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ON THE RAMIFIED OPTIMAL ALLOCATION PROBLEM

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Abstract. This paper proposes an optimal allocation problem with ramified transport technologies in a spatial economy. Ramified transportation is used to model network-like branching structures attributed to the economies of scale in group transportation. A social planner aims at finding an optimal allocation plan and an associated optimal allocation path to minimize the overall cost of transporting commodity from factories to households. This problem differentiates itself from existing ramified transport literature in that the distribution of production among factories is not fixed but endogenously determined as observed in many allocation practices. It is shown that due to the transport economies of scale, each optimal allocation plan corresponds equivalently to an optimal assignment map from households to factories. This optimal assignment map provides a natural partition of both households and allocation paths. We develop methods of marginal transportation analysis and projectional analysis to study the properties of optimal assignment maps. These properties are then related to the search for an optimal assignment map in the context of state matrix.

1. **Introduction.** One of the lasting interests in economics is to study the optimal resource allocation in a spatial economy. For instance, the well known Monge-Kantorovich transport problem aims at finding an efficient allocation plan or map for transporting some commodity from factories to households. This problem was pioneered by Monge [15] and advanced fully by Kantorovich [12] who won the Nobel prize in economics in 1975 for his seminal work on optimal allocation of resources. Recent advancement of this problem in mathematics can be found in Villani [18, 19] and references therein. The Monge-Kantorovich problem has also been applied to study other related economic problems, e.g., spatial firm pricing (Buttazzo and Carlier [5]), principal-agent problem (Figalli, Kim and McCann [10]), hedonic equilibrium models (Chiappori, McCann and Nesheim [7]; Ekeland [9]), matching and partition in labor market (Carlier and Ekeland [6]; McCann and Trokhimtchouk [14]).

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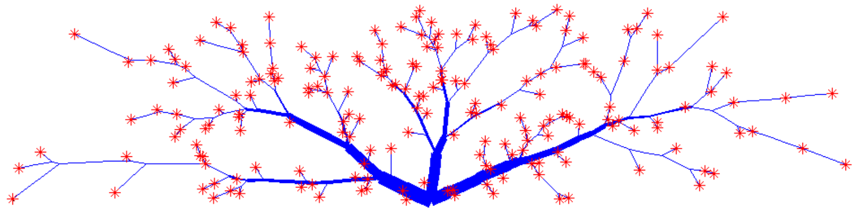


FIGURE 1. An optimal transport path with a ramified structure.

Recently, a research field known as *ramified optimal transportation* has grown out of the Monge-Kantorovich problem. Representative studies can be found for instance in Gilbert [11], Xia [20, 21, 22, 23, 24, 25, 26], Maddalena, Solimini and Morel [13], Bernot, Caselles and Morel [1, 2], Brancolini, Buttazzo and Santambrogio [3], Santambrogio [17], Devillanova and Solimini [8], Xia and Vershynina [27]. The ramified transport problem studies how to find an optimal transport path from sources to targets as illustrated in Figure 1 (Xia [26]). Different from the standard Monge-Kantorovich problem where the transportation cost is solely determined by a transport plan or map, the transportation cost in the ramified transport problem is determined by the actual transport path which transports the commodity from sources to targets. For example, in shipping two items from nearby cities to the same far away city, it may be less expensive to first bring them to a common location and put them on a single truck for most of the transport. In this case, a "Y-shaped" path is preferable to a "V-shaped" path. In general, an efficient transport system emerges in the form of a network with some branching structure. This network-like structure is attributed to the economies of scale in group transportation observed widely in both nature (e.g., trees, blood vessels, river channel networks, lightning) and efficiently designed transport systems of branching structures (e.g., railway configurations and postage delivery networks). An application of ramified optimal transportation in economics can be found in Xia and Xu [28], which showed that a well designed ramified transport system can improve the welfare of consumers in the system.

In this paper, we propose an optimal resource allocation problem where a planner chooses both an optimal allocation plan as well as an associated optimal transport path using ramified transport technologies. Both Monge-Kantorovich and ramified transport problems typically assume an exogenous fixed distribution on both sources and targets. However, in many resource allocation practices, the distribution on either sources or targets is not pre-determined but rather determined endogenously. For instance, in a *production allocation* problem, suppose there are k factories and ℓ households located in different places in some area. The demand for some commodity from each household is fixed. Nevertheless, the allocation of production among factories is not pre-determined but rather depends on the distribution of demands among households as well as their relative locations to factories. A planner needs to make an efficient allocation plan of production over these k factories to meet given demands from these ℓ households. With ramified transportation, the transportation cost of each production plan is determined by an associated optimal transport path from factories to households. Consequently, the planner needs to find an optimal production plan as well as an associated optimal transport path to minimize the total cost of distributing commodity from factories to households. Another example of similar nature exists in the following *storage arrangement* problem. Suppose there are k warehouses and ℓ factories located in different places in some area. Each factory has already produced some amount of commodity. However, the assignment of the commodity among warehouses is not pre-determined but instead relies on the distribution of output among factories as well as relative locations between factories and warehouses. Similarly, a planner needs to make an efficient storage arrangement as well as an associated optimal transport path for storing the produced commodity in the given k warehouses with minimum transportation cost.

This type of allocation problems is named as *ramified optimal allocation problem* formulated in Section 2. Throughout the following context, we will focus our discussion on the scenario of the production allocation problem. Little additional effort is needed to interpret results for other scenarios. We start with modeling a transport path from factories to households as a weighted directed graph, where the transportation cost on each edge of the graph depends linearly on the length of the edge but concavely on the amount of commodity moved on the edge. The motivation of a concave cost function comes from the observation of the economies of scale in group transportation. The more concave the cost function or the greater the magnitude of the transport economies of scale, the more efficient is to transport commodity in larger groups. We define the cost of an allocation plan as the minimum transportation cost of a transport path compatible with this plan. A planner needs to find an efficient allocation plan such that demands from households will be met in a least cost way. In this problem, the distribution of production over factories is not pre-determined as in either Monge-Kantorovich or ramified transport problem, but endogenously determined by the distribution of demands from households as well as their relative locations to factories.

We prove the existence of the ramified allocation problem in Section 3. It is shown that due to the transport economies of scale, under any optimal allocation plan, no two factories will be connected on any associated optimal allocation path. Consequently, any optimal allocation path can be decomposed into a set of mutually disjoint transport paths originating from each factory. As a result, each household will receive its commodity from only one factory under any optimal allocation plan. It implies that each optimal allocation plan corresponds equivalently to an optimal assignment map from households to factories. Thus, solving the ramified optimal allocation problem is equivalent to finding an optimal assignment map. This optimal assignment map is shown to provide a partition not only in households but also in the associated allocation path according to factories.

Because of the equivalence between an optimal allocation plan and an optimal assignment map, we can instead focus attention on studying the properties of optimal assignment maps in the ramified optimal allocation problem. In Section 4, we develop methods of marginal transportation analysis and projectional analysis to study properties of optimal assignment maps. The marginal transportation method extends the standard marginal analysis in economics into the analysis for transport paths. It builds upon an intuitive idea that a marginal change on an optimal allocation path should not reduce the existing minimal transportation cost. Using this method, we develop a criterion which relates the optimal assignment of a household with its relative location to factories and other households, as well as its demand and production at factories. In particular, it is shown that each factory has a nearby region such that a household living at this region will be assigned to the factory, where the size of this region depends positively on the demand of the household. In this case, a planner takes advantage of relative spatial locations between households and factories. Also, if an optimal assignment map assigns a household to some factory, then this household has a neighborhood area such that any household with a smaller demand living in this area will also be assigned to the same factory. Here, the planner utilizes the benefits in group transportation due to the economies of scale embedded in ramified transportation. The roles of spatial location and group transportation in resource allocation are further studied by a method of projectional analysis. We show that under an optimal assignment map, a household will be assigned to some factory only when either it lives close to the factory or it has some nearby neighbours assigned to the factory. In particular, when households and factories are located on two disjoint areas lying distant away from each other, the demand of households will solely be satisfied from their local factories.

An important application of the properties of optimal assignment maps is that they can shed light on the search for those maps. In Section 5, we develop a search method utilizing these properties in a notion of state matrix. A state matrix represents the information set of a planner during the search process for an optimal assignment map. Any zero entry u_{sh} in the matrix reflects that the planner has excluded the possibility of assigning household h to factory s under this map. When a state matrix has exactly one non-zero entry in each column, it completely determines an optimal assignment map by those non-zero entries. Our search method uses the properties of optimal assignment maps to update some non-zero entries with zeros in a state matrix. This method is motivated by the observation that via group transportation, the assignment of each household has a global effect on the allocation path as well as the associated assignment map. Thus, the planner can deduce more information about the optimal assignment map by exploiting the existing information from those zero entries of a state matrix. Each updated state matrix contains more zeros and thus more information than its pre-updated counterpart. This method is useful in the search for optimal assignment maps as each updating step increases the number of zero entries which in turn reduces the size of the restriction set of assignment maps in a large magnitude. In some non-trivial cases, it is shown that this method can exactly find an optimal assignment map.

Future Work. One natural extension of the ramified optimal allocation problem is to allow the locations of factories to vary which then gives rise to an optimal location problem. An analogous optimal location problem associated with Monge-Kantorovich transportation has been extensively studied as in Morgan and Bolton [16] and references therein. Meanwhile, it is also useful to extend the ramified allocation problem by generalizing the atomic measure representing households to a general Radon measure μ, not necessarily atomic. In particular, when μ represents the Lebesgue measure on a domain, a partition of μ given by an optimal assignment map may analogously lead to a partition of the domain. This consequently raises an optimal partition problem of dividing the given domain into k regions according to ramified transportation. Furthermore, our model considers the establishment of an optimal network by a social planner. The grouping nature in ramified transportation gives a space for strategic interdependence between network users. An interesting future work is to study the effect of strategic behaviours among households on the network formation.

2. **Ramified optimal allocation problem.** In this section, we describe the setting of the optimal allocation model with ramified optimal transportation.

2.1. **Ramified optimal transportation.** In a spatial economy, there are k factories and ℓ households located at $\mathbf{x} = \{x_1, x_{2,\cdots}, x_k\}$ and $\{y_1, y_2, \cdots, y_\ell\}$ in some area X, where X is a compact convex subset of a Euclidean space $\mathbb{R}^m$ endowed with the standard norm $\|\cdot\|$. In this model economy, there is only one commodity, and each household $j = 1, \cdots, \ell$ has a fixed demand $n_j > 0$ for the commodity. In the following context, we also use factory i (household j) to represent the factory at x_i (household at y_j) if no confusion arises.

For analytical convenience, we first represent households and factories as atomic measures. Recall that a measure $\mathbf{c}$ on X is atomic if $\mathbf{c}$ is a finite sum of Dirac measures with positive multiplicities, i.e., $\mathbf{c} = \sum_{i=1}^{s} c_i \delta_{z_i}$ for some integer $s \geq 1$ and some points $z_i \in X$ with $c_i > 0$ for each $i = 1, \cdots, s$. The mass of $\mathbf{c}$ is denoted by $m(\mathbf{c}) := \sum_{i=1}^{s} c_i$. We can thus represent the ℓ households as an atomic measure on X by

$$\mathbf{b} = \sum_{j=1}^{\ell} n_j \delta_{y_j}. \tag{2.1}$$

For each $i = 1, \cdots, k$, denote m_i as the units of the commodity produced at factory i located at x_i. Then, the k factories can be represented by another atomic measure on X by

$$\mathbf{a} = \sum_{i=1}^{k} m_i \delta_{x_i}. \tag{2.2}$$

In the study of transport problems, we usually assume $m(\mathbf{a}) = m(\mathbf{b})$, i.e., $\sum_{i=1}^{k} m_i = \sum_{j=1}^{\ell} n_j$, which simply means that supply equals demand in aggregate.

Next, we introduce the concept of transport path from $\mathbf{a}$ to $\mathbf{b}$ as in Xia [20].

Definition 2.1. Suppose $\mathbf{a}$ and $\mathbf{b}$ are two atomic measures on X of equal mass. A transport path from $\mathbf{a}$ to $\mathbf{b}$ is a weighted directed graph G consisting of a vertex set $V(G)$, a directed edge set $E(G)$ and a weight function $w : E(G) \to (0, +\infty)$ such that $\{x_1, x_2, ..., x_k\} \cup \{y_1, y_2, ..., y_\ell\} \subseteq V(G)$ and for any vertex $v \in V(G)$, there is a balance equation

$$\sum_{e \in E(G), e^- = v} w(e) = \sum_{e \in E(G), e^+ = v} w(e) + \begin{cases} m_i, \textit{if } \ v = x_i \textit{ for some } i = 1, ..., k \\ -n_j, \ \textit{if } \ v = y_j \textit{ for some } j = 1, ..., \ell \\ 0, \ \textit{otherwise} \end{cases} \tag{2.3}$$

where each edge $e \in E(G)$ is a line segment from the starting endpoint e^- to the ending endpoint e^+. Denote $Path(\mathbf{a}, \mathbf{b})$ as the space of all transport paths from $\mathbf{a}$ to $\mathbf{b}$.

Note that the balance equation (2.3) simply means the conservation of mass at each vertex. Now, we consider the transportation cost of a transport path. As observed in both nature and efficiently designed transport networks, there are economies of scale underlying group transportation. For this consideration, ramified optimal transport theories use a concave transportation cost function. Here, for the sake of simplicity, we only consider concave power functions instead of more general concave ones. The transportation cost of a transport path is defined as follows.

Definition 2.2. For each transport path $G \in Path(\mathbf{a}, \mathbf{b})$ and any $\alpha \in [0,1]$, the $\mathbf{M}_\alpha$ cost of G is defined by

$$\mathbf{M}_\alpha(G) := \sum_{e \in E(G)} [w(e)]^\alpha \, length(e). \tag{2.4}$$

The parameter α represents the magnitude of transport economies of scale. The smaller the α, the more efficient is to move the commodity in groups. Ramified optimal transportation studies how to find a transport path to minimize the $\mathbf{M}_\alpha$ cost, i.e.,

$$\min_{G \in Path(\mathbf{a},\mathbf{b})} \mathbf{M}_\alpha(G), \tag{2.5}$$

whose minimizer is called an optimal transport path from $\mathbf{a}$ to $\mathbf{b}$.

According to Xia [20, Proposition 2.1], an optimal transport path contains no cycles. Thus, without loss of generality, we assume that *all transport paths considered in the following context of this paper contain no cycles.*

The following example illustrates the effect of transport economies of scale on optimal transport paths. In a spatial economy, there are one factory $\mathbf{a} = \delta_O$ located at the origin and fifty households $\mathbf{b} = \sum_{j=1}^{50} \frac{1}{50} \delta_{y_j}$ of equal demand $n_j = \frac{1}{50}$ with their locations randomly generated. We use the numerical method proposed in Xia [26] to find optimal transport paths for three different values of α. As seen in Figure 2, when $\alpha = 1$, the optimal transport path is "linear" in the sense that the factory will ship the commodity directly to each household. When $\alpha < 1$, optimal transport paths become "ramified" or display some branching structures as a planner would like the commodity to be transported in groups in order to reap the benefits of transport economies of scale. Furthermore, by comparing for instance the width of the transport paths for $\alpha = 0.75$ and $\alpha = 0.25$, we observe that the smaller the α, the more likely the commodity will be transported in a larger scale.

For any atomic measures $\mathbf{a}$ and $\mathbf{b}$ on X of equal mass, define the minimum transportation cost as

$$d_\alpha(\mathbf{a}, \mathbf{b}) := \min \{\mathbf{M}_\alpha(G) : G \in Path(\mathbf{a}, \mathbf{b})\}. \tag{2.6}$$

As shown in Xia [20], d_α is indeed a metric on the space of atomic measures of equal mass. Also, for each $\lambda > 0$, it holds that

$$d_\alpha(\lambda \mathbf{a}, \lambda \mathbf{b}) = \lambda^\alpha d_\alpha(\mathbf{a}, \mathbf{b}). \tag{2.7}$$

Without loss of generality, we normalize both $\mathbf{a}$ and $\mathbf{b}$ to be a probability measure on X, i.e., $\sum_{i=1}^{k} m_i = \sum_{j=1}^{\ell} n_j = 1$.

