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HILBERT DISTANCES AND POSITIVE DEFINITE FUNCTIONS

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I. HILBERT DISTANCES

If $\mathfrak{S}$ is any topological space then a *continuous* complex function of two variables $f(P, Q)$ on $\mathfrak{S} \times \mathfrak{S}$ shall belong to class $\mathfrak{P}$ (positive definite functions) if

1) $f(P, Q) = \overline{f(Q, P)}$
2) $f(P, P) = f(Q, Q)$

and

3′) for any integer n, any points $P_1, \cdots, P_n$ and any complex numbers $\rho_1, \cdots, \rho_n$ the relation

$$(1) \qquad \sum_{i,j=1}^{n} f(P_i, P_j)\rho_i\bar{\rho}_j \geqq 0,$$

holds.

The wider class $\mathfrak{P}_0$ shall consist of all those functions for which, other assumptions being unchanged, the decisive inequality (1) is assumed to hold only subject to the restriction

$$(2) \qquad \sum_{i=1}^{n} \rho_i = 0.$$

Just how much wider the class $\mathfrak{P}_0$ actually is will be established under further assumptions. The *real* classes $\mathfrak{P}$ and $\mathfrak{P}_0$ shall consist of all real-valued functions of these classes.

We will call a function $\rho(P, Q)$ a distance function if $\rho(P, Q) = \rho(Q, P) \geqq 0$, $\rho(P, P) = 0$ and $\rho(P, Q) + \rho(Q, R) \geqq \rho(P, R)$. We call $\rho(P, Q)$ a *proper* distance if $P \neq Q$ implies $\rho(P, Q) > 0$. Introducing the real Hilbert space $\mathfrak{H}$ of sequences $\{x_n\}$, $\{y_n\}$ with the customary distance $(\sum_n (x_n - y_n)^2)^{1/2}$, we now call $\rho(P, Q)$ a *Hilbert distance* on $\mathfrak{S}$ if it is possible to map $\mathfrak{S}$ into $\mathfrak{H}$ in such a way that the value of $\rho(P, Q)$ shall be equal to the value of the latter distance for the transforms of P and Q. The following decisive criterion has been established by K. Menger and I. J. Schoenberg:[1] *if $\mathfrak{S}$ is separable, then $\rho(P, Q)$ is a Hilbert distance if and only if* $-\rho(P, Q)^2$ belongs to $\mathfrak{P}_0$. Also, Schoenberg discovered the fact that a function $f(P, Q)$ belongs to $\mathfrak{P}_0$ if and only if $e^{\lambda f(P,Q)}$ belongs to $\mathfrak{P}$ for each

[1] See I. J. Schoenberg, *Metric spaces and positive definite functions*, Trans. Amer. Math. Soc. 44 (1938), 522-536.

[2] Positive definite functions on non-compact but commutative groups have been analyzed recently in A. Powzner, Doklady U.S.S.R., 28 (1940) 294-295, and D. Raikow, Doklady U.S.S.R., 27 (1940), 324-327 and 28 (1940), 296-300.

positive λ, and he analyzed Hilbert distances on open euclidean and Hilbert spaces.

In the present note we will investigate Hilbert distances for some types of *general* but *compact* separable spaces. We will obtain new relations between the classes $\mathfrak{P}$ and $\mathfrak{P}_0$ and structural criteria in terms of expansions in orthogonal systems.[2]

II. Compact Spaces with Bounded Measure

Since $f(P, Q)$ is continuous, if $\mathfrak{S}$ is a finite interval, say, then condition 3′) for $\mathfrak{P}$ can be replaced by the following condition:

3) for each integrable $\rho(P)$ the relation

$$\int\int f(P, Q)\rho(P)\rho(Q)\, dP\, dQ \geqq 0 \tag{3}$$

holds;[3] and for $\mathfrak{P}_0$ relation (3) holds subject to the restriction

$$\int \rho(P)\, dP = 0. \tag{4}$$

This replacement of sums by integrals is admissible for more general spaces.

