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The Multidimensional Generalization of Quadrature Filters using Vector Field Theory

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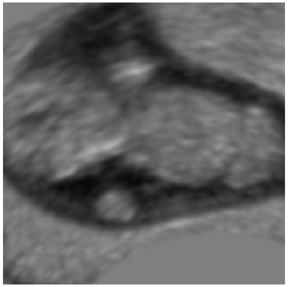
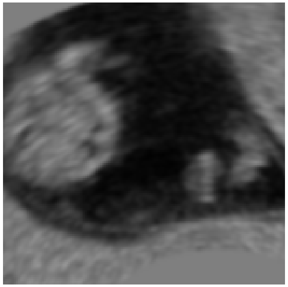
1 Introduction

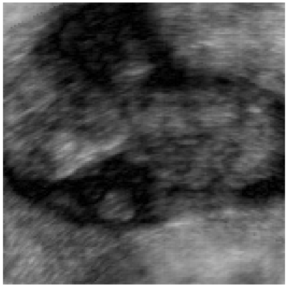
- What is this presentation about?. **Phase**
- Why is phase important?. **It carries structural information.**





$$F_1(\mathbf{u})e^{-i\varphi_1(\mathbf{u})} \Downarrow F_2(\mathbf{u})e^{-i\varphi_1(\mathbf{u})}$$

$$F_2(\mathbf{u})e^{-i\varphi_2(\mathbf{u})} \Downarrow F_1(\mathbf{u})e^{-i\varphi_2(\mathbf{u})}$$



Our interest is the **Local Phase**: spatial domain counterpart of the phase spectrum.

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2 Local Phase in 1D

- The local phase is the local decomposition of the signal according to its symmetries.
- In 1D the local phase is properly defined through the analytic signal [Gabor 1946].
- The analytic signal is constructed from the Hilbert transform.

Hilbert Transform

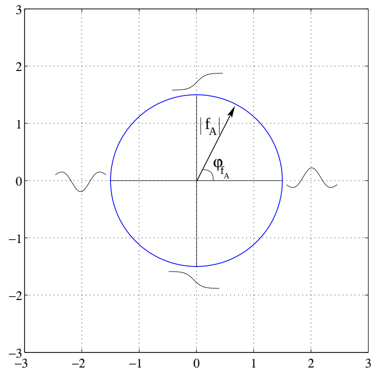
$$f_{\mathcal{H}}(y) = f(x) * \frac{1}{\pi x} \quad \rightleftharpoons \quad F_{\mathcal{H}}(u) = -i \cdot \operatorname{sign}(u) \cdot F(u) = -i \cdot \frac{u}{|u|} \cdot F(u)$$

Analytic Signal

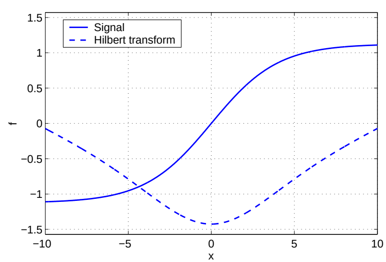
$$f_A(x) = f(x) + i \cdot \mathcal{H}[f(x)] = |f_A(x)| e^{i\varphi_{f_A(x)}} \quad \rightleftharpoons \quad F_A(u) = 2 \cdot \Pi(u) \cdot F(u),$$

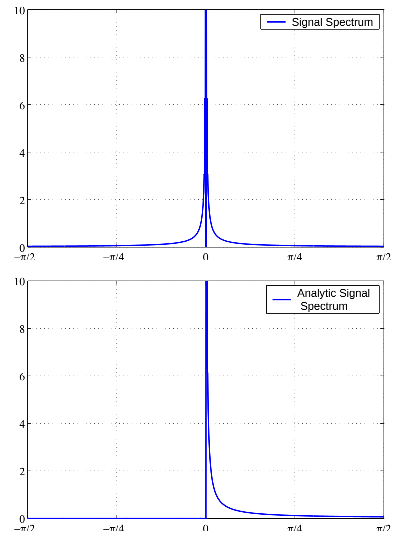
Interpretation

- Local Amplitude: $|f_A(x)|$
- Local Phase: $\varphi_{f_A(x)}$



Example





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3 Local Phase in nD

- In nD spaces we have a new degree of freedom: **orientation**.
- Some Hilbert transform extensions to nD:
  1. Total Hilbert transform [Stark 1971]
  2. Partial Hilbert transform [Knutsson 1995]
  3. Single Orthant analytic signal [Hanh 1992]
- Attempts lack of enough complexity: complex algebra is not enough.

