Elementary m-harmonic cardinal B-splines

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We generalize the notion of B-spline to the thin plate splines and to other d-dimensional polyharmonic splines as defined in [Duchon, [3]]; for regular nets, we give the main properties of these "B-splines": Fourier transform, decay when $\|x\| \to \infty$, stability, integration property, links between B-splines of different orders or of different dimensions and in particular link with the polynomial B-splines, approximation using B-splines... We show that, in some sense, B-splines may be considered as a regularized form of the Dirac distribution.

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0. Introduction

In one dimension, polynomial B-splines are very useful mainly because they are quasi-interpolants, i.e. for any set of data $(jh, y_j)_{j \in A}$ (where A is some interval of $\mathbb{Z}$), the function $\sigma = \sum_{j \in A} y_j B_j$ (where B_j is the B-spline *centered* at jh) follows the shape of the data and is quite easily computed.

In many dimensions, thin plate splines, and other d-dimensional polyharmonic splines are defined in [Duchon, [3]]. They are very useful because the set of the data does not need to be on a regular grid, and because of their minimizing property (see Section 1.1).

It seems to be of interest to define some d-dimensional polyharmonic splines which could enjoy most properties of usual polynomial B-splines. This is the aim of the present paper: we define some d-dimensional m-harmonic splines which have few knots (for example 13 knots for thin plate splines), and are a quite natural extension of usual polynomial B-splines; then we derive some properties concerning the "B-splines" themselves, such as bell shape, "integration", Fourier transform, or concerning some link between some "B-splines" of different order, or of different dimensions of space (for example some link between the "thin plate B-spline" and the cubic B-spline); lastly we derive some properties concerning "B-spline approximation", i.e. the polyharmonic spline defined by $\sigma = \sum_{j\in\mathbb{Z}^d} y_j B_j$, where $B_j = B(\cdot - jh)$ is the "B-spline" centered at jh.

Of course, work is much easier if the data are on a regular infinite grid; actually, quite a lot of work has already been done in that case for interpolation and quasi-interpolation, mainly by Madych and Nelson for m-harmonic spline interpolation (see [Madych, Nelson, [8]]), by Buhmann for interpolation by radial basis function (see [Buhmann, [1]]), and by Jackson for quasi-interpolation by radial basis function (see [Jackson, [7]]). Here we present "polyharmonic B-splines" and their properties on a regular infinite grid, (also called "cardinal grid"), and, as a consequence, we call them "cardinal B-splines".

The main idea is to build the B-splines by applying a mth iterated discrete version of the Laplacean operator to the fondamental solution of the mth iterated Laplacean; this has first been done by Nira Dyn and David Levin in order to improve the stability of some linear systems (see [Dyn, Levin, [4]]). Obviously, this is still valid for scattered data, but is not done here.

1. Definition and first properties of m-harmonic B-splines

1.1. NOTATIONS USED THROUGHOUT THE PAPER

Let $d \in \mathbb{N}^*$; hereafter we will work in $\mathbb{R}^d$.
m is an integer such that $m > d/2$.
For any $x \in \mathbb{R}^d$, x_j is the jth component of x, $\|x\|$ is its Euclidean norm.
For any $x \in \mathbb{R}^d$ and $X \in \mathbb{R}^d$, xX is the Euclidean scalar product of the vectors x and X.
$\mathcal{P}_k$ is the set of polynomials on $\mathbb{R}^d$ of total degree $\leqslant k$.
Δ is the usual Laplace operator: $\Delta = \sum_{j=1}^d \partial^2/\partial x_j^2$.
δ is the Dirac distribution.
A is some finite set of $\mathbb{R}^d$ such that the zero polynomial is the only one member of $\mathcal{P}_{m-1}$ that vanishes at all the points of A.
$\alpha \in \mathbb{N}^d$ is a multi-integer. We use standard notations for multi-integers:

$$\alpha = (\alpha_1, \ldots, \alpha_d); \quad |\alpha| = \alpha_1 + \ldots + \alpha_d; \quad \alpha! = \alpha_1!\alpha_2!\ldots\alpha_d!,$$

$$\text{and } D^\alpha = \frac{\partial^{\alpha_1}}{(\partial x_1)^{\alpha_1}} \frac{\partial^{\alpha_2}}{(\partial x_2)^{\alpha_2}} \cdots \frac{\partial^{\alpha_d}}{(\partial x_d)^{\alpha_d}}.$$

For any $f \in L^1(\mathbb{R}^d)$, we denote by $\hat{f}$, or $(f)\hat{}$ its Fourier transform, i.e.

$$\hat{f}(\omega) = (f)\hat{}(\omega) = \int_{\mathbb{R}^d} f(x)\, e^{-2i\pi x\omega}\, dx.$$

Moreover, we denote by $\mathscr{F}(u)$ the Fourier transform of the tempered distribution u, or of the tempered distribution associated with the function u.
Finally, we denote by $f * g$ the convolution product of the functions f and g (or of the distributions associated to the functions f and g), namely

$$(f*g)(x)=\int_{\mathbb{R}^d} f(t)g(x-t)\,\mathrm{d}t$$

We work on "m-harmonic splines" (also called "D^m-splines", or mth order splines", as they are defined in [Duchon, [3]], summarised in [Meinguet, [2]], and detailed, especially for an infinite number of knots, in [Rabut, [11a]]. For sake of completeness, we include their definition and an important property here:

DEFINITION

Let $m \in \mathbb{N}$ such that $m > d/2$.
Among all functions interpolating $(a, z_a)_{a\in A}$, the m-harmonic spline is the only one which minimizes the semi-norm

$$|f|_m=\left(\sum_{|\alpha|=m}\int_{\mathbb{R}^d}|D^\alpha f|^2\right)^{1/2}.$$

m is called the order of the spline.

PROPERTY

As proved in [Duchon, [3]], the m-harmonic spline interpolating the points $(a, z_a)_{a\in A}$ may be written as:

$$\sigma=\sum_{a\in A}\lambda_a v_{d,m}(\cdot-a)+p_{m-1}. \tag{1}$$

where
- $(\lambda_a)_{a\in A}$ are real numbers which satisfy $\forall p\in\mathscr{P}_{m-1}$, $\Sigma_{a\in A}\lambda_a p(a)=0$,
- p_{m-1} is a polynomial of degree at most $m-1$.
- $v_{d,m}$ is a solution of the equation $\Delta^m f=\delta$; for this reason, these splines are called "polyharmonic splines", or "m-harmonic splines" (see [Madych, [8]], [Rabut, [11a]] or "mth order splines". As shown for example in [Schwartz, [16], p. 47] $v_{d,m}$ may be written $v_{d,m}(x)=\|x\|^{2m-d}[C_{d,m}\ln\|x\|+D_{d,m}]$ where, if d is odd $C_{d,m}=0$ and $D_{d,m}=E_{d,m}$, and, if d is even $C_{d,m}=E_{d,m}$ and $D_{d,m}=0$, with

$$E_{d,m}=\frac{\Gamma(d/2)}{2^m\pi^{d/2}(m-1)!\prod_{\substack{i=0\\ i\neq m-d/2}}^{m-1}(2m-2i-d)}.$$

As a particular case, if $d=1$: $\alpha_{1,m}=1/2(2m-1)!$ and we get the usual polynomial splines of degree $2m-1$, written in a quite unusual, but equivalent, way. If $d=2$, $m=2$: $\alpha_{2,2}=1/8\pi$ and we get the so-called "thin plate splines", first presented in [Harder, [6]].

REMARK: "ORDER OF A SPLINE"

Note that we call "order" of the spline the value m such that the semi-norm $|f|_m$ is minimized by the spline. So, if $d=1$, the mth order spline is a piecewise polynomial function of degree $\leqslant 2m-1$ (not, as it is quite often used, $\leqslant m-1$). The reason of this notation is that it is the only one which is easily extendable to d-dimensional spaces. By this approach we do not obtain polynomial splines of even degree: they do not enjoy the minimizing property, and so they are not polyharmonic splines. However, as we will see in Section 6.5, this work is extendable to polynomial splines of even degree.

