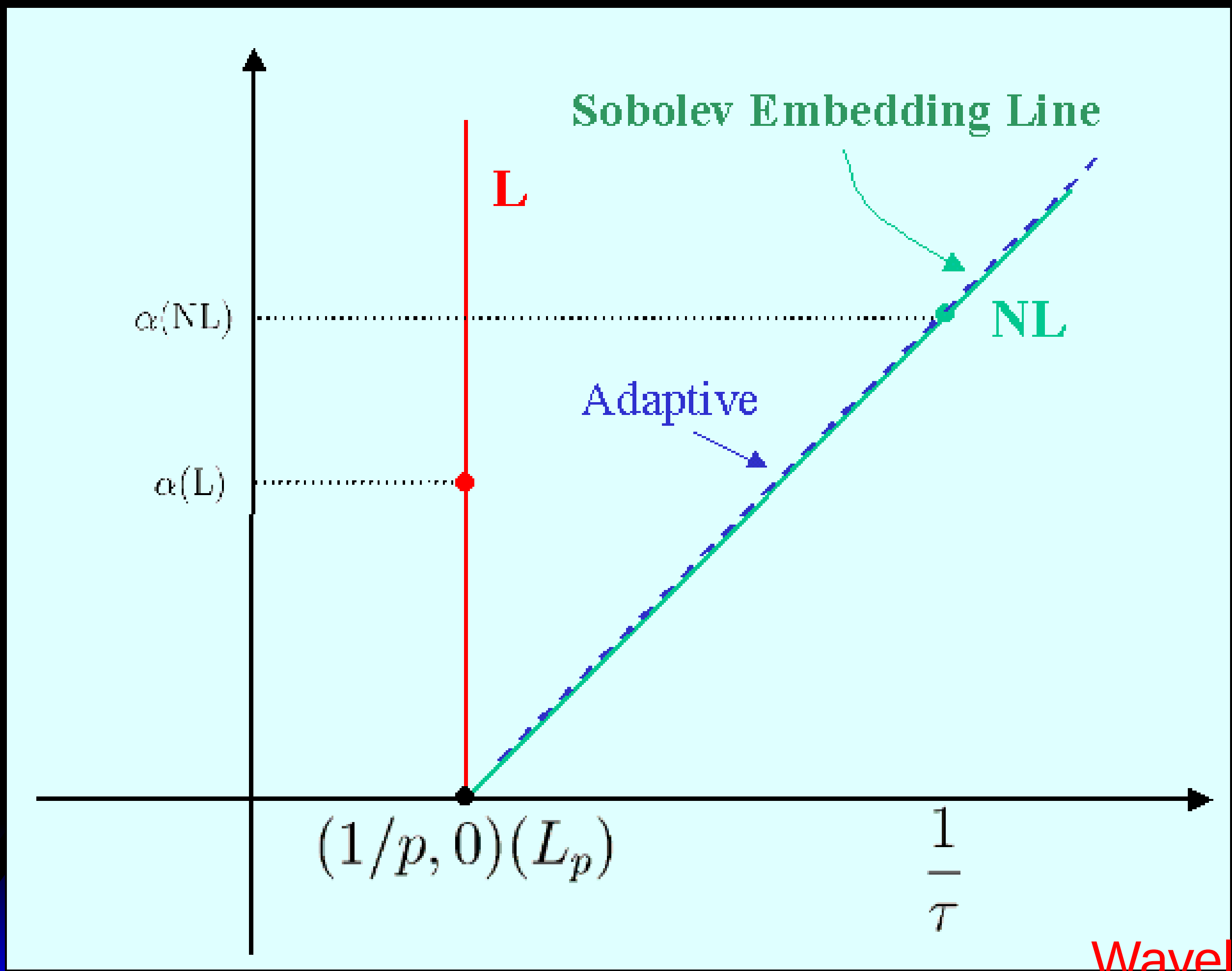
- $\mathcal{A}^{\alpha/d}_{\tau}(wavelets, L_p) = B^{\alpha}_{\tau}(L_p)$, $\frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}$

Multivariate Nonlinear Approximation Classes

Now $\frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}$



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- Wavelet basis is greedy
- Simple thresholding is a near best strategy

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- Other correlations in positions of big wavelet coefficients

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- Cohen-Dahmen-Daubechies-DeVore (Cohen-Daubechies-Gulleryuz-Orchard) If $f \in B^\alpha_\tau(L_\tau)$, $\frac{1}{\tau} < \frac{\alpha}{2} + \frac{1}{p}$ then f can be approximated to accuracy $C\|f\|_{B^\alpha_\tau(L_\tau)} n^{-\alpha/2}$ and the approximant can be encoded with n bits