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4 Hilbert Transform and Vector Field Theory

- Analytic functions related to harmonic functions.
- An harmonic function p satisfies Laplace's equation.
- 1D Hilbert transform relates the components of a 2D harmonic field $\mathbf{g} = \nabla p(x_1, x_2)$.

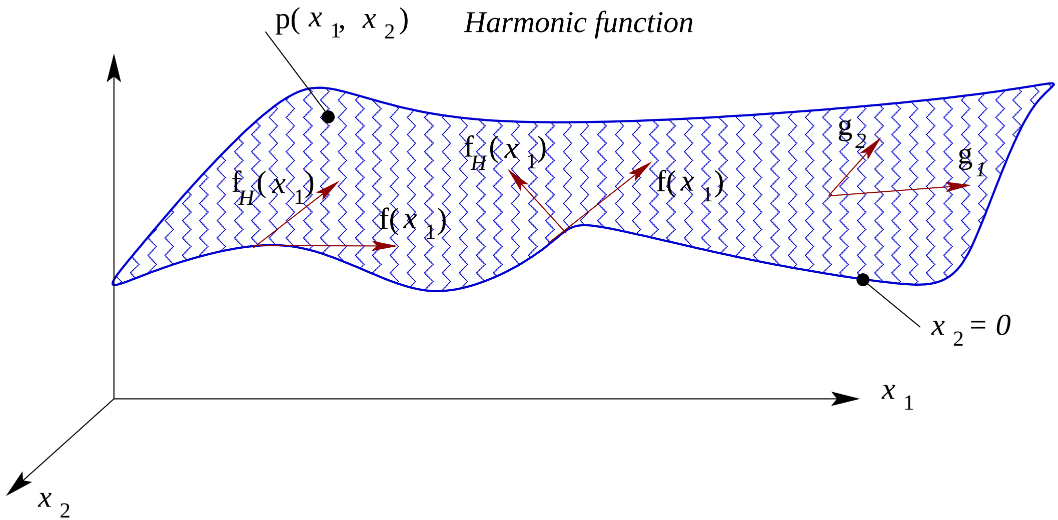
Hilbert transform derivation [Felsberg 2001]

- **Problem:** Laplace equation in the open domain $x_2 < 0$ with Neumann boundary conditions

$$\begin{cases} \Delta p(x_1, x_2) & = 0, \\ g_1(x_1, 0) & = f(x_1) \end{cases}$$

- **Statement:**

$$g_2(x_1, 0) = \mathcal{H}(g_1(x_1, 0)) = \mathcal{H}(f(x_1))$$



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- **Proof:**
  Rewriting Laplace equation in the transform domain for the coordinate x_1.

$$-4\pi^2 u_1^2 P(u_1,x_2) + \frac{\partial^2 P(u_1,x_2)}{\partial x_2^2} = 0.$$

The solution for $x_2 < 0$ is given by

$$P(u_1,x_2) = C(u_1)e^{2\pi|u_1|x_2}.$$

Then

$$G_1(u_1,x_2) = i2\pi u_1 P(u_1,x_2) \qquad G_2(u_1,x_2) = 2\pi|u_1|P(u_1,x_2).$$

G_1 related to G_2

$$G_1(u_1,x_2) = i\frac{u_1}{|u_1|}G_2(u_1,x_2) = H_1(u_1)G_2(u_1,x_2)$$

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5 Generalized Hilbert Transform: Riesz Transform