1.2. DEFINITION OF m-HARMONIC B-SPLINES

PRESENTATION

In order to define a particular d-dimensional spline function, written in the form of eq. (1), which could be considered as a generalisation of the polynomial B-splines, let us first have a particular look on the polynomial B-splines: let δ_h^{2i} be the operator defined for any real function f by:

$$\delta_h^2 f = f(\cdot + h) - 2f + f(\cdot - h) \text{ and } \delta_h^{2i} = \left(\delta_h^2\right)^i.$$

It is well known that $h^{-2}(\delta_h^2 f)(x)$ is a discrete approximation of $f''(x)$, and, from elementary calculus, that $h^{-2m}\delta_h^{2m}$ is a discrete approximation of $f^{(2m)}$. It is known, too, (see for example [Schoenberg, [12] p. 69]) that the mth order polynomial B-spline of step h (degree $2m-1$) can be written in the form $B_m = h \cdot h^{-2m}(\delta_h^{2m} v_{1,m}) = h^{1-2m}\delta_h^{2m} v_{1,m}$. So B_m is h times the discretization (with a step h) of the $2m$th derivative applied to an exact solution of $u^{(2m)} = \delta$.

In the same way, we define $B_{d,m}$, the "d-dimensional m-harmonic B-spline", by applying h^d times the discretization (with a step h) of Δ^m, applied to $v_{d,m}$, the exact solution of $\Delta^m u = \delta$.

NOTATIONS

Let e_j be the jth basis vector of $\mathbb{R}^d$.
Let $\delta_{h,j}^2$ be the operator defined for any f from $\mathbb{R}^d$ into $\mathbb{R}$ by

$$\delta_{h,j}^2 f = f(\cdot + he_j) - 2f + f(\cdot - e_j).$$

Let Δ_h be the operator defined by $\Delta_h = h^{-2}\Sigma_{j=1}^d \delta_{h,j}^2$, and $\Delta_h^m = (\Delta_h)^m$.

DEFINITION

We call mth order cardinal B-spline (in d-dimension, and with a step h) the function $B^h_{d,m}$ defined by

$$B^h_{d,m} = h^d \Delta^m_h v_{d,m}.$$

As, by definition of $v_{d,m}$, $\Delta^m v_{d,m} = \delta$, we can write $v_{d,m} = \Delta^{-m}\delta$, and so

$$B^h_{d,m} = h^d \Delta^m_h \Delta^{-m} \delta. \tag{2}$$

In the sequel, if there is no possible confusion, we will simply write $B_{d,m}$ instead of $B^h_{d,m}$. So:

$$B_{d,m} = h^d \Delta^m_h \Delta^{-m} \delta. \tag{2'}$$

REMARK

As it will be confirmed by many properties (see Remark 3), $h^{-d} B_{d,m} = \Delta^m_h \Delta^{-m} \delta$ is a bounded, continuous function which can be regarded as some regularized approximation (in $D^{-m}L^2(\mathbb{R}^d)$) of the Dirac distribution δ: since Δ^m_h is a discretisation of Δ^m, the operator Δ^m_h can be regarded as "trying" to cancel the effect of the operator Δ^{-m} (however without cancelling its regularisation effect).

1.3. FIRST PROPERTIES OF m-HARMONIC B-SPLINES: EXPLICITATION OF $B_{d,m}$

THEOREM 1.3

Let $A_{d,m} = \{\alpha h \in (\mathbb{Z}h)^d \colon |\alpha| \leqslant m\}$

(*i*) *There exists some real coefficients* $\lambda^{d,m}_a$ *such that, for any* $x \in \mathbb{R}^d$,

$$B_{d,m} = \sum_{a \in A_{d,m}} \lambda^{d,m}_a v_{d,m}(\cdot - a).$$

The coefficients $\lambda^{d,m}_a$ *enjoy the following properties*:

(ii) $\forall f \in \mathscr{F}(\mathbb{R}^d, \mathbb{R}), \quad \sum_{a \in A_{d,m}} \lambda^{d,m}_a f(\cdot - a) = h^d (\Delta^m_h f).$

(*iii*) *Let* $k \in \mathbb{N}$, $p \in \mathbb{P}_k$ *and let* q *be defined by* $q = \sum_{a \in A_{d,m}} \lambda^{d,m}_a p(\cdot - a)$. *Then if* $k < 2m$, $q = 0$, *and if* $k \geqslant 2m$, q *is a polynomial in* $\mathscr{P}_{k-2m}$.

Proof:

(i) and (ii) follow directly from the definition, (iii) is a particular case of (ii) when $f = p$. ■

COROLLARY 1.3

(*i*) *The coefficients* $\lambda^{d,m}_a$, *defined in Theorem* 1.3 *enjoy the relations*

$$\forall p \in \mathscr{P}_{2m-1}, \quad \sum_{a \in A_{d,m}} \lambda^{d,m}_a p(\cdot - a) = 0,$$

$$\forall p \in \mathscr{P}_{2m-1}, \quad \sum_{a \in A_{d,m}} \lambda^{d,m}_a p(a) = 0.$$

(*ii*) $B_{d,m}$ *is a polyharmonic spline function.*

Proof:
(i) is a particular case of Theorem 1.3 (iii), and (ii). comes from (i). and (1), which is a characteristic property of polyharmonic spline functions.

1.4. EXAMPLES AND FIGURES

We give here some specific examples in order to illustrate the definition and to help for the comprehension of it.

$d=1$: Obviously, $B_{1,m}$ is the usual mth order (degree $2m-1$) polynomial B-spline, with equidistant knots, in one dimension.

$d=2$, $m=2$: We call $B_{2,2}$ "thin plate B-spline". Only 13 of the $h^2 \cdot \lambda_a^{2,2}$ are non-zero; their values are given in the following table, where each $h^{-2}\lambda_a^{2,2}$ is at the place of the knot with which it is associated; so for example $\lambda_{(0,0)}^{2,2} = 20/h^2$:

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 2 & -8 & 2 & \\
1 & -8 & 20 & -8 & 1. \\
 & 2 & -8 & 2 & \\
 & & 1 & &
\end{array}
$$

$d=2$, $m=3$: The values of the 25 non zero $h^2\lambda_a^{3,2}$ are the following:

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 3 & -12 & 3 & & \\
 & 3 & -24 & 57 & -24 & 3 & \\
1 & -12 & 57 & -112 & 57 & -12 & 1. \\
 & 3 & -24 & 57 & -24 & 3 & \\
 & & 3 & -12 & 3 & & \\
 & & & 1 & & &
\end{array}
$$

$d=3$, $m=2$: The values of the 25 non zero $h^2\lambda_a^{2,3}$ are the following:

if $a_3 = \pm 2$: $\lambda_{(0,0,\pm 2)} = 1$,

if $a_3 = \pm 1$:
$$
\begin{array}{ccc}
 & 2 & \\
2 & -12 & 2, \\
 & 2 &
\end{array}
$$

if $a_3 = 0$:
$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 2 & -12 & 2 & \\
1 & -12 & 42 & -12 & 1. \\
 & 2 & -12 & 2 & \\
 & & 1 & &
\end{array}
$$

FIGURES

Perspective views and contour lines are presented in figs. 1, 2, 3 for $d=2$, and $m=2, 3, 4$. As in one dimension, we can see that the higher is m, the

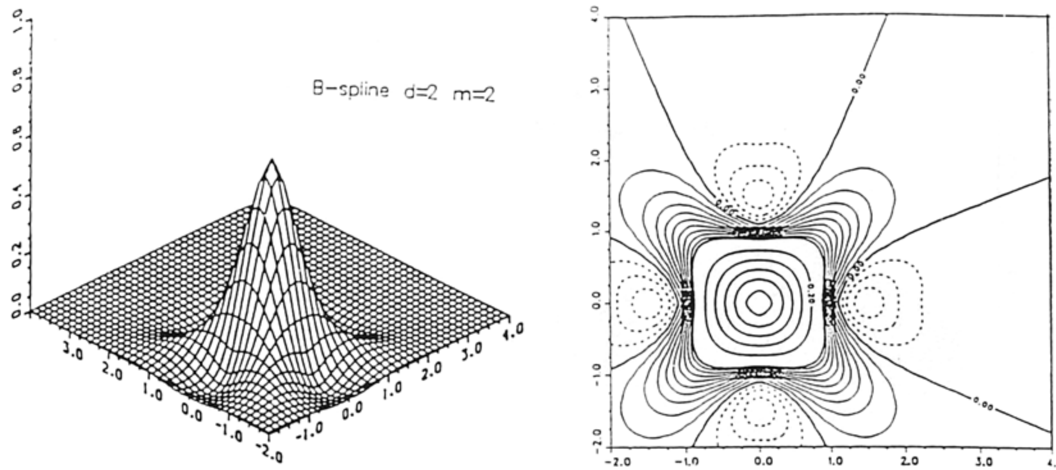

Fig. 1. Thin late B-spline.

flatter is the "B-spline". Unlike the case $d=1$, $B_{d,m}$ is not compactly supported, nor $B_{d,m}(x)$ is non negative for any $x \in \mathbb{R}^d$.

