# On Minimum/Maximum/All-Pass Decompositions in Time and Frequency Domains

Ramdas Kumaresan and Ashwin Rao

***Abstract*—We model a signal over $T$ seconds by a pole-zero model by considering its periodic extensions. Using this model, we decompose the signal into its minimum-phase (MinP)/maximum-phase (MaxP) and all-phase (AllP) components. A simple algorithm for this decomposition that does not seem to have a counterpart in the cepstral literature is presented. This decomposition leads to representing signals by their envelope and instantaneous frequency that takes on positive values only. Nonperiodic signals can be processed by appropriately windowing the signals.**

***Index Terms*—All-phase, maximum-phase, minimum-phase.**

## I. Introduction

Given the signal samples $x(n)$ $n = 0, 1, 2 \cdots, N-1$, a number of approaches to computing the minimum-phase (MinP)/maximum-phase (MaxP) and minimum-phase (MinP)/all-pass (AllP) components of $x(n)$ are well known [1]. These approaches include rooting the polynomial $X(z)$ to separate the components or computing the cepstrum, i.e., the inverse Fourier transform of the logarithm of the spectrum $X(e^{j\omega})$ of $x(n)$ [1]. These are collectively called homomorphic or logarithmic signal analysis [1]. In this correspondence, we propose a dual approach to logarithmic signal analysis based on processing the signal itself rather than its Fourier transform. This approach naturally leads to representation of signals by their envelope, instantaneous frequency, and zero-crossings.

In Section II, we consider complex-valued periodic signals and express them as a product of so-called elementary signals á la Voelcker [2]. This type of representation is analogous to that used in discrete-time systems theory, where the periodic frequency response of a system is characterized by a finite number of poles and zeros, except in our case, the poles and zeros are located in a complex-time or $\zeta$-plane. Using this signal model, expressions for the envelope, phase, and the instantaneous frequency (IF) of a signal may be derived [3], [4]. This representation also leads to a special class of signals whose instantaneous frequency is always positive. See also [2] and [5]. In Section III, we introduce a two-step algorithm to decompose an analytic periodic signal into two analytic signals: one completely described by its envelope and the other having a positive IF (PIF). This type of decomposition is different from those known in the cepstral literature [1]. By appropriately windowing signals, the proposed algorithm can be adapted to representing nonstationary signals by using envelopes/PIF's or equivalently by zero-crossings [6].

## II. Envelope and IF in Terms of a Signal Model

Consider a periodic analytic signal $s(t)$ with a period of $T$ seconds. Let $\Omega = 2\pi/T$. If $s(t)$ has finite bandwidth, it may be described by the following model for a sufficiently large $M$ over an interval of $T$ seconds.

$$s(t) = e^{j\omega_t t} \sum_{k=0}^{M} a_k e^{jk\Omega t}. \tag{1}$$

$\omega_t = K\Omega$ represents the lower band-edge of the spectrum of $s(t)$. $a_k$ are the complex amplitudes of the sinusoids $e^{jk\Omega t}$; $a_0 \neq 0$, and $a_M \neq 0$. By analytic continuation, we may write $s(t)$ in terms of a complex variable $\zeta$ (á la the traditional complex-frequency variable $z$) as $S(\zeta) = \zeta^{-K}(a_0 + a_1\zeta^{-1} + a_2\zeta^{-2} + \cdots + a_M\zeta^{-M}$. Thus, $s(t)$ is obtained by evaluating $S(\zeta)$ around the unit circle in the $\zeta$-plane, i.e., $\zeta = e^{-j\Omega t}$. Note that $s(t)$ [or $S(\zeta)$] is a polynomial in $e^{-j\Omega t}$ (or $\zeta$). We may factor this polynomial into its $M(= P + Q)$ factors and rewrite $s(t)$ as

$$s(t) = a_0 e^{j\omega_t t} \underbrace{\prod_{i=1}^{P}(1 - p_i e^{j\Omega t})}_{s_{\mathrm{MinP}}(t)} \underbrace{\prod_{i=1}^{Q}(1 - q_i e^{j\Omega t})}_{s_{\mathrm{MaxP}}(t).} \tag{2}$$

