The transition matrix for acoustic and elastic wave scattering in prolate spheroidal coordinates

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A spheroidal-coordinate-based transition matrix is derived for acoustic and elastic wave scattering. The formalism is based on Betti's third identity and an appropriately chosen set of vector spheroidal basis functions. Transition matrices are obtained for the scattering from an elastic inclusion in an elastic medium and in an inviscid fluid.

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INTRODUCTION

The transition matrix approach to acoustic and elastic wave scattering was introduced by Waterman[1–4] and further refined and developed by a number of authors.[5–8] While this approach has been successfully applied to scattering in the low $kL/2$ region (k = wavenumber, L = diameter of sphere circumscribing the scatterer), calculations have been computer limited to small aspect ratio scatterers.[9] As the aspect ratio of the scatterer increases, the system of equations generated by the T-matrix method becomes ill-conditioned and the number of terms required in the expansion of the surface traction and surface displacement of the scatterer increases dramatically.

The apparent source of this difficulty is the set of global basis functions used. For finite three-dimensional geometries, the T matrix has been based on the separable solutions of the vector Helmholtz equation in spherical coordinates, and the domain of convergence of expansions in these solutions is not appropriate for the description of elongated objects. From this point of view, prolate spheroidal coordinates are a more natural candidate for the description of scattering from large aspect ratio bodies. Bates and Wall[10] have successfully considered the scattering of scalar waves using the separable solutions to the scalar Helmholtz equation in this coordinate system, realizing a considerable advantage over the more conventional approach. Unfortunately the vector Helmholtz equation is not separable in spheroidal coordinates and it appears that it is impossible to analytically construct a Green's tensor for general waves in this coordinate system.[11,12] Naturally, this has inhibited investigations of scattering involving elastic media in spheroidal coordinates.

In the present paper, we resolve this difficulty. We do not attempt to construct a Green's tensor, but instead utilize Betti's third identity[13] to establish a spheroidal coordinate based T matrix. We do not rigorously establish the domain of convergence of the vector spheroidal expansions in the present paper. Instead we develop the formalism into a calculable form and obtain the transition matrices for the scattering from an elastic inclusion in an elastic medium and in an inviscid fluid. Briefly, the organization of this paper is as follows. In Secs. I–III, we define a set of global basis states in prolate spheroidal coordinates and use Betti's third identity to derive certain integral identities involving these basis states which serve as the foundation of our approach. In Secs. IV–VII, we obtain the mathematical representation of Huygen's principle and use this result to derive a spheroidal coordinate based T matrix for acoustic and elastic wave scattering. In Sec. VIII, we consider sound-hard and sound-soft scatterers, and in Sec. IX, we discuss the extension to other coordinate systems and to electromagnetic scattering. Appendix A is consigned to a brief discussion of the solutions to the scalar Helmholtz equation and the presentation of explicit forms for our global basis functions. Appendix B gives a convergence proof.

Throughout this paper, we use tensor notation. To prevent confusion, the specific greek indices $\alpha, \beta, \mu, \nu, \delta, \gamma$ are reserved for tensor indices, superscripts denoting contravariant indices and subscripts, convariant indices. Other indices, such as τ, σ, m, l, etc. are used to designate other degrees of freedom (e.g., the labeling of a particular global basis function). The metric tensor is denoted by

$$g_{\alpha\beta} = \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x_\mu}{\partial y^\beta},$$

$\epsilon^{\alpha\beta\mu}$ denotes the totally antisymmetric tensor of rank three and for shorthand, we use

$$\nabla_\alpha u_\beta = u_{\beta,\alpha}$$

to denote the covariant derivative of u_β. We remind the reader that the contravariant basis vectors $\mathbf{e}^\alpha$ are normalized according to

$$\mathbf{e}^\alpha \cdot \mathbf{e}^\beta = g^{\alpha\beta}.$$

I. VECTOR HELMHOLTZ EQUATION

Consider the two-dimensional elliptic coordinate system consisting of confocal ellipses and hyperbolas depicted in Fig. 1. The prolate spheroidal coordinate system (ξ, η, ϕ) can be formed by rotating this figure about the major axis of the ellipse. Surfaces of constant ξ define ellipsoids of revolution and those of constant η, hyperboloids. The angle ϕ has its usual role of designating axial orientation. The connection with rectangular coordinates is given by

$$x = f\sqrt{(\xi^2-1)(1-\eta^2)}\cos\phi,$$
$$y = f\sqrt{(\xi^2-1)(1-\eta^2)}\sin\phi, \qquad (1)$$
$$z = f\xi\eta,$$

where $2f$ is the interfocal distance.

It is well known that while the scalar Helmholtz equation is separable in prolate spheroidal coordinates, the vector

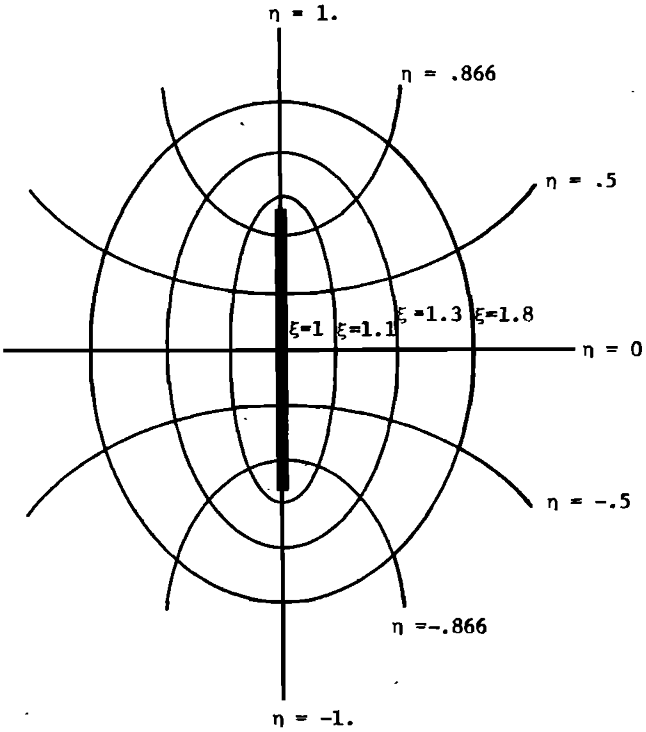

FIG. 1. The prolate spheroidal coordinate system.

Helmholtz equation is not. Consequently, the elastic wave equation,

$$(\lambda + 2\mu)\nabla(\nabla\cdot\mathbf{F}) - \mu\nabla\times(\nabla\times\mathbf{F}) = -\omega^2\rho\mathbf{F}, \tag{2}$$

has not been applied to elastic wave scattering in this coordinate representation and the selection of a convenient set of basis functions is open. Our selection, indicated below, is determined by general considerations. In the following, we separate the vector solution to Eq. (1) into longitudinal (irrotational) and transverse (solenoidal) parts

$$\mathbf{F} = \mathbf{F}_L + \mathbf{F}_T,$$
$$\nabla\times\mathbf{F}_L = 0, \tag{3}$$
$$\nabla\cdot\mathbf{F}_L = 0,$$

and consider each part separately. While it is possible to construct solutions with other methods, there is a simple, physically compelling reason for this procedure—the longitudinal and transverse elastic waves travel at different speeds and any other separation of the solution would result in waves of both velocities being part of both solutions. This separation is unique, provided that **F** is uniform, finite, continuous, and vanishes at infinity.[14]

Consider the longitudinal solution. An arbitrary irrotational vector may be written as the gradient (covariant derivative) of a scalar and it is trivial to verify that

$$u_\alpha = \nabla_\alpha\phi \tag{4}$$

is a solution of Eq. (2) if

$$(\nabla^2 + k_L^2)\phi = 0, \tag{5}$$
$$k_L^2 = \omega^2/c_L^2 = \omega^2\rho/(\lambda + 2\mu). \tag{6}$$

The transverse vector field has two degrees of freedom and hence, must depend upon two scalars. It is relatively easy to show that both

$$V_\alpha = g_{\alpha\beta}\epsilon^{\beta\nu\mu}\nabla_\nu(a_\mu\psi) \tag{7}$$

and

$$W_\alpha = \epsilon_{\alpha\beta\mu}g^{\beta\beta'}\nabla_{\beta'}\epsilon^{\mu\nu\nu'}\nabla_\nu(a_{\nu'}\chi) \tag{8}$$

are transverse solutions to Eq. (2), provided that

$$\nabla_\mu a_\nu = g_{\mu\nu}, \tag{9}$$
$$(\nabla^2 + k_T^2)\psi = (\nabla^2 + k_T^2)\chi = 0, \tag{10}$$
$$k_T^2 = \omega^2/c_T^2 = \omega^2\rho/\mu. \tag{11}$$