2. Fourier transform of m-harmonic cardinal B-splines

In this section, we study the Fourier transform of $B_{d,m}$; in one dimension, it is known (see for example [Schumaker, [15], p. 139]) that

$$\hat{B}_{1,m}(\omega) = \left(\frac{\sin \pi h \omega}{\pi h \omega} \right)^{2m}.$$

We generalize this formula to the function $B_{d,m}$ for $d \geqslant 2$, and examine some straightforward consequences of the so-obtained Fourier transform of $B_{d,m}$.

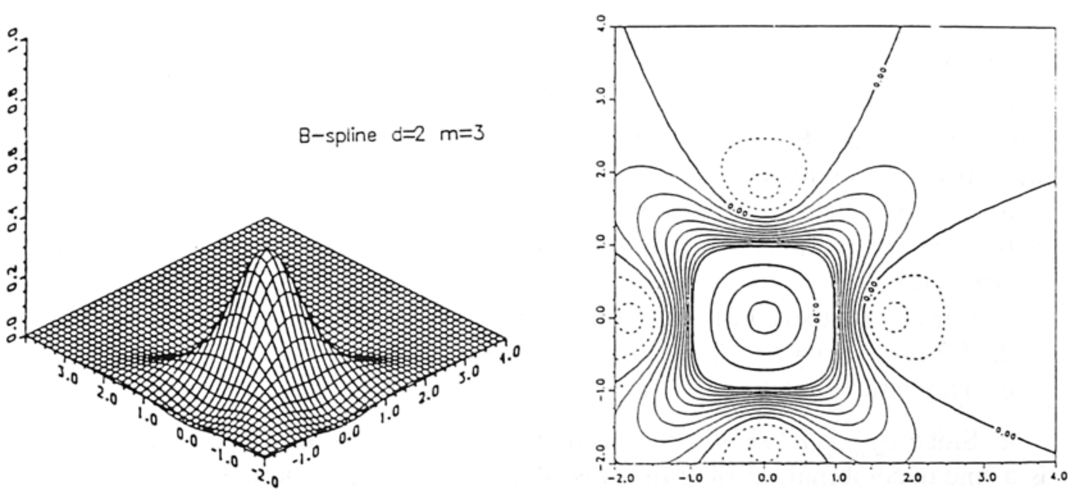

Fig. 2. Triharmonic B-spline.

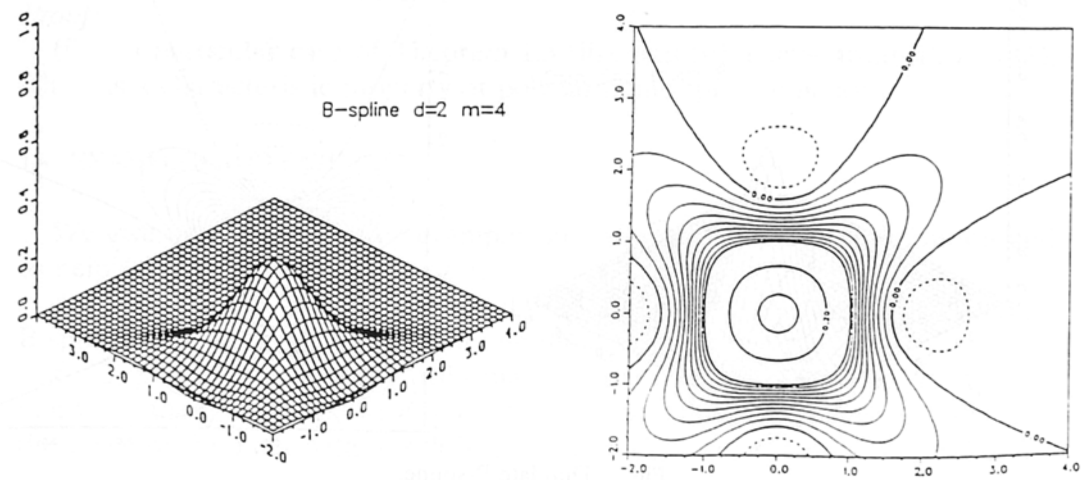

Fig. 3. Quadriharmonic B-spline.

THEOREM 2

(i) When $\| x \| \to \infty$, $B_{d,m}(x) = \mathscr{O}(\| x \|^{-d-2})$.

(ii) Let $\hat{B}_{d,m}$ *be the Fourier transform of* $B_{d,m}$. *Then*

$$\hat{B}_{d,m}(\omega) = h^d \left(\frac{\sum_{j=1}^{d} \sin^2(\pi h \omega_j)}{\| \pi h \omega \|^2} \right)^m, \tag{4}$$

which is, if we denote sin $\pi h \omega$ *the vector whose jth component is* sin $\pi h \omega_j$,

$$\hat{B}_{d,m}(\omega) = h^d \left(\frac{\| \sin \pi h \omega \|}{\| \pi h \omega \|} \right)^{2m}. \tag{5}$$

Proof

(i) The easiest way to prove (i) is to use the Fourier transform of $B_{d,m}$. But it is not obvious to see that the integral defining $\hat{B}_{d,m}$ converges. So we will first use $\mathscr{F}(B_{d,m})$, the Fourier transform of the distribution associated with $B_{d,m}$; we will prove successively:

a. $B_{d,m}$ is a "slowly increasing function".

b. For any tempered distribution u, $\mathscr{F}(\Delta_h^m u)(\omega) = (-4h^{-2})^m \| \sin \pi h \omega \|^{2m} \mathscr{F}(u)(\omega)$.

c. For any tempered distribution u, $\mathscr{F}(\Delta^m u)(\omega) = (-4\pi^2)^m \| \omega \|^{2m} \mathscr{F}(u)(\omega)$.

d. $\mathscr{F}(B_{d,m})(\omega) = h^d\, (\| \sin \pi h \omega \| / \| \pi h \omega \|)^{2m}$.

e. The assertion (i).

a. Since $v_{d,m}$ is a "slowly increasing function", this is also true for $B_{d,m}$, which is a linear combination of translates of $v_{d,m}$. So we can define $\mathscr{F}(B_{d,m})$ as the Fourier transform of the tempered distribution associated with $B_{d,m}$.

b. We have:

$$\mathcal{F}(\delta_{h,j}u)(\omega) = (e^{2i\pi h\omega_j} + e^{-2i\pi h\omega_j} - 2)\mathcal{F}(u)$$
$$= (2\cos 2\pi h\omega_j - 2)\mathcal{F}(u) = \left(-4\sin^2\pi h\omega_j\right)\mathcal{F}(u),$$

hence

$$\mathcal{F}(\Delta_h u)(\omega) = -4h^{-2}\left(\sum_{j=1}^{d}\sin^2\pi h\omega_j\right)\mathcal{F}(u),$$

and so

$$\mathcal{F}(\Delta_h^m u)(\omega) = (-4h^{-2})^m \|\sin \pi h\omega\|^{2m}\mathcal{F}(u)(\omega).$$

c. Let $v = \Delta^{-1}u$; then:

$$\mathcal{F}(u)(\omega) = \mathcal{F}(\Delta v) = \mathcal{F}\left(\sum_{j=1}^{d}\frac{\delta^2 v}{\delta x_j^2}\right) = \sum_{j=1}^{d}(2i\pi\omega_j)^2\mathcal{F}(v)$$
$$= -4\pi^2\|\omega\|^2\mathcal{F}(v)$$
$$= -4\pi^2\|\omega\|^2\mathcal{F}(\Delta^{-1}u)(\omega),$$