- (n+1)D extension of Laplace problem. $p(\mathbf{x}_{(n+1)}) : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$.
- Notation:

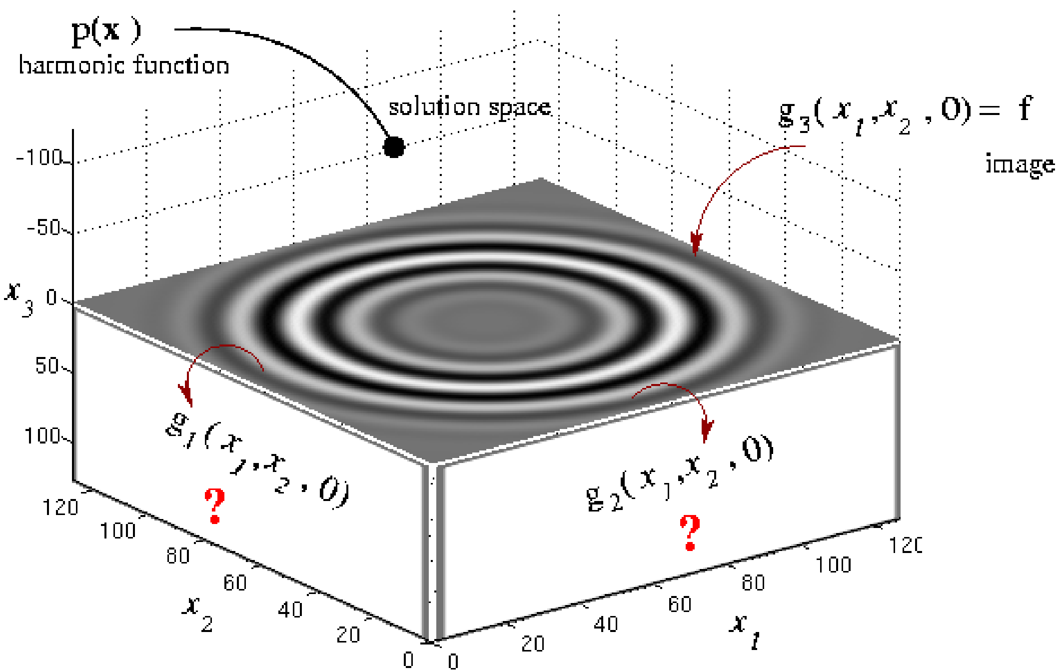
$$\begin{aligned}
\mathbf{x}_n &= [x_1, x_2, \cdots, x_n], \\
\mathbf{x}_{(n+1)} &= [x_1, x_2, \cdots, x_n, x_{n+1}] \\
\mathbf{g}(\mathbf{x}_{(n+1)}) &= [g_1(\mathbf{x}_{(n+1)}), g_2(\mathbf{x}_{(n+1)}), \ldots, g_{(n+1)}(\mathbf{x}_{(n+1)})] = \nabla p(\mathbf{x}_{(n+1)}) = \nabla p(\mathbf{x}_{(n+1)}).
\end{aligned}$$

- **Problem:** $(n+1)$D Laplace problem in the domain $x_{(n+1)} < 0$ with Neumann boundary condition

$$\begin{cases} \Delta p(\mathbf{x}_{(n+1)}) &= 0, \\ g_{(n+1)}(\mathbf{x}_n, 0) &= f(\mathbf{x}_n). \end{cases}$$

- **Statement:** nD Riesz transform relates the last component of **g** to the previous ones.

