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ON CERTAIN METRIC SPACES ARISING FROM EUCLIDEAN SPACES BY A CHANGE OF METRIC AND THEIR IMBEDDING IN HILBERT SPACE[1]

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1. W. A. Wilson ([9])[2] has recently investigated those metric spaces which arise from a metric space by taking as its new metric a suitable (one variable) function of the old one. He considered in particular the euclidean straight line R_1 whose metric $\delta = \overline{PP'}$ is changed to $\Delta = d(P, P') = \overline{PP'}^{\frac{1}{2}}$ and showed that this new metric space can be imbedded[3] in Hilbert space $\mathfrak{H}$. Here the old metric δ and the new metric Δ are connected by the relation $\Delta^2 = \delta$.

In an article soon to appear ([5]), John von Neumann and the author have determined all the functions $f(\delta)$ such that if R_1 is provided with the new metric Δ, defined by $\Delta^2 = f(\delta)$, $\delta = \overline{PP'}$, the new metric space thus arising shall be imbeddable in $\mathfrak{H}$. They are of the form

$$f(\delta) = \int_0^\infty \frac{\sin^2 (s\delta)}{s^2}\, d\alpha(s), \tag{1}$$

where $\alpha(s)$ is non-decreasing for $0 \leqq s < \infty$ and such that $\int_1^\infty s^{-2}\, d\alpha(s)$ exists.

Wilson's case $f(\delta) = \delta$ is included in the general formula on account of

$$\delta = \frac{2}{\pi} \int_0^\infty \frac{\sin^2 (s\delta)}{s^2}\, ds, \qquad (\delta \geqq 0). \tag{2}$$

In the present note Wilson's example is extended to higher dimensional euclidean spaces, its chief result being the following theorem.

THEOREM 1. *If we change the metric of the euclidean space R_m from the euclidean distance $\overline{PP'}$ to the new distance*

$$d(P, P') = \overline{PP'}^{\gamma}, \qquad (0 < \gamma < 1),[4] \tag{3}$$

the new space $\mathfrak{R}_m^{(\gamma)}$ thus arising may be imbedded isometrically in the Hilbert space $\mathfrak{H}$.

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[1] Presented to the American Mathematical Society, February 20, 1937.

[2] The numbers in square brackets refer to the list of references at the end of this note.

[3] Here and throughout this note the word *imbedding* is meant in the sense of *isometrical imbedding*.

[4] The case $\gamma = 1$ is trivial. The theorem does not hold for $\gamma = 0$, for the space $\mathfrak{R}_m^{(0)}$, with $d(P, P') = 1$ if $P \neq P'$ and $d(P, P) = 0$, is obviously not separable. The constant 1 is the best constant, for $\mathfrak{R}_m^{(\gamma)}$ is not a metric space if $\gamma > 1$.

2. As for all such imbedding problems into $\mathfrak{H}$, the proof of Theorem 1 is based on the following theorem of Menger ([4]).

A metric space $\mathfrak{R}$ *can be imbedded in* $\mathfrak{H}$ *if and only if* $\mathfrak{R}$ *is separable and every set of* $n + 1$ $(n = 2, 3, 4, \cdots)$ *distinct points of* $\mathfrak{R}$ *can be imbedded in* R_n.

Therefore, as our $\mathfrak{R}_m^{(\gamma)}$ is obviously separable, it suffices to show that any $n + 1$ distinct points $P_0, P_1, \cdots, P_n$, of $\mathfrak{R}_m^{(\gamma)}$ can be imbedded in R_n, i.e., there exist $n + 1$ points $Q_0, Q_1, \cdots, Q_n$, of R_n such that $Q_\mu Q_\nu = \overline{P_\mu P_\nu}^\gamma$ $(\mu, \nu = 0, 1, \cdots, n)$.[5] This "finite" imbedding problem is readily solved by means of the following theorem ([6], Theorem 1, p. 724).

The quantities $a_{\mu\nu}$ $(\mu, \nu = 0, 1, \cdots, n;\ a_{\mu\nu} = a_{\nu\mu} > 0$ if $\mu \neq \nu$, $a_{\mu\mu} = 0)$ *are the distances of* $n + 1$ *points* $Q_0, Q_1, \cdots, Q_n$, *of* R_n, *i.e.* $a_{\mu\nu} = \overline{Q_\mu Q_\nu}$, *if and only if the quadratic form*

$$F(x, x) = \tfrac{1}{2} \sum_{j,k=1}^{n} (a_{0j}^2 + a_{0k}^2 - a_{jk}^2) x_j x_k$$

is positive, i.e. always $\geqq 0$. *If this form is positive definite, the points* Q_μ *are the vertices of a n-simplex.*[6]

Our finite imbedding problem $a_{\mu\nu} = \overline{P_\mu P_\nu}^\gamma = Q_\mu Q_\nu$ is therefore contained (for $\alpha = 2\gamma$) in the following theorem.