so

$$\mathcal{F}(\Delta^{-1}u)(\omega) = (-4\pi^2)^{-1}\|\omega\|^{-2}\mathcal{F}(u)(\omega),$$

and finally,

$$\mathcal{F}(\Delta^{-m}u) = (-4\pi^2)^{-m}\|\omega\|^{-2m}\mathcal{F}(u).$$

d. From b. and c., we get

$$\mathcal{F}(B_{d,m})(\omega) = h^d(-4h^{-2})^m\|\sin \pi h\omega\|^{2m}(-4\pi^2)^{-m}\|\omega\|^{-2m}$$
$$= h^d\left(\frac{\|\sin \pi h\omega\|}{\|\pi h\omega\|}\right)^{2m}.$$

e. It is known (see for example [Vo Khac Koan, [17], p. 26] that, any tempered distribution u is decreasing at infinity faster than $\|x\|^{-k}$ if $\int_{\mathbb{R}^d}|D^\alpha\mathcal{F}(u)(\omega)|\,d\omega$ is finite for any $\alpha\in\mathbb{N}^d$ such that $|\alpha|\leqslant k$. By using the expansion of $\sin \pi h\omega_j$, we can easily see that $|\alpha|\leqslant d+1 \Rightarrow \int_{\mathbb{R}^d}|D^\alpha\mathcal{F}(u)(\omega)|\,d\omega<\infty$. So $B_{d,m}(x) = o(\|x\|^{-d-1})$. Writing $B_{d,m}(x)$ in the equivalent form $\sum_{a\in A}\lambda_a^{d,m}\|x-a\|^{2m-d}[C_{d,m}\ln(\|x-a\|/\|x\|) + D_{d,m}(\|x-a\|-\|x\|)/\|x-a\|]$ ($C_{d,m}$ and $D_{d,m}$ are defined in Section 1.1), and writing the expansion of $\ln(\|x-a\|/\|x\|)$ (or, if d is odd, of $(\|x-a\|-\|x\|)/\|x\|$) in $u_a = (2x-a, a)/\|x\|^2 = \mathcal{O}(\|x\|^{-1})$, we can see that $B_{d,m}(x) = \mathcal{O}(\|x\|^k)$ for some $k\in\mathbb{Z}$. Since $B_{d,m}(x) = o(\|x\|^{-d-1})$, we have $B_{d,m}(x) = \mathcal{O}(\|x\|^{-d-2})$. ■

COROLLARY 2

(i) $B_{d,m}$ is a totally positive function:

$$\forall t_1,\dots,t_k \in \mathbb{R}^d, \quad \forall \xi_1,\dots,\xi_k \in \mathbb{C}, \sum_{i,j=1}^{k} \xi_i \bar{\xi}_j B_{d,m}(t_i - t_j) \geqslant 0. \tag{6}$$

(ii) Stability condition:

Let $\|\cdot\|_2$ denote the standard norms on $l^2(\mathbb{Z}^d)$ and on $L^2(\mathbb{R}^d)$.

There exist positive constants C_1 and C_2, depending only on h, d, and m. such that, for any $y \in l^2(\mathbb{Z}^d)$,

$$C_1 \| y \|_2 \leqslant \left\| \sum_{j \in \mathbb{Z}^d} y_j B_{d,m}(\cdot/h - j) \right\|_2 \leqslant C_2 \| y \|_2. \tag{7}$$

(iii) Let m and m' be two integers such that $m > d/2$ and $m' > d/2$. Then:

$$B_{d,m+m'} = B_{d,m} * B_{d,m'}. \tag{8}$$

Proof

(i) It is known (see for example [Gelfand, Villenkin, [5]]) that (i) is equivalent to $\forall \omega \in \mathbb{R}^d$, $\hat{B}_{d,m}(\omega) \geqslant 0$.

(ii) As it is well known (see for example [Cohen, [2]]), (7) is equivalent to

$$\forall \omega \in \mathbb{R}^d, \quad 0 < C_1 \leqslant \sum_{j \in \mathbb{Z}^d} | \hat{B}_{d,m}((\omega + j)/h) |^2 \leqslant C_2. \tag{9}$$

So, let us prove (9). Now, using periodicity and symmetry, it is sufficient to prove (9) for any $\omega \in [-1/2h, 1/2h]^d$.

First, let us prove

$$h^{2d}(2/\pi)^{4m} \leqslant \sum_{j \in \mathbb{Z}^d} | \hat{B}_{d,m}((\omega + j)/h) |^2:$$

for any real number $\chi \in [-1/2h, 1/2h]$, we have $|\sin \pi h \chi| / |\pi h \chi| \geqslant 2/\pi$; so for any $k \in [1, d]$, $|\sin \pi h \omega_k| \geqslant (2/\pi) |\pi h \omega_k|$, hence $(\sum_{k=1}^{d} \sin^2 \pi h \omega_k) \geqslant (2/\pi)^2 (\sum_{k=1}^{d} (\pi h \omega_k)^2)$, which is $\hat{B}_{d,m}(\omega) \geqslant h^d (2/\pi)^{2m}$. Hence $| \hat{B}_{d,m}(\omega) |^2 \geqslant h^{2d}(2/\pi)^{4m}$, and so $\sum_{j \in \mathbb{Z}^d} | \hat{B}_{d,m}((\omega + j)/h) | \geqslant h^{2d}(2/\pi)^{4m}$.

Let us now prove the last inequality in (9): $B_{d,m}$ is a function which is continuous on $\mathbb{R}^d$ and which enjoys, for $j \neq 0$ and $\omega \in [-1/2h, 1/2h]^d$:

$$| \hat{B}_{d,m}((\omega + j)/h) |^2 \leqslant h^{2d} d^{2m} \| \pi(\omega + j) \|^{-4m} \leqslant h^{2d} d^{2m} \pi^{-4m} (\| j \| - 1/2)^{-4m}$$

Now, the series $\sum_{j \in \mathbb{Z}^d} (\| j \| - 1/2)^{-4m}$ is convergent as $4m > d$ (which is true as, by definition of m-harmonic splines, $m > d/2$), and so, the series $\sum_{j \in \mathbb{Z}^d} \| \hat{B}_{d,m}((\omega + j)/h) |^2$ is bounded by a numerical convergent series and therefore is continuous and bounded.

We can now conclude that (9) is valid, with $C_1 = h^{2d}(2/\pi)^{4m}$, and $C_2 = \hat{B}_{d,m}(0) + h^{2d} d^{2m} \pi^{-4m} \sum_{j \in \mathbb{Z}^d - \{0\}} (\| j \| - 1/2)^{-4m}$.

(iii) is a direct consequence of (4), as $B_{d,m+m'} = (\hat{B}_{d,m+m'}) = (\hat{B}_{d,m} \cdot \hat{B}_{d,m'}) = (\hat{B}_{d,m}) * (\hat{B}_{d,m'}) = B_{d,m} * B_{d,m'}$ ■

REMARKS

a. In one dimension, it is well known that $B_{1,n}(x) \geqslant 0$. This *is not* true for $d \geqslant 2$ and must be replaced by total positivity (property (6)). Note that, in one dimension, even degree polynomial splines are not totally positive functions... but they are not polyharmonic splines, as they do not enjoy the minimizing property (section 1.1).

b. Equation (7) is important as thanks to it, multiresolution analysis and wavelet decomposition for polyharmonic splines are possible (see [Micchelli et al., [10]]).

3. Integration properties

In this section, we study some properties of $B_{d,m}$ involving the integration or derivation of $B_{d,m}$: they show some strong link between different $B_{d,m}$ for different values of d or of m.
For any $x \in \mathbb{R}^d$, we denote $x^j = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d)$ the projection of x onto $\mathbb{R}^{d-1}$, obtained by removing the jth component of x.