$p_1, p_2, \cdots, p_P$ and $q_1, q_2, \cdots, q_Q$ denote the polynomial's roots; $p_i = |p_i|e^{j\theta_i}$, $q_i = |q_i|e^{j\phi_i}$. $p_i$ denotes roots inside the unit circle in the complex plane, and $q_i$ are outside the unit circle, assuming that there are no roots on the circle. That is, $|p_i| < 1$, and $|q_i| > 1$. Each factor of the form $(1 - p_i e^{j\Omega t})$ in the above is called an "elementary signal" [2]. The $p_i$ and $q_i$ are referred to as zeros of the signal $s(t)$. The above expressions, representing a bandlimited periodic signal, may be recognized as the counterpart of the frequency response of a finite impulse response (FIR) filter in discrete-time systems theory [1]. More generally, if $s(t)$ consists of an infinite number of spectral lines, then we can represent it to desired accuracy using sufficient number of poles and zeros, analogous to IIR filters. Voelcker called this way of modeling signals the "product representation of signals." We will primarily work with the all-zero models since they are easier to use.

$s_{\mathrm{MaxP}}(t)$ and $s_{\mathrm{MinP}}(t)$ are the direct counterparts of the frequency responses of the maximum- and minimum-phase FIR filters in discrete-time systems theory [1]. As in systems theory (see [1, Sec. 10.3]), the phase of the MinP signal is the Hilbert transform of its log envelope, that is, the MinP signal may be expressed in the form $e^{\alpha(t)+j\hat{\alpha}(t)}$. See [4] for details. $\hat{\alpha}(t)$ is the Hilbert transform of $\alpha(t)$. Similarly, since a maximum-phase (MaxP) signal has zeros outside the unit circle, it may be expressed as $e^{\beta(t)-j\hat{\beta}(t)}$. Thus, envelope or phase alone is sufficient to essentially characterize a MinP or a MaxP signal. [Along the same lines, an all-phase (AllP) analytic signal, which is the analog of an all-pass filter, would be of the form $e^{j\psi(t)}$.] Thus, $s(t)$ may be expressed as

$$s(t) = A_c e^{j\omega_c t} \underbrace{e^{\alpha(t)+j\hat{\alpha}(t)}}_{\mathrm{MinP}} \underbrace{e^{\beta(t)-j\hat{\beta}(t)}}_{\mathrm{MaxP}} \tag{3}$$

where the "hat" stands for Hilbert transform. $\omega_c$ is $Q\Omega$ (contributed by the linear-phase term from the MaxP signal) plus the arbitrary frequency translation $\omega_t$ shown in (2). $A_c$ is $a_0 \prod_{i=1}^{Q}(-q_i)$. See [7] for details. The expressions for $\alpha(t)$ and $\beta(t)$ are derived in [4]

$$\alpha(t) = \sum_{k=1}^{\infty} \sum_{i=1}^{P} -\frac{|p_i|^k}{k} \cos(k\Omega t + k\theta_i)$$

and

$$\beta(t) = \sum_{k=1}^{\infty} \sum_{i=1}^{Q} -\frac{1/|q_i|^k}{k} \cos(k\Omega t + k\phi_i). \tag{4}$$

Manuscript received July 17, 1998; revised May 12, 2000. This work was supported by the National Science Foundation under Grant CCR-9804050. The associate editor coordinating the review of this paper and approving it for publication was Dr. Sergio Barbarossa.

The authors are with the Department of Electrical Engineering, University of Rhode Island, Kingston, RI 02881 USA (e-mail: kumar@ele.uri.edu.).

Publisher Item Identifier S 1053-587X(00)07636-4.