Lemma 1. *Conditions* 3) *and* 3′) *are equivalent if* α), $\mathfrak{S}$ *is both separable and compact*, β) *there exists a Lebesque measure* dP *on* $\mathfrak{S}$ *for which* γ) *all Borel sets are measurable*, δ) *every open set has non-vanishing measure and* ϵ) *the total space has measure* 1.

Proof: We first observe that $\mathfrak{S}$ and $\mathfrak{S} \times \mathfrak{S}$ are each bi-compact, and that therefore every continuous function $f(P, Q)$ is uniformly continuous and bounded. In the proof M will be independent of ϵ. We will give the proof of the lemma for functions of $\mathfrak{P}_0$, for functions of $\mathfrak{P}$ it is even simpler. Let $f(P, Q)$ satisfy 3′) and let $\rho(P)$ be any function satisfying (4). Given $\epsilon > 0$ we partition $\mathfrak{S}$ into a finite number of Borel sets $\mathfrak{S}_1, \cdots, \mathfrak{S}_n$ such that $f(P, Q)$ oscillates by less than ϵ on each $\mathfrak{S}_i \times \mathfrak{S}_j$. Choosing an arbitrary point P_i in $\mathfrak{S}_i$ and putting $\rho_i = \int_{\mathfrak{S}_i} \rho(P)\, dP$ we obviously have (2) and

$$\left| \sum_{i,j=1}^{n} f(P_i, P_j)\rho_i\bar{\rho}_j - \int\int f(P, Q)\rho(P)\overline{\rho(Q)}\, dP\, dQ \right| < \epsilon M. \tag{5}$$

Letting $\epsilon \to 0$ we obtain (3). Conversely, if $f(P, Q)$ satisfies 3), and distinct points $\{P_i\}$ and numbers $\{\rho_i\}$ are given, and if (2) holds, then for $\epsilon > 0$, we pick a set of disjoint neighborhoods $\mathfrak{S}_i$ of P_i, such that $f(P, Q)$ oscillates by less than ϵ on each $\mathfrak{S}_i \times \mathfrak{S}_j$. Putting $\rho(P) = \dfrac{\rho_i}{\text{measure } \mathfrak{S}_i}$ for $P \in \mathfrak{S}_i$, and $\rho(P) = 0$ for $P \in \mathfrak{S} - (\mathfrak{S}_1 + \cdots + \mathfrak{S}_n)$, we have relation (2) and (5), and, by a limit, relation (1).

[3] Whenever the range of integration is not indicated it is the total set to which the variable refers.

In what follows we will always assume that $\mathfrak{S}$ satisfies the assumptions of the lemma.

THEOREM 1. *If $f(P, Q)$ belongs to $\mathfrak{P}$, if $\varphi(P)$ is continuous and c is a real constant, then*

$$g(P, Q) = f(P, Q) + \varphi(P) + \overline{\varphi(Q)} - c \tag{6}$$

belongs to $\mathfrak{P}_0$ except for property 2).

Conversely, if $g(P, Q)$ belongs to $\mathfrak{P}_0$, then there exists a continuous function $\varphi(P)$ and a real constant c such that

$$f(P, Q) = g(P, Q) - \varphi(P) - \overline{\varphi(Q)} + c \tag{7}$$

belongs to $\mathfrak{P}$, except for property 2), as for instance

$$\varphi(P) = \int g(P, Q)\, dQ \tag{8}$$

$$c = \int \varphi(P)\, dP = \iint g(P, Q)\, dP\, dQ. \tag{9}$$

PROOF: The first part follows from the fact that (4) implies

$$\iint \varphi(P)\rho(P)\overline{\rho(Q)}\, dP\, dQ = \int \varphi(P)\rho(P)\, dP \cdot \int \overline{\rho(Q)}\, dQ = 0; \tag{10}$$

the second part follows from the fact that (7), (8), (9) and $\lambda = \int \rho(P)\, dP$ implies

$$\iint f(P, Q)\rho(P)\overline{\rho(Q)}\, dP\, dQ = \iint g(P, Q)(\rho(P) - \lambda)\overline{(\rho(Q) - \lambda)}\, dP\, dQ.$$

We call a function $f(P, Q)$ measure invariant if $\int f(P, Q)\, dQ$ is independent of P. This leads to

THEOREM 2. *A function $g(P, Q)$ belongs to the measure invariant class $\mathfrak{P}_0$, if and only if it can be written in the form*

$$f(P, Q) - f(\Phi, \Phi) \tag{11}$$

where $f(P, Q)$ belongs to the measure invariant class $\mathfrak{P}$ and Φ is any point of $\mathfrak{S}$.