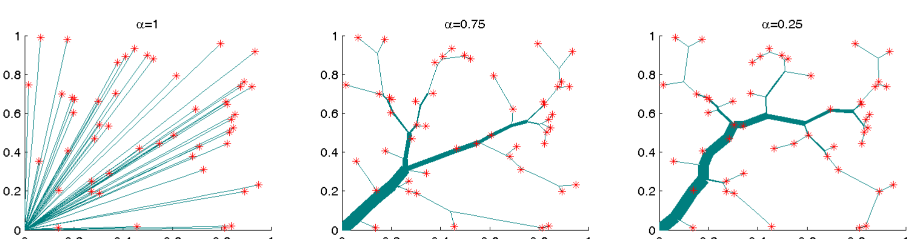


FIGURE 2. Examples of optimal transport paths.

2.2. **Compatibility between transport plan and path.** For the allocation problem to be described shortly, an important decision a planner needs to make is about the transport plan from factories to households. We first introduce the concept of transport plan commonly used in the Monge-Kantorovich transport literature (e.g., [12],[18]).

Definition 2.3. Suppose $\mathbf{a}$ and $\mathbf{b}$ are two atomic probability measures on X as in (2.2) and (2.1). A transport plan from $\mathbf{a}$ to $\mathbf{b}$ is an atomic probability measure

$$q = \sum_{i=1}^{k} \sum_{j=1}^{\ell} q_{ij} \delta_{(x_i, y_j)} \tag{2.8}$$

on the product space $X \times X$ such that for each i and j, $q_{ij} \geq 0$,

$$\sum_{i=1}^{k} q_{ij} = n_j \textit{ and} \sum_{j=1}^{\ell} q_{ij} = m_i. \tag{2.9}$$

Denote $Plan\,(\mathbf{a}, \mathbf{b})$ as the space of all transport plans from $\mathbf{a}$ to $\mathbf{b}$.

Now, as in Section 7.1 of Xia [20], we want to consider the compatibility between a transport path and a transport plan. Let G be a given transport path in $Path\,(\mathbf{a}, \mathbf{b})$. Since G contains no cycles, for each x_i and y_j, there exists at most one directed polyhedral curve g_{ij} on G from x_i to y_j. In other words, there exists a list of distinct vertices $V(g_{ij}) := \{v_{i_1}, v_{i_2}, \cdots, v_{i_h}\}$ in $V(G)$ with $x_i = v_{i_1}$, $y_j = v_{i_h}$, and each $\left[v_{i_t}, v_{i_{t+1}}\right]$ is a directed edge in $E(G)$ for each $t = 1, 2, \cdots, h-1$. For some pairs of (i, j), such a curve g_{ij} from x_i to y_j may not exist, in which case we set $g_{ij} = 0$ to denote the empty directed polyhedral curve.

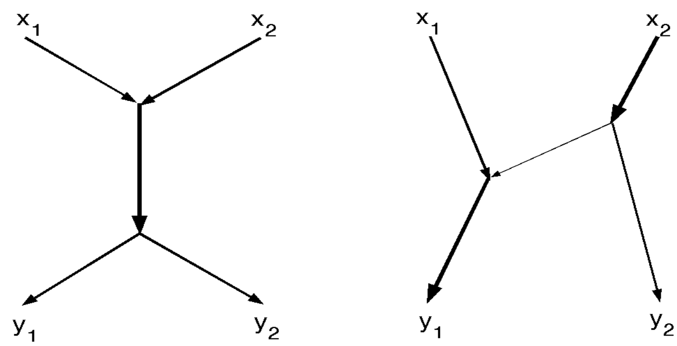
Definition 2.4. Let $G \in Path\,(\mathbf{a}, \mathbf{b})$ be a transport path and $q \in Plan\,(\mathbf{a}, \mathbf{b})$ be a transport plan. The pair (G, q) is compatible if $q_{ij} = 0$ whenever $g_{ij} = 0$ and as polyhedral chains,

$$G = \sum_{i=1}^{k} \sum_{j=1}^{\ell} q_{ij} \cdot g_{ij}, \tag{2.10}$$

where the product $q_{ij} \cdot g_{ij}$ denotes that q_{ij} units of commodity is moved along the polyhedral curve g_{ij} from factory i to household j.

Roughly speaking, compatibility conditions check whether a transport plan is realizable by a transport path. Given a transport plan, a planner must design a transport path which can support this plan. To see the concept more precisely, let $\mathbf{a} = \frac{1}{4}\delta_{x_1} + \frac{3}{4}\delta_{x_2}$ and $\mathbf{b} = \frac{5}{8}\delta_{y_1} + \frac{3}{8}\delta_{y_2}$, and consider a transport plan $q = \frac{1}{8}\delta_{(x_1,y_1)} + \frac{1}{8}\delta_{(x_1,y_2)} + \frac{1}{2}\delta_{(x_2,y_1)} + \frac{1}{4}\delta_{(x_2,y_2)} \in Plan\,(\mathbf{a}, \mathbf{b})$. It is straightforward to see from Figure 3 that q is compatible with G_1 but not G_2. This is because there is no directed curve g_{12} from factory 1 to household 2 in G_2.

2.3. **Ramified allocation problem.** Both Monge-Kantorovich and ramified transport problems typically assume an exogenous fixed distribution of production among factories. In this paper, we consider a scenario where this distribution is not fixed but instead endogenously determined. In other words, the atomic measure $\mathbf{a}$ which represents the k factories can have varying production level m_i at each factory i. This consideration is motivated by our observation that in many allocation practices as discussed in the introduction, the distribution of production among factories is not pre-determined but rather depends on the distribution of demands among households as well as their relative locations to factories.



(A) G_1 (B) G_2

FIGURE 3. Compatibility between transport plan and transport path.

Definition 2.5. Let $\mathbf{x}=\{x_1,x_{2,\cdots},x_k\}$ be a finite subset of X, and $\mathbf{b}$ be the atomic probability measure representing households defined in (2.1). An allocation plan from $\mathbf{x}$ to $\mathbf{b}$ is a probability measure

$$q=\sum_{i=1}^{k}\sum_{j=1}^{\ell}q_{ij}\delta_{(x_i,y_j)}$$

on $X\times X$ such that $q_{ij}\geq 0$ for each i, j and

$$\sum_{i=1}^{k}q_{ij}=n_j \text{ for each } j=1,\cdots,\ell.$$

Denote $Plan\,[\mathbf{x},\mathbf{b}]$ as the set of all allocation plans from $\mathbf{x}$ to $\mathbf{b}$.

Note that any allocation plan $q\in Plan\,[\mathbf{x},\mathbf{b}]$ corresponds to a transport plan q from $\mathbf{a}\,(q)$ to $\mathbf{b}$, where $\mathbf{a}\,(q)$ is the probability measure representing k factories defined as

$$\mathbf{a}\,(q):=\sum_{i=1}^{k}m_i\,(q)\,\delta_{x_i},\ \text{with}\ m_i\,(q)=\sum_{j=1}^{\ell}q_{ij},\ i=1,...,k. \tag{2.11}$$

In other words, $Plan\,[\mathbf{x},\mathbf{b}]$ is the union of $Plan\,(\mathbf{a},\mathbf{b})$ among all atomic probability measures $\mathbf{a}$ supported on $\mathbf{x}$.

Example 2.1. Any function $S:\{1,\cdots,\ell\}\to\{1,\cdots,k\}$ determines an allocation plan in $Plan\,[\mathbf{x},\mathbf{b}]$ as

$$q_S=\sum_{i=1}^{k}\sum_{j=1}^{\ell}q_{ij}\delta_{(x_i,y_j)}\text{ with }q_{ij}=\begin{cases} n_j, & \text{if } i=S\,(j)\\ 0, & \text{else}\end{cases}.$$

That is,

$$q_S=\sum_{j=1}^{\ell}n_j\delta_{\left(x_{(S(j))},y_j\right)}. \tag{2.12}$$

For a given allocation plan, we define the associated transportation cost as follows.

Definition 2.6. For any allocation plan $q \in Plan\,[\mathbf{x},\mathbf{b}]$ and $\alpha \in [0,1)$, the ramified transportation cost of q is

$$\mathbf{T}_\alpha (q) := \min \{\mathbf{M}_\alpha (G) : G \in Path\,(\mathbf{a}(q), \mathbf{b}), (G,q) \text{ compatible } \}, \tag{2.13}$$

where $\mathbf{M}_\alpha(\cdot)$ is defined in (2.4). An allocation plan $q^* \in Plan\,[\mathbf{x},\mathbf{b}]$ is optimal if

$$\mathbf{T}_\alpha (q^*) \le \mathbf{T}_\alpha (q) \text{ for any } q \in Plan\,[\mathbf{x},\mathbf{b}].$$

For each allocation plan q, as shown in Xia [20, Proposition 7.3], there exists a path $G_q \in Path\,(\mathbf{a}(q), \mathbf{b})$ such that G_q is compatible with q and $\mathbf{T}_\alpha(q) = \mathbf{M}_\alpha(G_q)$. Thus, the minimum value in (2.13) is achieved by G_q for each q. Now, we are ready to define the major problem in this paper.

Problem. *(Ramified Optimal Allocation Problem) Let X be a compact convex domain in $\mathbb{R}^m$. Given a finite subset $\mathbf{x} = \{x_1, x_{2,\cdots}, x_k\}$ in X, an atomic probability measure $\mathbf{b}$ on X defined in (2.1), and a parameter $\alpha \in [0,1)$. Find a minimizer of $\mathbf{T}_\alpha(q)$ among all allocation plans $q \in Plan\,[\mathbf{x},\mathbf{b}]$, i.e.,*

$$\min \ \{\mathbf{T}_\alpha (q) : q \in Plan\,[\mathbf{x},\mathbf{b}]\}. \tag{2.14}$$

3. **Characterizing optimal allocation plans.** In this section, we establish the existence result of the ramified optimal allocation problem. It is shown that any optimal allocation plan corresponds to an optimal assignment map from households to factories, which provides a partition in both households and transport paths. To characterize the properties of an optimal allocation plan, we first introduce the concept of an allocation path as follows:

Definition 3.1. An allocation path from $\mathbf{x}$ to $\mathbf{b}$ is a transport path $G \in Path\,(\mathbf{a},\mathbf{b})$ for some atomic probability measure $\mathbf{a}$ supported on $\mathbf{x}$. Denote $Path\,[\mathbf{x},\mathbf{b}]$ as the set of all allocation paths from $\mathbf{x}$ to $\mathbf{b}$. An allocation path $G^* \in Path\,[\mathbf{x},\mathbf{b}]$ is optimal if $\mathbf{M}_\alpha(G^*) \le \mathbf{M}_\alpha(G)$ for any $G \in Path\,[\mathbf{x},\mathbf{b}]$.

The following theorem presents a key property of an optimal allocation path.

Theorem 3.1. *For any allocation path $G \in Path\,[\mathbf{x},\mathbf{b}]$ from $\mathbf{x}$ to $\mathbf{b}$, there exists an allocation path $\tilde{G} \in Path\,[\mathbf{x},\mathbf{b}]$ such that $\mathbf{M}_\alpha\left(\tilde{G}\right) \le \mathbf{M}_\alpha(G)$, and for any $r \ne s \in \{1,\cdots,k\}$, x_r and x_s do not belong to the same connected component of $\tilde{G}$.*

Proof. Assume x_r and x_s belong to the same connected component of an allocation path $G = \{V(G),\ E(G),\ w : E(G) \to (0,+\infty)\}$, then there exists a polyhedra curve γ supported on G from x_r to x_s. We may list edges of γ as $\{\varepsilon_1 e_1, \cdots, \varepsilon_n e_n\}$ with $\varepsilon_i = \pm 1$ and $e_i \in E(G)$. Here, $\varepsilon_i = 1$ (or -1) if e_i has the same (or opposite) direction as γ. Let $\lambda = \min_{1\le i\le n} w(e_i) > 0$ and consider $G_t := G + t\gamma$ as polyhedral chains for $t = \pm\lambda$. Note that G_t is still in $Path\,[\mathbf{x},\mathbf{b}]$, and when $\alpha \in (0,1)$,

$$\mathbf{M}_\alpha(G_t) - \mathbf{M}_\alpha(G) = \sum_{i=1}^{n} \left[(w(e_i) + t\varepsilon_i)^\alpha - (w(e_i))^\alpha\right] length\,(e_i).$$

By the strict concavity of x^α, we have

$$(w(e_i) + \lambda)^\alpha + (w(e_i) - \lambda)^\alpha - 2(w(e_i))^\alpha < 0,$$

and thus

$$\begin{aligned} & \mathbf{M}_\alpha(G_\lambda) + \mathbf{M}_\alpha(G_{-\lambda}) - 2\mathbf{M}_\alpha(G) \\ = & \sum_{i=1}^{n} \left[(w(e_i)+\lambda)^\alpha + (w(e_i)-\lambda)^\alpha - 2(w(e_i))^\alpha\right] length\,(e_i) < 0. \end{aligned}$$

When $\alpha = 0$, note that

$$\mathbf{M}_\alpha (G) = \sum_{e \in E(G), w(e)>0} length\,(e)\,,$$

and then

$$\mathbf{M}_\alpha (G_\lambda) + \mathbf{M}_\alpha (G_{-\lambda}) - 2\mathbf{M}_\alpha (G) = - \sum_{w(e_i)=\lambda} length\,(e_i) < 0.$$

Therefore, when $\alpha \in [0, 1)$, it holds that

$$\mathbf{M}_\alpha (G_\lambda) + \mathbf{M}_\alpha (G_{-\lambda}) - 2\mathbf{M}_\alpha (G) < 0, \text{ i.e., } \min \{\mathbf{M}_\alpha (G_\lambda), \mathbf{M}_\alpha (G_{-\lambda})\} < \mathbf{M}_\alpha (G)\,.$$

Continue this process for G_λ (or $G_{-\lambda}$), we eventually find an allocation path $\tilde{G} \in Path\,[\mathbf{x}, \mathbf{b}]$ with desired properties. $\square$

This theorem says that no two factories will be connected on any optimal allocation path. Alternatively speaking, on an optimal allocation path, each single household will receive its commodity from only one factory, i.e., each household is assigned to one factory. This result is attributed to the economies of scale underlying ramified transport technologies. As seen in Section 2, an $\alpha \in [0, 1)$ implies the existence of transport economies of scale with transporting in groups being more cost efficient than transporting separately. Any allocation path on which some single household receives its commodity from two factories cannot be optimal because a planner would be able to reduce the transportation cost by transferring production of one factory to the other. This transfer makes the benefits of transport economies of scale more likely to be realized as this household's commodity is transported in a larger scale on the path.

By Theorem 3.1, each x_i determines a connected component $\tilde{G}_i$ of $\tilde{G}$. Thus, as polyhedral chains, $\tilde{G}$ can be decomposed as

$$\tilde{G} = \sum_{i=1}^{k} \tilde{G}_i \text{ with } \mathbf{M}_\alpha \left(\tilde{G}\right) = \sum_{i=1}^{k} \mathbf{M}_\alpha \left(\tilde{G}_i\right). \tag{3.1}$$

Note that each y_j is connected to a unique x_i on $\tilde{G}$. This defines a map $S : \{1, \cdots, \ell\} \to \{1, \cdots, k\}$ such that $S(j) = i$ if y_j is connected to x_i on $\tilde{G}$. Clearly, each $\tilde{G}_i$ is a transport path from $\mathbf{a}_i = \left(\sum_{j \in S^{-1}(i)} n_j\right) \delta_{x_i}$ to $\mathbf{b}_i = \sum_{j \in S^{-1}(i)} n_j \delta_{y_j}$.

Figure 4 illustrates a partition result in a spatial economy consisting of factories $\{x_1, x_2\}$ and households $\{y_1, \cdots, y_{145}\}$ with equal demand $n_j = \frac{1}{145}$. The figure shows a partition in both the allocation path and households: First, the allocation path is decomposed into two disjoint sub-transport paths originating from factory x_1 and x_2 respectively. Second, the 145 households are divided into two unconnected populations centered around the factory from which they receive the commodity.

The result that each household is assigned to one factory on an optimal allocation path motivates the following notion of assignment map.