Note that the zeros $q_i$ and $p_i$ are assumed to be outside and inside the unit circle, respectively. We will reflect the $q_i$ to inside the circle (as $1/q_i^*$) and cancel them using poles. Then, we group all the zeros inside the unit circle to form a different MinP signal and the zeros outside the circle and the poles that are their reflections inside the unit circle to form the all-phase or AllP part of the signal. That is

$$s(t) = \underbrace{A_c \prod_{i=1}^{P}(1 - p_i e^{j\Omega t}) \prod_{i=1}^{Q}\left(1 - \frac{1}{q_i^*}\, e^{j\Omega t}\right)}_{\text{MinP}} \cdot \underbrace{e^{j\omega_c t} \frac{\prod_{i=1}^{Q}(1 - q_i e^{j\Omega t})}{\prod_{i=1}^{Q}\left(1 - \frac{1}{q_i^*}\, e^{j\Omega t}\right)}}_{\text{AllP}}. \tag{5}$$

Equivalently, multiplying and dividing (3) by $e^{j2\hat{\beta}(t)}$ and collecting terms, we get

$$s(t) = \underbrace{A_c e^{\alpha(t)+\beta(t)+j(\hat{\alpha}(t)+\hat{\beta}(t))}}_{\text{MinP}} \underbrace{e^{j(\omega_c t - 2\hat{\beta}(t))}}_{\text{AllP}}. \tag{6}$$

This grouping of signals is, of course, analogous to well-known decomposition of a linear discrete-time system into minimum-phase and all-pass systems (see [1, Sec. 5.6]). Analogous to the fact that the group delay of the all-pass filters is always positive [1, Sec. 5.5], the IF of AllP part will always be positive (even if $\omega_t$, which is the frequency translation, is zero). We call this the positive IF (PIF). Thus, the PIF $\psi(t)$ of $s(t)$ is a positive function and is as follows:

$$\psi(t) = \omega_c - 2\dot{\hat{\beta}}(t). \tag{7}$$

The expression for $\hat{\beta}(t)$ is the same as that of $\beta(t)$ in (4) with cosine replaced by sine. Of course, we could also group the zeros outside the unit circle together to form a MaxP-AllP decomposition. If we can separate the MinP and the AllP components of the signal $s(t)$, then the MinP part conveys the AM information, i.e., $e^{\alpha(t)+\beta(t)}$ (or, equivalently, its logarithm $\alpha(t)+\beta(t)$) around the carrier $\omega_c$, and the AllP part conveys the PIF information $\psi(t)$.

Given $s(t)$ over a $T$-second interval, how do we separate the MinP and AllP components and compute the PIF of the signal? There are at least three traditional ways to separate the MinP and AllP components. First, one could find the Fourier coefficients of $s(t)$, then root the polynomial formed using the Fourier coefficients, i.e., find $p_i$ and $q_i$, and then group them as in (5) to separate the components. Second, one could compute the log-envelope of $s(t)$ [i.e., $\ln |s(t)|$], compute its Hilbert transform, and subtract it from the phase of $s(t)$. Third, we can use the block diagram in Oppenheim and Schafer [1, Fig. 12.7, p. 784] by replacing their $X(e^{j\omega})$ by $s(t)$. In this case, one computes the logarithm of $s(t)$ and keeps the causal part of its spectrum (i.e., spectrum corresponding to the positive frequencies) as the MinP part; the AllP part is obtained by dividing $s(t)$ by the MinP part as in [1]. However, there is an elegant way of achieving this decomposition, which we describe next [8]. It does not require explicit computation of the logarithm and phase unwrapping or the Hilbert transform or rooting of a polynomial.

## III. MinP/AllP Decomposition

In this section, we describe a simple algorithm to separate the MinP and AllP components. This is shown in Fig. 1. It consists of two parts. In the first part, which consists of the multiplier, an inverse signal generator (ISG), and an error minimization block, a model fitting procedure is used to remove the envelope of the signal $s(t)$.

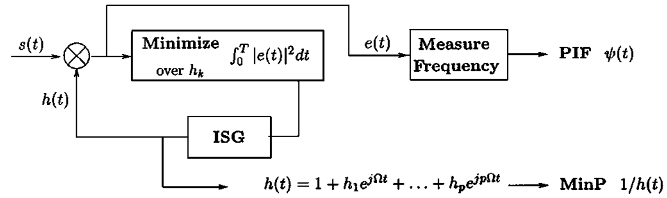


Fig. 1. LPSD algorithm: $1/h(t)$ corresponds to the MinP part of the signal $s(t)$. $\psi(t)$ corresponds to the IF of the AllP part of the signal $s(t)$.

