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FUNCTIONS DIFFERENTIABLE ON THE BOUNDARIES OF REGIONS[1]

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1. **Introduction.** Let the function $f(x_1, \cdots, x_n)$ be defined in the bounded region[2] R of n-space E, and suppose f has continuous m^{th} partial derivatives in R, i.e. f "is of class C^m" in R. If B is the boundary of R, how shall we decide whether f is of class C^m in $R + B$? If the derivatives of f take on boundary values on B, it would be natural to define the derivatives on B as the limit of their values in R. But it is easy to construct a region R and a function f such that the k^{th} partial derivatives of $f (0 < k \leqq m)$ are continuous in $R + B$, whereas at a certain boundary point P of B, f is not continuous;[3] it seems unreasonable in this case to say that f is of class C^m in $R + B$.

If it is possible to extend the definition of f throughout a region containing $R + B$ so that it has continuous m^{th} partial derivatives there, we may then surely say that f is of class C^m in $R + B$; this is the definition we shall use. We show in this note that, for certain regions, for a function to be of class C^m in the closed region, it is sufficient that the m^{th} partial derivatives be continuous on the boundary.

We shall use a one-dimensional notation, as in a paper by the author AE.[4] Thus $f_k(x) = f_{k_1 \ldots k_n}(x_1, \cdots, x_n)$, $(x' - x)^l = (x_1' - x_1)^{l_1} \cdots (x_n' - x_n)^{l_n}$, $l! = l_1! \cdots l_n!$, etc. We set $\sigma_k = k_1 + \cdots + k_n$. $r_{xx'}$ is the distance between the two points x and x'. The fundamental definition is: $f(x) = f_0(x)$ is of class C^m in A if functions $f_k(x)$ and $R_k(x'; x)$ $(\sigma_k \leqq m)$ exist in A such that

$$(1) \qquad f_k(x') = \sum_{\sigma_l \leqq m - \sigma_k} \frac{f_{k+l}(x)}{l!}(x' - x)^l + R_k(x'; x)$$

for each $k (\sigma_k \leqq m)$, and R_k has the following property. Given the point x^0 of A and an $\epsilon > 0$, there is a $\delta > 0$ such that if x and x' are any two points of A within δ of x^0, then

$$(2) \qquad | R_k(x'; x) | \leqq r_{xx'}^{m - \sigma_k} \epsilon.$$

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[2] The restriction that R be bounded is made merely for simplicity; the theorem holds equally well without it.

[3] We give an example with $m = 1$. In polar coordinates, let S be the spiral $r = \theta^{-\frac{1}{2}}$ $(\theta \geqq 0)$. We let R be a narrow region about S, and set $f(x, y) = F(r, \theta) = \log \theta$ for those values of r which are approximately $\theta^{-\frac{1}{2}}$. Differentiating, we find $\partial f/\partial x = -\theta^{-\frac{1}{2}} \sin \theta$ and $\partial f/\partial y = \theta^{-\frac{1}{2}} \cos \theta$ approximately in R. Hence $\partial f/\partial x$ and $\partial f/\partial y \to 0$ at the origin; but $f \to \infty$ there.

[4] Analytic extensions of differentiable functions defined in closed sets, Transactions of the American Mathematical Society, Vol. 36 (1934), pp. 63–89.

If $f(x)$ has continuous m^{th} partial derivatives in R, it is of class C^m in R, as we see by setting $f_{k_1 \ldots k_n} = \partial^{k_1 + \cdots + k_n} f / \partial x_1^{k_1} \cdots \partial x_n^{k_n}$ and using Taylor's formula. The converse also is true.

LEMMA 1. *If $f(x)$ is of class C^m in the closed set A, then its definition can be extended throughout E so it will be of class C^m there.*

This is proved in AE, Lemma 2.

2. **The property** P. We shall say a point set A has the property P if there is a number ω such that any two points x and y of A are joined by a curve C in A of length $L \leqq \omega\, r_{xy}$.