Definition 3.2. An assignment map is a function $S : \{1, \cdots, \ell\} \to \{1, \cdots, k\}$. Let $Map\,[\ell, k]$ be the set of all assignment maps. For any assignment map $S \in Map\,[\ell, k]$

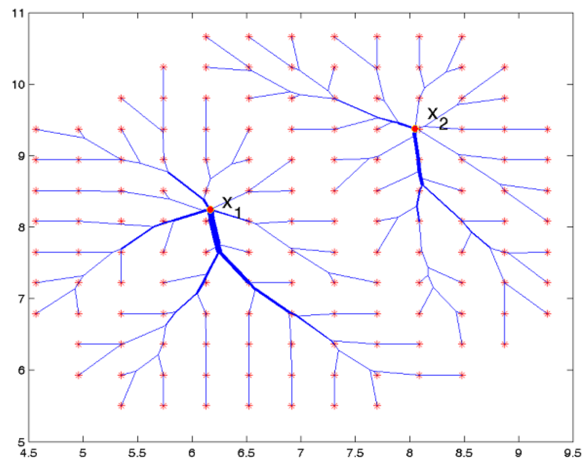


FIGURE 4. A partition in households and allocation path.

and $\alpha \in [0,1)$, define, for any given $\mathbf{x}$ and $\mathbf{b}$,

$$\mathbf{E}_\alpha(S;\mathbf{x},\mathbf{b}) := \sum_{i=1}^{k} d_\alpha(\mathbf{a}_i,\mathbf{b}_i) \text{ with } \mathbf{a}_i = \left(\sum_{j\in S^{-1}(i)} n_j\right)\delta_{x_i} \text{ and } \mathbf{b}_i = \sum_{j\in S^{-1}(i)} n_j \delta_{y_i}. \tag{3.2}$$

where d_α is the metric defined in (2.6). An assignment map $S^* \in Map[\ell,k]$ is optimal if

$$\mathbf{E}_\alpha(S^*;\mathbf{x},\mathbf{b}) \le \mathbf{E}_\alpha(S;\mathbf{x},\mathbf{b}) \text{ for any } S \in Map[\ell,k].$$

By Theorem 3.1, $\mathbf{M}_\alpha(G) \ge \mathbf{M}_\alpha\left(\tilde{G}\right) = \sum_{i=1}^k \mathbf{M}_\alpha\left(\tilde{G}_i\right) \ge \sum_{i=1}^k d_\alpha(\mathbf{a}_i,\mathbf{b}_i) = \mathbf{E}_\alpha(S;\mathbf{x},\mathbf{b})$. Therefore, we have the following corollary:

Corollary 3.1. *For any allocation path $G \in Path[\mathbf{x},\mathbf{b}]$ from $\mathbf{x}$ to $\mathbf{b}$, there exists an $S \in Map[\ell,k]$ such that $\mathbf{E}_\alpha(S;\mathbf{x},\mathbf{b}) \le \mathbf{M}_\alpha(G)$.*

Note that any assignment map $S \in Map[\ell,k]$ provides a partition among households $\mathbf{b} = \sum_{i=1}^k \mathbf{b}_i$ where all households in $\mathbf{b}_i$ given in (3.2) are assigned to a single factory located at x_i. For each i, let $G_i \in Path(\mathbf{a}_i,\mathbf{b}_i)$ be an optimal transport path which transports the commodity produced at factory i to households in $\mathbf{b}_i$. This yields an allocation path from $\mathbf{x}$ to $\mathbf{b}$

$$G_S = \sum_{i=1}^k G_i \in Path[\mathbf{x},\mathbf{b}], \tag{3.3}$$

which is compatible with the allocation plan q_S given in (2.12). Thus,

$$\mathbf{E}_\alpha(S;\mathbf{x},\mathbf{b}) = \sum_{i=1}^k d_\alpha(\mathbf{a}_i,\mathbf{b}_i) = \sum_{i=1}^k \mathbf{M}_\alpha(G_i) \ge \mathbf{M}_\alpha(G_S) \ge \mathbf{T}_\alpha(q_S). \tag{3.4}$$

By Corollary 3.1, any optimal allocation path is in the form of (3.3) with respect to its associated assignment map.

Now, we state the main results of this section as follows:

Theorem 3.2. *Given a subset* $\mathbf{x} = \{x_1, x_{2,\cdots}, x_k\}$ *in* X*, an atomic probability measure* $\mathbf{b}$ *as in (2.1), and a parameter* $\alpha \in [0, 1)$*.*

1. *(Existence) Each of the following minimization problems has a solution with*
$$\min_{S \in Map[\ell,k]} \mathbf{E}_\alpha (S; \mathbf{x}, \mathbf{b}) = \min_{G \in Path[\mathbf{x},\mathbf{b}]} \mathbf{M}_\alpha (G) = \min_{q \in Plan[\mathbf{x},\mathbf{b}]} \mathbf{T}_\alpha (q). \tag{3.5}$$
2. *An allocation path* $G \in Path[\mathbf{x}, \mathbf{b}]$ *is optimal if and only if there exists an optimal assignment map* $S \in Map[\ell, k]$ *such that* $G = G_S$ *for some* G_S *of* S *defined in (3.3).*
3. *An allocation plan* $q \in Plan[\mathbf{x}, \mathbf{b}]$ *is optimal if and only if there exists an optimal assignment map* $S \in Map[\ell, k]$ *such that* $q = q_S$*.*

Proof. Since $Map[\ell, k]$ is a finite set, it is obvious that problem
$$\min_{S \in Map[\ell,k]} \mathbf{E}_\alpha (S; \mathbf{x}, \mathbf{b})$$
has a solution. By Corollary 3.1 and (3.4), any optimal assignment map S provides an optimal allocation path G_S. This proves the first equality in (3.5) as well as the "if" part of (2). For the "only if" part of (2), suppose $G \in Path[\mathbf{x}, \mathbf{b}]$ is an optimal allocation path. By Theorem 3.1, $G = G_S$ for some $S \in Map[\ell, k]$. Then, the optimality of S follows from the optimality of G and the first equality in (3.5).

Now, suppose $G^* \in Path[\mathbf{x}, \mathbf{b}]$ is an optimal allocation path, i.e., G^* is an optimal transport path in $Path(\mathbf{a}^*, \mathbf{b})$ for some $\mathbf{a}^*$ supported on $\mathbf{x}$ with $\mathbf{M}_\alpha (G^*) = d_\alpha (\mathbf{a}^*, \mathbf{b})$. By Xia [20, Lemma 7.1], G^* has at least one compatible plan $q^* \in Plan(\mathbf{a}^*, \mathbf{b}) \subseteq Plan[\mathbf{x}, \mathbf{b}]$ and thus
$$\mathbf{T}_\alpha (q^*) \le \mathbf{M}_\alpha (G^*) = d_\alpha (\mathbf{a}^*, \mathbf{b}). \tag{3.6}$$
For any $q \in Plan[\mathbf{x}, \mathbf{b}]$, by the optimality of G^* in $Path[\mathbf{x}, \mathbf{b}]$ we have
$$\mathbf{T}_\alpha (q) \ge d_\alpha (\mathbf{a}(q), \mathbf{b}) \ge d_\alpha (\mathbf{a}^*, \mathbf{b}) \ge \mathbf{T}_\alpha (q^*).$$
This shows that
$$\mathbf{T}_\alpha (q^*) = \min \{\mathbf{T}_\alpha (q) : q \in Plan[\mathbf{x}, \mathbf{b}]\} \text{ and } \mathbf{T}_\alpha (q^*) = \mathbf{M}_\alpha (G^*). \tag{3.7}$$
Thus, q^* is a solution to the ramified optimal allocation problem (2.14). This shows the second equality in (3.5). Then, the equivalence in (3) follows from (3.5), (2) and (3.4). □

Theorem 3.2 shows that in the ramified optimal allocation problem, there exists an equivalence between an optimal allocation plan and an optimal assignment map. This result has an analogous counterpart in the Monge-Kantorovich problem, but has not been observed in the current literature of ramified transport problems. An implication of this theorem is that one can instead search for an optimal assignment map in order to find an optimal allocation plan. Each optimal assignment map $S \in Map[\ell, k]$ would give an optimal allocation plan $q_S \in Plan[\mathbf{x}, \mathbf{b}]$ as in (2.12). For this consideration, the following sections of the paper will focus on characterizing the various properties of optimal assignment maps.

4. **Marginal transportation analysis of optimal assignment maps.** In this section, we develop a method of marginal transportation analysis and use it to study the properties of optimal assignment maps.

We first formalize a concept of marginal transportation cost for a single-source transport system. Let $G \in Path(\mathfrak{m}(\mathbf{c})\, \delta_O, \mathbf{c})$ be a transport path from a single source O to an atomic measure $\mathbf{c}$ of mass $m(\mathbf{c})$. For any point p on the support of G, we set

$$\theta(p) := \begin{cases} w(e), & \text{if } p \text{ is in the interior of some edge } e \in E(G) \\ \mathfrak{m}(\mathbf{c}), & \text{if } p = O \\ w(e), & \text{if } p \in V(G) \setminus \{O\}, \text{ and } p = e^+ \text{ for some } e \in E(G) \end{cases}, \tag{4.1}$$

which represents the mass flowing through p, where e^+ denotes the ending endpoint of the edge $e \in E(G)$. Since G has a single source and contains no cycles, for any point p on G, there exists a unique polyhedral curve γ_p on G from O to p. Moreover, for any point $s \in \gamma_p$, it holds that

$$\mathfrak{m}(\mathbf{c}) \geq \theta(s) \geq \theta(p). \tag{4.2}$$

When the mass at p changes by an amount Δm with $\Delta m \geq -\theta(p)$, the mass flowing through each point of γ_p also changes by Δm. As a result, the corresponding incremental transportation cost is

$$\Delta C_G(p, \Delta m) := \mathbf{M}_\alpha(G + (\Delta m)\gamma_p) - \mathbf{M}_\alpha(G) = \int_{\gamma_p} (\theta(s) + \Delta m)^\alpha - (\theta(s))^\alpha \, ds. \tag{4.3}$$

The marginal transportation cost at p via G is defined by

$$MC_G(p) := \lim_{\Delta m \to 0} \frac{\Delta C_G(p, \Delta m)}{\Delta m} = \alpha \int_{\gamma_p} (\theta(s))^{\alpha - 1} \, ds.$$

Remark 4.1. As one of the referees has pointed out to the authors, the marginal transportation cost function $MC_G(p)$ is also called *the landscape function* in [4] and [17]. Using similar techniques and ideas in proving Proposition 4.1 and 4.2, authors of [4] and [17] have studied the Hölder regularity of the landscape function in the case of continuous measures. Nevertheless, for the purpose of this paper, we will show some results about the incremental transportation cost function $\Delta C_G(p, \Delta m)$ in the case of discrete measures.

The following proposition establishes some properties of the function $\Delta C_G(p, \Delta m)$ given in (4.3). Those properties are the key elements of marginal transportation analysis used to study optimal assignment maps later.

Proposition 4.1. *For any $G \in Path(\mathfrak{m}(\mathbf{c})\delta_O, \mathbf{c})$ and p on G, we have*

$$\Delta C_G(p, -\Delta m) = -\Delta C_{\tilde{G}}(p, \Delta m), \text{ for } \Delta m \in [-\theta(p), \theta(p)], \tag{4.4}$$

where $\tilde{G} = G - (\Delta m)\gamma_p$. Moreover, for any $\Delta m \geq 0$, we have

$$\begin{aligned} & [(\mathfrak{m}(\mathbf{c}) + \Delta m)^\alpha - \mathfrak{m}(\mathbf{c})^\alpha] \, length(\gamma_p) \\ \leq \; & \Delta C_G(p, \Delta m) \leq [(\theta(p) + \Delta m)^\alpha - \theta(p)^\alpha] \, length(\gamma_p). \end{aligned} \tag{4.5}$$

If, in addition, $G \in Path(\mathfrak{m}(\mathbf{c})\delta_O, \mathbf{c})$ is optimal in (2.5), then

$$d_\alpha((\mathfrak{m}(\mathbf{c}) + \Delta m)\delta_O, \mathbf{c} + (\Delta m)\delta_p) - d_\alpha(\mathfrak{m}(\mathbf{c})\delta_O, \mathbf{c}) \leq \Delta C_G(p, \Delta m) \tag{4.6}$$

for any $\Delta m \geq -\theta(p)$ with p on G.

Proof. It is easy to check that equality (4.4) follows directly from definition (4.3). For any $\Delta m \geq 0$, note that the function $f(t) := (t + \Delta m)^\alpha - t^\alpha$ is monotonically non-increasing on $t > 0$ when $0 \leq \alpha < 1$. By (4.2), we have

$$(\mathfrak{m}(\mathbf{c}) + \Delta m)^\alpha - (\mathfrak{m}(\mathbf{c}))^\alpha \leq (\theta(s) + \Delta m)^\alpha - (\theta(s))^\alpha \leq (\theta(p) + \Delta m)^\alpha - (\theta(p))^\alpha.$$

Thus, by (4.3), inequalities (4.5) hold.

When G is also optimal, we have $\mathbf{M}_\alpha(G) = d_\alpha(\mathfrak{m}(\mathbf{c})\delta_O, \mathbf{c})$. For any $\Delta m \geq -\theta(p)$, since $G + (\Delta m)\gamma_p \in Path((\mathfrak{m}(\mathbf{c}) + \Delta m)\delta_O, \mathbf{c} + (\Delta m)\delta_p)$, we have

$$\begin{aligned}\Delta C_G(p, \Delta m) &= \mathbf{M}_\alpha(G + (\Delta m)\gamma_p) - \mathbf{M}_\alpha(G)\\ &\geq d_\alpha((\mathfrak{m}(\mathbf{c}) + \Delta m)\delta_O, \mathbf{c} + (\Delta m)\delta_p) - d_\alpha(\mathfrak{m}(\mathbf{c})\delta_O, \mathbf{c}).\end{aligned}$$

□

Now, let $G \in Path[\mathbf{x}, \mathbf{b}]$ be any optimal allocation path. By Theorem 3.2, G must be in the form of (3.3), which is simply a disjoint union of single-sourced paths G_i's. For any p on G, there exists a unique i such that p is on G_i. Thus, we can define the corresponding $\theta_i(p)$ and $\Delta C_{G_i}(p, \Delta m)$ as in (4.1) and (4.3). Then, we set

$$\theta(p) := \theta_i(p) \text{ and } \Delta C_G(p, \Delta m) := \Delta C_{G_i}(p, \Delta m)$$

for any p on the support of G.

For any $\alpha \in (0,1)$ and $\sigma \geq \epsilon > 0$, define

$$\rho_\alpha(\sigma, \epsilon) := \left(\frac{\sigma}{\epsilon}\right)^\alpha - \left(\frac{\sigma}{\epsilon} - 1\right)^\alpha. \tag{4.7}$$

The function $\rho_\alpha(\sigma, \varepsilon)$ is decreasing in σ and increasing in ϵ; also $0 < \rho_\alpha(\sigma, \epsilon) \leq 1$ for any $\sigma \geq \epsilon > 0$. For $\alpha = 0$, set $\rho_0(\sigma, \epsilon) = 0$ when $\sigma > \epsilon > 0$ and $\rho_0(\sigma, \epsilon) = 1$ when $\sigma = \epsilon > 0$.

The following proposition includes a key result in marginal transportation analysis, and is used to prove Theorem 4.1 and 4.2.