Before defining our global basis functions, we briefly digress to consider the solutions obtained above and the meaning of the nonseparability of Eq. (2). Morse and Feshbach[14] have defined the separation of the vector Helmholtz equation to be a particular process of obtaining solutions which is much the same as that above, given that the transverse solutions can always be chosen so that the part of the field derived from one scalar will be tangential to the "radial coordinate = constant" surface, and that obtained from the other scalar, normal to it. When this scheme can be realized, the fitting of boundary conditions on that surface takes a particularly simple form. In the present case, these conditions would require that **a** be normal to the constant ξ surface. Since the most general solution to Eq. (8) is

$$\mathbf{a} = \mathbf{r} + \mathbf{c}, \tag{12}$$

where **r** is the (spherical) radius vector and **c** is a covariant constant (e.g., the unit vectors $\hat{i}, \hat{j}, \hat{k}$ in rectangular coordinates), this is clearly not possible. Thus, while we have obtained solutions to Eq. (2) in prolate spheroidal coordinates, the fitting of boundary conditions on a spheroidal surface will not be as simple as, for example, the fitting of boundary conditions on a spherical surface in spherical coordinates, but will in general involve infinite dimensional matrices. In this paper, we choose to generate the transverse solutions with $\mathbf{c} = 0$.

II. GLOBAL BASIS FUNCTIONS

Here, in complete analogy with spherical coordinates, the vector spheroidal partial wave solutions require four indices for their specification, τ, to distinguish among the one longitudinal and two transverse degrees of freedom, and σ, m, and l to specify the solution to the scalar Helmholtz equation. We define the vector basis functions corresponding to outgoing spheroidal waves with

$$(\psi_{1\sigma ml})_\alpha = A_{ml}(h_T)^{-1/2}g_{\alpha\beta}\epsilon^{\beta\mu\nu} \times\nabla_\mu[a_\nu he_{ml}(h_T,\xi)S_{\sigma ml}(h_T,\eta,\phi)], \tag{13}$$
$$(\psi_{2\sigma ml})_\alpha = (1/k_T)g_{\alpha\beta}\epsilon^{\beta\mu\nu}\nabla_\mu(\psi_{1\sigma ml})_\nu, \tag{14}$$
$$(\psi_{3\sigma ml})_\alpha = (1/k_L)\nabla_\alpha[he_{ml}(h_L,\xi)S_{\sigma ml}(h_L,\eta,\phi)], \tag{15}$$

where he_{ml} and $S_{\sigma ml}$ are the spheroidal counterparts of the spherical Hankel function of the first kind and the spherical harmonic, respectively, A_{ml} is an eigenvalue, and

$$h_L = k_L f, \quad h_T = k_T f. \tag{16}$$

Since these functions are much less familiar than the spherical solutions, in Appendix A we briefly discuss the properties of the scalar solutions and their defining equations and give explicit expressions for the $\{\psi_n\}$. Hence, we note in passing that in contrast to the spherical functions, the angular functions themselves depend upon the wavenumber.

The vector wavefunctions defined above, in general, are neither orthogonal among themselves, nor are the members of one set orthogonal to the members of another set. Nor is any simple separation into a sum of products of a radial part and mutually orthogonal "vector spheroidal harmonics" possible. This means that even where the explicit satisfaction of boundary conditions on a surface of constant ξ is concerned, one must deal with a set of coupled, infinite dimensional matrix equations. Physically, this means that a normal mode solution of the type possible with a sphere is not possible here—the modes are coupled to one another.

Fortunately, considerable simplification in the form of the $\{\psi_n\}$ does occur in the $\xi\to\infty$ limit and it turns out that this is essential to our derivation of the T-matrix formalism. The asymptotic forms of the basis states given in Appendix A are

$$\psi_{1\sigma ml}\to he_{ml}(h_T,\xi)\mathbf{C}_{\sigma ml}(h_T,\eta,\phi), \tag{17}$$

$$\psi_{2\sigma ml}\to\frac{1}{h_T}\frac{\partial}{\partial\xi}he_{ml}(h_T,\xi)\mathbf{B}_{\sigma ml}(h_T,\eta,\phi), \tag{18}$$

$$\psi_{3\sigma ml}\to\frac{1}{h_L}\frac{\partial}{\partial\xi}he_{ml}(h_L,\xi)\mathbf{A}_{\sigma ml}(h_L,\eta,\phi), \tag{19}$$

where we have defined

$$\mathbf{A}_{\sigma ml}(h_L,\eta,\phi)=S_{\sigma ml}(h_L,\eta,\phi)\hat{\xi}, \tag{20}$$

$$\mathbf{B}_{\sigma ml}(h_T,\eta,\phi) =\frac{1}{A_{ml}(h_T)^{1/2}}\left((1-\eta^2)^{1/2}\frac{\partial}{\partial\eta}S_{\sigma ml}(h_T,\eta,\phi)\hat{\eta} +\frac{1}{(1-\eta^2)^{1/2}}\frac{\partial}{\partial\phi}S_{\sigma ml}(h_T,\eta,\phi)\hat{\phi}\right), \tag{21}$$

$$\mathbf{C}_{\sigma ml}(h_T,\eta,\phi) =\frac{1}{A_{ml}(h_T)^{1/2}}\left(\frac{1}{(1-\eta^2)^{1/2}}\frac{\partial}{\partial\phi}S_{\sigma ml}(h_T,\eta,\phi)\hat{\eta} -(1-\eta^2)^{1/2}\frac{\partial}{\partial\eta}S_{\sigma ml}(h_T,\eta,\phi)\hat{\phi}\right), \tag{22}$$

and where we have introduced the unit vectors

$$\hat{\xi}=(g_{11})^{1/2}\mathbf{e}^1,\quad \hat{\eta}=(g_{22})^{1/2}\mathbf{e}^2,\quad \hat{\phi}=(g_{33})^{1/2}\mathbf{e}^3. \tag{23}$$

Naturally the limit

$$he_{ml}(h,\xi)\to(1/h\xi)\exp\{i[h\xi-(l+1)\pi/2]\} \tag{24}$$

is to be understood in these equations. We also note for future reference that the surface tractions on a constant ξ surface (i.e., $\hat{n}=\hat{\xi}$)

$$\mathbf{t}(\psi_n)=\lambda\nabla\cdot\psi_n\hat{n}+\mu\hat{n}\cdot(\nabla\psi_n+\psi_n\nabla) \tag{25}$$

have the simple asymptotic limits

$$\mathbf{t}(\psi_{1\sigma mn})=\frac{\mu}{f}\frac{\partial}{\partial\xi}he_{ml}(h_T,\xi)\mathbf{C}_{\sigma ml}(h_T,\eta,\phi), \tag{26}$$

$$\mathbf{t}(\psi_{2\sigma ml})=-\mu k_T he_{ml}(h_T,\xi)\mathbf{B}_{\sigma ml}(h_T,\eta,\phi), \tag{27}$$

$$\mathbf{t}(\psi_{3\sigma mn})=-(\lambda+2\mu)k_L he_{ml}(h_T,\xi)\mathbf{A}_{\sigma ml}(h_L,\eta,\phi). \tag{28}$$