THEOREM 3

$B_{d,m}$ enjoys the following properties:

(i) Integration in one direction:

$$\int_{\mathbb{R}} B_{d,m}(x)\, \mathrm{d}x_j = hB_{d-1,m}(x^j). \tag{10}$$

(ii) Integration over the whole space:

$$\int_{\mathbb{R}^d} B_{d,m}(x)\, \mathrm{d}x = h^d. \tag{11}$$

(iii) Derivation: for $k \in \mathbb{N}$, $k < m - d/2$, we have:

$$\Delta^k B_{d,m} = \Delta_h^k B_{d,m-k}. \tag{12}$$

(iv) Convolution with some $p \in \mathscr{P}_1$: Let $p \in \mathscr{P}_1$, then,

$$\int_{\mathbb{R}^d} p(t) B_{d,m}(\cdot - t)\, \mathrm{d}t = h^d p. \tag{13}$$

(v) Convolution with some $\Delta^m f$: Let $f \in \mathscr{W}^{2m,\infty}(\mathbb{R}^d)$; then for any $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} B_{d,m}(x-t) \Delta^m f(t)\, \mathrm{d}t = h^d (\Delta_h^m f)(x). \tag{14}$$

(vi) Convolution with some $f \in \mathscr{C}^2(\mathbb{R}^d)$: Let $f \in \mathscr{C}^2(\mathbb{R}^d)$, then for any $x \in \mathbb{R}^d$, there exists some $C \geqslant 0$ such that:

$$\left| \int_{\mathbb{R}^d} h^{-d} B_{d,m}(x-t) f(t) \,\mathrm{d}t - f(x) \right| \leqslant Ch^2. \tag{15}$$

Proof

(i) From eq. (4), we obviously have

$$\hat{B}_{d,m}(X_1, \ldots, X_{j-1}, 0, X_{j+1}, \ldots, X_d) = h\hat{B}_{d-1,m}(X^j),$$

i.e.

$$\int_{\mathbb{R}^d} B_{d,m}(x) \,\mathrm{e}^{2\mathrm{i}\pi x^j \cdot X^j} \,\mathrm{d}x = h\hat{B}_{d-1,m}(X^j),$$

and thus

$$\int_{\mathbb{R}^{d-1}} \left[\int_{\mathbb{R}} B_{d,m}(x) \,\mathrm{d}x_j \right] \mathrm{e}^{2\mathrm{i}\pi x^j \cdot X^j} \,\mathrm{d}x^j = \widehat{hB}_{d-1,m}(X^j),$$

which means that $\int_{\mathbb{R}} B_{d,m}(x) \,\mathrm{d}x_j$ and $hB_{d-1,m}$ are both continuous functions from $\mathbb{R}^{d-1}$ into $\mathbb{R}$ which have same Fourier transform; so they are equal.

(ii) Obviously,

$$\int_{\mathbb{R}^d} B_{d,m}(x) \,\mathrm{d}x = \lim_{\|X\| \to 0} \hat{B}_{d,m}(X) = h^d.$$

(iii)

$$\begin{aligned} \Delta^k B_{d,m} &= \Delta^k \left(h^d \Delta_h^m \Delta^{-m} \delta \right) = h^d \Delta_h^m \Delta^{-m+k} \delta \\ &= \Delta_h^k \left(h^d \Delta_h^{m-k} \Delta^{-m+k} \delta \right) = \Delta_h^k B_{d,m-k}. \end{aligned}$$

(iv) First, let us prove that $\forall p \in \mathscr{P}_1$, $\int_{\mathbb{R}^d} p(t) B_{d,m}(t) \,\mathrm{d}t = h^d p(0)$:
 * if $p \in \mathscr{P}_0$, it is exactly (ii);
 * if $p(t) = a_i t_i$ for some $i \in [1, d]$ and $a_i \in \mathbb{R}$: $p(t) B_{d,m}(t)$ is then an odd function in t_i; its integral is null.
 * for any $p \in \mathscr{P}_1$: $p(t) = b + \sum_{i=1}^d a_i t_i$, and so $\int_{\mathbb{R}^d} p(x) B_{d,m}(x) \,\mathrm{d}x = b = p(0)$.
 Then we get (iv):

$$\int_{\mathbb{R}^d} p(t) B_{d,m}(t-x) \,\mathrm{d}t = \int_{\mathbb{R}^d} p(t+x) B_{d,m}(t) \,\mathrm{d}x = p(x).$$

(v) We have $\int_{\mathbb{R}^d} B_{d,m}(x-t) \Delta^m f(t) \,\mathrm{d}t = (B_{d,m} * \Delta^m f)(x)$, hence we have

$$B_{d,m} * (\Delta^m f) = (\Delta^m B_{d,m}) * f = \left(h^d \Delta_h^m \delta \right) * f = h^d \cdot \Delta_h^m (\delta * f) = h^d \cdot \Delta_h^m f.$$

(vi) Applying (v) to a function g such that $\Delta^m g = f$, we obtain (13) by using the fact that, for any $x \in \mathbb{R}^d$, there exists $C \geqslant 0$ such that $| \Delta_h^m g(x) - \Delta^m g(x) | \leqslant Ch^2$.

COROLLARY 3

There holds, in the sense of distributions:

$$\lim_{h \to 0} h^{-d} B_{d,m} = \delta. \tag{16}$$

Proof

From theorem 3(vi), for any $\phi \in \mathscr{D}$ ($\mathscr{D}$ is the usual set of infinitely derivable, compactly supported functions from $\mathbb{R}^d$ to $\mathbb{R}$),

$$\lim_{h \to 0} \langle \phi, h^{-d} B_{d,m} \rangle = \phi(0) = \langle \phi, \delta \rangle$$

(here $\langle \cdot , \cdot \rangle$ denotes the duality product), so we have (16).

REMARK 3

B-splines as a regularization of the Dirac distribution

The properties of $h^{-d}B_{d,m}$ shown in this Section confirm that $h^{-d}B_{d,m}$ can be regarded as some regularized approximation (in $D^{-m}L^2(\mathbb{R}^d)$) of the Dirac distribution:

a. Theorem 3(iv) shows that, applied to any polynomial $p \in \mathscr{P}_1$, $h^{-d}B_{d,m}$ has the same effect than the Dirac distribution has, applied to any function f.

b. Theorem 3(v) shows that for any $f \in \mathscr{C}^2(\mathbb{R}^d)$, $h^{-d}B_{d,m}$ has the same effect as the Dirac distribution has, with an error being in $Ah^2 \| D^2 f \|$.

c. Corollary 3 shows that the smaller h is, the "nearer" $h^{-d}B_{d,m}$ is from the Dirac distribution (while staying in $D^{-m}L^2(\mathbb{R}^d)$).

4. B-spline approximation

We now use the notation $B^h_{d,m}$ instead of $B_{d,m}$, since h appears explicitly in the formulae.

The aim of this section is to show that $B^h_{d,m}$ is a "quasi-interpolant", and to study the properties of the associated quasi-interpolation. General results about quasi-interpolation are established in the more general context of radial basis functions in [Jackson, [7]].

DEFINITIONS

(i) Given some vector $y \in \mathbb{R}^{\mathbb{Z}^d}$, $y = (y_j)_{j \in \mathbb{Z}^d}$ such that for any j in $\mathbb{Z}^d$, $|y_j| \leqslant C(1 + \| j \|)^{2-\epsilon}$ (where C and ϵ are some non-negative real valued numbers), let

$$a_{m,h} y = \sum_{j \in \mathbb{Z}^d} y_j B^h_{d,m}(\cdot - jh). \tag{17}$$

$a_{m,h} y$ is called "m-harmonic B-spline approximation of the points $(jh, y_j)_{j \in \mathbb{Z}^d}$", or simply "B-spline approximation of y".

(ii) For any function f from $\mathbb{R}^d$ into $\mathbb{R}$, such that, near $\|x\|$ infinite, $|f(x)| \leqslant \mathscr{O}(\|x\|^{2-\epsilon}$ (where ϵ is some non-negative real number), and for any bounded open subset of $\mathbb{R}^d$, say Ω, let

$$a_{m,h}f = \sum_{j\in\mathbb{Z}^d} f(jh)B^h_{d,m}(\cdot - jh) \tag{18}$$

and

$$a_{m,h,\Omega}f = \sum_{t\in(\mathbb{Z}h)^d\cap\Omega} f(t)B_{d,m}(\cdot - t). \tag{19}$$

We now study the properties of the so defined operators $a_{m,h}$ and $a_{m,h,\Omega}$.