Proposition 4.2. *Suppose $G \in Path[\mathbf{x}, \mathbf{b}]$ is an optimal allocation path as given in (3.3). Let p be a point on G_s for some $s \in \{1, \cdots, k\}$ and $\Delta m \in (0, \theta(p)]$. For any p^* on G with $\gamma_p \cap \gamma_{p^*}$ having zero length, we have*

$$\Delta C_G(p, -\Delta m) + (\Delta m)^\alpha \|p - p^*\| + \Delta C_G(p^*, \Delta m) \geq 0, \tag{4.8}$$

and

$$\|p - p^*\| + \rho_\alpha(\theta(p^*) + \Delta m, \Delta m)\, length(\gamma_{p^*}) \geq \rho_\alpha(\mathfrak{m}(\mathbf{b}_s), \Delta m)\, length(\gamma_p). \tag{4.9}$$

In particular, for any $i \in \{1, \cdots, k\}$,

$$\|p - x_i\| \geq \rho_\alpha(\mathfrak{m}(\mathbf{b}_s), \Delta m)\, length(\gamma_p) \geq \rho_\alpha(\mathfrak{m}(\mathbf{b}_s), \Delta m)\|p - x_s\|. \tag{4.10}$$

Moreover, suppose p^* is on G_i with $i \neq s$ and $\Delta m \leq \theta(p^*)$, then

$$\|p - p^*\| + \frac{\rho_\alpha(\theta(p^*) + \Delta m, \Delta m)}{\rho_\alpha(\mathfrak{m}(\mathbf{b}_i), \Delta m)}\|p^* - x_i\| \geq \rho_\alpha(\mathfrak{m}(\mathbf{b}_s), \Delta m)\|p - x_s\|. \tag{4.11}$$

Proof. Let $\hat{G} = G - (\Delta m)\gamma_p + (\Delta m)[p, p^*] + (\Delta m)\gamma_{p^*}$, where $[p, p^*]$ denotes the line segment from p to p^*. Then, when the intersection of polyhedral curves $\gamma_p \cap \gamma_{p^*}$ has length zero, we have

$$\begin{aligned}&\Delta C_G(p, -\Delta m) + (\Delta m)^\alpha \|p - p^*\| + \Delta C_G(p^*, \Delta m)\\ =\ &\int_{\gamma_p}\left[(\theta(s) - \Delta m)^\alpha - \theta(s)^\alpha\right] ds + (\Delta m)^\alpha \|p - p^*\|\\ &+ \int_{\gamma_{p^*}}\left[(\theta(s) + \Delta m)^\alpha - \theta(s)^\alpha\right] ds\\ \geq\ &\mathbf{M}_\alpha\left(\hat{G}\right) - \mathbf{M}_\alpha(G) \geq 0, \text{ by the optimality of } G.\end{aligned}$$

To prove (4.9), we observe that

$$\begin{aligned}
& (\Delta m)^{\alpha}\left[\|p-p^*\|+\rho_{\alpha}\left(\theta\left(p^*\right)+\Delta m,\Delta m\right) length\left(\gamma_{p^*}\right)\right] \\
= \quad & (\Delta m)^{\alpha}\|p-p^*\|+\left[\left(\theta\left(p^*\right)+\Delta m\right)^{\alpha}-\left(\theta\left(p^*\right)\right)^{\alpha}\right] length\left(\gamma_{p^*}\right) \\
\geq \quad & (\Delta m)^{\alpha}\|p-p^*\|+\Delta C_G\left(p^*,\Delta m\right), \text{ by (4.5)} \\
\geq \quad & -\Delta C_G\left(p,-\Delta m\right)=\Delta C_{\tilde{G}}\left(p,\Delta m\right), \text{ by (4.8) and (4.4)} \\
\geq \quad & \left[\left(\mathfrak{m}\left(\mathbf{b}_s\right)\right)^{\alpha}-\left(\mathfrak{m}\left(\mathbf{b}_s\right)-\Delta m\right)^{\alpha}\right] length\left(\gamma_p\right), \text{ by (4.5)} \\
= \quad & (\Delta m)^{\alpha}\rho_{\alpha}\left(\mathfrak{m}\left(\mathbf{b}_s\right),\Delta m\right)\int_{\gamma_p} ds \geq (\Delta m)^{\alpha}\rho_{\alpha}\left(\mathfrak{m}\left(\mathbf{b}_s\right),\Delta m\right)\|p-x_s\|.
\end{aligned}$$

In particular, when $p^*=x_i$ for some i, (4.9) becomes (4.10) as $length\left(\gamma_{p^*}\right)=0$. Now, suppose p^* is on G_i with $i\neq s$. Apply (4.10) to p^*, we have

$$\frac{\|p^*-x_i\|}{\rho_{\alpha}\left(\mathfrak{m}\left(\mathbf{b}_i\right),\Delta m\right)} \geq length\left(\gamma_{p^*}\right).$$

Therefore, by (4.9),

$$\begin{aligned}
& \|p-p^*\|+\frac{\rho_{\alpha}\left(\theta\left(p^*\right)+\Delta m,\Delta m\right)}{\rho_{\alpha}\left(\mathfrak{m}\left(\mathbf{b}_i\right),\Delta m\right)}\|p^*-x_i\| \\
\geq \quad & \|p-p^*\|+\rho_{\alpha}\left(\theta\left(p^*\right)+\Delta m,\Delta m\right) length\left(\gamma_{p^*}\right) \\
\geq \quad & \rho_{\alpha}\left(\mathfrak{m}\left(\mathbf{b}_s\right),\Delta m\right) length\left(\gamma_p\right) \geq \rho_{\alpha}\left(\mathfrak{m}\left(\mathbf{b}_s\right),\Delta m\right)\|p-x_s\|.
\end{aligned}$$

□

Inequality (4.8) follows intuitively by a standard marginal argument. For convenience of illustration, let p denote the location y_j of household j who is connected to factory s by some curve γ_p. A planner will find it not optimal to choose an allocation path G such that $\Delta C_G\left(y_j,-\Delta m\right)+(\Delta m)^{\alpha}\|y_j-p^*\|+\Delta C_G\left(p^*,\Delta m\right)<0$ for p^* on some G_i. It is because in this case the planner has a less costly alternative by transferring Δm amount of production from factory s to factory i and transporting this additional Δm units of commodity from factory i first to a stopover point p^* via the curve γ_{p^*} and then directly from p^* to household j. It is clear that this strategy will send the same amount of commodity to household j as before. However, by the inequality, the reduction in transportation cost $-\Delta C_G\left(p,-\Delta m\right)$ on the curve γ_p exceeds its increase counterpart $\Delta C_G\left(p^*,\Delta m\right)+(\Delta m)^{\alpha}\|y_j-p^*\|$, which cannot be the case for an optimal allocation path.

By means of Proposition 4.2, we obtain the following results regarding the properties of optimal assignment maps.

Theorem 4.1. *Suppose $S\in Map\left[\ell,k\right]$ is an optimal assignment map. Let $j\in\{1,\cdots,\ell\}$ and $s\in\{1,\cdots,k\}$. If*

$$y_j\in\Gamma_S^s\left(n_j\right):=\left\{z\in\mathbb{R}^m:\min_{i\neq s}\|z-x_i\|<\rho_{\alpha}\left(\mathfrak{m}\left(\mathbf{b}_s\right),n_j\right)\|z-x_s\|\right\}, \tag{4.12}$$

then $S\left(j\right)\neq s$. If

$$y_j\in\Omega_S^s\left(n_j\right):=\left\{z\in\mathbb{R}^m:\|z-x_s\|<\min_{i\neq s}\rho_{\alpha}\left(\mathfrak{m}\left(\mathbf{b}_i\right),n_j\right)\|z-x_i\|\right\}, \tag{4.13}$$

then $S\left(j\right)=s$.

Proof. Assume $y_j \in \Gamma_S^s(n_j)$ but $S(j) = s$. Let $G = G_S$ be an allocation path as in (3.3). By Theorem 3.2, $G \in \mathit{Path}[\mathbf{x}, \mathbf{b}]$ is an optimal allocation path. Clearly, y_j is on G_s when $S(j) = s$. Apply (4.10) to $p = y_j$ and $\Delta m = n_j$, we have

$$\min_{i \in \{1,\cdots,k\}} \|y_j - x_i\| \geq \rho_\alpha(\mathfrak{m}(\mathbf{b}_s), n_j) \|y_j - x_s\|,$$

which contradicts (4.12).

On the other hand, for each $i \neq s$, if $y_j \in \Omega_S^s(n_j)$, then

$$\min_{i^* \neq i} \|y_j - x_{i^*}\| \leq \|y_j - x_s\| < \rho_\alpha(\mathfrak{m}(\mathbf{b}_i), n_j) \|y_j - x_i\|.$$

By (4.12), $y_j \in \Gamma_S^i(n_j)$ and $S(j) \neq i$ for such $i \neq s$. Thus, $S(j) = s$. $\square$

Intuitively speaking, inequality in (4.12) says that if household j locates "closer" to some factory other than factory s, then the planner will not assign it to factory s. Here the relative closeness is weighted by a number $\rho_\alpha(\mathfrak{m}(\mathbf{b}_s), n_j)$. When the production $m(\mathbf{b}_s)$ at factory s is low, due to the transport economies of scale, one would expect that the planner would less likely assign household j to factory s. This predication is justified by Theorem 4.1 because in this case, inequality in (4.12) becomes more likely to hold as $\rho_\alpha(\mathfrak{m}(\mathbf{b}_s), n_j)$ becomes higher. The later part (4.13) of the theorem states a special case that if household j is located uniformly closer to a factory s than to other factories, then it will be assigned to factory s under any optimal assignment map.

We now give a geometric description of sets $\Gamma_S^s(n_j)$ and $\Omega_S^s(n_j)$. It can be verified that for any constant $C \in (0,1)$, the set

$$\begin{aligned} &\{x \in \mathbb{R}^m : \|x - x_i\| < C\|x - x_s\|\} \\ &= B\left(x_i + \tfrac{C^2}{1-C^2}(x_i - x_s), \tfrac{C}{1-C^2}\|x_i - x_s\|\right), \end{aligned} \tag{4.14}$$

where $B(x, r)$ denotes the open ball $\{z \in \mathbb{R}^m : \|z - x\| < r\}$. By (4.14), relation (4.12) says that geometrically, if household j lies in the union of $k-1$ open balls

$$\Gamma_S^s(n_j) = \bigcup_{i \neq s} B\left(x_i + \frac{(w_{sj})^2}{1-(w_{sj})^2}(x_i - x_s), \frac{w_{sj}}{1-(w_{sj})^2}\|x_s - x_i\|\right),$$

where $w_{sj} = \rho_\alpha(\mathfrak{m}(\mathbf{b}_s), n_j)$, then $S(j) \neq s$. Also, (4.13) says that if household j lies in the intersection of $(k-1)$ balls

$$\Omega_S^s(n_j) = \bigcap_{i \neq s} B\left(x_s + \frac{(w_{ij})^2}{1-(w_{ij})^2}(x_s - x_i), \frac{w_{ij}}{1-(w_{ij})^2}\|x_i - x_s\|\right) \tag{4.15}$$

for some s, then $S(j) = s$.

Since $m(\mathbf{b}_s) \leq m(\mathbf{b}) = 1$ and the function $\rho_\alpha(\cdot, n_j)$ is decreasing, we have $\rho_\alpha(1, n_j) \leq \rho_\alpha(\mathfrak{m}(\mathbf{b}_s), n_j)$ and thus

$$\Gamma^s(n_j) \quad : \quad = \left\{z : \min_{i \neq s} \|z - x_i\| < \rho_\alpha(1, n_j)\|z - x_s\|\right\} \subseteq \Gamma_S^s(n_j), \tag{4.16}$$

$$\Omega^s(n_j) \quad : \quad = \left\{z : \|z - x_s\| < \rho_\alpha(1, n_j) \min_{s \neq i} \|z - x_i\|\right\} \subseteq \Omega_S^s(n_j) \tag{4.17}$$

for any optimal assignment map S. Note that the set $\Gamma^s(n_j)$ (or $\Omega^s(n_j)$) is still the union (or intersection) of $k-1$ balls in $\mathbb{R}^m$, and is independent of S. For example, sets $\Omega^s(n_j)$ with $n_j = 0.8$ and $n_j = 0.5$ are displayed in Figure 5, where $x_1 = (0,0), x_2 = (2,0)$ and $x_3 = (1,2)$.

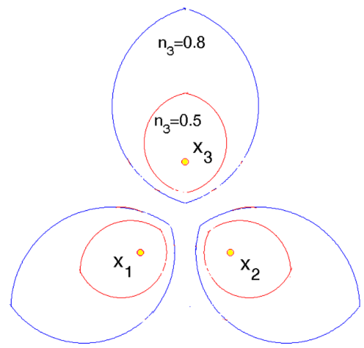


FIGURE 5. An example of set $\Omega^s(n_j)$ with $n_j = 0.8$ (blue) and $n_j = 0.5$ (red) when $\alpha = 1/2$.

By Theorem 4.1, (4.16) and (4.17), we have the following corollary.

Corollary 4.1. *For any optimal assignment map $S \in Map[\ell, k]$ and $s \in \{1, \cdots, k\}$, if $y_j \in \Gamma^s(n_j)$, then $S(j) \neq s$. If $y_j \in \Omega^s(n_j)$, then $S(j) = s$.*

This corollary shows that if household j falls into the region $\Omega^s(n_j)$ of some factory s, then it will be assigned to this factory under any optimal assignment map S. As a result, if all households belong to the union of regions $\Omega^s(n_j)$ of factories s except factory i, then factory i will not be used. Note that as $\rho_\alpha(1, n_j)$ is increasing in n_j, the size of the region $\Omega^s(n_j)$ increases with n_j as shown in Figure 5.

Theorem 4.2. *Suppose $S \in Map[\ell, k]$ is an optimal assignment map, h and $j \in \{1, \cdots, \ell\}$ with $n_j \leq n_h$. If $S(h) = s^* \neq s$ for some $s \in \{1, \cdots, k\}$ and*

$$y_j \in \Gamma_S^{s,h}(n_j) := \left\{ z \in \mathbb{R}^m \;\middle|\; \begin{array}{l} \|z - y_h\| + \frac{\rho_\alpha(n_h + n_j, n_j)}{\rho_\alpha(\mathfrak{m}(\mathbf{b}_{s^*}), n_j)} \|y_h - x_{s^*}\| \\ < \rho_\alpha(\mathfrak{m}(\mathbf{b}_s), n_j) \|z - x_s\| \end{array} \right\}, \tag{4.18}$$

then $S(j) \neq s$. If $S(h) = s$ for some $s \in \{1, \cdots, k\}$ and

$$y_j \in \Omega_S^{s,h}(n_j) := \bigcap_{i \neq s, \mathfrak{m}(\mathbf{b}_i) \geq n_j} \left\{ z \in \mathbb{R}^m \;\middle|\; \begin{array}{l} \|z - y_h\| + \frac{\rho_\alpha(n_h + n_j, n_j)}{\rho_\alpha(\mathfrak{m}(\mathbf{b}_s), n_j)} \|y_h - x_s\| \\ < \rho_\alpha(\mathfrak{m}(\mathbf{b}_i), n_j) \|z - x_i\| \end{array} \right\}, \tag{4.19}$$

then $S(j) = s$.

Proof. In the first scenario, assume $S(h) = s^* \neq s$, $y_j \in \Gamma_S^{s,h}(n_j)$ but $S(j) = s$. Then

$$\begin{aligned} & \|y_j - y_h\| + \frac{\rho_\alpha(n_h + n_j, n_j)}{\rho_\alpha(\mathfrak{m}(\mathbf{b}_{s^*}), n_j)} \|y_h - x_{s^*}\| \\ \geq\ & \|y_j - y_h\| + \frac{\rho_\alpha(\theta(y_h) + n_j, n_j)}{\rho_\alpha(\mathfrak{m}(\mathbf{b}_{s^*}), n_j)} \|y_h - x_{s^*}\| \text{ as } \theta(y_h) \geq n_h, \\ \geq\ & \rho_\alpha(\mathfrak{m}(\mathbf{b}_s), n_j) \|y_j - x_s\|, \text{ by (4.11)}, \end{aligned}$$

a contradiction with $y_j \in \Gamma_S^{s,h}(n_j)$. Thus, $S(j) \neq s$.

Now, in the second scenario, assume $S(h) = s$, $y_j \in \Omega_S^{s,h}(n_j)$ but $S(j) = i^*$ for some $i^* \neq s$. Then,

$$\begin{aligned}
& \|y_j - y_h\| + \frac{\rho_\alpha(n_h + n_j, n_j)}{\rho_\alpha(\mathfrak{m}(\mathbf{b}_s), n_j)} \|y_h - x_s\| \\
\geq \quad & \|y_j - y_h\| + \frac{\rho_\alpha(\theta(y_h) + n_j, n_j)}{\rho_\alpha(\mathfrak{m}(\mathbf{b}_s), n_j)} \|y_h - x_s\| \text{ as } \theta(y_h) \geq n_h, \\
\geq \quad & \rho_\alpha(\mathfrak{m}(\mathbf{b}_{i^*}), n_j) \|y_j - x_{i^*}\|, \text{ by (4.11)},
\end{aligned}$$

a contradiction with $y_j \in \Omega_S^{s,h}(n_j)$ as $m(\mathbf{b}_{i^*}) \geq n_j$ and $i^* \neq s$. Thus, $S(j) = s$. $\square$

The first part of Theorem 4.2 says that if some household h is not assigned to factory s, then any other nearby household j (i.e., within the neighborhood region $\Gamma_S^{s,h}(n_j)$) with a smaller demand will also not be assigned to factory s. The second part says that if some household h is assigned to factory s, then any other nearby household j (i.e., within the neighborhood region $\Omega_S^{s,h}(n_j)$) with a smaller demand will also be assigned to factory s. These findings agree with the intuition that grouping with nearby households of large demand would make it more likely to reap the benefits of transport economies of scale.