The explicit calculation of the surface tractions $\mathbf{t}(\psi_n)$ is quite lengthy and we shall not give it here. Instead, we obtain Eqs. (26)–(28) indirectly through a simple observation. In the $\xi\to\infty$ limit, the eccentricity ($e=1/\xi$) of the constant-ξ surface becomes zero. Thus, there must be the exact formal correspondence

$$\begin{aligned} &h\xi\to kr,\\ &he_{ml}(h,\xi)\to h_l(kr),\\ &S_{\sigma ml}(h,\xi,\phi)\to y_{\sigma ml}(\theta,\phi),\\ &(\mathbf{A}_{\sigma ml},\mathbf{B}_{\sigma ml},\mathbf{C}_{\sigma ml})\to\text{vector spherical harmonics,} \end{aligned} \tag{29}$$

between Eqs. (26)–(28) and the asymptotic $r\to\infty$ limit of their spherical counterpart. This correspondence is predicated on an exact correspondence between the asymptotic limits of the $\{\psi_n\}$ and their spherical counterparts. In fact this is the rationale for the choice $c=0$ in Eq. (12) and the normalization encountered in Eqs. (20)–(22). While utilizing this formal similarity, we point out one important difference between the above results and those obtained in spherical coordinates—the **A**, **B**, **C** vectors do not share the orthogonality properties of the vector spherical harmonics. For example, consider two members of the $\{\mathbf{B}_n\}$. Using an integration by parts and the defining differential equation for $S_{\sigma ml}$ (see Appendix A) we obtain

$$\begin{aligned}
&\int_0^{2\pi}d\phi\int_{-1}^{1}d\eta\,\mathbf{B}_{\sigma ml}(h_T,\eta,\phi)\cdot\mathbf{B}_{\sigma' m'l'}(h_T,\eta,\phi)\\
&=[A_{ml}(h_T)A_{m'l'}(h_T)]^{-1/2}\int_0^{2\pi}d\phi\int_{-1}^{1}d\eta\left((1-\eta^2)\frac{\partial}{\partial\eta}S_{\sigma ml}(h_T,\eta,\phi)\frac{\partial}{\partial\eta}S_{\sigma' m'l'}(h_T,\eta,\phi)\right.\\
&\quad\left.+\frac{1}{1-\eta^2}\frac{\partial}{\partial\phi}S_{\sigma ml}(h_T,\eta,\phi)\frac{\partial}{\partial\phi}S_{\sigma' m'l'}(h_T,\eta,\phi)\right)\\
&=-[A_{ml}(h_T)A_{m'l'}(h_T)]^{-1/2}\int_0^{2\pi}d\phi\int_{-1}^{1}d\eta\,S_{\sigma ml}(h_T,\eta,\phi)\left(\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta}S_{\sigma' m'l'}(h_T,\eta,\phi)\right.\\
&\quad\left.+\frac{1}{1-\eta^2}\frac{\partial^2}{\partial\phi^2}S_{\sigma' m'l'}(h_T,\eta,\phi)\right)\\
&=[A_{ml}(h_T)A_{m'l'}(h_T)]^{-1/2}\int_0^{2\pi}d\phi\int_{-1}^{1}d\eta\,[A_{m'l'}(h_T)-h_T^2\eta^2]S_{\sigma ml}(h_T,\eta,\phi)S_{\sigma' m'l'}(h_T,\eta,\phi)\\
&=\delta_{\sigma\sigma'}\delta_{mm'}\left(\delta_{ll'}-\frac{[A_{ml}(h_T)A_{ml'}(h_T)]^{-1/2}}{[A_{ml}(h_T)A_{ml'}(h_T)]^{1/2}}h_T^2\int_{-1}^{1}d\eta\,\eta^2S_{ml}(h_T,\eta)S_{ml'}(h_T,\eta)\right)\\
&\equiv\delta_{\sigma\sigma'}\delta_{mm'}\Omega_{ll'}^{m}(h_T).
\end{aligned} \tag{30}$$

Only in the limit $h_T \to 0$ does $\Omega^{m}_{ll'}(h_T)$ become diagonal. For the sake of completeness, we give a compendium of these integrals below.

$$\int_0^{2\pi} d\phi \int_{-1}^{1} d\eta\, \mathbf{A}_{\sigma ml}(h_L,\eta,\phi)\cdot\mathbf{A}_{\sigma' m' l'}(h_L,\eta,\phi) = \delta_{\sigma\sigma'}\delta_{mm'}\delta_{ll'}, \tag{31}$$

$$\int_0^{2\pi} d\phi \int_{-1}^{1} d\eta\, \mathbf{B}_{\sigma ml}(h_T,\eta,\phi)\cdot\mathbf{B}_{\sigma' m' l'}(h_T,\eta,\phi) = \delta_{\sigma\sigma'}\delta_{mm'}\Omega^{m}_{ll'}(h_T), \tag{32}$$

$$\int_0^{2\pi} d\phi \int_{-1}^{1} d\eta\, \mathbf{C}_{\sigma ml}(h_T,\eta,\phi)\cdot\mathbf{C}_{\sigma' m' l'}(h_T,\eta,\phi) = \delta_{\sigma\sigma'}\delta_{mm'}\Omega^{m}_{ll'}(h_T), \tag{33}$$

$$\int_0^{2\pi} d\phi \int_{-1}^{1} d\eta\, \mathbf{B}_{\sigma ml}(h_T,\eta,\phi)\cdot\mathbf{C}_{\sigma' m' l'}(h_T,\eta,\phi) = 0, \tag{34}$$

$$\int_0^{2\pi} d\phi \int_{-1}^{1} d\eta\, \mathbf{A}_{\sigma ml}(h_L,\eta,\phi)\cdot\mathbf{B}_{\sigma' m' l'}(h_T,\eta,\phi) = 0, \tag{35}$$

$$\int_0^{2\pi} d\phi \int_{-1}^{1} d\eta\, \mathbf{A}_{\sigma ml}(h_L,\eta,\phi)\cdot\mathbf{C}_{\sigma' m' l'}(h_T,\eta,\phi) = 0. \tag{36}$$

III. BETTI'S IDENTITY

As discussed in the previous section, the members of our basis set $\{\psi_n\}$ are in general not orthogonal even on a surface of constant ξ, and in fact from the explicit forms given in Appendix A, it is clear that the inner products

$$\int d\Omega\, \psi_n \cdot \psi_{n'}$$

will be quite complicated. Nonetheless, it is possible to establish a set of simple integral identities which are of great practical importance in developing the T-matrix formalism. First we establish a theorem whose utility in deriving the spherical cordinate based T-matrix formalism was first noticed by Waterman,[3] and later utilized in an elegant fashion by Pao.[7]

Theorem: Let u_α and v_α be solutions to be time-independent vector Helmholtz equation in some volume V bounded by a surface or collection of surfaces, denoted by S, and let $\Upsilon^{\alpha\beta}$ denote the corresponding stress tensor. Then

$$\oint_s ds\, n_\alpha [\Upsilon^{\alpha\beta}(u)v_\beta - \Upsilon_{\alpha\beta}(v)u_\beta] = 0, \tag{37}$$

where $\mathbf{n} = n_\alpha \mathbf{e}^\alpha$ is a unit vector pointing out of the volume, normal to S.

The proof is straightforward. Using the definition of the stress tensor

$$\Upsilon^{\alpha\beta}(u) = \lambda u^{\gamma}_{,\gamma} g^{\alpha\beta} + \mu(u_{\gamma,\delta} + u_{\delta,\gamma})g^{\gamma\alpha}g^{\delta\beta} \tag{38}$$

and noting that the equation of motion, Eq. (2), can be rewritten as

$$\Upsilon^{\alpha\beta}_{,\alpha}(u) = -\omega^2 \rho u^\beta, \tag{39}$$

we find

$$\begin{aligned}\nabla_\alpha [\Upsilon^{\alpha\beta}(u)v_\beta - \Upsilon^{\alpha\beta}(v)u_\beta] &= \Upsilon^{\alpha\beta}_{,\alpha}(u)v_\beta + \Upsilon^{\alpha\beta}(u)v_{\beta,\alpha} - \Upsilon^{\alpha\beta}_{,\alpha}(v)u_\beta - \Upsilon^{\alpha\beta}(v)u_{\beta,\alpha}\\ &= \mu(u_{\gamma,\delta} + u_{\delta,\gamma})g^{\gamma\alpha}g^{\delta\beta}v_{\beta,\alpha}\\ &\quad - \mu v_{\delta,\gamma}g^{\beta\delta}g^{\alpha\gamma}(u_{\beta,\alpha} + u_{\alpha,\beta}) = 0.\end{aligned} \tag{40}$$

Applying the divergence theorem

$$\int_v F^{\alpha}_{,\alpha}\, dv = \oint_s n_\alpha F^\alpha\, ds \tag{41}$$

to the expression above leads to the desired result. Using the definition of the surface traction

$$t(u)_\alpha = n^\beta \Upsilon_{\beta\alpha}(u), \tag{42}$$

we can rewrite our result as

$$\oint_s ds[\mathbf{t}(u)\cdot\mathbf{v} - \mathbf{t}(v)\cdot\mathbf{u}] = 0, \tag{43}$$

which is a particular form of Betti's third identity. With the above theorem, we establish a crucial lemma.