We now assume the existence of a fixed transitive group $\mathfrak{C}$ of continuous transformations of $\mathfrak{S}$ into itself, and we assume that our measure is invariant under the operations of $\mathfrak{C}$. We call a function $f(P, Q)$ group invariant if for all elements s of $\mathfrak{C}$, $f(sP, sQ) = f(P, Q)$, where sP is the transform of the point P under the transformation s. In particular $f(sP, sP) = f(P, P)$. Now if P and Q are any two points, since our group is transitive there exists an element s such that $Q = sP$. Thus from now onward property 2) of $\mathfrak{P}$ will be automatically fulfilled for any function $f(P, Q)$ which is group invariant. In order to emphasize the independence of $f(P, P)$ of the special point P we will designate it by $f(\Phi, \Phi)$ where Φ is any fixed point which is chosen appropriately. For instance if $\mathfrak{S}$ itself is a group it is customary to take for Φ the identity element of the group.

If $f(P, Q)$ is group invariant we also have

$$\varphi(P) = \int f(P, Q)\, dQ = \int f(sP, sQ)\, dQ = \int f(sP, sQ)\, d(sQ)$$

$$= \int f(sP, Q)\, dQ = \varphi(sP),$$

and this implies that $\varphi(P)$ is a constant. In other words, if $f(P, Q)$ is group invariant it is also measure invariant. This leads to

THEOREM 3. *A function $g(P, Q)$ belongs to the group invariant class $\mathfrak{P}_0$ if and only if it can be written in the form* (11) *where $f(P, Q)$ is group invariant and belongs to $\mathfrak{P}$.*

A function $\rho(P, Q)$ is a group invariant Hilbert distance if and only if it can be represented in the form

$$\sqrt{f(\Phi, \Phi) - f(P, Q)},$$

where $f(P, Q)$ is group invariant and belongs to the real class $\mathfrak{P}$. The function $\rho(P, Q)$ is a proper distance if and only if $f(P, Q) < f(\Phi, \Phi)$ for $P \neq Q$.

III. Group Invariant Functions

In the present section we will supplement Theorem 3 by statements involving expansions in orthonormal systems. An orthonormal system $\{\varphi_n(P)\}$ is of course defined by the property

$$\int \varphi_m(P)\overline{\varphi_n(P)}\, dP = \delta_{mn};$$

the orthonormal system we will consider will automatically consist of *continuous* functions and it will be *complete* both in the space of continuous functions and in the space of integrable functions.

The simplest space $\mathfrak{S}$ is the torus $0 \leqq x < 2\pi$ with the group of translations on it. Any continuous function $f(x, y)$ on $\mathfrak{S} \times \mathfrak{S}$ has a Fourier expansion

$$\sum_{m,n} a_{m,n} e^{imx} e^{-iny}. \tag{12}$$

Group invariance means that $f(x, y) = f(x + s, y + s)$ and this implies

$$a_{m,n} = a_{m,n} e^{i(m-n)s}.$$

Thus, $a_{m,n} = 0$ if $m \neq n$, and (12) has the form

$$\sum_{-\infty}^{\infty} a_m e^{im(x-y)}. \tag{13}$$

Now $f(x, y)$ belongs to $\mathfrak{P}$ if and only if a_m is real and $\geqq 0$, and $\sum a_m < \infty$; $f(x, y)$ is real if and only if $a_{-m} = a_m$. Thus, by Theorem 3, $\rho(x, y)$ is a Hilbert distance if and only if it is the square root of an expression

$$(14) \quad \sum_{m=1}^{\infty} a_m(2 - e^{im(x-y)} - e^{-i(x-y)}) = \sum_{m=1}^{\infty} a_m(e^{imx} - e^{imy})(e^{-imx} - e^{-imy})$$

where

$$(15) \qquad a_m \geqq 0 \quad \text{and} \quad \sum_m a_m < \infty.$$

If we prefer to look upon $\rho(x, y)$ as a function of the one variable $x - y$ it is more appropriate to write (15) in the form

$$(16) \qquad 2\sum_{m=1}^{\infty} a_m(1 - \cos m(x - y)) = 4\sum_{m=1}^{\infty} a_m\left(\sin \frac{m}{2}(x - y)\right)^2.$$