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- Bits to quantize coefficients
- Bits to encode positions of wavelets
- number of quantization bits determined by decay of rearranged wavelet coefficients: the α in Besov regularity
- If n coefficients are taken then need at least (and at most) n bits to achieve distortion $n^{-\alpha}$

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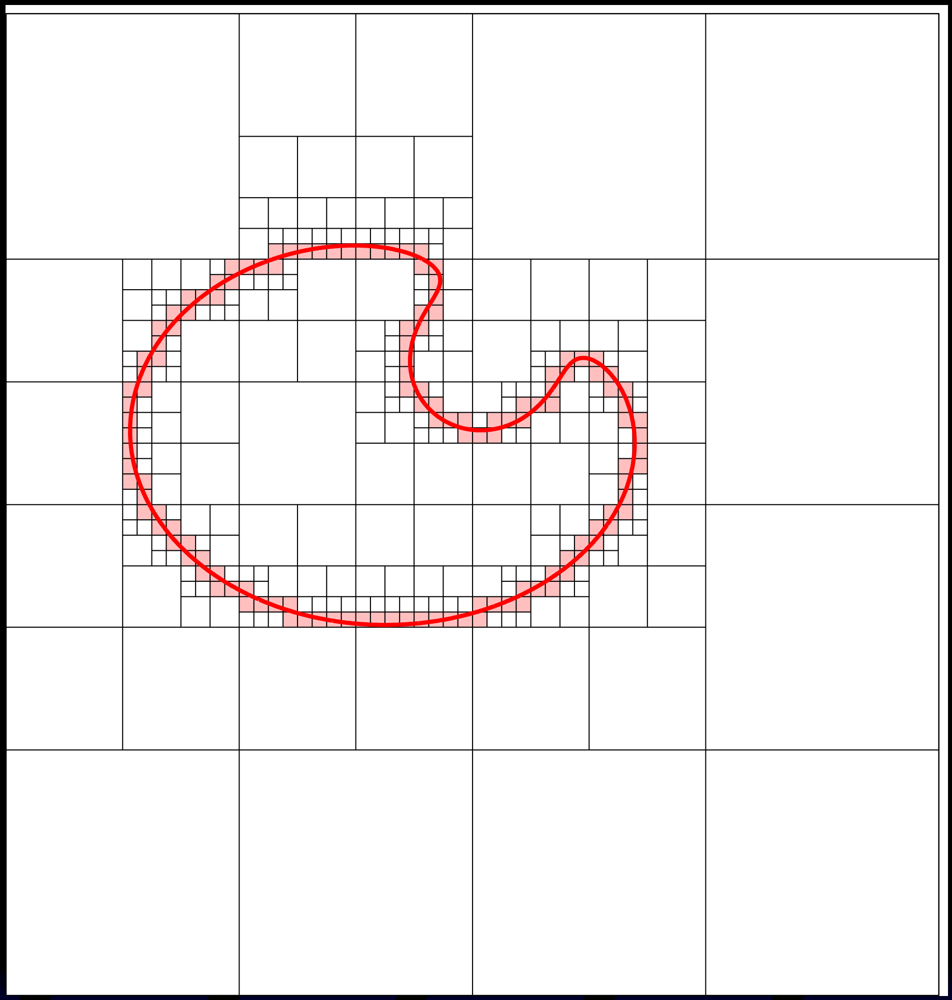
Horizon approximation