Lemma: Let $\psi_{\tau n}$ and $\text{Re}\,\psi_{\tau n}$, $n = (\sigma,m,l)$ be the outgoing and regular solutions to the time-independent vector Helmholtz equation, respectively, and let S be an arbitrary closed surface. Then

$$\oint_s ds[\mathbf{t}(\text{Re}\,\psi_{\tau n})\cdot\text{Re}\,\psi_{\tau' n'} - \mathbf{t}(\text{Re}\,\psi_{\tau' n'})\cdot\text{Re}\,\psi_{\tau n}] = 0, \tag{44}$$

$$\oint_s ds[\mathbf{t}(\psi_{\tau n})\cdot\psi_{\tau' n'} - \mathbf{t}(\psi_{\tau' n'})\cdot\psi_{\tau n}] = 0, \tag{45}$$

$$\oint_s ds[\mathbf{t}(\psi_{\tau n})\cdot\text{Re}\,\psi_{\tau' n'} - \mathbf{t}(\text{Re}\,\psi_{\tau' n'})\cdot\psi_{\tau n}] = i\delta_{\tau\tau'}\delta_{\sigma\sigma'}\delta_{mm'}O^{\tau m}_{ll'}, \tag{46}$$

where

$$\begin{aligned}O^{1m}_{ll'} &= O^{2m}_{ll'} = (-1)^{(l-l')/2}(\mu/k_T)\Omega^{m}_{ll'}(h_T),\\ O^{3m}_{ll'} &= [(\lambda + 2\mu)/k_L]\delta_{ll'}.\end{aligned} \tag{47}$$

The proof of this lemma follows. Let S be an arbitrary imaginary surface drawn between two constant ξ surfaces, S_- and S_+ (see Fig. 2). For the first equation, we use the theorem with $u = \text{Re}\,\psi_{\tau n}$, $v = \text{Re}\,\psi_{\tau' n'}$. Since these functions are regular within S_-, we have

$$\oint_{S_-} ds[\mathbf{t}(\text{Re}\,\psi_{\tau n})\cdot\text{Re}\,\psi_{\tau' n'} - \mathbf{t}(\text{Re}\,\psi_{\tau' n'})\cdot\text{Re}\,\psi_{\tau n}] = 0. \tag{48}$$

As S_- may be deformed to S without encountering any singularities, Eq. (44) follows.

For the third equation, we apply our theorem to the volume bounded by S_+ and S_∞ with $\mathbf{u} = \psi_{\tau n}$ and $\mathbf{v} = \text{Re}\,\psi_{\tau' n'}$. Consistently choosing $\hat{n}$ to be in the $+\xi$ direction, we have

$$\left(\oint_{S_\infty} - \oint_{S_+}\right) ds[\mathbf{t}(\psi_{\tau n})\cdot\text{Re}\,\psi_{\tau' n'} - \mathbf{t}(\text{Re}\,\psi_{\tau' n'})\cdot\psi_{\tau n}] = 0, \tag{49}$$

and since we cross no singularities by extending S_∞ towards $\xi \to \infty$ and S_+ towards S, we find

$$\oint_s ds[\mathbf{t}(\psi_{\tau n})\cdot\text{Re}\,\psi_{\tau' n'} - \mathbf{t}(\text{Re}\,\psi_{\tau' n'})\cdot\psi_{\tau n}] = \lim_{\xi\to\infty}\oint_{S_\infty} ds[\mathbf{t}(\psi_{\tau n})\cdot\text{Re}\,\psi_{\tau' n'} - \mathbf{t}(\text{Re}\,\psi_{\tau' n'})\cdot\psi_{\tau n}], \tag{50}$$

which can easily be evaluated by examining the asymptotic limits of $\psi_{\tau n}$ and $\tau(\psi_{\tau n})$ already established. Using

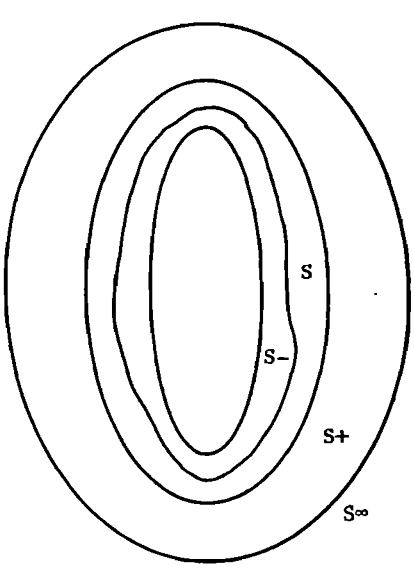

FIG. 2. The surfaces used in the proof of the lemma.

$$ds = f^2\sqrt{(\xi^2-1)(\xi^2-\eta^2)}\,d\phi\,d\eta,$$
$$\overset{\xi\to\infty}{\longrightarrow} f^2\xi^2\,d\phi\,d\eta, \qquad (51)$$

and the asymptotic forms of je_{ml} and he_{ml} (see Appendix A), easily gives the desired result, e.g.,

$$\lim_{\xi\to\infty}\oint_{S_\infty} ds[\mathbf{t}(\psi_{ln})\cdot\text{Re}\,\psi_{l'n'} - \mathbf{t}(\text{Re}\,\psi_{l'n'})]$$
$$= (\mu/f)\lim_{\xi\to\infty} f^2\xi^2 W[je_{m'l'}(h_T,\xi),he_{ml}(h_T,\xi)]$$
$$\times\int d\Omega\,\mathbf{C}_{\sigma ml}\cdot\mathbf{C}_{\sigma' m'l'}$$
$$= i(-1)^{(l-l')/2}(\mu/k_T)\Omega^{m}_{ll'}(h_T)\delta_{\sigma\sigma'}\delta_{mm'}, \qquad (52)$$

where

$$W(je_{ml},he_{ml}) = je_{ml'}he'_{ml} - he_{ml}\,je'_{ml'}. \qquad (53)$$

We complete the proof of the lemma by noting that integrals involving two outgoing wavefunctions [e.g., Eq. (45)] may be treated as in the foregoing example, only now we have

$$\lim_{\xi\to\infty}\xi^2 W(he_{ml'},he_{ml})$$

appearing, which certainly vanishes.

Equations (44)–(46) are the new results needed to derive a spheroidal coordinate based T matrix.

IV. HUYGEN'S PRINCIPLE

Consider the scattering from a region bounded by the closed surface S shown in Fig. 2. We assume that the scattering medium in general has different elastic properties than the surrounding host medium and differentiate between the parameters and basis functions of the two regions by affixing a superscript ^{0} to those of the scatterer. In this paper, we shall not attempt to rigorously establish the domain of convergence of the vector spheroidal expansions of the incident, scattered, and refracted waves. Instead, we assume that the results establishing the convergence of the vector spherical expansions in some region bounded by a circumscribing (or inscribing) spherical surface can be taken over directly in the present case by replacing the spherical surface by a prolate spheroidal surface. This is likely assured by the uniform convergence of the scalar spheroidal eigenfunction expansions to the scalar functions ϕ, ψ, and χ defined in Eqs. (4), (7), and (8) in the aforementioned domains, and we shall return to this point in a future paper.

The incident wave $\mathbf{u}^i$ is assumed known and the series

$$\mathbf{u}^i = \sum_n a_n\,\text{Re}\,\boldsymbol{\psi}_n \qquad (54)$$

to be uniformly convergent for $\xi < \xi_\infty$, given that there are no sources within this boundary. The scattering region serves as a source for the scattered wave $\mathbf{u}^s$ and we assume that the representation

$$\mathbf{u}^s = \sum_n f_n\boldsymbol{\psi}_n \qquad (55)$$

converges for ξ greater than or equal to the radial coodinate of the smallest circumscribed prolate spheroidal surface. Likewise, the series for the refracted wave

$$\mathbf{u}^r = \sum_n b_n\,\text{Re}\,\boldsymbol{\psi}^0_n \qquad (56)$$

is presumed to be uniformly convergent at least on and within the largest spheroid which can be inscribed in the scattering region.[15] The exact regions of convergence of these series can, of course, be established once the coefficients have been determined.