THEOREM 4

(*i*) *For all x in $\mathbb{R}^d$,*

$$\sum_{j\in\mathbb{Z}^d} B^h_{d,m}(x - jh) = 1.$$

As a particular case,

$$\sum_{j\in\mathbb{Z}^d} B^h_{d,m}(jh) = 1.$$

(*ii*) *B-spline approximation is reproducing $\mathscr{P}_1$; that is to say:*

$$\forall p \in \mathscr{P}_1,\ a_{m,h}p = p. \tag{20}$$

(*iii*) *Let $k \in \mathbb{N}$ so that $k < m - d/2$. Then $\Delta^k a_{m,h}y$ is a m-harmonic cardinal spline; more precisely:*

$$\Delta^k(a_{m,h}y) = \Delta^k_h(a_{m-k,h}y). \tag{21}$$

(*iv*) *Let Ω be a bounded open subset of $\mathbb{R}^d$.*
Let $\Omega_\epsilon = \{t \in \Omega: |t - t'| \leqslant \epsilon \Rightarrow t' \in \Omega\}$ for some $\epsilon > 0$.
Let $l \in \mathbb{N}^*$ and $f \in \mathscr{C}^l(\mathbb{R}^d)$.
Suppose all derivatives of f, up to order l, be bounded on $\mathbb{R}^d$. Then:

$$\|a_{m,h}f - f\|_\infty \underset{h\to 0}{=} \begin{cases} \mathscr{O}(h) & \text{if} \quad l = 1 \\ \mathscr{O}(h^2|\ln h|) & \text{if} \quad l \geqslant 2 \end{cases}, \tag{22}$$

$$\|a_{m,h,\Omega}f - f\|_{\infty,\Omega_\epsilon} \underset{h\to 0}{=} \begin{cases} \mathscr{O}(h) & \text{if} \quad l = 1 \\ \mathscr{O}(h^2|\ln h|) & \text{if} \quad l \geqslant 2 \end{cases} \tag{23}$$

Proof

(i) is a consequence of Poisson's formula:

$$\forall x \in \mathbb{R}^d,\ \sum_{j\in\mathbb{Z}^d} B^h_{d,m}(x + jh) = \sum_{k\in\mathbb{Z}^d} \hat{B}^h_{d,m}(k/h)\, \mathrm{e}^{2\mathrm{i}\pi k\cdot x/h}.$$

Now, if $k \neq 0$, $\hat{B}_{d,m}(k/h) = 0$, and if $k = 0$, $\hat{B}_{d,m}(k/h)\, e^{2i\pi k \cdot x/h} = 1$. So we get (i).

(ii) is a consequence of [Jackson, [7], p. 63]; here is a direct proof of (20): Any $p \in \mathscr{P}_1$ can be written as $p(x) = q_0 + q(x)$, where $q_0 \in \mathbb{R}$ and q is a linear function of x (and so $q(jh) = q(x) - q(x - jh)$). Then:

$$(a_{m,h}p)(x) = (q_0 + q(x)) \sum_{j \in \mathbb{Z}^d} B^h_{d,m}(x - jh) - \sum_{j \in \mathbb{Z}^d} q(x - jh) B^h_{d,m}(x - jh)$$

and from (i), we get

$$(a_{m,h}p)(x) = p(x) - \sum_{j \in \mathbb{Z}^d} q(x - jh) B^h_{d,m}(x - jh).$$

Now, from Poisson formula,

$$\sum_{j \in \mathbb{Z}^d} q(x - jh) B^h_{d,m}(x - jh) = h^{-d} \sum_{j \in \mathbb{Z}^d} \widehat{qB}^h_{d,m}(j/h)\, e^{2i\pi j \cdot x/h}.$$

Setting $q(x) = \sum_{l=1}^d a_l x_l$, we get

$$\widehat{qB}^h_{d,m} = \sum_{l=1}^{d} \frac{a_l}{-2i\pi} \frac{\partial \hat{B}^h_{d,m}}{\partial X_l},$$

hence, by using (4): $\forall j \in \mathbb{Z}^d$, $\widehat{qB}^h_{d,m}(j/h) = 0$. Therefore we obtain $\sigma_{m,h}(x) = 0$.

(iii) is a direct consequence of the definition of $a_{m,h}y$ and $a_{m-k,h}y$, and of Theorem 3(iii).

(iv) is a direct consequence of some results on quasi-interpolants proved by I.J. Jackson [Jackson, [7], p. 91 and p. 104].

REMARKS

(i) Theorem 4 shows that $B^h_{d,m}$ is a "quasi-interpolant", as:

a. If data $(jh, y_j)_{j \in \mathbb{Z}^d}$ lie in a plan (i.e. $\exists p \in \mathscr{P}_1$: $\forall j \in \mathbb{Z}^d$, $y_j = p(jh)$), then $a_{m,h}y$ reproduces this plan.

b. For any $f \in \mathscr{C}^1(\mathbb{R}^d)$, all first derivatives of f being bounded, $\| a_{m,h}f - f \|_\infty \xrightarrow[h \to 0]{} 0$, and so $a_{m,h}f$ is some polyharmonic spline approximating f; consequently, $a_{m,h}y$ is some polyharmonic spline aproximating the data $(jh, y_j)_{j \in \mathbb{Z}^d}$.

(ii) It is of interest to note that thanks to the decay of $B^h_{d,m}(x)$ when $\| x \| \to \infty$, it is only necessary, for numerical purposes, to compute a finite number of terms of the infinite summation defining $a_{m,h}y$. In practical applications, especially for thin plate B-spline approximation, the number of terms needed to be numerically evaluated is quite low: computing $(a_{m,h}y)(x)$ needs about 25 terms for each x if we require an error less than 10^{-3}, about 49 terms for an error less than 10^{-4} (these values are given for thin plate splines and for y such that $\| y \|_\infty \leqslant 1$).

5. Consequences on interpolation

Given $y \in \mathbb{R}^{\mathbb{Z}^d}$, a spline function σ is interpolating y if σ interpolates the points $(jh, y_j)_{j \in \mathbb{Z}^d}$, i.e. if $\forall j \in \mathbb{Z}^d$, $\sigma(jh) = y_j$.

Results about such an interpolation are established in the more general context of radial basis functions in [Buhmann, [1]], and for polyharmonic splines in [Madych, [8]]. The aim of this section is to add some results concerning the Fourier Transform of the further defined Lagrangean polyharmonic spline and to write any interpolatng spline in terms of translates of $B^h_{d,m}$ functions.

5.1. NOTATIONS

We first need the following result, in [Buhmann, [8]]:

There exists one and only one m-harmonic spline, which we will denote by $L^h_{d,m}$, such that

$$\left.\begin{array}{ll} a. & L^h_{d,m}(0) = 1;\ \forall j \in \mathbb{Z}^d - \{0\},\ L^h_{d,m}(jh) = 0 \\ b. & L^h_{d,m}(x) \text{ decays exponentially when } \|x\| \to \infty \end{array}\right\}. \tag{23}$$

The so-defined $L^h_{d,m}$ is called "Lagrangean m-harmonic cardinal spline".

We denote b the vector defined by: $\forall j \in \mathbb{Z}^d$, $b_j = B^h_{d,m}(jh)$, and by ϵ the vector defined by: $\epsilon_0 = 1$, $\epsilon_j = 0$ if $j \in \mathbb{Z}^d - \{0\}$. $l^1(\mathbb{Z}^d)$ is the set of vectors u such that $\sum_{j \in \mathbb{Z}^d} |u_j|$ is finite.

For any $u \in l^1(\mathbb{Z}^d)$ and any $v \in l^1(\mathbb{Z}^d)$, we denote by $u * v$ the convolution of u and v:

$$(u * v)_k = \sum_{j \in \mathbb{Z}^d} u_j v_{k-j}.$$

For any $u \in l^1(\mathbb{Z}^d)$, we denote by $\hat{u}$ the function defined by ($\omega \in \mathbb{R}^d$):

$$\hat{u}(\omega) = \sum_{j \in \mathbb{Z}^d} u_j \, \mathrm{e}^{-2\mathrm{i}\pi jh\omega}.$$

5.2. FOURIER TRANSFORM OF $L_{d,m}$

THEOREM 5.2

Let $L^h_{d,m}$ be defined above by (23), $\hat{L}^h_{d,m}$ the Fourier transform of $L^h_{d,m}$. Let b and $\hat{b}$ be defined as above. Then:

(*i*)

$$\hat{L}^h_{d,m} = \frac{\hat{B}^h_{d,m}}{\hat{b}} = \frac{h^d \hat{B}^h_{d,m}}{\sum_{j \in \mathbb{Z}^d} \hat{B}^h_{d,m}(\cdot - j/h)}. \tag{24}$$

(*ii*) $\forall\omega\in\mathbb{R}^d$, $\hat{L}^h_{d,m}(\omega)\geqslant 0$. *As a consequence*, $L^h_{d,m}$ *is a totally positive function.*

(*iii*) $\hat{L}^h_{d,m}(0)=h^d$, *and so* $\int_{\mathbb{R}^d}L^h_{d,m}(x)\,dx=h^d$.