This result can be extended to our spaces in general in terms of expansions into "generalized spherical harmonics" as given by E. Cartan and H. Weyl.[4] If $\mathfrak{S}$ is a space satisfying lemma 1 with a transitive group $\mathfrak{C}$ of motions then there exists on $\mathfrak{S}$ a complete orthonormal system of continuous functions of the following description:

(i) corresponding to each k, $k = 0, \pm 1, \pm 2, \pm 3, \cdots$ there exist a finite rectangular system of functions $\varphi_{k,m\mu}(P)$ with

$$(17) \qquad \begin{aligned} m &= 1, \cdots, l, \; l = l(k); \\ \mu &= 1, \cdots, h, \; h = h(k), \end{aligned}$$

and a quadratic system of functions $u_{k,m\alpha}(P)$ on $\mathfrak{C}$ with $m, \alpha = 1, \cdots, l = l(k)$, such that

$$(18) \qquad \varphi_{k,m\mu}(sP) = \sum_{\alpha=1}^{l} u_{k,m\alpha}(s)\varphi_{k,\alpha\mu}(P).$$

(ii) For each k, $u_{k,m\alpha}(s)$ is an irreducible unitary representation of $\mathfrak{C}$, and for different values of k the representations are inequivalent.

(iii) for $k = 0$, $l = h = 1$, and $\varphi_{0,11}(P) = 1$, $u_{0,11}(s) = 1$; also $l(-k) = l(k)$, $h(-k) = h(k)$, and

$$\varphi_{-k,m\mu}(P) = \overline{\varphi_{k,m\mu}(P)}, \qquad u_{-k,m\alpha}(s) = \overline{u_{k,m\alpha}(s)}.$$

Now the system of functions

$$\varphi_{p,m\mu}(P) \cdot \overline{\varphi_{q,n\nu}(Q)}$$

is an orthonormal system on $\mathfrak{S} \times \mathfrak{S}$, and thus $f(P, Q)$ has an expansion

$$(19) \qquad \sum_{p,m,\mu,q,n,\nu} a_{pq,m\mu,n\nu}\, \varphi_{p,m\mu}(P)\overline{\varphi_{q,n\nu}(Q)}.$$

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[4] H. Weyl, *Harmonics on homogeneous manifolds*, Annals of Math., 35 (1934), 486–494.

Replacing P and Q by sP and sQ respectively and using (18) we obtain as a necessary and sufficient condition for group invariance the system of relations

$$a_{pq,\alpha\mu,\beta\nu} = \sum_{m=1}^{l(p)} \sum_{n=1}^{l(q)} a_{pq,m\mu,n\nu} u_{p,m\alpha}(s) \overline{u_{q,n\beta}(s)}$$

and this is equivalent with

$$\sum_{n=1}^{l(q)} a_{pq,\alpha\mu,n\nu} u_{q,\beta n}(s)' = \sum_{m=1}^{l(p)} a_{pq,m\mu,\beta\nu} u_{p,m\alpha}(s). \tag{20}$$

Since the functions $\{u(s)\}$ are linearly independent the comparison of coefficients on both sides of (20) will show that $a_{pq,m\mu,n\nu}$ vanishes if $p \neq q$ or if $m \neq n$, and that for $p = q$ and $m = n$ its value is independent of m. This leads to writing (19) in the form

$$\sum_{k=-\infty}^{\infty} \left(\sum_{\mu,\nu=1}^{h(k)} a_{k,\mu\nu} \varphi_{k,\mu\nu}(P, Q) \right) \tag{21}$$

where

$$\varphi_{k,\mu\nu}(P, Q) = \sum_{m=1}^{l(k)} \varphi_{k,m\mu}(P) \overline{\varphi_{k,m\nu}(Q)}. \tag{22}$$