LEMMA 2. *If R has the property P, so has $R + B$.*

Let x and y be points of $R + B$, and suppose first that y is in B while x is in R. Let $y_1, y_2, \cdots$ be a sequence of points of R such that $r_{y_s y} < 2^{-s-2}\, r_{xy}$. As $r_{xy_1} < 3r_{xy}/2$, there is a curve C_0 joining x and y_1 of length $< 3\omega\, r_{xy}/2$. In general, $r_{y_s y_{s+1}} < 2^{-s-1}\, r_{xy}$, and hence there is a curve C_s joining y_s and y_{s+1} of length at most $2^{-s-1}\,\omega\, r_{xy}$. From the curves $C_0, C_1, C_2, \cdots$ we pick out a curve C joining x to y; its length is

$$L < 3\omega r_{xy}/2 + 2^{-2}\,\omega r_{xy} + 2^{-3}\,\omega r_{xy} + \cdots = 2\omega r_{xy}.$$

If both x and y are in B, we take a point x' with $r_{x'x} < r_{xy}/2$, find curves joining it to x and y, of lengths $L_1 < \omega r_{xy}$ and $L_2 < 3\omega r_{xy}$ respectively, as above; from these we pick out a curve C joining x and y, of length $L < 4\omega r_{xy}$. Thus the number $\omega' = 4\omega$ will do for $R + B$. Evidently any number $\omega' > \omega$ will do.

3. **A remainder formula.** We prove here that *if* (1) *holds on the rectifiable curve C of length L with end points x^* and x', then $R_k(x'; x^*)(\sigma_k \leqq m)$ is given by the Stieltjes integral*[4a]

$$R_k(x'; x^*) = - \sum_{\sigma_l = m - \sigma_k} \frac{1}{l!} \int_0^L [f_{k+l}(x(s)) - f_{k+l}(x^*)]\, d(x' - x(s))^l, \tag{3}$$

where s denotes the length of that part of C between x^* and the variable point $x(s)$. If we carry out the differentiation indicated, we may write this as a sum of contour integrals

$$R_k(x'; x^*) = \sum_{\sigma_l = m - \sigma_k} \sum_{\sigma_j = 1} \int_{x^*}^{x'} [f_{k+l}(x) - f_{k+l}(x^*)] \frac{(x' - x)^{l-j}}{(l-j)!} (dx)^j. \tag{4}$$

In the case $n = 2$, $k = 0$, this equation, written out in full, is

$$\begin{aligned} R_{00}(x', y'; x^*, y^*) = \sum_{i+j=m} \int_{(x^*, y^*)}^{(x', y')} & [f_{ij}(x, y) - f_{ij}(x^*, y^*)] \\ & \cdot \left[\frac{(x' - x)^{i-1}(y' - y)^j}{(i-1)!\, j!} dx + \frac{(x' - x)^i (y' - y)^{j-1}}{i!\,(j-1)!} dy \right]. \end{aligned} \tag{5}$$

[4a] An example shows that no such formula holds for all curves.

To prove (3), let $x^0 = x^*, x^1, \cdots, x^p = x'$ be the end points of a subdivision of C. If in equation (6.3) of AE we subtract $f_k(x'')$ from both sides and change x, x', x'' to x^{i-1}, x^i, x' respectively, we find

$$R_k(x'; x^{i-1}) - R_k(x'; x^i) = \sum_{\sigma_l \leqq m - \sigma_k} \frac{R_{k+l}(x^i; x^{i-1})}{l!}(x' - x^i)^l. \tag{6}$$

As $R_k(x'_h; x^i_h) = 0$, summing over i gives

$$R_k(x'; x^*) = \sum_{\sigma_l \leqq m - \sigma_k} \frac{1}{l!} \sum_{i=1}^{p} R_{k+l}(x^i; x^{i-1})(x' - x^i)^l. \tag{7}$$

We show first that as the norm of the subdivision tends to zero, the terms on the right with $\sigma_l < m - \sigma_k$ tend to zero. By (2), given an $\eta > 0$, we can take the norm so small that

$$| R_j(x^i; x^{i-1}) | < r_{x^{i-1} x^i} \eta \qquad (\sigma_j < m).$$

As $| x'_h - x^i_h | \leqq L$, we find for $\sigma_l < m - \sigma_k$

$$\left| \sum_{i=1}^{p} R_{k+l}(x^i; x^{i-1})(x' - x^i)^l \right| < \eta L^{\sigma_l} \sum_{i=1}^{p} r_{x^{i-1} x^i} \leqq \eta L^{\sigma_l + 1}.$$

as η is arbitrary, the statement is proved.

Now take any l with $\sigma_l = m - \sigma_k$. We have

$$\sum_{i=1}^{p} R_{k+l}(x^i; x^{i-1})(x' - x^i)^l = \sum_{i=1}^{p} [f_{k+l}(x^i) - f_{k+l}(x^{i-1})]\,(x' - x^i)^l$$

$$= - \sum_{i=1}^{p-1} [f_{k+l}(x^i) - f_{k+l}(x^*)]\,[(x' - x^{i+1})^l - (x' - x^i)^l].$$

Putting this in (7) and passing to the limit as the norm of the subdivision tends to zero, we obtain (3).