This is achieved by minimizing the energy of an error signal $e(t)(= h(t)s(t))$. The energy of $e(t)$ is defined as follows:

$$\int_0^T |e(t)|^2\, dt = \int_0^T |s(t)h(t)|^2\, dt. \tag{8}$$

$h(t)$ is a signal generated by the ISG using the formula $h(t) = 1 + \sum_{k=1}^{p} h_k e^{jk\Omega t}$. $\Omega = 2\pi/T$. In other words, the ISG generates a low-pass periodic signal. The error is minimized by choosing the coefficients $h_k$. Those who familiar with model-based spectral analysis will immediately recognize the duality between this method and the "autocorrelation method" of linear prediction [9], [10]. In the autocorrelation method, a discrete-time FIR filter called an inverse filter or predicton-error filter with frequency response $H(e^{j\omega})$ (with first coefficient held at unity) is used to flatten the envelope of a spectrum $X(e^{j\omega})$ of a sequence $x(n)$ by minimizing the error $\int_0^{2\pi} |X(e^{j\omega})H(e^{j\omega})|^2\, d\omega$. This is an exact analog of (8). Analogous to the autocorrelation method, the error in (8) is a measure of the flatness of the envelope of $e(t)$. In addition, minimizing the error in (8) amounts to performing linear prediction on the Fourier coefficients of the signal $s(t)$, and hence, we called it linear prediction in spectral domain (LPSD) in earlier work [8].

Similar to the MinP property of the prediction-error filter used in linear prediction [10], minimizing $\int_0^T |e(t)|^2\, dt$ results in a $h(t)$ that is a MinP signal (having all its signal zeros inside the unit-circle). This is true even if the envelope of $s(t)$ goes to zero at some instants between 0 and $T$ seconds and for any order $p \geq 1$. The significance of this MinP property is that, as we already mentioned, $h(t)$'s log-envelope and phase are Hilbert transforms. Because the error minimization is performed to flatten $s(t)$'s envelope, if the value of $p$ is chosen sufficiently large [analogous to using a high order moving average (MA) model to approximate an autoregressive (AR) spectrum], then $h(t)$ will be given by

$$h(t) \approx e^{-(\alpha(t)+\beta(t))} e^{-j(\hat{\alpha}(t)+\hat{\beta}(t))}. \tag{9}$$

Thus, $(1/h(t))$ is the desired approximation to $s(t)$'s MinP component, hence, the name "inverse signal" for $h(t)$. Consequently, the error signal $e(t)$ will be $e(t) \approx A_c e^{j(\omega_c t - 2\hat{\beta}(t))}$ and, hence, is an approximation to the AllP component of $s(t)$. In the second part, which is denoted in Fig. 1 as "measure frequency," the PIF is computed as $(\dot{e}(t)/|e(t)|)$ or $(d\angle e(t)/dt)$.

The decomposition or LPSD algorithm involves solving only a linear system of equations. Let $s[n]$ $n = 0, 1, \cdots, N-1$ denote samples of the given signal. Let $\Omega = (2\pi/N)$ be the assumed fundamental frequency. By replacing $h(t)$ and $e(t)$ by their respective sampled versions, we have

$$e[n] = s[n] + \sum_{k=1}^{p} h_k s[n] e^{jk\Omega n}. \tag{10}$$

We minimize $\sum_{n=0}^{N-1} |e[n]|^2$ by choosing the best set of $h_k$. This is a linear least squares problem. Typically, $p$ is much smaller than $N$.