We can now express the properties of $\mathfrak{P}$ in terms of the coefficients $a_{k,\mu\nu}$. Property 1) of $\mathfrak{P}$ simply means that for each k, the matrix

$$| a_{k,\mu\nu} |_{\mu,\nu=1,\cdots,h(k)} \tag{23}$$

is hermitian. Property 2) is automatically fulfilled, and property 3) means that (23) is non-negative definite. In fact putting in (3)

$$\rho(P) = \sum_{\mu=1}^{h} x_\mu \overline{\varphi_{k,m\mu}(P)}$$

we obtain

$$\sum_{\mu,\nu=1}^{h} a_{k,\mu\nu} x_\mu \overline{x_\nu} \geqq 0. \tag{24}$$

Conversely if $\rho(P)$ is a finite sum of the form

$$\sum_{p,m,\mu} x_{p,m\mu} \overline{\varphi_{p,m\mu}(P)} \tag{25}$$

the left side of (3) is

$$\sum_{p,m} \left(\sum_{\mu,\nu=1}^{h} a_{p,\mu\nu} x_{p\,m\mu} \overline{x_{p,m\nu}} \right)$$

and this is $\geqq 0$ if (24) holds. But finite sums of the form (25) are dense in the family of all function $\rho(P)$ and thus (24) implies property 3). Finally by property (iii) of $\{\varphi(P)\}$, $f(P, Q)$ is real if and only if

$$\overline{a_{k,\nu\mu}} = a_{-k,\mu\nu}. \tag{26}$$

Thus a function $f(P, Q)$ of the real class $\mathfrak{P}$ has an expansion of the form

$$a_0 + \sum_{k=1}^{\infty}\left(\sum_{\mu,\nu=1}^{h(k)} a_{k,\mu\nu}(\varphi_{k,\mu\nu}(P, Q) + \overline{\varphi_{k,\mu\nu}(P, Q)})\right) \tag{27}$$

where each matrix (23) is a semi-definite hermitian matrix. Before proceeding we require the following

Lemma 2. *There exists an array of real numbers* $r_{n,k}$, $n = 1, 2, \cdots, k = 0, \pm 1, \pm 2, \cdots$, *with the following properties*: (i) $0 \leqq r_{n,k} \leqq 1$, (ii) $\lim_{n\to\infty} r_{n,k} = 1$, (iii) *for each n only a finite number of coefficients* $r_{n,k}$ *is* $\neq 0$, *and what is decisive,* (iv) *if* (21) *is the expansion of a continuous function* $f(P, Q)$, *then the sequence of functions*

$$f_n(P, Q) = \sum_{k=-\infty}^{\infty} r_{n,k}\left(\sum_{\mu,\nu=1}^{h} a_{k,\mu\nu}\varphi_{k,\mu\nu}(P, Q)\right) \tag{28}$$

is uniformly convergent towards $f(P, Q)$ *as* $n \to \infty$.

The proof of the lemma can be carried out along familiar lines and will be omitted.[5]

Now, if (23) is hermitian the number

$$\lambda_k = \sum_{\mu,\nu=1}^{h} a_{k,\mu\nu}\varphi_{k,\mu\nu}(\Phi, \Phi)$$

is $\geqq 0$ and

$$\left|\sum_{\mu,\nu=1}^{h} a_{k,\mu\nu}\varphi_{k,\mu\nu}(P, Q)\right| \leqq \lambda_k ;$$

therefore

$$0 \leqq \sum_{k=-\infty}^{\infty} r_{n,k}\lambda_k \leqq f_n(\Phi, \Phi) \leqq M$$

where M is independent of n. Letting $n \to \infty$ we obtain $\sum_k \lambda_k \leqq M$, and we hence conclude that the series (21) and (27) are absolutely and uniformly convergent. Therefore we have for

$$f(\Phi, \Phi) - f(P, Q) \equiv \tfrac{1}{2}f(P, P) + \tfrac{1}{2}f(Q, Q) - f(P, Q)$$

the series

$$\sum_{k=1}^{\infty}\left(\sum_{\mu,\nu=1}^{h(k)} a_{k,\mu\nu}\psi_{k,\mu\nu}(P, Q)\right) \tag{29}$$

where

$$\psi_{k,\mu\nu}(P, Q) = \sum_{m=1}^{l(k)} (\varphi_{k,m\mu}(P) - \varphi_{k,m\mu}(Q))(\overline{\varphi_{k,m\nu}(P)} - \overline{\varphi_{k,m\nu}(Q)}). \tag{30}$$