5. Projectional analysis of optimal assignment maps.

As seen in the previous section, under an optimal assignment map, a household will be assigned to some factory if it lives close to the factory (Theorem 4.1) or it has some nearby neighbours assigned to the factory (Theorem 4.2). In this section, we will show a reverse result (Theorem 5.1) using a method of projectional analysis.

Throughout this section, we consider the orthogonal projection from $\mathbb{R}^m$ to a fixed line $L = \{p + tv : t \in \mathbb{R}\}$ represented by a fixed point $p \in \mathbb{R}^m$ and a unit vector v. Under this map, each point $z \in \mathbb{R}^m$ is mapped to $p + \pi(z)v$ with

$$\pi(z) = \langle z - p, v \rangle, \tag{5.1}$$

where $\langle \cdot, \cdot \rangle$ stands for the standard inner product in $\mathbb{R}^m$. For instance, when $p = (0, \cdots, 0) \in \mathbb{R}^m$, and $v = (1, 0, \cdots, 0) \in \mathbb{R}^m$, for each $z = (z_1, \cdots, z_m)$, $\pi(z) = z_1$ gives the first coordinate of z. For any real number $R > 0$, let

$$L_R = \{z \in \mathbb{R}^m : \|z - p - \pi(z)v\| \leq R\} \tag{5.2}$$

be the tubular neighborhood of the line L of radius R.

5.1. Preliminary lemmas.

We start with some lemmas regarding properties of a single-sourced transport system. These lemmas will play an important role in establishing Theorem 5.1 later. Suppose

$$\mathbf{c} = \sum_{j \in \Theta} c_j \delta_{z_j}, \ c_j > 0 \text{ for } j \text{ in a finite set } \Theta \tag{5.3}$$

is an atomic measure on $X \subseteq \mathbb{R}^m$, and $\bar{B}_R(O_1)$ is the closed ball in $\mathbb{R}^{m-1}$ centered at the origin O_1 with radius R.

Lemma 5.1. *If* $\mathbf{c}$ *in (5.3) is supported on the ball* $\bar{B}_R(O_1) \times \{0\}$, *then for any point* $P = (p_1, p_2, \cdots, p_m) \in \bar{B}_R(O_1) \times \mathbb{R} \subseteq \mathbb{R}^m$, *it holds that*

$$\mathfrak{m}(\mathbf{c})^\alpha |p_m| \leq d_\alpha(\mathbf{c}, \mathfrak{m}(\mathbf{c})\delta_P) \leq \mathfrak{m}(\mathbf{c})^\alpha (|p_m| + CR)$$

where

$$C = \frac{\sqrt{m-1}}{2^{1-(m-1)(1-\alpha)} - 1} + 1. \tag{5.4}$$

Proof. By a normalization process and (2.7), without loss of generality, we may assume that $\mathbf{c}$ is a probability measure with $\mathfrak{m}(\mathbf{c}) = 1$. Since $d_\alpha(\mathbf{c}, \delta_P)$ is a decreasing function in $\alpha \in [0,1]$, it holds that

$$d_\alpha(\mathbf{c}, \delta_P) \geq d_1(\mathbf{c}, \delta_P) = \sum_{j\in\Theta} c_j \|z_j - P\| \geq \sum_{j\in\Theta} c_j |p_m| = |p_m|.$$

On the other hand, since $\mathbf{c}$ is supported on the $m-1$ dimensional ball $\bar{B}_R(O_1)\times\{0\}$, by Xia [20, Theorem 3.1],

$$d_\alpha(\mathbf{c}, \delta_{O_1}) \leq \frac{\sqrt{m-1}}{2^{1-(m-1)(1-\alpha)}-1}R.$$

By the triangle inequality,

$$d_\alpha(\mathbf{c}, \delta_P) \leq d_\alpha(\mathbf{c}, \delta_{O_1}) + d_\alpha(\delta_{O_1}, \delta_P) \leq \frac{\sqrt{m-1}}{2^{1-(m-1)(1-\alpha)}-1}R + R + |p_m| = |p_m| + CR.$$

□

Lemma 5.2. *If $\mathbf{c}$ in (5.3) is supported in the semi-tube $\bar{B}_R(O_1)\times(-\infty,0]$ as shown in Figure 6, then for any point $P = (p_1, p_2, \cdots, p_m) \in \bar{B}_R(O_1)\times(0,\infty) \subseteq \mathbb{R}^m$, and $Q = (q_1, q_2, \cdots, q_m) \in \bar{B}_R(O_1)\times\mathbb{R}$, it holds that*

$$d_\alpha(\mathbf{c}, \mathfrak{m}(\mathbf{c})\delta_P) - d_\alpha(\mathbf{c}, \mathfrak{m}(\mathbf{c})\delta_Q) \geq \mathfrak{m}(\mathbf{c})^\alpha (|p_m| - CR - |q_m|),$$

where C is the constant given in (5.4).

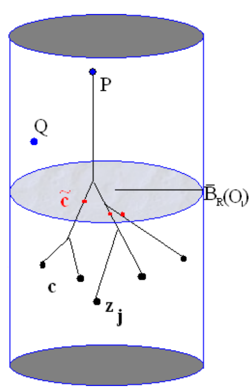


FIGURE 6. Transportation of a measure $\mathbf{c}$ located in a semi-tube region to a point P located on the other side of $\bar{B}_R(O_1)\times\{0\}$.

Proof. Let $G \in Path(\mathbf{c}, \mathfrak{m}(\mathbf{c})\delta_P)$ be an optimal transport path. Since both $\mathbf{c}$ and δ_P are supported in the convex set $\bar{B}_R(O_1)\times(-\infty,\infty)$, the optimal transport path G is also contained in $\bar{B}_R(O_1)\times(-\infty,\infty)$. Since G contains no cycles, there exists a unique polyhedral curve γ_{z_j} from P to z_j for each j. Because P and z_j are located on different sides of $\bar{B}_R(O_1)\times\{0\}$, each γ_{z_j} will intersect with the $(m-1)$-dimensional ball $\bar{B}_R(O_1)\times\{0\}$ at least once. Let v_j be the last intersection point of γ_{z_j} with $\bar{B}_R(O_1)\times\{0\}$, and define

$$\tilde{\mathbf{c}} = \sum_{j\in\Theta} c_j \delta_{v_j},$$

which is supported on the ball $\bar{B}_R(O_1) \times \{0\}$. By the optimality of G, we have

$$d_\alpha(\mathbf{c}, \mathfrak{m}(\mathbf{c})\delta_P) = d_\alpha(\mathbf{c}, \tilde{\mathbf{c}}) + d_\alpha(\tilde{\mathbf{c}}, \mathfrak{m}(\mathbf{c})\delta_P). \tag{5.5}$$

Therefore, by the triangle inequality, (5.5) and Lemma 5.1,

$$\begin{aligned} & d_\alpha(\mathbf{c}, \mathfrak{m}(\mathbf{c})\delta_P) - d_\alpha(\mathbf{c}, \mathfrak{m}(\mathbf{c})\delta_Q) \\ \geq \; & d_\alpha(\mathbf{c}, \mathfrak{m}(\mathbf{c})\delta_P) - [d_\alpha(\mathbf{c}, \tilde{\mathbf{c}}) + d_\alpha(\tilde{\mathbf{c}}, \mathfrak{m}(\mathbf{c})\delta_Q)] \\ = \; & d_\alpha(\tilde{\mathbf{c}}, \mathfrak{m}(\mathbf{c})\delta_P) - d_\alpha(\tilde{\mathbf{c}}, \mathfrak{m}(\mathbf{c})\delta_Q) \\ \geq \; & \mathfrak{m}(\mathbf{c})^\alpha |p_m| - \mathfrak{m}(\mathbf{c})^\alpha (CR + |q_m|). \end{aligned}$$

□

Corollary 5.1. *Suppose* $\mathbf{c}$ *in (5.3) is supported in a tubular neighborhood* L_R *of the line* L *as given in (5.2) and* P *is a point in* L_R *with*

$$\pi(P) \geq \max_{j \in \Theta} \pi(z_j).$$

For any point $Q \in L_R$*, and any* t_1 *with* $\pi(P) \geq t_1 \geq \max_{j\in\Theta} \pi(z_j)$*, it holds that*

$$d_\alpha(\mathbf{c}, \mathfrak{m}(\mathbf{c})\delta_P) - d_\alpha(\mathbf{c}, \mathfrak{m}(\mathbf{c})\delta_Q) \geq \mathfrak{m}(\mathbf{c})^\alpha (|\pi(P) - t_1| - CR - |\pi(Q) - t_1|), \tag{5.6}$$

where C *is the constant given in (5.4).*

Proof. By means of a rigid motion in $\mathbb{R}^m$, we could transform the tubular neighborhood L_R of the line L to a tubular neighborhood of the z_m-axis $\{z = (0, 0, \cdots, 0, z_m)\}$ and the point $p + t_1 v$ on L to the origin O_1. Then, under this rigid motion, inequality (5.6) follows from Lemma 5.2. □

Lemma 5.3. *Let* c *be an atomic measure as given in (5.3) and* O *be a fixed point in* X*. If the set* Θ *is decomposed as the disjoint union of two nonempty subsets (as illustrated in Figure 7)*

$$\Theta = \Theta_1 \amalg \Theta_2,$$

then, for any optimal transport path $G \in Path(\mathfrak{m}(\mathbf{c})\delta_O, \mathbf{c})$*, there exist a vertex point* $P \in V(G)$ *and a decomposition of each* Θ_i*:*

$$\Theta_i = \tilde{\Theta}_i \amalg \bar{\Theta}_i \text{ with } \tilde{\Theta}_i \text{ nonempty, } i = 1, 2$$

such that G *can be decomposed as*

$$G = G_1 + G_2 + G_3 \tag{5.7}$$

where for each $i = 1, 2$*,* G_i *is an optimal transport path from* $\mathfrak{m}(\tilde{\mathbf{c}}_i)\delta_P$ *to* $\tilde{\mathbf{c}}_i$ *for*

$$\tilde{\mathbf{c}}_i = \sum_{j \in \tilde{\Theta}_i} c_j \delta_{z_j}, \tag{5.8}$$

G_3 *is an optimal transport path from* $\mathfrak{m}(\mathbf{c})\delta_O$ *to* $\bar{\mathbf{c}} + (\mathfrak{m}(\tilde{\mathbf{c}}_1) + \mathfrak{m}(\tilde{\mathbf{c}}_2))\delta_P$ *for*

$$\bar{\mathbf{c}} = \sum_{j \in \bar{\Theta}_1 \cup \bar{\Theta}_2} c_j \delta_{z_j},$$

and $\{G_i\}_{i=1}^3$ *are pairwise disjoint except at* P*. Moreover, (5.7) implies*

$$\mathbf{M}_\alpha(G) = \mathbf{M}_\alpha(G_1) + \mathbf{M}_\alpha(G_2) + \mathbf{M}_\alpha(G_3) \tag{5.9}$$

and by the optimality of G*, it follows*

$$\begin{aligned} d_\alpha(\mathfrak{m}(\mathbf{c})\delta_O, \mathbf{c}) = \; & d_\alpha(\mathfrak{m}(\tilde{\mathbf{c}}_1)\delta_P, \tilde{\mathbf{c}}_1) + d_\alpha(\mathfrak{m}(\tilde{\mathbf{c}}_2)\delta_P, \tilde{\mathbf{c}}_2) \\ & + d_\alpha(\mathfrak{m}(\mathbf{c})\delta_O, \bar{\mathbf{c}} + (\mathfrak{m}(\tilde{\mathbf{c}}_1) + \mathfrak{m}(\tilde{\mathbf{c}}_2))\delta_P). \end{aligned} \tag{5.10}$$

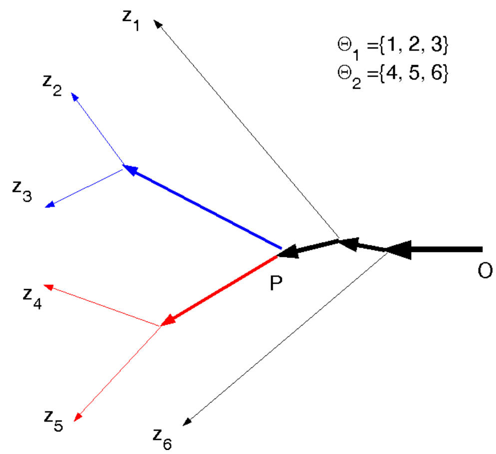


FIGURE 7. In this example, $\Theta = \{1,2,3,4,5,6\}, \Theta_1 = \{1,2,3\}, \Theta_2 = \{4,5,6\}, \bar{\Theta}_1 = \{2,3\}, \bar{\Theta}_2 = \{4,5\}$. G_1 is in blue color, G_2 is in red color, and the rest black part is G_3.

Proof. For any z on the support of G, since G is a transport path from a single source O and contains no cycles, there exists a unique curve γ_z on G from O to z. As a result, it holds that

$$\text{if } \tilde{z} \text{ lies on } \gamma_z \text{ for some } z, \text{ then } \gamma_{\tilde{z}} \text{ is the part of } \gamma_z \text{ from } O \text{ to } \tilde{z}. \tag{5.11}$$

Now, let set Γ_i be the union of all curves γ_{z_j} with $j \in \Theta_i$ for $i = 1,2$, and set $\Gamma = \Gamma_1 \cap \Gamma_2$. By (5.11), if $z \in \Gamma$, then $\gamma_z \subseteq \Gamma$. This shows that Γ is a connected subset of the support of G containing O. Since Γ contains no cycles, it is a contractible set containing O. Then, by calculating the Euler characteristic number of Γ, we have either $\Gamma = \{O\}$ or Γ has at least two endpoints (i.e., vertices of degree 1).

If $\Gamma = \{O\}$, then set $P = O$, and $\tilde{\Theta}_i = \Theta_i$ for $i = 1,2$. If $\Gamma \neq \{O\}$, pick P to be an endpoint of Γ with $P \neq O$. Since $P \in \Gamma \subseteq \Gamma_i$, set $\tilde{\Theta}_i := \left\{ j \in \Theta_i : P \in \gamma_{z_j} \right\} \neq \emptyset$, for $i = 1,2$. For any $j \in \tilde{\Theta}_i$, P divides the curve γ_{z_j} into two parts: $\gamma_{z_j}^{(1)}$ from O to P and $\gamma_{z_j}^{(2)}$ from P to z_j. Since P is an endpoint of Γ, we have $\left(\gamma_{z_j}^{(2)} \setminus \{P\}\right) \cap \Gamma = \emptyset$. For $i = 1,2$, define $\tilde{\mathbf{c}}_i$ using (5.8) and denote the part of G from $\mathfrak{m}(\tilde{\mathbf{c}}_i)\,\delta_P$ to $\tilde{\mathbf{c}}_i$ by G_i. The rest of G is denoted by $G_3 = G - (G_1 + G_2)$. Then, by construction, $\{G_i\}_{i=1}^3$ are pairwise disjoint except at P, and thus (5.9) holds. By the optimality of G, each G_i must also be optimal for $i = 1,2,3$, which yields (5.10). □

5.2. **Main theorem of the section.** The following theorem states that: under an optimal assignment map, a household will be assigned to some factory only when either it lives close to the factory or it has some nearby neighbours assigned to the factory. In the first situation, a planner takes advantage of relative spatial locations between households and factories; in the second situation, the planner takes advantage of group transportation due to the economies of scale embedded in ramified transport technologies.