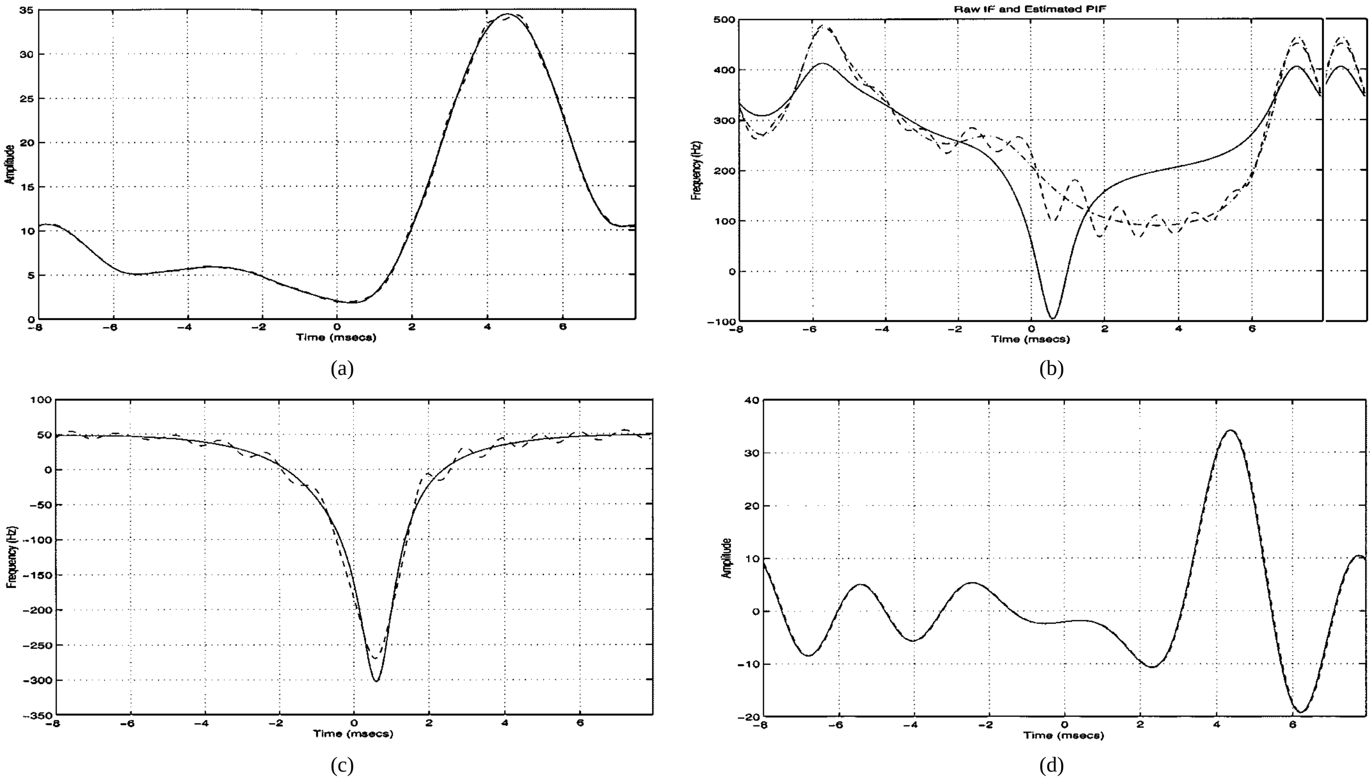


Fig. 2. (a) Estimated envelope $1/|h(t)|$ obtained using LPSD ($p = 12$ coefficients) (dashed line) along with the true envelope $|s(t)|$ (shown in solid line); (b) Plot of the signal's raw IF (solid) that goes negative; dashed line refers to estimate of PIF; the true PIF is also displayed (dashed-dotted). The estimated PIF can be made to match the true PIF more closely by choosing a larger $p$. (c) Estimate of $\dot{\hat{\alpha}}[n]$ (dashed line) along with true $\dot{\hat{\alpha}}[n]$, plotted as solid line. (d) Real part of the reconstructed signal synthesized by using the separated components (dashed line) along with the real part of the original signal $s(t)$ (solid line); they match closely.