[5] See S. Bochner and J. v. Neumann, *Almost periodic functions in groups*, Trans. Amer. Math. Soc., 37 (1935), 21–50, esp. Part III; H. Weyl, loc. cit., p. 498–499.

Altogether, on the basis of Theorem 3 the following theorem can now be easily verified.

THEOREM 4. *A non-negative function* $\rho(P, Q)$ *is a group invariant Hilbert distance if and only if its square is an absolutely and uniformly convergent series of the form* (29) *in which such matrix* (23) *is non-negative hermitian and* $\psi_{k,\mu\nu}$ *has the value* (30).

We observe that our series (29) is a generalization of series (14).

THEOREM 5. *The Hilbert distance of Theorem* 4 *is certainly a proper distance if each matrix* (23) *is strictly positive definite.*

PROOF: In fact, if (23) is strictly positive then $\rho(P, Q)$ can only vanish if for each k, m, α

$$\varphi_{k,m\alpha}(P) - \varphi_{k,m\alpha}(Q) = 0.$$

But the latter fact implies that every continuous function on $\mathfrak{S}$ assumes equal values on the points P and Q, and this can only happen if P and Q are identical.

IV. SOME SPECIFIC CASES

Formulas (27) and (29) are greatly simplified if for all values of k we have $h(k) = 1$. In this case we can write

$$f(P, Q) = a_0 + \sum_{k=1}^{\infty} a_k(\varphi_k(P, Q) + \overline{\varphi_k(P, Q)})$$

where

$$\varphi_k(P, Q) + \overline{\varphi_k(P, Q)} = \sum_{m=1}^{l(k)} (\varphi_{k,m}(P)\overline{\varphi_{k,m}(Q)} + \overline{\varphi_{k,m}(P)}\varphi_{k,m}(Q)); \tag{31}$$

also the expression

$$\sum_{k=1}^{\infty} a_k(2\varphi_k(\Phi, \Phi) - \varphi_k(P, Q) - \overline{\varphi_k(P, Q)}) \tag{32}$$

for $\rho(P, Q)^2$ shows a strong resemblance to the left side of (16). The resemblance is most pronounced if $\mathfrak{S}$ is the $(m - 1)$-dimensional unit sphere in m-dimensional Euclidean space. In this case (31) is but for a numerical factor depending on k (and m) the expression

$$T_k^{(\lambda)}(\cos \vartheta)$$

where $T_k^{(\lambda)}$ is an ultraspherical polynomial, $\lambda = m - 2/2$, and ϑ is the geodesic distance between the points P, Q. The expression

$$\sum_{k=0}^{\infty} b_k T_k^{(\lambda)}(\cos \vartheta)$$

with $b_k \geqq 0$ and $\sum_k b_k T_k^{(\lambda)}(1) < \infty$, for the most general positive definite function depending on the geodesic distance alone, has been given previously by Schoenberg.[6]

More special than the assumption $h(k) = 1$ is the assumption $l(k) = 1$. It corresponds to the case of the group $\mathfrak{C}$ being an Abelian group, and in this case we simply have

$$f(P, Q) = \sum_{k=0}^{\infty} a_k(\varphi_k(P)\overline{\varphi_k(Q)} + \overline{\varphi_k(P)}\varphi_k(Q))$$

with $a_k \geqq 0$.