Let $S \in Map\,[\ell, k]$ be an optimal assignment map. For each $i \in \{1, \cdots, k\}$, define

$$\Psi_i = \{y_h : S(h) = i\} \cup \{x_i\} \tag{5.12}$$

and

$$R_i = \max \{\|z - p - \pi(z) v\| : z \in \{y_h : S(h) = i\} \cup \{x_1, \cdots, x_k\}\}. \tag{5.13}$$

Theorem 5.1. *Suppose $S \in Map[\ell, k]$ is an optimal assignment map. If $S(j) = i$ for some $i \in \{1, \cdots, k\}$ and $j \in \{1, \cdots, \ell\}$, then there exists $z \in \Psi_i \setminus \{y_j\}$, such that*

$$0 < |\pi(y_j) - \pi(z)| \leq 2CR_i + \min_{s \neq i} |\pi(x_s) - \pi(y_j)| \tag{5.14}$$

and $\pi(z)$ is between $\pi(y_j)$ and $\pi(x_i)$, where C is the constant given in (5.4) and R_i is the constant given in (5.13).

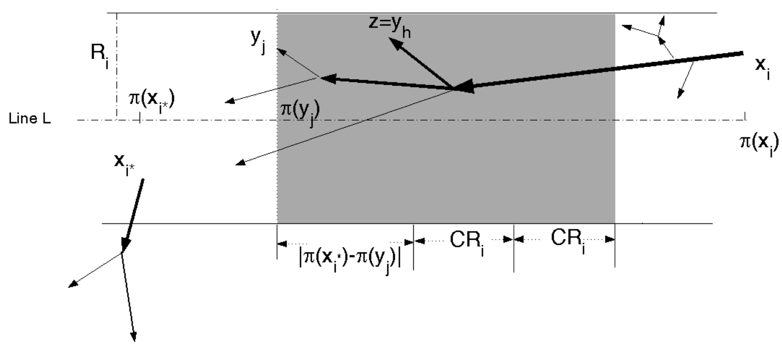


FIGURE 8. Theorem 5.1 says that if the household at y_j is assigned to the factory at x_i under an optimal assignment map S, then either x_i is located in the shadowed region, or S also assigns another household at y_h located in the shadowed region to the factory at x_i.

Proof. **Step 1:** Without loss of generality, we may assume that $\pi(y_j) \leq \pi(x_i)$. Let $|\pi(x_{i^*}) - \pi(y_j)| = \min_{s \neq i} |\pi(x_s) - \pi(y_j)|$ for some $i^* \neq i$. We want to prove (5.14) by contradiction. Assume for any $z \in \Psi_i \setminus \{y_j\}$ with $\pi(y_j) < \pi(z) \leq \pi(x_i)$,

$$|\pi(y_j) - \pi(z)| > 2CR_i + |\pi(x_{i^*}) - \pi(y_j)|,$$

i.e., $\pi(z) - \pi(y_j) > 2CR_i + |\pi(x_{i^*}) - \pi(y_j)|$. Then, there exists a real number t_2 such that

$$\pi(z) > t_2 > \pi(y_j) + 2CR_i + |\pi(x_{i^*}) - \pi(y_j)| \tag{5.15}$$

whenever $\pi(y_j) < \pi(z) \leq \pi(x_i)$. In particular, since $x_i \in \Psi_i$, (5.15) yields

$$\pi(x_i) > t_2 > \pi(y_j) + 2CR_i + |\pi(x_{i^*}) - \pi(y_j)|. \tag{5.16}$$

As a result, $S^{-1}(i)$ can be expressed as the disjoint union of two sets:

$$\Theta_1 := \left\{h \in S^{-1}(i) : \pi(y_h) \leq \pi(y_j)\right\} \text{ and } \Theta_2 := \left\{h \in S^{-1}(i) : \pi(y_h) > t_2\right\}.$$

Clearly, $j \in \Theta_1$.

Step 2: If $\Theta_2 = \emptyset$ as shown in Figure 9, then $S^{-1}(i) = \Theta_1$ and thus

$$\pi(x_i) \geq \pi(y_j) \geq \max_{h \in S^{-1}(i)} \pi(y_h).$$

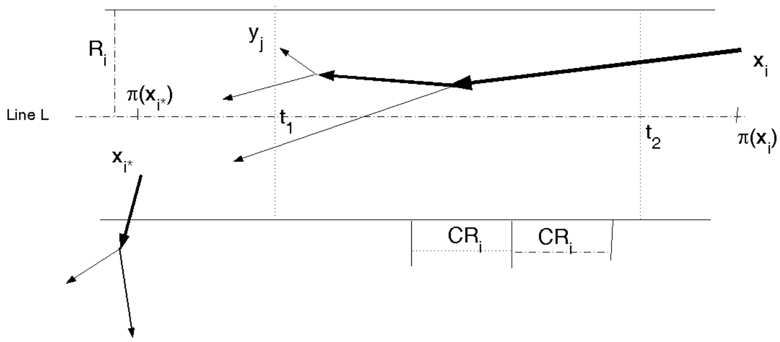


FIGURE 9. If $\Theta_2 = \emptyset$, then it is better to assign $\mathbf{b}_i$ to another factory at x_{i^*} other than that at x_i.

Let $\mathbf{a}_i$ and $\mathbf{b}_i$ be given as in (3.2). By Corollary 5.1 with $t_1 = \pi(y_j)$ and (5.16), we have

$$\begin{aligned} & d_\alpha\left(\mathbf{b}_i, \mathfrak{m}(\mathbf{b}_i)\delta_{x_i}\right) - d_\alpha\left(\mathbf{b}_i, \mathfrak{m}(\mathbf{b}_i)\delta_{x_{i^*}}\right) \\ \geq\ & \mathfrak{m}(\mathbf{b}_i)^\alpha \left(\left|\pi(x_i) - \pi(y_j)\right| - CR_i - \left|\pi(x_{i^*}) - \pi(y_j)\right|\right) \\ \geq\ & \mathfrak{m}(\mathbf{b}_i)^\alpha \left(\pi(x_i) - \pi(y_j) - CR_i - \left|\pi(x_{i^*}) - \pi(y_j)\right|\right) > \mathfrak{m}(\mathbf{b}_i)^\alpha CR_i > 0, \end{aligned}$$

a contradiction with the optimality of S. Thus, $\Theta_2 \neq \emptyset$.

Let $G_i \in Path(\mathbf{a}_i, \mathbf{b}_i)$ be an optimal transport path. Then, by Theorem 3.2 and the optimality of S, $G = \sum_i G_i$ is an optimal allocation path. Since both Θ_1 and Θ_2 are nonempty, by setting $\Theta = S^{-1}(i) = \Theta_1 \amalg \Theta_2, O = x_i$ and $c = \mathbf{b}_i$ in Lemma 5.3, there exists a point $P \in V(G_i)$ such that G_i can be decomposed as $G_i = G_i^{(1)} + G_i^{(2)} + G_i^{(3)}$ with

$$\mathbf{M}_\alpha(G_i) = \mathbf{M}_\alpha\left(G_i^{(1)}\right) + \mathbf{M}_\alpha\left(G_i^{(2)}\right) + \mathbf{M}_\alpha\left(G_i^{(3)}\right). \tag{5.17}$$

Here, for $h = 1, 2$, $G_i^{(h)}$ is an optimal transport path from $\mathfrak{m}\left(\mathbf{b}_i^{(h)}\right)\delta_P$ to $\mathbf{b}_i^{(h)}$ for some positive atomic measures $\mathbf{b}_i^{(h)}$ with $spt\left(\mathbf{b}_i^{(h)}\right) \subseteq \left\{y_{\tilde{h}} : \tilde{h} \in \Theta_h\right\}$ and

$$G_i^{(3)} \in Path\left(\mathfrak{m}(\mathbf{b}_i)\delta_{x_i}, \mathbf{b}_i - \left(\mathbf{b}_i^{(1)} + \mathbf{b}_i^{(2)}\right) + \left(\mathfrak{m}\left(\mathbf{b}_i^{(1)}\right) + \mathfrak{m}\left(\mathbf{b}_i^{(2)}\right)\right)\delta_P\right).$$

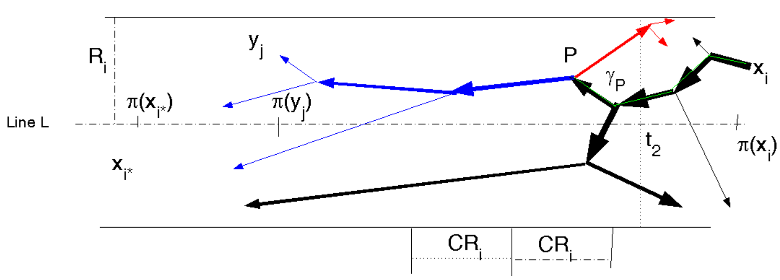


FIGURE 10. If $\pi(P) \geq t_2 - CR_i$, then it is better to assign $\mathbf{b}_i^{(1)}$ to another factory x_{i^*} other than that at x_i.

Step 3: If $\pi(P) \geq t_2 - CR_i$ as shown in Figure 10, then we can modify G into another allocation path $\tilde{G}$ by just replacing the corresponding transport path from factory i to households $\mathbf{b}_i^{(1)}$ with an optimal transport path from factory i^* to $\mathbf{b}_i^{(1)}$. More precisely, we replace G_i by

$$\tilde{G}_i = \tilde{G}_i^{(1)} + \tilde{G}_i^{(2)} + \tilde{G}_i^{(3)}, \tag{5.18}$$

where $\tilde{G}_i^{(1)}$ is an optimal transport path from $\mathfrak{m}\left(\mathbf{b}_i^{(1)}\right)\delta_{x_{i^*}}$ to $\mathbf{b}_i^{(1)}$, $\tilde{G}_i^{(2)} = G_i^{(2)}$ and

$$\tilde{G}_i^{(3)} = G_i^{(3)} - \mathfrak{m}\left(\mathbf{b}_i^{(1)}\right)\gamma_P, \tag{5.19}$$

where γ_P is the curve on G from x_i to P. Equation (5.18) and (5.19) imply respectively

$$\mathbf{M}_\alpha\left(\tilde{G}_i\right) \leq \mathbf{M}_\alpha\left(\tilde{G}_i^{(1)}\right) + \mathbf{M}_\alpha\left(G_i^{(2)}\right) + \mathbf{M}_\alpha\left(\tilde{G}_i^{(3)}\right) \text{ and } \mathbf{M}_\alpha\left(\tilde{G}_i^{(3)}\right) \leq \mathbf{M}_\alpha\left(G_i^{(3)}\right).$$

Consequently, by (5.17),

$$\begin{aligned}
& \mathbf{M}_\alpha(G_i) - \mathbf{M}_\alpha\left(\tilde{G}_i\right) \\
\geq\ & \mathbf{M}_\alpha\left(G_i^{(1)}\right) - \mathbf{M}_\alpha\left(\tilde{G}_i^{(1)}\right) + \left(\mathbf{M}_\alpha\left(G_i^{(3)}\right) - \mathbf{M}_\alpha\left(\tilde{G}_i^{(3)}\right)\right) \\
\geq\ & \mathbf{M}_\alpha\left(G_i^{(1)}\right) - \mathbf{M}_\alpha\left(\tilde{G}_i^{(1)}\right) \\
=\ & d_\alpha\left(\mathfrak{m}\left(\mathbf{b}_i^{(1)}\right)\delta_P, \mathbf{b}_i^{(1)}\right) - d_\alpha\left(\mathfrak{m}\left(\mathbf{b}_i^{(1)}\right)\delta_{x_{i^*}}, \mathbf{b}_i^{(1)}\right),
\end{aligned}$$

where the last equality follows from the optimality of both $G_i^{(1)}$ and $\tilde{G}_i^{(1)}$.

Since $\alpha < 1$, for $\tilde{G} = \sum_{s \neq i} G_s + \tilde{G}_i$, we have

$$\mathbf{M}_\alpha\left(\tilde{G}\right) \leq \sum_{s \neq i} \mathbf{M}_\alpha(G_s) + \mathbf{M}_\alpha\left(\tilde{G}_i\right).$$

Due to the optimality of G, equation (3.1) says

$$\mathbf{M}_\alpha(G) = \sum_{s \neq i} \mathbf{M}_\alpha(G_s) + \mathbf{M}_\alpha(G_i).$$

As a result,

$$\begin{aligned}
& \mathbf{M}_\alpha(G) - \mathbf{M}_\alpha\left(\tilde{G}\right) \geq \mathbf{M}_\alpha(G_i) - \mathbf{M}_\alpha\left(\tilde{G}_i\right) \\
\geq\ & d_\alpha\left(\mathfrak{m}\left(\mathbf{b}_i^{(1)}\right)\delta_P, \mathbf{b}_i^{(1)}\right) - d_\alpha\left(\mathfrak{m}\left(\mathbf{b}_i^{(1)}\right)\delta_{x_{i^*}}, \mathbf{b}_i^{(1)}\right) \\
\geq\ & \mathfrak{m}\left(\mathbf{b}_i^{(1)}\right)^\alpha\left(|\pi(P) - \pi(y_j)| - CR_i - |\pi(x_{i^*}) - \pi(y_j)|\right), \text{ by Corollary 5.1} \\
\geq\ & \mathfrak{m}\left(\mathbf{b}_i^{(1)}\right)^\alpha\left(\pi(P) - \pi(y_j) - CR_i - |\pi(x_{i^*}) - \pi(y_j)|\right) \\
\geq\ & \mathfrak{m}\left(\mathbf{b}_i^{(1)}\right)^\alpha\left(t_2 - \pi(y_j) - 2CR_i - |\pi(x_{i^*}) - \pi(y_j)|\right) > 0, \text{ by (5.15)}.
\end{aligned}$$

Thus, $\mathbf{M}_\alpha(G) > \mathbf{M}_\alpha\left(\tilde{G}\right)$, which contradicts the optimality of G, and thus inequality (5.14) hold.

Step 4: If $\pi(P) < t_2 - CR_i$ as in Figure 11, then let Q be the first point of γ_P with $\pi(Q) = t_2$. We can modify G into another allocation path $\bar{G}$ by just replacing the corresponding transport path from the point Q to households $\mathbf{b}_i^{(2)}$ with an optimal transport path from Q to $\mathbf{b}_i^{(2)}$. More precisely, we replace G_i by

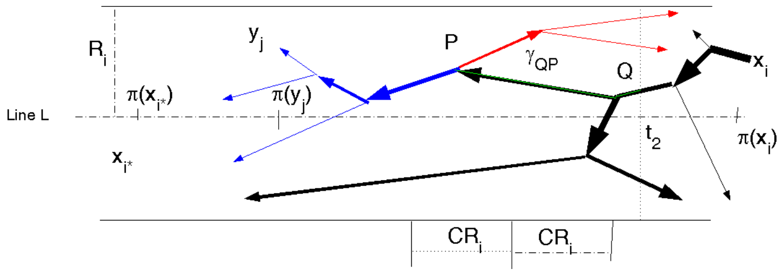


FIGURE 11. If $\pi(P) < t_2 - CR_i$, then it is better to transport $\mathbf{b}_i^{(2)}$ to the point Q directly than indirectly via the point P.

$\bar{G}_i = \bar{G}_i^{(1)} + \bar{G}_i^{(2)} + \bar{G}_i^{(3)}$ where $\bar{G}_i^{(1)} = G_i^{(1)}$, $\tilde{G}_i^{(2)}$ is an optimal transport path from $\mathfrak{m}\left(\mathbf{b}_i^{(2)}\right)\delta_Q$ to $\mathbf{b}_i^{(2)}$, and $\tilde{G}_i^{(3)} = G_i^{(3)} - \mathfrak{m}\left(\mathbf{b}_i^{(2)}\right)\gamma_{QP}$ where γ_{QP} is the part of the curve γ_P from Q to P. Similar arguments as in the previous case show that

$$\begin{aligned}&\mathbf{M}_\alpha(G) - \mathbf{M}_\alpha(\bar{G}) \geq d_\alpha\left(\mathfrak{m}\left(\mathbf{b}_i^{(2)}\right)\delta_P, \mathbf{b}_i^{(2)}\right) - d_\alpha\left(\mathfrak{m}\left(\mathbf{b}_i^{(2)}\right)\delta_Q, \mathbf{b}_i^{(2)}\right)\\ \geq\ & \mathfrak{m}\left(\mathbf{b}_i^{(2)}\right)^\alpha (t_2 - \pi(P) - CR_i) > 0, \text{ by Corollary 5.1.}\end{aligned}$$

Thus $\mathbf{M}_\alpha(G) > \mathbf{M}_\alpha(\bar{G})$, which contradicts the optimality of G. Therefore, inequality (5.14) must hold. □

5.3. **Applications of Theorem 5.1.** The following corollary states a scenario when a factory is located far away from the community of households, a planner will never assign any production to this factory under any optimal assignment map.