(*iv*) *As a particular case when* $d=1$:

$$\hat{L}^h_{1,m}(\omega)=\left(\frac{\sin\pi h\omega}{\pi h\omega}\right)^{2m}\cdot\frac{h}{B_{1,m}(0)+2\sum_{j=1}^{m}B_{1,m}(jh)\cos(2\pi\omega jh)},$$

which is, for cubic splines:

$$\hat{L}^h_{1,2}(\omega)=\left(\frac{\sin\pi h\omega}{\pi h\omega}\right)^{4}\cdot\frac{h}{1-(2/3)\sin^2\pi h\omega},$$

and, for quintic splines:

$$\hat{L}^h_{1,3}(\omega)=\left(\frac{\sin\pi h\omega}{\pi h\omega}\right)^{6}\cdot\frac{h}{1-\sin^2\pi h\omega+(2/15)\sin^4\pi h\omega}.$$

Proof

(i) In [Madych, [8]], it is proved that every m-harmonic cardinal spline σ satisfies the relation

$$\sigma=\sum_{j\in\mathbb{Z}^d}f(jh)L^h_{d,m}(\cdot-jh).$$

Applying that Theorem to $B^h_{d,m}$, we get:

$$B^h_{d,m}=\sum_{j\in\mathbb{Z}^d}B^h_{d,m}(jh)L^h_{d,m}(\cdot-jh).$$

So:

$$\hat{B}^h_{d,m}=\sum_{j\in\mathbb{Z}^d}B^h_{d,m}(jh)\,\mathrm{e}^{-2\mathrm{i}\pi jh\cdot}\hat{L}^h_{d,m}=\hat{L}^h_{d,m}\sum_{j\in\mathbb{Z}^d}B^h_{d,m}(jh)\,\mathrm{e}^{2\mathrm{i}\pi jh\cdot}$$

$$=\hat{L}^h_{d,m}\hat{b}.$$

and so

$$\hat{L}^h_{d,m}=\hat{B}^h_{d,m}/\hat{b}.$$

The second equation is then obtained directly by Poisson's formula:

$$\hat{b}=\sum_{j\in\mathbb{Z}^d}B^h_{d,m}(jh)\,\mathrm{e}^{-2\mathrm{i}\pi jh\cdot}=h^{-d}\sum_{j\in\mathbb{Z}^d}\hat{B}^h_{d,m}(\cdot-j/h).$$

(ii), (iii) are simple consequences of (i).

(iv) is a consequence of (i) and of the expression of $\hat{B}^h_{d,m}$ (5). ■

REMARK

From (i) with (5), we find

$$\hat{L}^h_{d,m} = \frac{h^d}{\sum_{j\in\mathbb{Z}^d}\left(\frac{\|h\cdot\|}{\|h\cdot -j\|}\right)^{2m}} = \frac{h^d}{1+\sum_{j\in\mathbb{Z}^d-\{0\}}\left(\frac{\|\cdot\|}{\|\cdot -j/h\|}\right)^{2m}}.$$

This expression is taken by Buhmann and Madych as the definition of $L^h_{d,m}$.

5.3. EXPRESSION OF A SPLINE IN TERMS OF TRANSLATES OF $B_{d,m}$

Let us remind the following result, due to Madych: for any s of polynomial growth (i.e. there exist some real constants c and α such that $|s_j| \leqslant c(1+|j|^\alpha)$, there exist one an only one m-harmonic spline of polynomial growth (i.e. there exist some real constants c and α such that $\sigma(x) \leqslant c(1+\|x\|)^\alpha$) interpolating s; it has a unique representation in terms of translates of $L^h_{d,m}$, namely

$$\sigma = \sum_{j\in\mathbb{Z}^d} s_j L^h_{d,m}(\cdot - jh). \tag{25}$$

The aim of this section is to write σ in terms of translates of $B^h_{d,m}$ functions, if $s \in l^1(\mathbb{Z}^d)$.

THEOREM 5.3

Let $s \in l^1(\mathbb{Z}^d)$. Let σ be the m-harmonic spline interpolating s, defined as above. Then:

(i) There exists a unique vector c such that

$$\sigma = \sum_{j\in\mathbb{Z}^d} c_j B^h_{d,m}(\cdot - jh). \tag{26}$$

Furthermore, we have, for any j in $\mathbb{Z}^d$:

$$c_j = \int_{[0,\,1/h]^d} \frac{\hat{s}(\omega)}{\hat{b}(\omega)} \mathrm{e}^{2\mathrm{i}\pi jh\omega}\, \mathrm{d}\omega. \tag{27}$$

(ii) Let a be the vector defined by ($j \in \mathbb{Z}^d$):

$$a_j = \int_{[0,\,1/h]^d} \frac{1}{\hat{b}(\omega)} \mathrm{e}^{2\mathrm{i}\pi jh\omega}\, \mathrm{d}\omega. \tag{28}$$

Then:

$$L^h_{d,m} = \sum_{j\in\mathbb{Z}^d} a_j B^h_{d,m}(\cdot - jh). \tag{29}$$

Furthermore,

$$c = a * s, \tag{30}$$

$$s = c * b, \tag{31}$$

$$\epsilon = a * b. \tag{32}$$

Proof

(i) First, let us prove the unicity of c: suppose σ is written as in (26), and let us then prove (27):
From (26), we get

$$\hat{\sigma}(\omega) = \sum_{j\in\mathbb{Z}^d} c_j\, \mathrm{e}^{2i\pi jh\omega}\hat{B}^h_{d,m}(\omega)$$
$$= \hat{B}^h_{d,m}(\omega)\hat{c}(\omega), \tag{33}$$

and, from (25),

$$\hat{\sigma}(\omega) = \sum_{j\in\mathbb{Z}^d} s_j\, \mathrm{e}^{-2i\pi jh\omega}\hat{L}^h_{d,m}(\omega)$$
$$= \hat{L}^h_{d,m}(\omega)\hat{s}(\omega). \tag{34}$$

Comparing (33) and (34), we get, using (24):

$$\hat{c}(\omega) = \frac{\hat{s}(\omega)}{\hat{b}(\omega)}, \tag{35}$$

and so (27).
Now, to prove (26), we can go backwards: from (27) we have (35), and from (35), (34) and (24), we get (31), and so (26).

(ii) (28) and (29) are (27) and (26) for $s = \epsilon$.
Let us prove (30): from (25) and (29), we get

$$\sigma = \sum_{j\in\mathbb{Z}^d} s_j\left(\sum_{k\in\mathbb{Z}^d} a_k B^h_{d,m}(\cdot - kh - jh)\right)$$
$$= \sum_{l\in\mathbb{Z}^d}\left(\sum_{j\in\mathbb{Z}^d} s_j a_{l-j}\right) B^h_{d,m}(\cdot - lh).$$

(30) is then a consequence of the unicity of the decomposition of σ in terms of translates of $B^h_{d,m}$.
(31) is a direct consequence of (26):

$$s_k = \sigma(kh) = \sum_{j\in\mathbb{Z}^d} c_j B^h_{d,m}(kh - jh) = \sum_{j\in\mathbb{Z}^d} c_j b_{k-j}.$$

(32) is (31) when $s = \epsilon$.