THEOREM 2. *If* $P_0, P_1, \cdots, P_n$, *are distinct points of a euclidean space* R_m $(m \geqq 1)$, *the quadratic form*

$$F^{(\alpha)}(x, x) = \tfrac{1}{2} \sum_{j,k=1}^{n} (\overline{P_0 P_j}^\alpha + \overline{P_0 P_k}^\alpha - \overline{P_j P_k}^\alpha) x_j x_k \quad (0 < \alpha < 2) \tag{4}$$

is positive definite.[7]

Note that in order to prove Theorem 1 we need only to know that $F^{(\alpha)}(x, x) \geqq 0$. Its positive definiteness means that in order to imbed into $\mathfrak{H}$ any $n + 1$ distinct points of $\mathfrak{R}_m^{(\gamma)}$ we need fully all dimensions of a n-dim. subspace of $\mathfrak{H}$, i.e. a R_n.

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[5] L. M. Blumenthal ([2], Corollary, p. 402) proved the following result. *If* $P_i (i = 0, 1, 2, 3)$ *are four points of a metric space* $\mathfrak{R}$, *for any nonnegative number* γ, *not exceeding* $\tfrac{1}{2}$, *there exist four points* $Q_i (i = 0, 1, 2, 3)$ *of* R_3 *such that* $\overline{Q_i Q_j} = \{d(P_i, P_j)\}^\gamma (i, j = 0, 1, 2, 3)$.

This result is *not* contained in our present problem, for the distances $d(P_i, P_j)$ are not assumed in Blumenthal's theorem to be the edges of a euclidean tetrahedron. If this assumption is added, as for instance by assuming $\mathfrak{R}$ to be a euclidean space, Blumenthal conjectures that the inequality $0 \leqq \gamma \leqq \tfrac{1}{2}$ of his theorem may be replaced by $0 \leqq \gamma \leqq 1$ (loc. cit., concluding remark of section 4, p. 403). Theorem 2 below proves this conjecture and extends it from four points to $n + 1$ points.

[6] This elementary theorem is in substance identical with the well known correspondence between lattices of points and positive definite quadratic forms. See H. Minkowski, *Gesammelte Abhandlungen*, vol. 1, pp. 243–254, where also references to Gauss and Dirichlet are found. For an imbedding problem of arithmetical nature see I. J. Schoenberg, [7].

[7] Communicating the proof of Theorem 2 to Prof. G. Szegö, my letter and one of his crossed each other; in his letter Prof. Szegö proves independently and in a different way Theorem 2 for $\alpha = 1$ and $m = 1, 2$ and 3. An extension of his proof to arbitrary α $(0 < \alpha < 2)$ is obvious, but not an extension to all dimensions m.

3. Let us pass now to the proof of Theorem 2. Although this theorem is algebraic in nature, at least for rational values of α, an algebraic proof would probably be difficult and complicated. The following proof is elementary but uses transcendental means. To simplify notations we prove it first for $m = 3$, i.e., $P_0, P_1, \cdots, P_n$, are points in ordinary 3-space.

Consider the following function of three real variables

$$\Omega(u, v, w) = \frac{1}{4\pi} \cdot \iint_{\xi^2+\eta^2+\zeta^2=1} e^{i(u\xi+v\eta+w\zeta)}\, d\sigma = \mathfrak{M}\{e^{i(u\xi+v\eta+w\zeta)}\},$$

which is the mean value of the function $e^{i(u\xi+v\eta+w\zeta)}$ over the spherical shell $\xi^2 + \eta^2 + \zeta^2 = 1$. $\Omega(u, v, w)$ is obviously invariant with respect to rigid rotations around the origin and is therefore a function of $r = (u^2 + v^2 + w^2)^{\frac{1}{2}}$ only, which we denote by $\Omega(r)$. Now

$$\Omega(r) = \Omega(0, 0, r) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} e^{ir\cos\theta} \sin\theta\, d\varphi\, d\theta = \tfrac{1}{2} \int_0^{\pi} e^{ir\cos\theta} \sin\theta\, d\theta = \frac{\sin r}{r},$$

hence

$$\Omega(r) = \frac{\sin r}{r} = \mathfrak{M}\{e^{i(u\xi+v\eta+w\zeta)}\} \quad (r = (u^2 + v^2 + w^2)^{\frac{1}{2}}). \tag{5}$$