To obtain expressions for the coefficients a_n and f_n, we apply Eq. (43) to the volume V bounded by S and S_+, i.e.,

$$\oint_{S_+} ds[\mathbf{t}(\mathbf{u})\cdot\mathbf{v} - \mathbf{t}(\mathbf{v})\cdot\mathbf{u}] = \oint_S ds[\mathbf{t}(\mathbf{u})\cdot\mathbf{v} - \mathbf{t}(\mathbf{v})\cdot\mathbf{u}]. \qquad (57)$$

On the outer boundary S_+, we may use the uniformly convergent series for the total displacements and tractions

$$\mathbf{u} = \sum_n a_n\,\text{Re}\,\boldsymbol{\psi}_n + \sum_n f_n\boldsymbol{\psi}_n, \qquad (58)$$

$$\mathbf{t}(\mathbf{u}) = \sum_n a_n\mathbf{t}(\text{Re}\,\boldsymbol{\psi}_n) + \sum_n f_n\mathbf{t}(\boldsymbol{\psi}_n). \qquad (59)$$

Choosing first $\mathbf{v} = \boldsymbol{\psi}_n$ and then $\mathbf{v} = \text{Re}\,\psi_n$ in conjunction with Eqs. (58) and (59), and evaluating the left-hand side of Eq. (57) with our lemma, we find

$$-i\sum_n O_{n'n}a_{n'} = \oint_S ds[\mathbf{t}_+\cdot\boldsymbol{\psi}_n - \mathbf{t}(\boldsymbol{\psi}_n)\cdot\mathbf{u}_+], \qquad (60)$$

$$i\sum_n O_{n'n}f_{n'} = \oint_S ds[\mathbf{t}_+\cdot\text{Re}\,\boldsymbol{\psi}_n - \mathbf{t}(\text{Re}\,\boldsymbol{\psi}_n)\cdot\mathbf{u}_+]. \qquad (61)$$

Here $\mathbf{t}_+$ and $\mathbf{u}_+$ denote the exterior (to the volume bounded by S) boundary values of the surface traction and displacement on S.

In like fashion, to obtain expressions for the b_n's, we apply Eq. (43) to the region bounded by S and S_-,

$$\oint_{S_-} ds[\mathbf{t}(\mathbf{u})\cdot\mathbf{v} - \mathbf{t}(\mathbf{v})\cdot\mathbf{u}] = \oint_S ds[\mathbf{t}(\mathbf{u})\cdot\mathbf{v} - \mathbf{t}(\mathbf{v})\cdot\mathbf{u}]. \qquad (62)$$

At the inner surface S_-, we are assured that Eq. (56) and

$$\mathbf{t}(u^r) = \sum_n b_n \mathbf{t}(\mathrm{Re}\,\psi_n^0) \tag{63}$$

are uniformly convergent. Evaluating Eq. (62) by utilizing the expressions in conjunction with first $\mathbf{v} = \mathrm{Re}\,\psi_n^0$ and then $\mathbf{v} = \psi_n^0$, we obtain

$$0 = \oint_s ds[\mathbf{t}_- \cdot \mathrm{Re}\,\psi_n^0 - \mathbf{t}(\mathrm{Re}\,\psi_n^0)\cdot \mathbf{u}_-], \tag{64}$$

$$-i\sum_{n'} O^0_{n'n} b_{n'} = \oint_s ds[\mathbf{t}_- \cdot \psi_n^0 - \mathbf{t}(\psi_n^0)\cdot \mathbf{u}_-], \tag{65}$$

where $\mathbf{t}_-$ and $\mathbf{u}_-$ denote the interior boundary values of the traction and the displacement.

Equations (60) and (61) and Eqs. (64) and (65) constitute the mathematical representation of Huygen's principle exterior to, and interior to the scattering region, respectively. They differ from their spherical counterparts by the nondiagonal nature of the $O_{nn'}$, which in turn, is simply a reflection of the mode coupling encountered in all finite geometries other than the sphere.

V. T MATRIX FOR AN ELASTIC INCLUSION IN AN ELASTIC SOLID

Consider the scattering from an elastic inclusion in an elastic solid. We assume the surfaces of the scatterer and host medium to be perfectly welded, i.e.,

$$\mathbf{t}_+ = \mathbf{t}_- \quad \text{on } s, \tag{66}$$

$$\mathbf{u}_+ = \mathbf{u}_- \quad \text{on } s. \tag{67}$$

Making these replacements, the equations for the incident and scattered wave coefficients become

$$-i\sum_{n'} O_{n'n} a_{n'} = \oint_s ds[\mathbf{t}_- \cdot \psi_n - \mathbf{t}(\psi_n)\cdot \mathbf{u}_-], \tag{68}$$

$$i\sum_{n'} O_{n'n} f_{n'} = \oint_s ds[\mathbf{t}_- \cdot \mathrm{Re}\,\psi_n - \mathbf{t}(\mathrm{Re}\,\psi_n)\cdot \mathbf{u}_-]. \tag{69}$$

The unknown boundary values of the surface traction and displacement on S we expand according to

$$\mathbf{u}_- = \sum_n b_n\, \mathrm{Re}\,\psi_n^0, \tag{70}$$

$$\mathbf{t}_- = \sum_n b_n \mathbf{t}(\mathrm{Re}\,\psi_n^0). \tag{71}$$

In Appendix B, we show that Huygen's principle is a necessary and sufficient condition for the uniform convergence of these series on and within S, given that the expansion for $\mathbf{u}$ [Eq. (56)] is uniformly convergent in some region interior to S.

Inserting Eqs. (70) and (71) into Eqs. (68) and (69) and defining

$$R_{nn'} = \oint_s ds[\mathbf{t}(\mathrm{Re}\,\psi_{n'}^0)\cdot \psi_n - \mathbf{t}(\psi_n)\cdot \mathrm{Re}\,\psi_{n'}^0], \tag{72}$$

$$\hat{R}_{nn'} = \oint_s ds[\mathbf{t}(\mathrm{Re}\,\psi_{n'}^0)\cdot \mathrm{Re}\,\psi_n - \mathbf{t}(\mathrm{Re}\,\psi_n)\cdot \mathrm{Re}\,\psi_{n'}^0], \tag{73}$$

we obtain

$$-i\sum_{n'n} O_{n'n} a_{n'} = \sum_{n'} R_{nn'} b_{n'}, \tag{74}$$

$$i\sum_{n'n} O_{n'n} f_{n'} = \sum_{n'} \hat{R}_{nn'} b_{n'}. \tag{75}$$

Note that both ψ_n and $\psi_{n'}$ carry four indices, i.e., $n = (\tau, \sigma, m, l)$, so that $O_{nn'}$, $R_{nn'}$, and $\hat{R}_{nn'}$ are square matrices and hence we can write

$$f_n = \sum_{n'} T_{nn'} a_{n'}, \tag{76}$$

where $T_{nn'}$ is, by definition, the T matrix. In matrix notation, T is given by

$$T = -\hat{Q}Q^{-1}, \tag{77}$$

$$Q = O^{-1}R, \tag{78}$$

$$\hat{Q} = O^{-1}\hat{R}, \tag{79}$$

where we have used the fact that O is symmetric.

From Eqs. (77)–(79), we see that, as compared to the spherical result, we must generate and invert one additional matrix, O. However, this presents no special difficulty, as this real, symmetric matrix is trivial to calculate and it is not ill-conditioned.

VI. T MATRIX FOR A FLUID-LOADED ELASTIC SOLID

Consider next the scattering from an elastic body immersed in an inviscid fluid. The boundary conditions on the surface between the scatterer and fluid are

$$\hat{n}\cdot \mathbf{u}_+ = \hat{n}\cdot \mathbf{u}_-, \tag{80}$$

$$\hat{n}\cdot \mathbf{t}_+ = \hat{n}\cdot \mathbf{t}_-, \tag{81}$$

$$\hat{n}\times \mathbf{t}_- = 0, \tag{82}$$

where $\hat{n}$ is a unit vector pointing out of the volume enclosed by S. In an inviscid fluid ($\mu \to 0$), there are no shear degrees of freedom and O becomes diagonal [see Eq. (47)],

$$O_{nn'} = (\lambda / k)\delta_{nn'}. \tag{83}$$

Likewise, the stress tensor simplifies considerably, i.e.,

$$\mathbf{t}(u) \to \lambda \nabla\cdot \mathbf{u}\hat{n}, \tag{84}$$

and we can write

$$\mathbf{t}_+ \cdot \psi_n - \mathbf{t}(\psi_n)\cdot u_+ = \hat{n}\cdot \mathbf{t}_+ \hat{n}\cdot \psi_n - \lambda \nabla\cdot \psi_n \hat{n}\cdot \mathbf{u}_+, \tag{85}$$

etc. Thus, in the present case, Eqs. (60) and (61) reduce to

$$-ia_n = \frac{k}{\lambda}\oint_s ds[\hat{n}\cdot \mathbf{t}_- \hat{n}\cdot \psi_n - \lambda \nabla\cdot \psi_n \hat{n}\cdot \mathbf{u}_-], \tag{86}$$