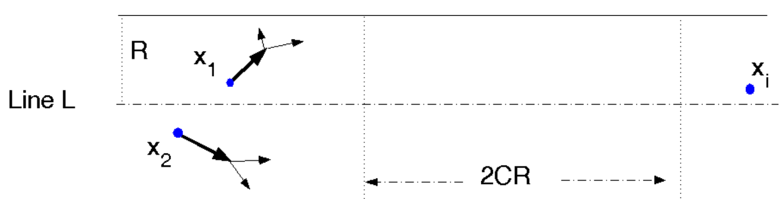


FIGURE 12. If a factory at x_i is located far away from the community of households, then $S^{-1}(i)$ will be empty under any optimal assignment map S.

Corollary 5.2. *Suppose for some $i \in \{1, \cdots, k\}$,*

$$|\pi(x_i) - \pi(y_j)| > 2CR + \min_{s \neq i} |\pi(x_s) - \pi(y_j)| \tag{5.20}$$

for each $j = 1, \cdots, \ell$, where C is the constant given in (5.4) and

$$R = \max\{\|z - p - \pi(z)v\| : z \in \{y_1, \cdots, y_\ell\} \cup \{x_1, \cdots, x_k\}\}. \tag{5.21}$$

Then, $S^{-1}(i) = \emptyset$ for any optimal assignment map $S \in Map[\ell, k]$.

Proof. Assume there exists $j \in \{1, \cdots, \ell\}$ with $S(j) = i$. Without loss of generality, we may assume $\pi(x_i) \geq \pi(y_j)$. Thus, $\max\{\pi(y_h) : \pi(x_i) \geq \pi(y_h), S(h) = i\} = \pi(y_{j^*})$ for some $j^* \in S^{-1}(i)$. For this j^*, by Theorem 5.1, there exists a $z \in \Psi_i$

with $\pi(y_{j^*}) < \pi(z) \le \pi(x_i)$ satisfying (5.14). By the maximality of $\pi(y_{j^*})$, $z \ne y_h$ for any $y_h \in \Psi_i$. Thus, $z = x_i$ and by (5.20),

$$\pi(x_i) > \pi(y_{j^*}) + 2CR + \min_{s \ne i} |\pi(x_s) - \pi(y_{j^*})| \ge \pi(y_{j^*}) + 2CR_i + \min_{s \ne i} |\pi(x_s) - \pi(y_{j^*})|,$$

a contradiction with (5.14). □

The next corollary shows an "autarky" situation: if households and factories are located on two disjoint areas lying distant away from each other, then the demand of households will solely be satisfied from factories within the same area.

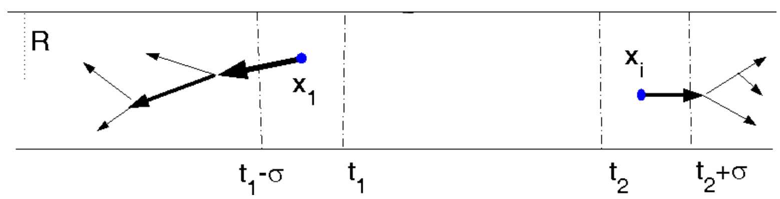


FIGURE 13. An "autarky" situation.

Corollary 5.3. *Suppose (t_1, t_2) is an interval on $\mathbb{R}$ with*

$$\{\pi(x_1), \cdots, \pi(x_k), \pi(y_1), \cdots, \pi(y_\ell)\} \cap (t_1, t_2) = \emptyset \tag{5.22}$$

and $t_2 > t_1 + 2CR$ for C and R given in (5.4) and (5.21). Let $\sigma = t_2 - t_1 - 2CR > 0$. If

$$\{\pi(x_1), \cdots, \pi(x_k)\} \cap (t_1 - \sigma, t_1] \ne \emptyset \text{ and } \{\pi(x_1), \cdots, \pi(x_k)\} \cap [t_2, t_2 + \sigma) \ne \emptyset,$$

then for any optimal assignment map $S \in Map[\ell, k]$ and interval $I = (-\infty, t_1]$ or $[t_2, \infty)$, we have for $j = 1, \cdots, \ell$,

$$\pi(y_j) \in I \iff \pi\left(x_{S(j)}\right) \in I.$$

Proof. It is sufficient to prove that if $\pi(x_i) \in [t_2, \infty)$ for some i, then the set $\{\pi(y_h) : S(h) = i, \pi(y_h) \le t_1\}$ must be empty. Indeed, if not, pick

$$\pi(y_j) = \max\{\pi(y_h) : S(h) = i, \pi(y_h) \le t_1\}$$

for some j. By Theorem 5.1, there exists $z \in \Psi_i$ such that $\pi(y_j) < \pi(z) \le \pi(x_i)$, and

$$\pi(z) - \pi(y_j) \le 2CR_i + \min_{i^* \ne i} |\pi(x_s) - \pi(y_j)| < 2CR_i + t_1 - \pi(y_j) + \sigma \le t_2 - \pi(y_j).$$

Thus, $\pi(z) < t_2$. On the other hand, the maximality of $\pi(y_j)$ and (5.22) yield $\pi(z) \ge t_2$, a contradiction. □

As a direct application of Corollary 5.3, the next corollary states that households living in a relatively isolated area are more likely to receive their commodity from local factories.

Corollary 5.4. *Let $t_1^- < t_2^- < t_1^+ < t_2^+$ be real numbers with*

$$t_2^- = t_1^- + 2CR + \sigma \text{ and } t_2^+ = t_1^+ + 2CR + \sigma, \sigma > 0,$$

where the constants C and R are given in (5.4) and (5.21). If

$$\{\pi(x_1), \cdots, \pi(x_k), \pi(y_1), \cdots, \pi(y_\ell)\} \cap \left(\left(t_1^-, t_2^-\right) \cup \left(t_1^+, t_2^+\right)\right) = \emptyset,$$

$$\{\pi(x_1), \cdots, \pi(x_k)\} \cap \left[t_2^-, t_1^+\right] = \{\pi(x_i)\} \tag{5.23}$$

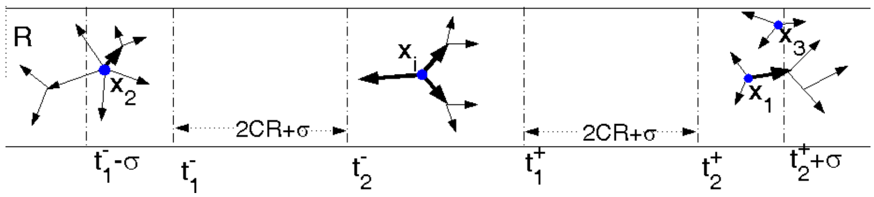


FIGURE 14. Households living nearby a local factory at x_i receive their commodity from the factory.

for some $i \in \{1, \cdots, k\}$, *and*

$$\{\pi(x_1), \cdots, \pi(x_k)\} \cap (t_1^- - \sigma, t_1^-] \neq \emptyset, \ \{\pi(x_1), \cdots, \pi(x_k)\} \cap [t_2^+, t_2^+ + \sigma) \neq \emptyset,$$

then for any optimal assignment map $S \in Map[\ell, k]$,

$$S^{-1}(i) = \left\{ j : \pi(y_j) \in \left[t_2^-, t_1^+\right] \right\}.$$

Proof. For any $j \in S^{-1}(i)$, using $t_1 = t_1^-$, $t_2 = t_2^-$ in Corollary 5.3, and the fact $\pi(x_i) \in [t_2^-, \infty)$, we have $\pi(y_j) \in [t_2^-, \infty)$. Similarly, using $t_1 = t_1^+$, $t_2 = t_2^+$ in Corollary 5.3, and the fact $\pi(x_i) \in (-\infty, t_1^+]$, we have $\pi(y_j) \in (-\infty, t_1^+]$. Thus, $\pi(y_j) \in \left[t_2^-, t_1^+\right]$. This shows that $S^{-1}(i) \subseteq \left\{j : \pi(y_j) \in \left[t_2^-, t_1^+\right]\right\}$. On the other hand, for any j with $\pi(y_j) \in \left[t_2^-, t_1^+\right]$, we have $\pi(y_j) \in [t_2^-, \infty)$ and $\pi(y_j) \in (-\infty, t_1^+]$. Using Corollary 5.3 again, we have $\pi\left(x_{S(j)}\right) \in [t_2^-, \infty)$ and $\pi\left(x_{S(j)}\right) \in (-\infty, t_1^+]$. Thus, $\pi\left(x_{S(j)}\right) \in \left[t_2^-, t_1^+\right]$. By (5.23), $S(j) = i$. Therefore, $\left\{j : \pi(y_j) \in \left[t_2^-, t_1^+\right]\right\} \subseteq S^{-1}(i)$. □

6. **State matrix.** In this section, we show that the properties of optimal assignment maps explored in the previous sections can shed light on the search for those maps. The analysis is built upon a notion of state matrix defined as follows.

Definition 6.1. Let $U = (u_{sh})$ be a $k \times \ell$ matrix with $u_{sh} \in \{0, 1\}$. The matrix U is called

1. a state matrix for an optimal assignment map S if $S(h) \neq s$ whenever $u_{sh} = 0$.
2. a uniform state matrix if U is a state matrix for any optimal assignment map.

A state matrix could be regarded as an information set of a planner during the search process for optimal assignment maps. An entry $u_{sh} = 0$ (or $u_{sh} = 1$) simply denotes that the planner has (or has not) excluded the possibility of assigning household h to factory s. Recall that finding an optimal assignment map is to minimize the function $E_\alpha(S; \mathbf{x}, \mathbf{b})$ over the set $Map[\ell, k]$ whose cardinality is k^ℓ. Any zero entry of a state matrix U for an optimal assignment map S may exclude as many as $k^{\ell-1}$ assignment maps in $Map[\ell, k]$ from being S. The more zero entries in a state matrix U, the more information about S is contained in U. Consequently, we aim at finding a state matrix U for S with as many zero entries as possible, using properties of optimal assignment maps studied in the previous sections. When U has exactly one non-zero entry in each column, S is completely determined by those non-zero entries in U.

The idea of state matrix is motivated by the observation that via group transportation with ramified transport technologies, assignment of each household has a global effect on the allocation path as well as the associated assignment map.

Thus, a planner can deduce more information about the optimal assignment map by exploiting the existing information embedded in the zero entries of a state matrix.

We first explore the implication of Theorem 4.1 on the search for optimal assignment maps in the context of state matrix. For any state matrix $U = (u_{ij})$, we consider a $k \times \ell$ matrix

$$W_U = (w_{ij}(U))$$

where $w_{ij}(U) = \rho_\alpha(w_i(U), n_j)$ with $w_i(U) := \sum_{h=1}^{\ell} u_{ih} n_h$, and the function ρ_α is given in (4.7). Here, $w_i(U)$ denotes the maximum amount of commodity produced at factory i one could conjecture using the existing information in the state matrix U.

For any state matrix U, define

$$\begin{aligned} \Gamma^s(U; n_j) \quad &: \quad = \bigcup_{i \neq s} \{z \in \mathbb{R}^m : \|z - x_i\| < w_{sj}(U) \|z - x_s\|\} \text{ and} \\ \Omega^s(U; n_j) \quad &: \quad = \bigcap_{i \neq s} \{z \in \mathbb{R}^m : \|z - x_s\| < w_{ij}(U) \|z - x_i\|\}, \end{aligned}$$

for any $s = 1, \cdots, k$ and $j = 1, \cdots, \ell$. By (4.14), each $\Gamma^s(U; n_j)$ (or $\Omega^s(U; n_j)$) is the union (or intersection) of $k-1$ open balls.

Example 6.1. The matrix

$$U^{(0)} = (u_{ij}) \text{ with } u_{ij} = 1 \text{ for any } i \text{ and } j \tag{6.1}$$

is a uniform state matrix. Then, $W_{U^{(0)}} = \left(w_{ij}\left(U^{(0)}\right)\right)$ with

$$w_{ij}\left(U^{(0)}\right) = \rho_\alpha\left(\sum_{h=1}^{\ell} n_h, n_j\right) = \rho_\alpha(1, n_j)$$

which is independent of i. Here, $\Gamma^s\left(U^{(0)}; n_j\right) = \Gamma^s(n_j)$ and $\Omega^s\left(U^{(0)}; n_j\right) = \Omega^s(n_j)$, where $\Gamma^s(n_j)$ and $\Omega^s(n_j)$ are given in (4.16) and (4.17).

Example 6.2. Let $S \in Map[\ell, k]$ be an optimal assignment map. Define

$$U_S = (u_{ij}) \text{ with } u_{ij} = \begin{cases} 1, & \text{if } S(j) = i \\ 0, & \text{else} \end{cases}. \tag{6.2}$$

Then, $W_{U_S} = (w_{ij}(U_S))$ with $w_{ij}(U_S) = \rho_\alpha\left(\sum_{S(h)=i} n_h, n_j\right) = \rho_\alpha(\mathfrak{m}(\mathbf{b}_i), n_j)$. Note that $\Gamma^s(U_S; n_j) = \Gamma^s_S(n_j)$ and $\Omega^s(U_S; n_j) = \Omega^s_S(n_j)$ where $\Gamma^s_S(n_j)$ and $\Omega^s_S(n_j)$ are given in (4.12) and (4.13).

Definition 6.2. Given two $k \times \ell$ real matrices $U = (u_{ij})$ and $\tilde{U} = (\tilde{u}_{ij})$, we define

1. $U \geq \tilde{U}$ if $u_{ij} \geq \tilde{u}_{ij}$ for each i and j.
2. $U \gneq \tilde{U}$ if $U \geq \tilde{U}$ but $U \neq \tilde{U}$.

Proposition 6.1. *Let U and $\tilde{U}$ be two state matrices for an optimal assignment map S. If $U \geq \tilde{U}$, then*

$$W_U \leq W_{\tilde{U}} \tag{6.3}$$

and

$$x_s \in \Omega^s(U; n_j) \subseteq \Omega^s\left(\tilde{U}; n_j\right) \text{ and } \Gamma^s(U; n_j) \subseteq \Gamma^s\left(\tilde{U}; n_j\right). \tag{6.4}$$

Proof. For each i, since $U \geq \tilde{U}$,

$$w_i(U) = \sum_{h=1}^{\ell} u_{ih} n_h \geq \sum_{h=1}^{\ell} \tilde{u}_{ih} n_h = w_i\left(\tilde{U}\right).$$

Then, since $\rho_\alpha(\cdot, n_j)$ is decreasing, it follows that

$$w_{ij}(U) = \rho_\alpha(w_i(U), n_j) \leq \rho_\alpha\left(w_i\left(\tilde{U}\right), n_j\right) = w_{ij}\left(\tilde{U}\right)$$

for each i and j. By definition, we have both (6.3) and (6.4). $\square$

Let U be a state matrix for an optimal assignment map S. By definitions of $U^{(0)}$ in (6.1) and U_S in (6.2), it follows that

$$U^{(0)} \geq U \geq U_S. \tag{6.5}$$

Thus, by (6.4), we have

$$\Gamma^s\left(U^{(0)}; n_j\right) \subseteq \Gamma^s(U; n_j) \subseteq \Gamma^s(U_S; n_j)$$

and

$$\Omega^s\left(U^{(0)}; n_j\right) \subseteq \Omega^s(U; n_j) \subseteq \Omega^s(U_S; n_j).$$

These relations, together with Theorem 4.1, immediately imply the following proposition.