Let $P_0 = (0, 0, 0)$, $P_j = (u_j, v_j, w_j)$, indicate the coordinates of our points in R_3. For $s \geqq 0$ we have

$$\Omega(s\cdot\overline{P_\mu P_\nu}) = \Omega(s\cdot\sqrt{(u_\mu - u_\nu)^2 + (v_\mu - v_\nu)^2 + (w_\mu - w_\nu)^2})$$
$$= \mathfrak{M}\{e^{is[(u_\mu-u_\nu)\xi+(v_\mu-v_\nu)\eta+(w_\mu-w_\nu)\zeta]}\},$$

whence

$$\Omega(s\cdot\overline{P_\mu P_\nu}) = \mathfrak{M}\{e^{is(u_\mu\xi+v_\mu\eta+w_\mu\zeta)}\cdot e^{-is(u_\nu\xi+v_\nu\eta+w_\nu\zeta)}\} \qquad (s \geqq 0). \tag{6}$$

On the other hand we have (as is readily seen by substituting st^{-1} for s in the integral) for $0 < \alpha < 2$

$$t^\alpha = c(\alpha)\cdot \int_0^\infty \frac{1 - \Omega(ts)}{s^2} s^{1-\alpha}\, ds = c(\alpha)\cdot \int_0^\infty \{1 - \Omega(ts)\} s^{-1-\alpha}\, ds \tag{7}$$
$$(0 < \alpha < 2; t > 0),$$

where

$$c(\alpha) = 1 \Big/ \int_0^\infty \{1 - \Omega(s)\} s^{-1-\alpha}\, ds \qquad (0 < \alpha < 2).$$

We may now express our hermitian form $F^{(\alpha)}(x, \bar{x})$ as follows

$$\tfrac{1}{2} \sum_{j,k=1}^{n} \{\overline{P_0P_j}^\alpha + \overline{P_0P_k}^\alpha - \overline{P_jP_k}^\alpha\} x_j \bar{x}_k$$
$$= \frac{c(\alpha)}{2} \int_0^\infty s^{-1-\alpha} \sum_1^n \{1 - \Omega(s\cdot\overline{P_0P_j}) - \Omega(s\cdot\overline{P_0P_k}) + \Omega(s\cdot\overline{P_jP_k})\} x_j \bar{x}_k \cdot ds, \tag{8}$$

where $x_1, \cdots, x_n$, are arbitrary complex numbers. Writing

$$x_0 = -\sum_1^n x_j$$

we have

$$\sum_1^n \{1 - \Omega(s\cdot\overline{P_0P_j}) - \Omega(s\cdot\overline{P_0P_k}) + \Omega(s\cdot\overline{P_jP_k})\}x_j\bar{x}_k$$

$$= \left|\sum_1^n x_j\right|^2 - \sum_1^n \Omega(s\cdot\overline{P_0P_j})x_j\cdot\sum_1^n \bar{x}_k - \sum_1^n \Omega(s\cdot\overline{P_0P_k})\bar{x}_k\cdot\sum_1^n x_j + \sum_1^n \Omega(s\cdot\overline{P_jP_k})x_j\bar{x}_k$$

$$= |x_0|^2 + \bar{x}_0\cdot\sum_1^n \Omega(s\cdot\overline{P_0P_j})x_j + x_0\cdot\sum_1^n \Omega(s\cdot\overline{P_0P_k})\bar{x}_k + \sum_1^n \Omega(s\cdot\overline{P_jP_k})x_j\bar{x}_k$$

$$= \sum_{\mu,\nu=0}^n \Omega(s\cdot\overline{P_\mu P_\nu})x_\mu\bar{x}_\nu,$$

which, in view of (6), is equal to

$$\mathfrak{M}\left\{\left| x_0 + \sum_{j=1}^n x_j e^{is(u_j\xi+v_j\eta+w_j\zeta)}\right|^2\right\}.$$

Now (8) becomes

$$(9)\qquad \tfrac{1}{2}\sum_{j,k=1}^n (\overline{P_0P_j}^\alpha + \overline{P_0P_k}^\alpha - \overline{P_jP_k}^\alpha)x_j\bar{x}_k = \frac{c(\alpha)}{2}\cdot\int_0^\infty s^{-1-\alpha}\cdot\mathfrak{M}\left\{\left|x_0 + \sum_{j=1}^n x_j e^{is(u_j\xi+v_j\eta+w_j\zeta)}\right|^2\right\}ds \geqq 0.$$

Here we have the equality sign if and only if

$$(10)\qquad x_0 + \sum_{j=1}^n x_j e^{is(u_j\xi+v_j\eta+w_j\zeta)} = 0$$

holds identically in s and the direction cosines ξ, η, ζ. As the points (u_j, v_j, w_j) are all different and none is at the origin, a direction (ξ, η, ζ) can be found for which the inner products $u_j\xi + v_j\eta + w_j\zeta$ $(j = 1, \cdots, n)$ are all different and none is zero, and now (10) implies that $x_1, \cdots, x_n$ must all vanish. Theorem 2 is thus completely proved for $m = 3$, hence also for $m = 1$ and $m = 2$.