$$if_n = \frac{k}{\lambda}\oint_s ds[\hat{n}\cdot \mathbf{t}_- \hat{n}\cdot \mathrm{Re}\,\psi_n - \lambda \nabla\cdot \mathrm{Re}\,\psi_n \hat{n}\cdot \mathbf{u}_-], \tag{87}$$

where we have incorporated the boundary conditions, Eqs. (80) and (81). We now use the expansion equations (70) and (71) for the unknowns $\mathbf{u}_-$ and (in this case) $\hat{n}\cdot \mathbf{t}_-$ obtaining

$$-ia_n = \sum_{n'} R_{nn'} b_{n'}, \tag{88}$$

$$if_n = \sum_{n'} \hat{R}_{nn'} b_{n'}, \tag{89}$$

with

$$R_{nn'} = \frac{k}{\lambda}\oint_s ds[\hat{n}\cdot \mathbf{t}(\mathrm{Re}\,\psi_{n'}^0)\hat{n}\cdot \psi_n - \lambda \nabla\cdot \psi_n \hat{n}\cdot \mathrm{Re}\,\psi_{n'}^0], \tag{90}$$

$$\hat{R}_{nn'} = R_{nn'}(\psi_n \to \mathrm{Re}\,\psi_n). \tag{91}$$

Here, however, R is not a square matrix and we cannot straightforwardly invert Eq. (88). Bostrom[8] has shown the correct way to proceed for the spherical-coordinate based T matrix, and we follow his treatment here. Utilizing Eqs. (80) and (82), Eq. (64) may be rewritten as

$$0 = \oint_s ds\,[\hat{n}\cdot\mathbf{t}_- \hat{n}\cdot\mathrm{Re}\,\psi_n^0 - \hat{n}\cdot\mathbf{t}(\mathrm{Re}\,\psi_n^0)\hat{n}\cdot\mathbf{u}_+ + \mathbf{t}(\mathrm{Re}\,\psi_n^0)\cdot\hat{n}\times(\hat{n}\times\mathbf{u}_-)]. \tag{92}$$

Expanding the normal component of the surface field $\hat{n}\cdot\mathbf{u}_+$ with

$$\hat{n}\cdot\mathbf{u}_+ = \sum_n c_n \hat{n}\cdot\mathrm{Re}\,\psi_n \tag{93}$$

and $\mathbf{u}_-$ and $\mathbf{t}_-$ with Eqs. (70) and (71), we obtain

$$0 = -\sum_{n'} M_{nn'} c_{n'} + \sum_{n'} P_{nn'} b_n, \tag{94}$$

$$M_{nn'} + \oint_s ds\,\hat{n}\cdot\mathbf{t}(\mathrm{Re}\,\psi_n^0)\hat{n}\cdot\mathrm{Re}\,\psi_{n'}, \tag{95}$$

$$P_{nn'} = \oint_s ds\,[\hat{n}\cdot\mathbf{t}(\mathrm{Re}\,\psi_{n'}^0)\hat{n}\cdot\mathrm{Re}\,\psi_n^0 + \mathbf{t}(\mathrm{Re}\,\psi_n^0)\cdot\hat{n}\times(\hat{n}\times\mathrm{Re}\,\psi_{n'}^0)]. \tag{96}$$

The point behind this selective use of the boundary conditions is that it may not be possible to differentiate Eq. (93) in the normal direction. Combining Eqs. (88), (89), and (94), in matrix notation we have

$$-ia = RP^{-1}Mc, \tag{97}$$

$$if = \hat{R}P^{-1}Mc, \tag{98}$$

where a and c are column matrices, and R, P^{-1}, and M are $n\times 3n$, $3n\times 3n$, and $3n\times n$, respectively. Thus, in complete analogy with the spherical case, we have

$$T = -\hat{Q}Q^{-1}, \tag{99}$$

$$Q = RP^{-1}M, \tag{100}$$

$$\hat{Q} = Q(R\to\hat{R}). \tag{101}$$

VII. ENERGY CONSERVATION AND TIME REVERSAL INVARIANCE

In spherical coordinates, the dual requirements of energy conservation and time reversal invariance lead to a transition matrix which is both unitary and symmetric. As a practical matter, these two properties greatly ease the numerical effort involved in a typical calculation, leading both to improved numerical convergence for the T-matrix procedure and to a reduction of the number of terms which must be calculated for any given truncation of the T matrix. In the following, we determine the consequences of these invariance principles for the spheroidal coordinate based T matrix.

Consider the requirement of energy conservation first. For plane waves, the energy-flux vector averaged over one period of oscillation is[3]

$$\langle P_\alpha\rangle = \tfrac{1}{2}\omega\,\mathrm{Im}[\Upsilon_{\alpha\beta}(u)u^{\beta *}], \tag{102}$$

where

$$\mathbf{u} = \sum_n (a_n \mathrm{Re}\,\psi_n + f_n\psi_n) \tag{103}$$

is the total field. From Eq. (40) we see that

$$\nabla_\alpha\langle P^\alpha\rangle = 0, \tag{104}$$

and

$$\oint_s ds\,\hat{n}\cdot\langle\mathbf{P}\rangle = \mathrm{const}, \tag{105}$$

where S is any closed surface, then follows from the divergence theorem. Equation (105) is a statement of the conservation of energy; for plane waves, the average total energy flux through any closed surface is constant. In the present circumstance, that constant will be zero, as long as the surface does not enclose any sources or sinks of energy. In which case, we have

$$\sum_{n,n'}\Big(a_n a_{n'}^*\oint_s ds\,[\mathbf{t}(\mathrm{Re}\,\psi_n)\cdot\mathrm{Re}\,\psi_{n'}^* - \mathbf{t}(\mathrm{Re}\,\psi_{n'}^*)\cdot\mathrm{Re}\,\psi_n] + a_n f_{n'}^*\oint_s ds\,[\mathbf{t}(\mathrm{Re}\,\psi_n)\cdot\psi_{n'}^* - \mathbf{t}(\psi_{n'}^*)\cdot\mathrm{Re}\,\psi_n] + f_n a_{n'}^*\oint_s ds\,[\mathbf{t}(\psi_n)\cdot\mathrm{Re}\,\psi_{n'}^* - \mathbf{t}(\mathrm{Re}\,\psi_{n'}^*)\cdot\psi_n] + f_n f_{n'}^*\oint_s ds\,[\mathbf{t}(\psi_n)\cdot\psi_{n'}^* - \mathbf{t}(\psi_{n'}^*)\cdot\psi_n]\Big) = 0. \tag{106}$$

Noting that

$$\psi_n^* = 2\,\mathrm{Re}\,\psi_n - \psi_n, \tag{107}$$

and using the arbitrary nature of the a_n, we obtain

$$T^+O + OT = -2T^+OT. \tag{108}$$

This is the "unitarity" condition for the spheroidal coordinate-based transition matrix. In terms of the scattering matrix

$$S = 1 + 2T \tag{109}$$

which relates the outgoing waves to incoming waves, Eq. (108) becomes

$$S^+OS = O. \tag{110}$$

Thus, strictly speaking, the S matrix in spheroidal coordinates is unitary only when O is proportional to the unit matrix (i.e., when the host medium is an inviscid fluid).

Consider next the consequences of time-reversal invariance. To obtain these constraints, it is convenient to rewrite the total field in terms of outgoing $\{\psi_n\}$ and incoming $\{\psi_n^*\}$ fields, i.e.,

$$\mathbf{u} = \sum_n (a_n'\psi_n^* + f_n'\psi_n), \tag{111}$$

where, by definition,

$$f_n' = \sum_{n'} S_{nn'} a_{n'}'. \tag{112}$$

Time reversal invariance requires that the time-reversed field

$$\mathbf{u}^* = \sum_n (f_n'^*\mathbf{u}_n^* + a_n'^*\psi_n) \tag{113}$$

also be a solution to the Helmholtz equation, and that

$$a_n'^{*} = \sum_{n'} S_{nn'} f_n'^{*}. \quad (114)$$

Thus

$$f_n' = \sum_{n'} S_{nn'} \left(\sum_{n''} S^{*}_{n'n''} f'_{n''} \right), \quad (115)$$

i.e.,

$$SS^{*} = 1. \quad (116)$$

Equations (110) and (116) are oddly incongruent relations. While it is possible to define an S matrix, S', which satisfies the unitarity condition

$$S'S'^{+} = 1, \quad (117)$$

it is unlikely that S' will also satisfy Eq. (116).[16] However, we note that for an inviscid fluid, 0 becomes a multiple of the unit matrix and the S matrix is both unitary and symmetric.