The symbol sP is a real multiplication of P by s if $\mathfrak{S}$ is a group space (in which case we write x, y, etc. instead of P, Q, etc.) and $\mathfrak{C}$ is its group of left transformations $x \to sx$. Our assumptions concerning $\mathfrak{S}$ are now that $\mathfrak{S}$ is a compact separable group, the invariant measure on it being now uniquely determined. In this case, $h(k) = l(k)$ and but for the factor $l(k)^{1/2}$, $\varphi_{k,\mu\nu}(x)$ is $u_{k,\mu\nu}(x)$. Since

$$\sum_{m=1}^{l} u_{k,m\mu}(x)\overline{u_{k,m\nu}(y)} = \sum_{m=1}^{l} u_{k,\nu m}(y^{-1})u_{k,m\mu}(x)$$

we see that, except for the factor $l(k)$,

$$\varphi_{k,\mu\nu}(x, y) = u_{k,\nu\mu}(y^{-1}x),$$

and hence we obtain the following result.

THEOREM 6. *If $\mathfrak{C}$ is a compact separable group, then $f(x, y)$ is a left invariant member of $\mathfrak{P}$ if and only if $f(x, y) = f(y^{-1}x)$ where*

$$f(t) = \sum_{k=-\infty}^{\infty} \sum_{\mu,\nu=1}^{l(k)} b_{k,\mu\nu}u_{k,\mu\nu}(t); \tag{33}$$

the matrix $|b_{k,\mu\nu}|$ being non-negative hermitian and the series (33) in k being absolutely convergent.

In order to obtain right invariant functions we have to put $t = yx^{-1}$. Our function is invariant on both sides if $f(s^{-1}y^{-1}xs) = f(y^{-1}x)$ that is if $f(t)$ is a class function. For class functions the expansion (33) depends only on the group characters

$$\chi_k(t) = \sum_{u=1}^{l} u_{k,\mu\mu}(t).^7$$

Thus we obtain

THEOREM 7. *If $\mathfrak{C}$ is a compact separable group, then $f(x, y)$ is a group invariant member of $\mathfrak{P}$, if and only if*

$$f(x, y) = \sum_{k=-\infty}^{\infty} b_k\chi_k(yx^{-1})$$

[6] I. J. Schoenberg, *On positive definite functions on spheres.* Bull. Amer. Math. Soc., 46 (1940), p. 888.

[7] See Bochner-von Neumann, loc. cit.

where $b_k \geqq 0$, *and* $\sum_k b_k\chi_k(1) < \infty$; *and* $\rho(x, y)$ *is a group invariant Hilbert distance if and only if*

$$\rho(x, y)^2 = \sum_{k=1}^{\infty} c_k\left(1 - \frac{\chi_k(yx^{-1}) + \chi_k(xy^{-1})}{2l(k)}\right)$$

with $c_k \geqq 0$; *the function* $\rho(x, y)$ *is certainly a proper distance if all* c_k *are* > 0.

V. Isometric Imbedding of Riemannian Space into Hilbert Space

In the present section we will draw a conclusion from Theorem 5. We assume that our space $\mathfrak{S}$ is a coordinate space of class C_r, $r \geqq 2$, (continuous partial derivatives of order $\leqq r$), or C_∞ (derivatives of every order) or C_ω (analytic coordinates), and we further assume explicitly that the functions $\{\varphi_{k,m\mu}(P)\}$ belong to the same class. By choosing the coefficients of the matrix (23) sufficiently small we can obtain a series (29) whose sum will belong to the same class on $\mathfrak{S} \times \mathfrak{S}$. Now, by a general theorem,[8] such a distance can be generated by a Riemannian metric. Altogether we have the following

Theorem 8. *If* $\mathfrak{S}$ *is a compact differentiable manifold (of arbitrary dimension and) of class* C_r, $r \geqq 2$, *or* C_∞, *or* C_ω, *if* $\mathfrak{C}$ *is a fixed transitive group of homomorphisms on it, and if the corresponding generalized spherical harmonics* $\{\varphi_{k,m\mu}(P)\}$ *belong to the same class, then there exists on* S *a group invariant positive definite Riemannian metric of class* C_{r-2}, *or* C_∞, *or* C_ω *for which the given space can be isometrically imbedded in real Hilbert space.*

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[8] S. Bochner, *Differentiable and Riemann metric*, Duke Math. Jour., 4 (1938), 51–54.