4. An extension of the proof to any value of m is now obvious. All we have to do is to repeat the above argument with the function

$$(11)\qquad \Omega_m(r) = \mathfrak{M}\{e^{i(u_1\xi_1+\cdots+u_m\xi_m)}\}, \qquad r = (u_1^2 + \cdots + u_m^2)^{\frac{1}{2}},$$

that is, the mean value of $e^{i(u_1\xi_1+\cdots+u_m\xi_m)}$ over the spherical shell

$$\xi_1^2 + \cdots + \xi_m^2 = 1.$$

Thus for $m = 1$ we have

$$\Omega_1(r) = \tfrac{1}{2}(e^{ir} + e^{-ir}) = \cos r$$

and for $m = 2$

$$\Omega_2(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(u_1 \cos\theta + u_2 \sin\theta)}\, d\theta = J_0(r)$$

where $J_0(r)$ is the Bessel function.

To settle the matter there remain only two essential details to be checked. First, that the improper integral in the formula

$$t^\alpha = c_m(\alpha)\int_0^\infty \frac{1 - \Omega_m(ts)}{s^2} s^{1-\alpha}\, ds \qquad (0 < \alpha < 2; t > 0), \tag{12}$$

which is the analogue of (7), actually converges for $0 < \alpha < 2$. Second, that the factor

$$c_m(\alpha) = 1 \Big/ \int_0^\infty \{1 - \Omega_m(s)\} s^{-1-\alpha}\, ds$$

on the right side of (12) is defined and positive for $0 < \alpha < 2$. Both facts were obvious in the case of (7), for $\Omega_3(r) = \sin r/r$ enjoys the properties

$$-1 \leqq \Omega_3(r) \leqq 1, \qquad \Omega_3(r) = 1 - \tfrac{1}{6}r^2 + \cdots.$$

In order to establish similar properties of $Q_m(r)$, we remark that by m-dimensional polar coordinates we readily find

$$\Omega_m(r) = \int_0^\pi e^{ir\cos\theta} \sin^{m-2}\theta\, d\theta \Big/ \int_0^\pi \sin^{m-2}\theta\, d\theta. \tag{13}$$

Expansion of the exponential integrand in its power series shows that $\Omega_m(r)$ is a real and even entire function of r. The remark that $|\Omega_m(r)| \leqq \Omega_m(0) = 1$ completes the argument.

Incidentally, (13) gives the expansion

$$\Omega_m(r) = 1 - \frac{r^2}{2\cdot m} + \frac{r^4}{2\cdot 4\cdot m(m+2)} - \frac{r^6}{2\cdot 4\cdot 6\cdot m(m+2)(m+4)} + \cdots, \tag{14}$$

from which the following expression in terms of Bessel functions of the first kind becomes apparent

$$\Omega_m(r) = \Gamma\left(\frac{m}{2}\right)\left(\frac{2}{r}\right)^{\frac{1}{2}(m-2)} J_{\frac{1}{2}(m-2)}(r) \qquad (m = 1, 2, 3, \cdots). \tag{15}$$

5. To John von Neumann are due the following consequences of our previous results.

THEOREM 3. *Let $\mathfrak{H}_\gamma$ denote the metric space obtained from the Hilbert space $\mathfrak{H}$ by replacing its metric $d(x, x') = \| x - x' \|$ by $d_\gamma(x, x') = \| x - x' \|^\gamma$ $(0 < \gamma \leqq 1)$.*

1. *$\mathfrak{H}_\gamma$ may be imbedded in $\mathfrak{H}$.*
2. *If $0 < \gamma \leqq \delta \leqq 1$, $\mathfrak{H}_\gamma$ may be imbedded in $\mathfrak{H}_\delta$.*