Clearly, the solution depends only on the magnitude of $s[n]$. $h[n]$ can then be reconstructed by computing $h[n] = 1 + \sum_{k=1}^{p} h_k e^{jk\Omega n}$. $s_{\rm MinP}[n]$ can then be computed as $(1/h[n])$; the log-envelope and phase of $s_{\rm MinP}[n]$ correspond to $\alpha[n] + \beta[n]$ and $\hat{\alpha}[n] + \hat{\beta}[n]$, respectively. The PIF $\omega_c - 2\dot{\hat{\beta}}[n]$ can be found as the IF of the error signal $e[n]$ using any standard IF estimator such as the phase difference between neighboring samples.

The LPSD algorithm attempts to flatten the envelope of the signal $s(t)$ by using an adaptive amplitude demodulator. This process not only eliminates the AM but also *automatically* removes from the phase of $s(t)$, which is a quantity equal to the Hilbert transform of the log envelope of $s(t)$. This causes the IF of $e(t)$ to be positive. Instead, if we simply "complex clip" $s(t)$, i.e., obtain $s(t)/|s(t)|$, then its phase derivative (the traditional IF) will not always be positive. Second, the MinP property of $h(t)$ guarantees that the envelope approximation $1/|h(t)|$ will never equal zero. Further, MinP signals will have their energy concentrated over a relatively small region in the spectral domain analogous to a MinP filter, which has its impulse response peaking close to the origin. It is also possible to use the LPSD algorithm to achieve a MinP-MaxP (instead of MinP-AllP) decomposition of $s(t)$. See [4]. Third, an important advantage of the LPSD algorithm is that it achieves the separation of the MinP and AllP components without explicitly rooting a polynomial or computing the logarithm or Hilbert transform of the signal $s(t)$.

### *A. Simulation Results*

A signal $s(t)$ consisting of seven [$M = 6$ in (1)] harmonically related complex exponentials with frequencies 0 Hz, 62.5 Hz, $\cdots$, up to 375 Hz, with amplitudes 1.0000, 3.3330, 5.8910, 7.3820, 7.0350, 8.8136, and 3.3600, respectively, and whose respective phases (in radians) were 0, $-1.7822$, 2.8095, 1.5068, $-0.2974$, $-2.2036$, and 1.1969 was synthesized. $s(t)$ corresponds to a mixed phase (MixP) signal consisting of two ($= P$) zeros inside and four ($= Q$) zeros outside the unit circle. The signal is periodic with a period of 16 ms (62.5 Hz fundamental frequency) and has a carrier frequency of 249 Hz (corresponding to its MaxP component's translation $Q\Omega$, $\omega_c = 2\pi \times Q \times 62.5$, and $\omega_t = 0$). The signal was sampled at 16 kHz. The signal samples were fed to the LPSD algorithm described in the previous subsection. The coefficients $h_k$ were computed, and $h(t)$ (actually its samples) was synthesized. For the case of 12 coefficients [i.e., $p = 12$ in (10)], the estimated envelope given by $1/|h(t)|$ is shown (dashed line) in Fig. 2(a). Only one period (16 ms) of the envelope is shown. The true envelope (solid line) given by $|s(t)|$ is also shown. They match closely. In Fig. 2(b), we have plotted the signal's raw IF [which is obtained by differencing the phase angles of adjacent samples of the signal $s(t)$] in a solid line. Note that the raw IF could go negative. On the other hand, the PIF (i.e., $\omega_c - 2\dot{\hat{\beta}}[n]$) computed by differencing the phases of the neighboring samples of the error signal $e(t)$ stays positive (dashed line), as it should. In Fig. 2(b), the true PIF (dashed-dotted line) that is superposed on to the estimated PIF is also plotted. The true PIF was obtained, for the purpose of comparison, by using the roots of the polynomial in (1) and synthesizing the AllP signal given in (5) and determining its IF. $\omega_c$ was estimated as the mean of PIF, and $\dot{\hat{\beta}}[n]$ was estimated by subtracting $\omega_c$'s estimate from the PIF. Further, $\dot{\hat{\alpha}}[n]$ was computed by subtracting the estimate of $\dot{\hat{\beta}}[n]$ from the MinP signal's ($1/h(t)$'s) IF; the solid line in Fig. 2(c) corresponds to the separated $\dot{\hat{\alpha}}[n]$; it matches closely with the true one (which is obtained using the signal's roots) shown as dashed-dotted line. In Fig. 2(d), we have displayed the real part of the signal reconstructed using the separated MinP and MaxP components using a solid line; the dashed line corresponds to one period of the real part of the original signal $s(t)$; they match exactly. In all of the above cases, the estimated components can be made to match the true ones more closely by increasing the order $p$. In summary, given a signal $s(t)$, its various components (MinP/MaxP/AllP), which are actually multiplied components, can be separated using simple linear techniques without resorting to rooting or phase unwrapping algorithms.