REMARKS

a. (32) shows that a may be obtained by deconvolution of the vector ϵ by the vector b; numerically this is quite easily done by using Fast Fourier Transform.

b. (26) and (29) give the expression of $L^h_{d,m}$ and of the spline σ interpolating some vector $s \in l^1(\mathbb{R}^d)$ in terms of translates of $B^h_{d,m}$. Since $B^h_{d,m}$ is easily computable, these formulae enable us to compute $L^h_{d,m}$ and σ. Furthermore, thanks to the decay of $B^h_{d,m}(x-jh)$ when $\|x-jh\| \to \infty$, it is numerically sufficient to compute only a finite number of terms of the sums (26) or (29).

6. Conclusion

We have defined functions $B_{d,m}$ which can be regarded as an extension, to all polyharmonic splines, of the usual polynomial B-spline in one dimension. Let us summarize some of the main properties of these $B_{d,m}$.

6.1. PROPERTIES WHICH ARE NOT SATISFIED BY $B_{d,m}$ IF $d \geqslant 2$

Some properties satisfied by polynomial B-splines are not satisfied by polyharmonic B-splines if $d \geqslant 2$; such are:
a. Positivity: The relation $\forall x \in \mathbb{R}^d$, $B_{d,m}(x) \geqslant 0$, which is true if $d=1$, is *false* if $d \geqslant 2$. It is a quite important property for its consequences (the most important consequence is that the B-spline approximation of some data is, in one dimension, within the convex hull of these data). However, we can notice that the negative part of the function is never very important (for example, numerically we have $(B_{2,2}(x)) > -0{,}04$); furthermore since the interpolating cardinal spline is not a positive function, we can think that positivity is not essential. Moreover, in [Rabut, [11c]], it is shown that other interesting polyharmonic spline functions may be regarded as B-splines too, and are not positive functions...

However $B_{d,m}$ is a *totally positive function* (Theorem 2(ii)), as is $L^h_{d,m}$ (Theorem 5a(ii)). It seems to us that total positivity is the intrinsic property of polyharmonic B-splines (note that even degree polynomial B-splines are not totally positive... neither are they polyharmonic splines).
b. Compact support: Polynomial B-splines are compactly supported... and polyharmonic B-splines *are not* if $d \geqslant 2$. Let us notice that polyharmonic splines are linear combination of $v_{d,m}(x-jh)$, therefore it is impossible to obtain a non-zero compact-supported spline (when $d \geqslant 2$) and there is no possibility to extend the property of compact support to dimension $d \geqslant 2$. Nevertheless, we must notice that the value of $B_{d,m}(x)$, even for quite small $\|x\|$ is very low ($B_{2,2}(3, 0) \simeq 0.003$, $B_{2,2}(4, 0) \simeq -0.0006$ for example), and that $B_{d,m}(x) = \mathcal{O}(1/\|x\|^{-d-2})$; so $B_{d,m}$ can be called *"numerically compactly supported function"*, and as we saw in Section 4, a series as $\sum_{j \in \mathbb{Z}^d} y_j B_{d,m}(ih)$ needs only a few terms to be computed.

So, the general shape of these functions does extend in a quite appropriate way the shape of polynomial B-splines.

6.2. PROPERTIES SATISFIED BY $B_{d,m}$

a. Their definition (Section 1.2) is very close to the definition of polynomial B-splines.

b. They are "bell-shaped" functions... but actually this is a direct consequence of total positivity.

c. As they are reproducing $\mathscr{P}_1$ and "Bell-shaped", they are quasi-interpolant functions: the spline function $a_{m,h}y = \sum_{j \in \mathbb{Z}^d} y_j B_{d,m}(\cdot - jh)$ follows the shape of the data $(jh, y_j)_{j \in \mathbb{R}^d}$, and, if $\forall j \in \mathbb{Z}^d$, $y_j = f(jh)$, then $a_{m,h}y \underset{h \to 0}{\longrightarrow} f$. So, it is quite natural to use $B^h_{d,m}$ in a similar way as we usually use polynomial B-splines.

d. Fourier transform and convolution relation $B_{d,m+m'} = B_{d,m} * B_{d,m'}$ are quite simple and pretty extensions of the Fourier transform and of the convolution relation of the polynomial B-splines.

e. The property linking up $B_{d,m}$ to $B_{d-1,m}$, and, as a consequence of it, $B_{d,m}$ to $B_{1,m}$, is another illustration of the very close intrinsic link between $B_{d,m}$ and $B_{1,m}$. Furthermore derivation property linking $B_{d,m}$ to $B_{d,m-k}$ is a pretty extension of the same relation in one dimension.

f. Last but not least: cardinal splines can be written in terms of translates of $B_{d,m}$, and, for sufficiently decreasing data, the coefficients of the interpolating spline are explicitly known (Theorem 5b).

6.3. B-SPLINES AS REGULARISATION OF THE DIRAC DISTRIBUTION

As shown in the remark of Section 3, B-splines can be regarded as some regularization, in $D^{-m}L^2$, of the Dirac distribution, multiplied by some dimensional coefficient.

6.4. PROPOSITION OF TERMINOLOGY

According to the terminology used by Schoenberg about interpolation or approximation for equidistant data, and by Madych about splines, we propose:

a. To call "mth order polyharmonic splines", or "m-harmonic splines" all splines minimising $|f|_m$ under some discrete interpolation condition, as they are m-harmonic functions of $\mathbb{R}^d - A$, where A is a discrete subset of $\mathbb{R}^d$.

b. To add the qualifier "cardinal" when all the knots are in $(\mathbb{Z}h)^d$ (even if h is not equal to 1 and even if only a finite number of knots have corresponding non-zero coefficients).

c. To add the letter "B" to the splines defined as some regularisation of the Dirac distribution.

d. Since the ones defined in the present paper are obtained by the most elementary discretization of the Laplacean operator (and since other discretizations give other spline functions which warrant attention too, as shown in [Rabut, [11c]]), we propose to add the word "elementary" to them.

So we propose, as we used in this paper, to call the functions $B_{d,m}$ "elementary m-harmonic cardinal B-splines", or shortly when no confusion is possible, "elementary B-splines", or even "B-splines".

6.5. EXTENSIONS

First, let us mention that Nira Dyn and David Levin were the first ones to apply a discretization of bilaplacian to the function $f(x) = \| x \|^2 \ln \| x \|^2$. They did so to improve the condition number of the linear system to be solved in order to interpolate some data ([Dyn, [4]]); but as they took no care of the coefficient $E_{2,2} = 1/8\pi$ (see Section 1.1) most of the properties shown here were not satisfied.

In one dimension, for even degree polynomial splines, all the work is directly extendable, when using $(\delta_h f)(x) = f(x + h/2) - f(x - h/2)$; B-spline of degree $2k$ is then $M^{2k+1}(x) = h(h^{-1}\delta_h D^{-1})^{2k+1}\delta$ where D is the derivation operator. As it is known, the Fourier transform is then

$$\hat{M}^{2k+1}(\omega) = \left(\frac{\sin \pi h \omega}{\pi h \omega} \right)^{2k+1},$$

which is not positive, and so M^{2k+1} is not a totally positive function. All other properties are valid.

m-harmonic splines are also defined for non-integer values of $m > d/2$ (by using Sobolev spaces, and Fourier Transform, see [Duchon, [3]], or [Rabut, [11a]]). In that situation too, it is possible to define "B-splines"; we do it directly by its Fourier transform

$$B^h_{d,m}(\omega) = h^d \left(\frac{\| \sin \pi h \omega \|}{\| \pi h \omega \|} \right)^{2m},$$

and main properties remain valid (however these functions $B^h_{d,m}$ do not seem numerically interesting, as long to evaluate); for more details, see [Rabut, [11a], §II-1].

It is possible to define B-splines in the same way for other geometrical grids, such as for example hexagonal grids. It is also possible to define B-splines in the same way for scattered data, and most of the properties seem to be fulfilled in that case; but we have not yet succeeded in proving them. However, we still hope to prove properties similar to the ones available for non regular data in one dimension.

In [Rabut, [11c]], we use other discretizations of Δ^m to get other m-harmonic B-splines, which have some specific properties (such as, for example the associated B-spline approximation is reproducing $\mathcal{P}_k$ for $k \leqslant 2m - 1$); by the way this confirms that B-splines may be regarded as some approximation of the Dirac distribution.

At last, let us mention that it is possible to compute B-spline approximation, or spline interpolation, of data on a cardinal grid by subdivision techniques. That is quickly presented in [Micchelli et al. [10]].

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