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Example: Wedgelets

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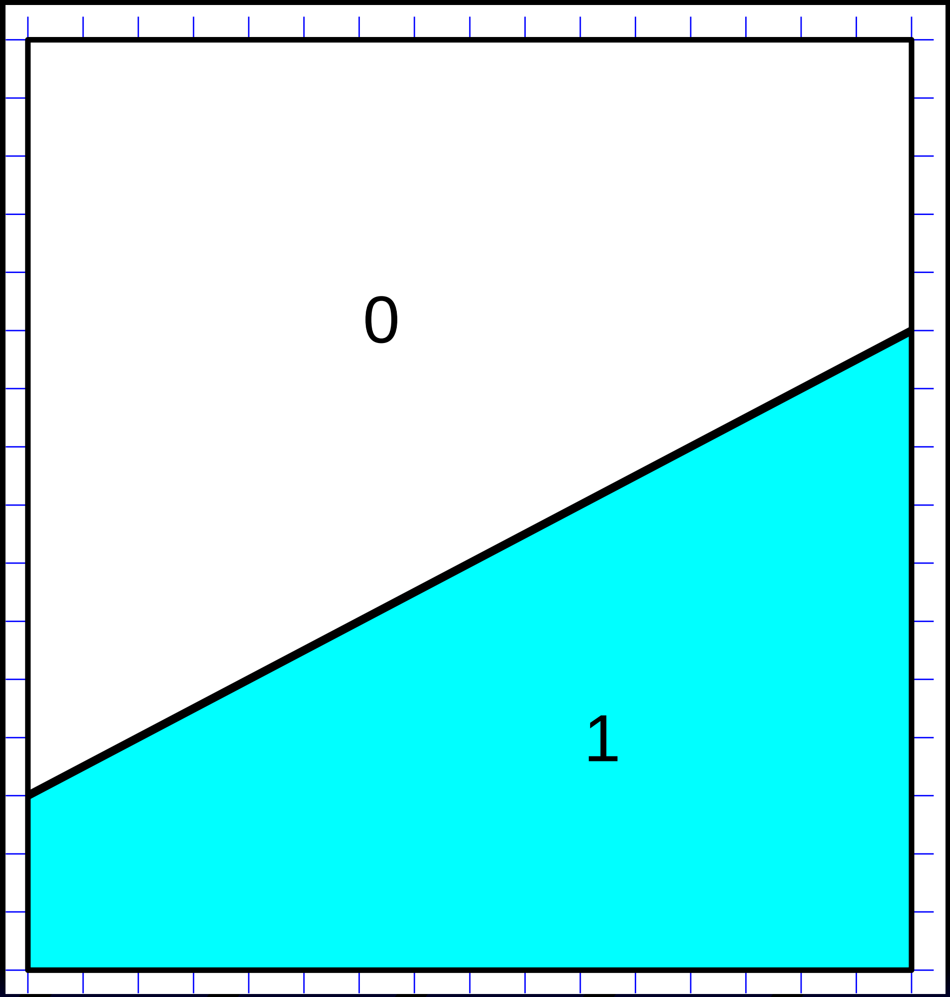
Picture of a wedgelet



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- By comparison wavelets give error $n^{-1/2}$: $f \in \mathrm{BV} \subset \mathcal{A}^1(\mathrm{nonlinear}, \mathrm{L}_2)$. Fourier gives error $n^{-1/4}$: $f \in B^{1/2}_{\infty}(L_2) = \mathcal{A}^{1/2}(Fourier, L_2)$

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- Greedy Algorithm converges (L. Jones)

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- Characterize approximation classes for this type of approximation?

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- Ornate wavelet tree with label on how wavelet coefficients to be computed
- interior nodes compute wavelet coefficients in standard way

BRW Continued

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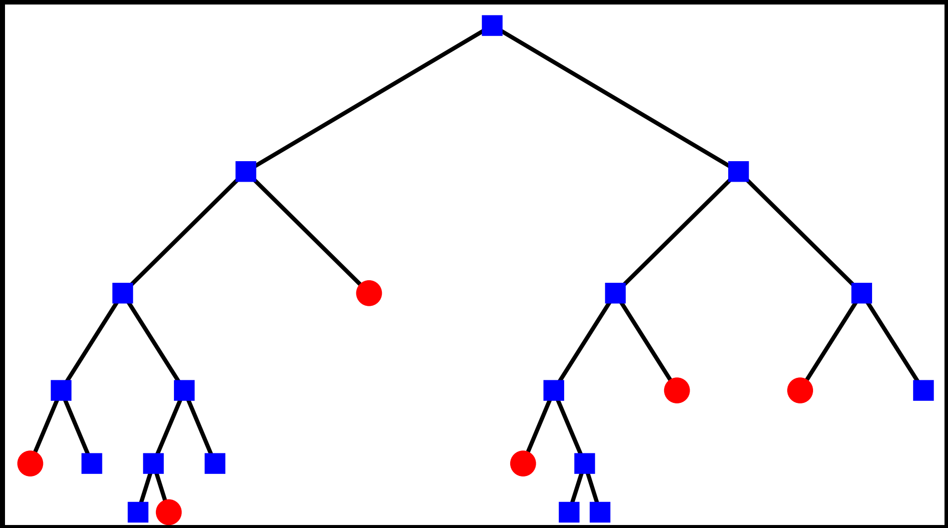
- If leave is ornated with wedgelet, all coefficients below this node are computed as wavelet coefficients of that wedgelet (wedgeprint)

Wedgelet-wavelet tree

red = Wedgelets   blue = Wavelets



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- Similarly have an error $e(n, I)$ for enconding the wavelet coefficient for interior nodes using n nodes

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- Can we dynamically find best tree and best bit allocation