The first statement is essentially equivalent to Theorem 1 on the basis of

Menger's theorem. Indeed, as any $n + 1$ points of $\mathfrak{H}_\gamma$ may be imbedded in $\mathfrak{H}$, the whole of $\mathfrak{H}_\gamma$ may thus be imbedded. This result may be stated analytically as follows: There exists a function $\varphi_\gamma(x)$, defined for all $x \in \mathfrak{H}$, with $\varphi_\gamma(x) \in \mathfrak{H}$, such that

$$\| \varphi_\gamma(x) - \varphi_\gamma(x') \| = \| x - x' \|^\gamma \qquad (x \in \mathfrak{H},\ x' \in \mathfrak{H}).$$

As for the second statement, which contains the first as a special case ($\delta = 1$), consider $\mathfrak{H}_{\gamma/\delta}$. A function $\varphi_{\gamma/\delta}(x)$ which performs its imbedding in $\mathfrak{H}$ satisfies, as just mentioned, the identity

$$\| \varphi_{\gamma/\delta}(x) - \varphi_{\gamma/\delta}(x') \| = \| x - x' \|^{\gamma/\delta} \qquad (x \in \mathfrak{H},\ x' \in \mathfrak{H}),$$

whence

$$\| \varphi_{\gamma/\delta}(x) - \varphi_{\gamma/\delta}(x') \|^\delta = \| x - x' \|^\gamma \qquad (x \in \mathfrak{H},\ x' \in \mathfrak{H}).$$

Hence $\varphi_{\gamma/\delta}(x)$ $(x \in \mathfrak{H})$, which imbeds $\mathfrak{H}_{\gamma/\delta}$ in $\mathfrak{H}$, at the same time performs the imbedding of $\mathfrak{H}_\gamma$ in $\mathfrak{H}_\delta$.

6. Let us finally state and prove the following corollary of Theorem 2.

THEOREM 4. *If $P_0, P_1, \cdots, P_n$ are distinct points of the euclidean space R_n, the quadratic form*

$$\sum_{i,k=0}^{n} \overline{P_i P_k}^\alpha x_i x_k \qquad (0 < \alpha < 2)$$

is non-singular and its canonical representation contains one positive and n negative squares.

The sole difficulty consists in proving that the determinant

$$\det \| \overline{P_i P_k}^\alpha \|_{0,n} \neq 0.$$

Let us show that

$$\operatorname{sgn} \det \| \overline{P_i P_k}^\alpha \|_{0,n} = (-1)^n \qquad (n \geqq 1). \tag{16}$$

Now perform in R_n an ordinary inversion by reciprocal radii with respect to the sphere of center P_0 and radius $r = 1$, and let $Q_1, \cdots, Q_n$ be the transforms of the points $P_1, \cdots, P_n$ by this inversion.[8] Consider the determinant of order $n + 1$

$$D = \begin{vmatrix} 0 & 1 \\ 1 & \overline{Q_i Q_k}^\alpha \end{vmatrix} \qquad (i, k = 1, 2, \cdots, n).$$

On the one hand we find (compare Blumenthal [1], p. 424) by suitable subtractions of lines and columns

$$D = (-1)^n \det \| \overline{Q_1 Q_r}^\alpha + \overline{Q_1 Q_s}^\alpha - \overline{Q_r Q_s}^\alpha \| \qquad (r, s = 2, 3, \cdots, n),$$

[8] This inversion was suggested by the equivalence under inversion between the triangle inequality and Ptolemy's inequality of elementary geometry. See J. Hadamard [3], pp. 228–229.

hence

$$\text{sgn}\, D = (-1)^n \tag{17}$$

by Theorem 2. On the other hand we have (since $r = 1$) by an elementary property of inversion

$$\overline{Q_i Q_k} = \overline{P_i P_k}/(\overline{P_0 P_k} \cdot \overline{P_0 P_i}), \qquad (i, k = 1, \cdots, n),$$

whence

$$D = \begin{vmatrix} 0 & 1 \\ 1 & \overline{P_i P_k}^\alpha / \overline{P_0 P_i}^\alpha \cdot \overline{P_0 P_k}^\alpha \end{vmatrix} = (\overline{P_0 P_1} \cdots \overline{P_0 P_n})^{-2\alpha} \cdot \begin{vmatrix} 0 & \overline{P_0 P_k}^\alpha \\ \overline{P_0 P_i}^\alpha & \overline{P_i P_k}^\alpha \end{vmatrix}.$$

Hence (17) implies (16). The theorem now follows from the classical theory of the signature of quadratic forms.

A special case of Theorem 4, where $\alpha = 1$ and $P_0, \cdots, P_n$ are equidistant points on a straight line, was discussed from the point of view of Toeplitz matrices by G. Szegö [8].

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