## IV. Discussion and Conclusion

Traditionally, MinP/MaxP/All-Pass decompositions are applied to filter transfer functions, i.e., in the frequency domain. In this correspondence, we have proposed the dual of this decomposition in the time domain. In Section III, we also presented a simple algorithm to achieve this decomposition. The duality between the frequency and time domains can be extended further. If $H(z)$ is a MinP filter of order $p$, then the symmetric/antisymmetric polynomials $H^*(1/z^*) \pm z^{-n}H(z)$ for any $n \geq p$ will have all their roots on the unit circle $|z| = 1$, and they are interlaced. These roots are called line spectrum frequencies (LSFs) [11] and are sufficient to represent $H(z)$. The LSF's are used extensively in speech coding. Analogously, in the time domain, the signals $h(t)e^{-j(n/2)\Omega t} \pm h^*(t)e^{j(n/2)\Omega t}$ (for $n \geq p$) will have interlaced zero crossings that are sufficient to uniquely identify $h(t)$. In other words, $|1/h(t)|$, which models the envelope of a signal $s(t)$, can be represented by these zero crossings. These ideas are explored in more detail in [6] and for real-valued signals in [12]. Although we have considered only periodic signals, the extension of the algorithm to any signal requires appropriate time windowing.

## References


[1] A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1989, pp. 718–721.

[2] H. B. Voelcker, “Toward a unified theory of modulation—Part I: Phase-envelope relationships,” *Proc. IEEE*, vol. 54, pp. 340–354, 1966.

[3] R. Kumaresan, “An inverse signal approach to computing the envelope of a real valued signal,” *IEEE Signal Processing Lett.*, pp. 256–259, Oct. 1998.

[4] R. Kumaresan and A. Rao, “Model-based approach to envelope and positive-instantaneous frequency of signals and application to speech,” *J. Acoust. Soc. Amer.*, vol. 105, pp. 1912–1924, Mar. 1999.

[5] B. Picinbono, “On instantaneous amplitude and phase of signals,” *IEEE Trans. Signal Processing*, vol. 45, pp. 552–560, 1997.

[6] R. Kumaresan and Y. Wang, “On the relationship between line spectrum pairs and zero-crossings of band-pass signals,” *IEEE Trans. Speech Audio Processing*, to be published.

[7] A. Rao, “Signal analysis using product expansions inspired by the auditory periphery,” Ph.D. dissertation, Univ. Rhode Island, Kingston, RI, June 1997.

[8] R. Kumaresan and A. Rao, “An algorithm for decomposing an analytic signal into AM and positive FM components,” in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, Seattle, WA, May 1998, pp. III:1561–1564.

[9] J. I. Makhoul, “Linear prediction: A tutorial review,” *Proc. IEEE*, vol. 63, pp. 561–580, Apr. 1975.

[10] S. M. Kay, *Modern Spectral Estimation: Theory and Application*. Englewood Cliffs, NJ: Prentice-Hall, 1987.

[11] F. Itakura, “Line spectrum representation of linear predictive coefficients of speech signal,” *J. Acoust. Soc. Amer.*, vol. 57, p. S35, 1975.

[12] R. Kumaresan and Y. Wang, “A new real-zero conversion algorithm,” in *Proc. ICASSP*, Istanbul, Turkey, June 2000.