Lemma 3. *Let $f(x)$ be of class C^m on the curve C of length L with end points x^* and x'. If*

$$| f_k(x) - f_k(x^*) | < \epsilon \qquad (\sigma_k = m, x \text{ on } C), \tag{8}$$

then

$$| R_k(x'; x^*) | < n(m+1)^n L^{m-\sigma_k} \epsilon^5 \qquad (\sigma_k \leqq m). \tag{9}$$

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[5] The numerical factor $n(m+1)^n$ may of course be replaced by the factor

$$\sum_{\sigma_l = m - \sigma_k} \sum_{\sigma_j = 1} \frac{1}{(l-j)!} = \frac{n^{m-\sigma_k}}{(m - \sigma_k - 1)!}.$$

This follows at once from (4) when we note that the first sum contains at most $(m+1)^n$ terms, the second sum contains n terms, and

$$\left| \int_{x^*}^{x'} \frac{(x'-x)^{l-j}}{(l-j)!} (dx)^j \right| < \int_0^L L^{\sigma_l - 1}\, ds = L^{\sigma_l}.$$

4. **Functions of class C^m in $R+B$.** One more lemma will lead us to the main theorem of the paper.

LEMMA 4. *If R has the property P and $f_l(x)$ is uniformly continuous in $R(\sigma_l = m)$, then $f_k(x)$ is also $(\sigma_k < m)$.*

Assuming this is true for values of k such that $\sigma_k > s$ $(0 \leqq s < m)$, we shall prove it for any k with $\sigma_k = s$. By hypothesis, $f_{k+l}(x)$ is uniformly continuous for $\sigma_l > 0$; from (1), we see that it is sufficient to show that for every $\epsilon > 0$ there is a $\delta > 0$ such that if x' and x^* are any two points of R with $r_{x'x^*} < \delta$, then $| R_k(x'; x^*) | < \epsilon$. Take $\delta' < 1$ so that $| f_k(x) - f_k(x^*) | < \epsilon/[n(m+1)^n]$ if $r_{xx^*} < \delta'$ and $\sigma_k = m$. Set $\delta = \delta'/\omega$, and take any two points x', x^* of R with $r_{x'x^*} < \delta$. There is a curve C joining them of length $L \leqq \omega r_{x'x^*} < \delta'$; if x is on C, then $r_{xx^*} < \delta'$. As $L < 1$, Lemma 3 gives $| R_k(x'; x^*) | < \epsilon$, as required.

THEOREM. *Let the region R have the property P, and let $f(x_1, \cdots, x_n)$ be of class C^m in R. If $\partial^{k_1 + \cdots + k_n} f / \partial x_1^{k_1} \cdots \partial x_n^{k_n}$ $(k_1 + \cdots + k_n = m)$ can be defined on the boundary B of R so that it is continuous in $R+B$, then the definition of f can be extended throughout space so that it is of class C^m there.*

By the last lemma, $f_k(x)$ can be defined in B so that it is continuous in $R+B$ $(\sigma_k \leqq m)$. We must show that $f_0(x) = f(x)$ is of class C^m in $R+B$; the theorem then follows from Lemma 1. Given an $\epsilon > 0$, take $\delta' < 1$ so that $| f_k(x) - f_k(x^*) | < \epsilon/[n(m+1)^n \omega'^m]$ $(\sigma_k = m)$ if x and x^* are in $R+B$ and $r_{xx^*} < \delta'$; set $\delta = \delta'/\omega'$. Now let y' and y^* be any two points of $R+B$ with $r_{y'y^*} < \delta$. By Lemma 2, there is a curve C' joining them of length $L' \leqq \omega' r_{y'y^*} < \delta'$; all of C' except possibly its end points lies in R. If x' and x^* are interior points of C', and C, of length L, is that part of C' joining them, then, as $L < L'$, Lemma 3 gives

$$\frac{| R_k(x'; x^*) |}{r_{y'y^*}^{m-\sigma_k}} < n(m+1)^n \frac{L^{m-\sigma_k}}{r_{y'y^*}^{m-\sigma_k}} \frac{\epsilon}{n(m+1)^n \omega'^m} < \epsilon.$$

As $f_k(x)$ is continuous in $R+B$, (1) shows that $R_k(x'; x^*)$ is also. Hence, letting $x' \to y'$ and $x^* \to y^*$ in the above inequality, we obtain (2) for $R_k(y'; y^*)$, as required.

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