VIII. SPHEROIDAL INCLUSIONS

To further illustrate the differences between the spherical- and spheroidal-coordinate based transition matrices, we calculate the T matrices for the scattering from sound-soft and sound-hard constant-ξ (spheroidal) inclusions.

Consider first the scattering from a sound-soft spheroidal inclusion embedded in an elastic medium. Using the vanishing of the surface traction

$$t_+ = t_- = 0, \quad (118)$$

and expanding the surface displacement in regular basis functions

$$\mathbf{u}_+ = \sum_n c_n \, \mathrm{Re}\, \psi_n, \quad (119)$$

Eq. (60) becomes

$$-i \sum_{n'} O_{n'n} a_{n'} = -\sum_{n'} c_{n'} \oint_s ds\, \mathbf{t}(\psi_n)\cdot \mathrm{Re}\, \psi_{n'}, \quad (120)$$

or

$$a_n = -i \sum_{n',n''} O^{-1}_{nn'} P_{n'n''} C_{n''}, \quad (121)$$

$$P_{n'n''} = \oint_s ds\, \mathbf{t}(\psi_{n'})\cdot \mathrm{Re}\, \psi_{n''}. \quad (122)$$

In a similar fashion, Eq. (61) yields

$$f_n = i \sum_{n',n''} O^{-1}_{nn'} \hat{P}_{n'n''} c_{n''}, \quad (123)$$

$$\hat{P}_{n'n''} = \oint_s ds\, \mathbf{t}(\mathrm{Re}\, \psi_{n'})\cdot \mathrm{Re}\, \psi_{n''}. \quad (124)$$

Thus, the T matrix

$$T = -(O^{-1}\hat{P})(O^{-1}P)^{-1} \quad (125)$$

couples incident and outgoing partial wave coefficients over all l consistent with parity considerations (i.e., there is no coupling between even and odd l).

Likewise, for a rigid spheroid in an elastic medium,

$$\hat{n}\cdot\mathbf{u}_+ = \hat{n}\cdot\mathbf{u}_- = 0, \quad (126)$$

and we expand the surface traction according to

$$\mathbf{t}_+ = \sum_n c_n \mathbf{t}(\mathrm{Re}\, \psi_n), \quad (127)$$

obtaining

$$T = -(O^{-1}\hat{R})(O^{-1}R)^{-1}, \quad (128)$$

$$R_{nn'} = \oint_s ds\, \mathbf{t}(\mathrm{Re}\, \psi_{n'})\cdot \psi_n, \quad (129)$$

$$\hat{R}_{nn'} = \oint_s ds\, \mathbf{t}(\mathrm{Re}\, \psi_{n'})\cdot \mathrm{Re}\, \psi_n, \quad (130)$$

which once again couples all l consistent with parity considerations.

This coupling, anticipated in Sec. II, is a consequence of the nonseparable (in the sense of Morse and Feshbach[14]) nature of the vector Helmholtz equation in spheroidal coordinates.

In the limit that the shear modulus of the host medium vanishes, these results simplify considerably. Using

$$O_{nn'} \xrightarrow{\mu\to 0} (\lambda/k)\delta_{nn'} \quad (131)$$

and

$$\oint_s ds\, \mathbf{t}(\psi_{\tau n})\cdot \mathrm{Re}\, \psi_{\tau' n'}$$

$$\xrightarrow{\mu\to 0} \lambda \oint_s ds\, \nabla\cdot\psi_{3n} \hat{n}\cdot \mathrm{Re}\, \psi_{3n'}$$

$$= -\lambda f(\xi^2 - 1) he_{ml}(h,\xi) \frac{\partial}{\partial \xi} je_{ml}(h,\xi), \quad (132)$$

(recall that the spheroidal harmonics of the wavenumber are orthonormal) etc., we find that the transition matrices

$$(\text{sound-soft}) T_{nn'} - \frac{je_{ml}(h,\xi)}{he_{ml}(h,\xi)} \delta_{nn'}, \quad (133)$$

$$(\text{sound-hard}) T_{nn'} = -\frac{\partial je_{ml}(h,\xi)/\partial\xi}{\partial he_{ml}(h,\xi)/\partial\xi} \delta_{nn'}, \quad (134)$$

both become diagonal in this limit.

IX. DISCUSSION

In the foregoing, we have derived a transition matrix formalism for acoustic and elastic wave scattering in prolate spheroidal coordinates. The two problems which remain are (1) the completeness and domain of convergence of the expansions in $\{\psi_n\}$ and $\{\mathrm{Re}\, \psi_n\}$ must be rigorously established, and (2) the practicality and soundness of the approach must be demonstrated.

With regard to (1), the formal demonstration of convergence and completeness is likely just a straightforward extension of existing work on spherical expansions and this line is presently being pursued. With regard to (2) however, we note that the generation of the spheroidal functions is a nontrivial matter,[17] due in part to the lack of recursion relations among these functions. As a first practical calculation, in a subsequent paper we will apply this formalism to the acoustic scattering from an elastic prolate spheroid immersed in water. This should provide a good proving ground for determining the efficacy of the approach for large aspect ratio scatterers.

The present work can be trivially extended to electromagnetic scattering by deleting all reference to the irrotational degree of freedom (i.e., $\psi_{3\sigma ml}$). Further, as much of our formalism has been written covariantly, the extension to other coordinate systems in which the scalar Helmholtz equation is separable is straightforward.

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APPENDIX A

In this appendix, we briefly review the solutions of the scalar Helmholtz equation in spheroidal coordinates and give explicit expressions for the vector basis functions defined in this paper.[18] Let

$$(x^1,x^2,x^3)=(x,y,z),$$
$$(y^1,y^2,y^3)=(\xi,\eta,\phi). \tag{A1}$$

Then the convariant metric tensor is given by

$$g_{11}=f^2[(\xi^2-\eta^2)/(\xi^2-1)], \tag{A2}$$
$$g_{22}=f^2[(\xi^2-\eta^2)/(1-\eta^2)], \tag{A3}$$
$$g_{33}=f^2(\xi^2-1)(1-\eta^2), \tag{A4}$$
$$g_{ij}=0,\quad i\neq j, \tag{A5}$$

and the contravariant tensor (no sum over i), by

$$g^{ii}=1/g_{ii};\quad g^{ij}=0,\quad i\neq j. \tag{A6}$$

We note for future reference that the covariant expressions for the gradient, divergence, and curl in this coordinate system are

$$\text{grad}\,\psi=\left(\frac{\partial\psi}{\partial\xi}\right)\mathbf{e}^1+\left(\frac{\partial\psi}{\partial\eta}\right)\mathbf{e}^2+\left(\frac{\partial\psi}{\partial\phi}\right)\mathbf{e}^3, \tag{A7}$$

$$\text{div}\,\mathbf{A}=\frac{1}{f^2(\xi^2-\eta^2)}\times\left(\frac{\partial}{\partial\xi}[(\xi^2-1)A_1]+\frac{\partial}{\partial\eta}[(1-\eta^2)A_2]+\frac{(\xi^2-\eta^2)}{(\xi^2-1)(1-\eta^2)}\frac{\partial}{\partial\phi}A_3\right), \tag{A8}$$

$$\text{curl}\,\mathbf{A}=\frac{1}{f^3(\xi^2-\eta^2)}\times\left[\left(\frac{\partial}{\partial\eta}A_3-\frac{\partial}{\partial\phi}A_2\right)\mathbf{e}_1+\left(\frac{\partial}{\partial\phi}A_1-\frac{\partial}{\partial\xi}A_3\right)\mathbf{e}_3+\left(\frac{\partial}{\partial\xi}A_2-\frac{\partial}{\partial\eta}A_1\right)\mathbf{e}_3\right]. \tag{A9}$$

Separating the time-independent Helmholtz equation

$$(\nabla^2+k^2)\psi=0 \tag{A10}$$

with

$$\psi=R_{ml}(h,\xi)S_{ml}(h,\eta)\Phi_m(\phi) \tag{A11}$$

leads to the three ordinary differential equations

$$\left(\frac{d}{d\xi}(\xi^2-1)\frac{d}{d\xi}-\frac{m^2}{\xi^2-1}+h^2\xi^2\right)R_{ml}(h,\xi)=A_{ml}(h)R_{ml}(h,\xi), \tag{A12}$$

$$\left(\frac{d}{d\eta}(1-\eta^2)\frac{d}{d\eta}-\frac{m^2}{1-\eta^2}-h^2\eta^2\right)S_{ml}(h,\eta)=-A_{ml}(h)S_{ml}(h,\eta), \tag{A13}$$

$$\frac{d^2}{d\phi^2}\Phi_m(\phi)=-m^2\Phi_m(\phi), \tag{A14}$$

where $h=fk$ and where $A_{ml}(h)$ denotes the eigenvalue. Notice that Eqs. (A12) and (A13) are identical, except for the range of the arguments, the first involving $(1,\infty)$ and the second $(-1,+1)$. These two equations possess regular singular points at ± 1 and an irregular singular point at infinity.