2D case → 3D Laplace equation



- **Proof:**
  Let us perform the calculation in the frequency domain for $\mathbf{x}_n$, keeping x_{n+1} coordinate in the spatial domain

$$\frac{\partial^2 P(\mathbf{u}_n, x_{n+1})}{\partial^2 x_{n+1}} = 4\pi^2 |\mathbf{u}_n|^2 P(\mathbf{u}_n, x_{n+1})$$

The solution for $x_{(n+1)}$

$$P(\mathbf{u}_n, x_{n+1}) = C(\mathbf{u}_n) e^{2\pi|\mathbf{u}_n| x_{n+1}}$$

$C(\mathbf{u}_n)$ is a function independent of x_{n+1}

$$\begin{cases} G_k(\mathbf{u}_n, x_{n+1}) & = i2\pi u_k P(\mathbf{u}_n, x_{n+1}) \qquad 1 \leq k \leq n \\ G_{n+1}(\mathbf{u}_n, x_{n+1}) & = \frac{\partial P}{\partial x_{n+1}} = 2\pi|\mathbf{u}_n| C(\mathbf{u}_n) P(\mathbf{u}_n, x_{n+1}). \end{cases}$$

The relation between G_k and G_{n+1}

$$G_k(\mathbf{u}_n, x_{n+1}) = i \frac{u_k}{|\mathbf{u}_n|} O(\mathbf{u}_n) G_{n+1}(\mathbf{u}_n, x_{n+1}) \qquad 1 \leq k \leq n,$$

From this result is possible to construct a vector function $\mathbf{F}_{\mathcal{R}}(\mathbf{u}) : \mathbb{R}^n \to \mathbb{R}^n$ such as

$$\mathbf{F}_{\mathcal{R}}(\mathbf{u}) = [G_1(\mathbf{u}, 0), G_2(\mathbf{u}, 0), \cdots, G_n(\mathbf{u}, 0)]^T .$$

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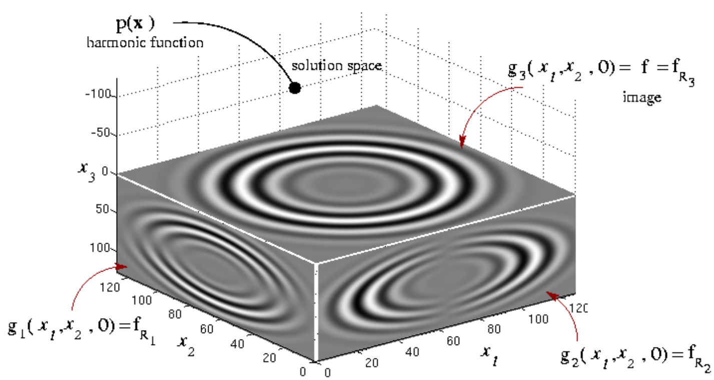
Riesz transform of f in the Fourier domain:

$$\mathbf{F}_{\mathcal{R}}(\mathbf{u}) = i\frac{\mathbf{u}}{|\mathbf{u}|}O(\mathbf{u})F(\mathbf{u})$$

Riesz transform of f in the spatial domain:

$$\mathbf{f}_{\mathcal{R}}(\mathbf{x}) = -\frac{\mathbf{x}}{2\pi|\mathbf{x}|^n} * o(\mathbf{x}) * f(\mathbf{x})$$

2D case → 3D Riesz Transform



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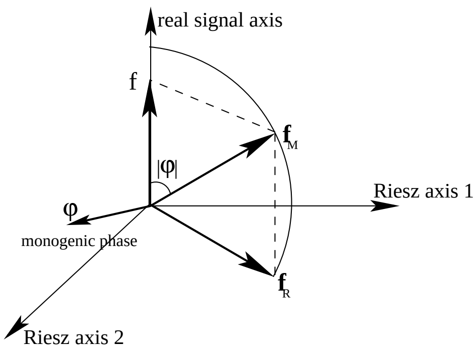
6 Monogenic Signal

- nD extension of analytic signal.
- Embedding of nD signal $f(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ in a (n+1)D dimensional space

$$\mathbf{f}_M : \mathbb{R}^n \to \mathbb{R}^{(n+1)} \qquad \mathbf{f}_M(\mathbf{x}) = [-\mathbf{f}_{\mathcal{R}}(\mathbf{x}), f(\mathbf{x})]^T$$