Proposition 6.2. *Let $U = (u_{sj})$ be a state matrix for an optimal assignment map $S \in Map[\ell, k]$. For some s and j,*

1. *if $y_j \in \Gamma^s(U; n_j)$, then $S(j) \neq s$;*
2. *if $y_j \in \Omega^s(U; n_j)$, then $S(j) = s$.*

Corollary 6.1. *Suppose U is a state matrix for an optimal assignment map $S \in Map[\ell, k]$. Let $\widehat{U}^{(1)} = \left(\widehat{u}_{ij}^{(1)}\right)$ be a $k \times \ell$ matrix with*

$$\widehat{u}_{ij}^{(1)} = \begin{cases} 0, & \text{if } y_j \in \Gamma^i(U; n_j) \\ u_{ij}, & \text{else} \end{cases}, \tag{6.6}$$

Then, $\widehat{U}^{(1)}$ is also a state matrix for S with $U \geq \widehat{U}^{(1)}$.

Proof. If $\widehat{u}_{ij}^{(1)} = 0$, then either $u_{ij} = 0$ or $y_j \in \Gamma^i(U; n_j)$. In the first case, since U is a state matrix for S, by definition, $S(j) \neq i$. In the second case, by Proposition 6.2, $S(j) \neq i$. Thus, $\widehat{U}^{(1)}$ is also a state matrix for S with $U \geq \widehat{U}^{(1)}$. $\square$

We now explore the implication of Theorem 4.2 on the search for optimal assignment maps. Suppose U is a state matrix for an optimal assignment map S. If $u_{sh} = 0$ for some $h \in \{1, \cdots, \ell\}$ and $s \in \{1, \cdots, k\}$, then for each $j \neq h$ with $n_j \leq n_h$, we consider the set

$$\Gamma^{s,h}(U; n_j) := \{z \in \mathbb{R}^m : \|z - y_h\| + \Lambda(U) < w_{sj}(U) \|z - x_s\|\}, \tag{6.7}$$

where

$$\Lambda(U) = \max\left\{\frac{\rho_\alpha(n_h + n_j, n_j)}{w_{ij}(U)} \|y_h - x_i\| : \text{ for } i \in \{1, \cdots, k\} \text{ with } u_{ih} = 1\right\}.$$

Lemma 6.1. *Let U and $\tilde{U}$ be two state matrices for an optimal assignment map S. If $U \geq \tilde{U}$, then*

$$\Gamma^{s,h}(U; n_j) \subseteq \Gamma^{s,h}\left(\tilde{U}; n_j\right) \tag{6.8}$$

for any $s \in \{1, \cdots, k\}$, $h \in \{1, \cdots, \ell\}$ with $u_{sh} = 0$, and $n_j \leq n_h$ for $j \neq h$.

Proof. For each i, if $\tilde{u}_{ih} = 1$, then $u_{ih} = 1$ as $U \geq \tilde{U}$. By (6.3), we have $w_{ij}(U) \leq w_{ij}\left(\tilde{U}\right)$. Thus,

$$\begin{aligned}
\Lambda\left(\tilde{U}\right) &= \max\left\{\frac{\rho_\alpha(n_h + n_j, n_j)}{w_{ij}\left(\tilde{U}\right)}\|y_h - x_i\| : \text{for } i \text{ with } \tilde{u}_{ih} = 1\right\} \\
&\leq \max\left\{\frac{\rho_\alpha(n_h + n_j, n_j)}{w_{ij}(U)}\|y_h - x_i\| : \text{for } i \text{ with } \tilde{u}_{ih} = 1\right\} \\
&\leq \max\left\{\frac{\rho_\alpha(n_h + n_j, n_j)}{w_{ij}(U)}\|y_h - x_i\| : \text{for } i \text{ with } u_{ih} = 1\right\} = \Lambda(U).
\end{aligned}$$

Consequently, (6.8) follows from (6.7). □

As a result, by (6.5),

$$\Gamma^{s,h}(U_0; n_j) \subseteq \Gamma^{s,h}(U; n_j) \subseteq \Gamma^{s,h}(U_S; n_j) = \Gamma_S^{s,h}(n_j),$$

where $\Gamma_S^{s,h}(n_j)$ is given in (4.18). The following proposition and its associated corollary follow from Theorem 4.2.

Proposition 6.3. *Suppose U is a state matrix for an optimal assignment map $S \in Map[\ell, k]$. If $y_j \in \Gamma^{s,h}(U; n_j)$ for some $h \neq j$ with $u_{sh} = 0$ and $n_j \leq n_h$, then $S(j) \neq s$.*

Corollary 6.2. *Suppose U is a state matrix for an optimal assignment map $S \in Map[\ell, k]$. Let $\widehat{U}^{(2)} = \left(\widehat{u}_{sj}^{(2)}\right)$ be a $k \times \ell$ matrix with*

$$\widehat{u}_{sj}^{(2)} = \begin{cases} 0, & \text{if } y_j \in \Gamma^{s,h}(U; n_j) \text{ for some } h \neq j \text{ with } u_{sh} = 0 \text{ and } n_j \leq n_h \\ u_{sj}, & \text{else} \end{cases}, \tag{6.9}$$

Then, $\widehat{U}^{(2)}$ is also a state matrix for S with $U \geq \widehat{U}^{(2)}$.

We now explore the implication of Theorem 5.1 on the search for optimal assignment maps. Suppose U is a state matrix for an optimal assignment map S. For each $i \in \{1, \cdots, k\}$, let

$$\Psi_i(U) = \{y_j : u_{ij} = 1\} \cup \{x_i\}.$$

Clearly, $\Psi_i(U) \supseteq \Psi_i(U_S) = \Psi_i$ where Ψ_i is defined in (5.12).

Now, for $\pi : \mathbb{R}^m \to \mathbb{R}$ given in (5.1), we define

$$R_i = \max\{\|z - p - \pi(z)v\| : z \in \Psi_i(U)\}. \tag{6.10}$$

Without loss of generality, we may assume that

$$\Psi_i(U) = \{y_{j_h} : h = 1, \cdots, N_i\} \cup \{x_i\} \text{ with } \pi(y_{j_1}) \leq \pi(y_{j_2}) \leq \cdots \leq \pi\left(y_{j_{N_i}}\right). \tag{6.11}$$

Proposition 6.4. *Suppose $U = (u_{sj})$ is a state matrix for an optimal assignment map $S \in Map[\ell, k]$. For each $i \in \{1, \cdots, k\}$, let $h \in \{1, \cdots, N_i\}$ and $i^* \in \{1, \cdots, k\}$. If*

$$\min\left\{\pi\left(y_{j_{h+1}}\right) - \pi\left(y_{j_h}\right), \pi\left(x_i\right) - \pi\left(y_{j_h}\right)\right\} > 2CR_i + \left|\pi\left(x_{i^*}\right) - \pi\left(y_{j_h}\right)\right|, \quad (6.12)$$

where C and R_i are the constants given in (5.4) and (6.10) respectively, then $S(j_t) \neq i$ for any $t \leq h$. Similarly, if

$$\min\left\{\pi\left(y_{j_h}\right) - \pi\left(y_{j_{h-1}}\right), \pi\left(y_{j_h}\right) - \pi\left(x_i\right)\right\} > 2CR_i + \left|\pi\left(x_{i^*}\right) - \pi\left(y_{j_h}\right)\right|, \quad (6.13)$$

then $S(j_t) \neq i$ for any $t \geq h$.

Proof. Assume (6.12) holds but $S(j_{t^*}) = i$ for some $t^* \leq h$. Without loss of generality, we may assume that $\pi(y_{j_{t^*}}) = \max\{\pi(y_{j_t}) : t \leq h, S(j_t) = i\}$. Note that $\pi(y_{j_{t^*}}) \leq \pi(y_{j_h}) < \pi(x_i)$ by (6.11) and (6.12). Then, by Theorem 5.1, there exists a $z \in \Psi_i \backslash \{y_{j_{t^*}}\}$ with $\pi(y_{j_{t^*}}) < \pi(z) \leq \pi(x_i)$ such that

$$0 < \pi(z) - \pi(y_{j_{t^*}}) \leq 2CR_i + \left|\pi(x_{i^*}) - \pi(y_{j_{t^*}})\right|. \quad (6.14)$$

By the maximality of $\pi(y_{j_{t^*}})$ and $z \in \Psi_i \backslash \{y_{j_{t^*}}\}$, we know $\pi(y_{j_h}) < \pi(z)$. Thus, by the ordering in (6.11), we have

$$\min\left\{\pi\left(y_{j_{h+1}}\right), \pi\left(x_i\right)\right\} \leq \pi(z). \quad (6.15)$$

Therefore,

$$\begin{aligned}
& \pi(z) - \pi(y_{j_{t^*}}) = \pi(z) - \pi(y_{j_h}) + \pi(y_{j_h}) - \pi(y_{j_{t^*}}) \\
\geq \; & \min\left\{\pi\left(y_{j_{h+1}}\right) - \pi\left(y_{j_h}\right), \pi\left(x_i\right) - \pi\left(y_{j_h}\right)\right\} + \pi(y_{j_h}) - \pi(y_{j_{t^*}}), \text{ by (6.15)} \\
> \; & 2CR_i + \left|\pi(x_{i^*}) - \pi(y_{j_h})\right| + \pi(y_{j_h}) - \pi(y_{j_{t^*}}), \text{ by (6.12)} \\
\geq \; & 2CR_i + \left|\pi(x_{i^*}) - \pi(y_{j_{t^*}})\right|,
\end{aligned}$$

a contradiction with (6.14). This proves (6.12). Similar arguments give (6.13). □

For each $i = \{1, \cdots, k\}$, and $\lambda \in \mathbb{R}$, denote

$$I_i(\lambda) := \begin{cases} (-\infty, \lambda], & \text{if } \lambda \leq \pi(x_i) \\ [\lambda, \infty), & \text{if } \lambda > \pi(x_i) \end{cases}.$$

Then, for each $\pi : \mathbb{R}^m \to \mathbb{R}$ given in (5.1), we define

$$\tilde{\Gamma}^i_\pi(U; n_j) := \left\{z \in \mathbb{R}^m : \pi(z) \in I_i(\pi(y_{j_h})) \text{ for some } j_h \text{ satisfying (6.12) or (6.13)}\right\}.$$

Corollary 6.3. *Suppose U is a state matrix for an optimal assignment map $S \in Map[\ell, k]$. Let $\widehat{U}^{(3)} = \left(\widehat{u}^{(3)}_{ij}\right)$ be a $k \times \ell$ matrix with*

$$\widehat{u}^{(3)}_{ij} = \begin{cases} 0, & \textit{if } y_j \in \tilde{\Gamma}^i_\pi(U; n_j) \textit{ for some } \pi \\ u_{ij}, & \textit{else} \end{cases}, \quad (6.16)$$

Then, $\widehat{U}^{(3)}$ is also a state matrix for S with $U \geq \widehat{U}^{(3)}$.

Remark 6.1. Depending on the spatial locations of households and factories, for each fixed $i \in \{1, \cdots, k\}$, a planner may choose π to be one of the standard coordinate functions in $\mathbb{R}^m$, i.e., $\pi(z_1, \cdots, z_m) = z_t$ for some fixed $1 \leq t \leq m$. In this case, (6.12) and (6.13) may be simply expressed in terms of coordinates of x_i's and y_j's. Another reasonable choice is to set $\pi(z) = \langle z - p_i, v_i\rangle$, where

$$(p_i, v_i) \in \arg\min\left\{\max_{z \in \Psi_i(U)} \|z - p - \langle z - p, v\rangle v\| : p, v \in \mathbb{R}^m \text{ with } \|v\| = 1\right\}.$$

This will minimize R_i given in (6.10), because the line passing through p_i in direction v_i, i.e., $\{p_i+tv_i\colon t \in \mathbb{R}\}$ provides the least supremum norm approximation for $\Psi_i(U)$ in $\mathbb{R}^m$.

Given a state matrix U for an optimal assignment map S, we have used results from previous sections to provide three updated state matrices $\hat{U}^{(j)}$, $j=1,2,3$, for U. The next proposition, whose proof follows directly from the definition of state matrix, makes it possible to combine them together into a further updated state matrix.

Proposition 6.5. *Suppose $U=(u_{ij})$ and $\bar{U}=(\bar{u}_{ij})$ are two state matrices for an optimal assignment map $S \in Map[\ell,k]$. Then, the matrix $\tilde{U}=(\tilde{u}_{ij})$ with $\tilde{u}_{ij}=\min\{u_{ij},\bar{u}_{ij}\}$ for all i and j is also a state matrix for S.*

Proposition 6.5 says that one could deduce more information from any two existing state matrices regarding the optimal assignment map. Using this proposition, we immediately have the following corollary.

Corollary 6.4. *Suppose U is a state matrix for an optimal assignment map $S \in Map[\ell,k]$. For each i and j, define $\widehat{u}_{ij}=\min\left\{\widehat{u}^{(1)}_{ij},\widehat{u}^{(2)}_{ij},\widehat{u}^{(3)}_{ij}\right\}$, where $\widehat{u}^{(1)}_{ij}$, $\widehat{u}^{(2)}_{ij}$ and $\widehat{u}^{(3)}_{ij}$ are given in (6.6), (6.9) and (6.16) respectively. Then, $\widehat{U}=(\widehat{u}_{ij})$ is also a state matrix for S with $U \geq \widehat{U}$.*

This idea of updating a state matrix U into another state matrix $\widehat{U}$ as in Corollary 6.4 can be implemented iteratively to obtain an even further updated state matrix. Given any initial state matrix U (e.g. $U=U^{(0)}$ as in (6.1)) for an optimal assignment map S. For each $n=0,1,2,\cdots$, define $U_{n+1}=\widehat{U_n}$ with $U_0=U$. This gives a non-increasing sequence of $k\times\ell$ matrices $\{U_n\}$ whose entries are either 0 or 1. Hence, there exists an $N\geq 1$ such that $U_0 \gneqq U_1 \gneqq \cdots \gneqq U_{N-1}=U_N=U_{N+1}=\cdots$. We denote this U_N as U^*. Clearly, the matrix U^* is still a state matrix for S with $U \geq U^*$ and $\widehat{U^*}=U^*$.

This updated state matrix U^* contains more information about S than the initial state matrix U_0 because U^* contains more zero entries. In some non-trivial cases as shown in the following example, U^* may have exactly one non-zero entry in each column. In such a situation, U^* completely determines the optimal assignment map S.

Proposition 6.6. *Let U be a uniform state matrix (e.g., $U=U^{(0)}$ as in (6.1)), and suppose that $n_1 \geq n_2 \geq \cdots \geq n_\ell$. If for each $j=1,\cdots,\ell$,*

$$y_j \in \bigcap_{\substack{u_{sj}=1\\ s\neq s_j}} \left(\Gamma^s(U;n_j) \cup \bigcup_{\substack{1\leq h\leq j-1\\ s_h\neq s}} \Gamma^{s,h}(U;n_j) \right) \tag{6.17}$$

for some $s_j \in \{1,\cdots,k\}$, then $S:\{1,\cdots,\ell\}\to\{1,\cdots,k\}$ given by $S(j)=s_j$ is the optimal assignment map.

Proof. It is sufficient to show that for any optimal assignment map S, it holds that $S(j)=s_j$ for any j. Indeed, for any $s\neq s_j$, if $u_{sj}=0$, then $S(j)\neq s$ because U is a state matrix for S. If $u_{sj}=1$, then by assumption (6.17), either $y_j\in\Gamma^s(U;n_j)$ or $\Gamma^{s,h}(U;n_j)$ for some $h<j$ with $s_h\neq s$. If $y_j\in\Gamma^s(U;n_j)$, then by Proposition 6.2, $S(j)\neq s$. If $y_j\in\Gamma^{s,h}(U;n_j)$ for some $h<j$ with $s_h\neq s$, then either $S(h)\neq s_h$ or

$S(h) = s_h \neq s$. In the later case, since $y_j \in \Gamma^{s,h}(U; n_j) \subseteq \Gamma_S^{s,h}(n_j)$ and $n_j \leq n_h$, by Theorem 4.2, we still have $S(j) \neq s$. Thus, in all cases for any $s \neq s_j$, we know

$$\text{either } S(j) \neq s \text{ or } S(h) \neq s_h \text{ for some } h < j. \tag{6.18}$$

Consequently, when $j = 1$, we always have $S(1) \neq s$ for any $s \neq s_1$, and thus $S(1) = s_1$. Using (6.18) again, we get $S(2) \neq s$ for any $s \neq s_2$, which yields $S(2) = s_2$. Repeating this process leads to the conclusion that $S(j) = s_j$ for any $j \in \{1, \cdots, \ell\}$. □

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