Interpretation



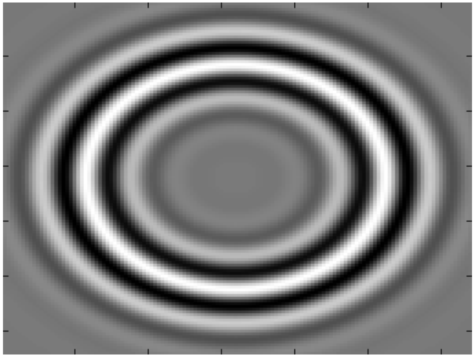
Local Amplitude

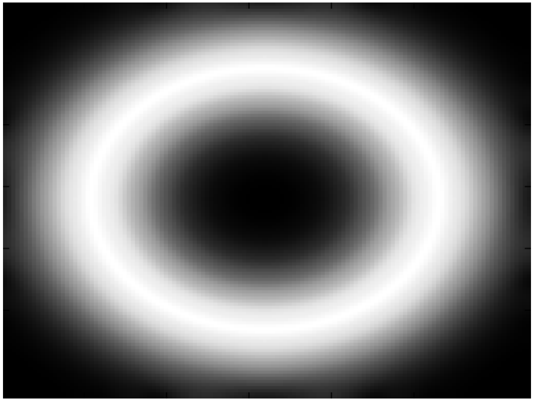
$$\|\mathbf{f}_M(\mathbf{x})\| = \sqrt{f^2(\mathbf{x}) + \|\mathbf{f}_R(\mathbf{x})\|^2}.$$

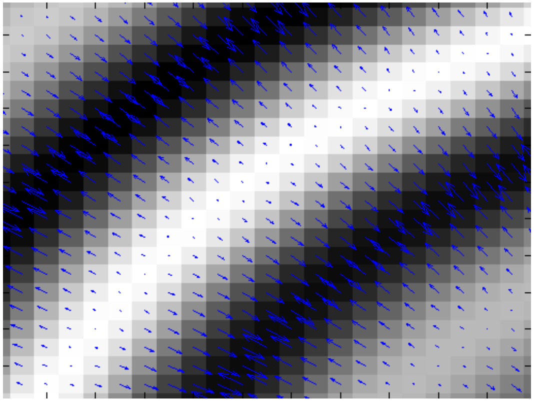
Local Phase Vector

$$\varphi(\mathbf{x}) = \frac{\mathbf{f}_{\mathcal{R}}(\mathbf{x})}{\|\mathbf{f}_{\mathcal{R}}(\mathbf{x})\|} \arctan\left(\frac{\|\mathbf{f}_{\mathcal{R}}(\mathbf{x})\|}{f(\mathbf{x})}\right)$$

Example





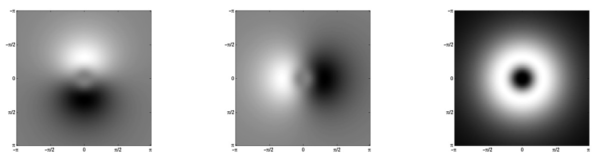


7 Generalized Quadrature Filters

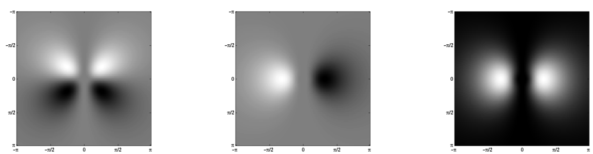
- How to estimate analytic signal? → quadrature filters [Granlund and Knutsson 1995].
- How to estimate monogenic signal? → Generalized quadrature filters [Knutsson 2003]
- Spherical separable: $\mathbf{Q}(\mathbf{u}) = R(|\mathbf{u}|)\mathbf{D}(\mathbf{u})$

$$\mathbf{D}(\mathbf{u}) = (\hat{\mathbf{u}}^T \hat{\mathbf{n}})^{2a} \begin{pmatrix} -i\frac{\mathbf{u}}{\|u\|} \\ 1 \end{pmatrix}$$

- $a = 0$: *spherical QF* [Felsberg 2001]



- $a = 1$: *loglets QF* [Knutsson 2003]



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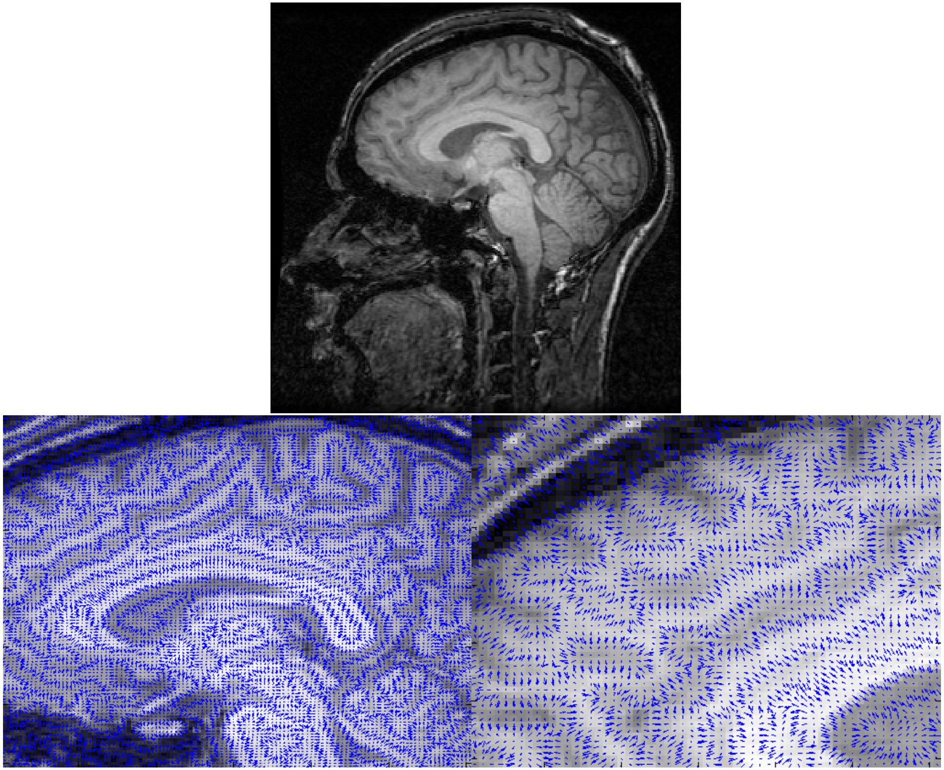
8 Results

MRI



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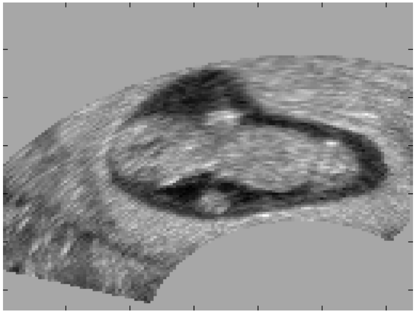
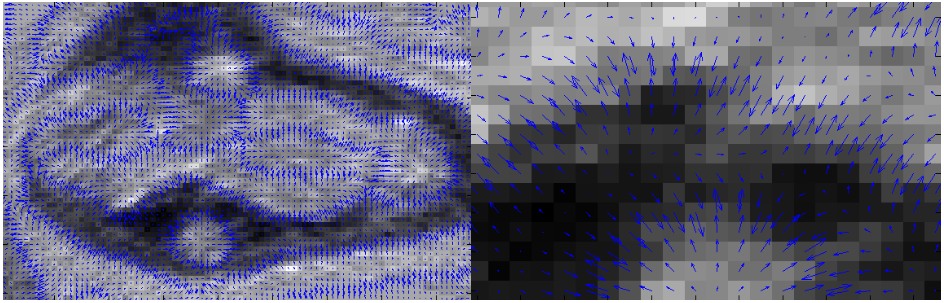
Ultrasound





9 Conclusion

- nD generalization of Hilbert Transform.
- Local phase analysis of nD signals
- Applications:
  - Phase based registration
  - Phase based segmentation
  - Local Structure Tensor Estimation