The regular (je_{ml}) and irregular (ne_{ml}) radial functions are normalized according to

$$je_{ml}(h,\xi)\overset{\xi\to\infty}{\longrightarrow}(1/h\xi)\cos[h\xi-(l+1)\pi/2], \tag{A15}$$

$$ne_{ml}(h,\xi)\overset{\xi\to\infty}{\longrightarrow}(1/h\xi)\sin[h\xi-(l+1)\pi/2], \tag{A16}$$

and their Wronskian is given by

$$W(je_{ml},ne_{ml})=1/h(\xi^2-1), \tag{A17}$$

for $h\neq 0$. With he_{ml} we denote the spheroidal equivalent of the Hankel function

$$he_{ml}(h,\xi)=je_{ml}(h,\xi)+ine_{ml}(h,\xi), \tag{A18}$$

which corresponds to outgoing spheroidal waves.

The regular angular functions, normalized to

$$S_{ml}(h,\eta)\overset{\eta\to 1}{\longrightarrow}P_l^m(\eta) \tag{A19}$$

(P_l^m is the associated Lengendre function) satisfy the orthogonality relations

$$\int_{-1}^{1}d\eta\,S_{ml}(h,\eta)S_{ml'}(h,\eta)=A_{ml}(h)\delta_{ll'}. \tag{A20}$$

We do not concern ourselves with the irregular angular functions here.

The solution to the ϕ equation is

$$\Phi_m(\phi)=\begin{cases}\cos m\phi\\ \sin m\phi.\end{cases} \tag{A21}$$

For future reference, we define the "spheroidal harmonic"

$$S_{\sigma ml}(h,\eta,\phi)=\left(\frac{\epsilon_m}{2\pi}\frac{1}{A_{ml}}(h)\right)^{1/2}\times S_{ml}(h,\eta)\begin{cases}\cos m\phi, & \sigma=e\\ \sin m\phi, & \sigma=0,\end{cases} \tag{A22}$$

normalized such that

$$\int_0^{2\pi} d\phi \int_{-1}^{1} d\eta\, S_{\sigma ml}(h,\eta,\phi) S_{\sigma' m' l'}(h,\eta,\phi) = \delta_{\sigma\sigma'}\delta_{mm'}\delta_{ll'}. \tag{A23}$$

We note that the spheroidal functions defined above do not possess recursion relations of the kind which exist, for example, among three contiguous Legendre polynomials or spherical Bessel functions. This is an added complication where both formal manipulations and numerical calculations are concerned.

Consider next the vector solutions. Transforming **r** to prolate spheroidal coordinates with

$$a_i(\xi,\eta,\phi) = \frac{\partial x^j}{\partial y_i} b_j(x,y,z),$$
$$b_j(x,y,z) = x_j, \tag{A24}$$

gives

$$a_1 = f^2\xi, \quad a_2 = f^2\eta, \quad a_3 = 0. \tag{A25}$$

Letting $\chi_n = R_{ml}S_{\sigma ml}$ denote a solution to Eq. (A11) with $k = k_T$, i.e., $n = (\sigma,m,l)$, we find

$$A_{ml}^{1/2}\psi_{1n} = -\frac{f\eta}{\xi^2-1}\frac{\partial}{\partial\phi}\chi_n \mathbf{e}^1 + \frac{f\xi}{1-\eta^2}\frac{\partial}{\partial\phi}\chi_n\mathbf{e}^2 + \frac{f(\xi^2-1)(1-\eta^2)}{\xi^2-\eta^2}\left(\eta\frac{\partial}{\partial\xi}\chi_n - \xi\frac{\partial}{\partial\eta}\chi_n\right)\mathbf{e}^3 \tag{A26}$$

and

$$\begin{aligned} k_T A_{ml}^{1/2}\psi_{2n} &= \Big[\frac{\partial}{\partial\xi}\chi_n + \xi h_T^2\chi_n + \frac{1}{\xi^2-\eta^2}\Big((\xi^2-1)\frac{\partial}{\partial\xi}\chi_n + \xi(A_{ml} - h_T^2\xi^2)\chi_n\Big) \\ &\quad - \frac{2\xi^2(\xi^2-1)}{(\xi^2-\eta^2)^2}\frac{\partial}{\partial\xi}\chi_n + \frac{\eta(1-\eta^2)}{\xi^2-\eta^2}\frac{\partial^2}{\partial\xi\partial\eta}\chi_n - \frac{2\xi\eta(1-\eta^2)}{(\xi^2-\eta^2)^2}\frac{\partial}{\partial\eta}\chi_n + \frac{m^2\xi}{(\xi^2-1)(\xi^2-\eta^2)}\chi_n\Big]\mathbf{e}^1 \\ &\quad + \Big[\frac{\eta}{\xi^2-\eta^2}(h_T^2\xi^2 - A_{ml})\chi_n + \frac{2\eta\xi(\xi^2-1)}{(\xi^2-\eta^2)^2}\frac{\partial}{\partial\xi}\chi_n + \frac{1}{\xi^2-\eta^2}\Big((3\xi^2-1)\frac{\partial}{\partial\eta}\chi_n + \xi(\xi^2-1)\frac{\partial^2}{\partial\eta\partial\xi}\chi_n\Big) \\ &\quad - \frac{2\xi^2(\xi^2-1)}{(\xi^2-\eta^2)^2}\frac{\partial}{\partial\eta}\chi_n + \frac{m^2\eta}{(1-\eta^2)(\xi^2-\eta^2)}\chi_n\Big]\mathbf{e}^2 + \Big(\frac{\partial}{\partial\phi}\chi_n + \frac{\xi(\xi^2-1)}{\xi^2-\eta^2}\frac{\partial^2}{\partial\xi\partial\phi}\chi_n + \frac{\eta(1-\eta^2)}{\xi^2-\eta^2}\frac{\partial^2}{\partial\eta\partial\phi}\chi_n\Big)\mathbf{e}^3 \end{aligned} \tag{A27}$$

[notice $A_{ml} = A_{ml}(h_T)$] and for the irrotational field, using Φ_n to denote a solution to Eq. (A11) with $k = k_L$, we obtain

$$k\psi_{3n} = \frac{\partial}{\partial\xi}\Phi_n\mathbf{e}^1 + \frac{\partial}{\partial\eta}\Phi_n\mathbf{e}^2 + \frac{\partial}{\partial\phi}\Phi_n\mathbf{e}^3. \tag{A28}$$

APPENDIX B

The implications of Huygen's principle for the convergence and differentiability of the interior expansion, Eq. (56), have been consider by Waterman[1,3] and Pao[7] for the spherical-coordinate based T matrix. The additional complications in the present case are minor. First we state the theorem.

Theorem: Huygen's principle is a necessary and sufficient condition for the convergence and differentiability of the expansion of the interior fields

$$\mathbf{u} = \sum_n b_n \operatorname{Re}\psi_n^0 \tag{B1}$$

throughout the interior of the scatterer and on the interior of the surface of the scatterer.

This theorem is proved as follows. First we assume that $\mathbf{u}_-$ and its covariant derivatives possess different expansions, i.e.,

$$\mathbf{u}_- = \sum_n c_n \operatorname{Re}\psi_n^0, \tag{B2}$$

$$\mathbf{t}_- = \sum_n d_n\, \mathbf{t}(\operatorname{Re}\psi_n^0). \tag{B3}$$

Substituting these expansions in Eqs. (64) and (65), we obtain

$$0 = \sum_n (\hat{R}_{nn'}d_{n'} - \hat{R}_{n'n}c_{n'}), \tag{B4}$$

$$-i\sum_{n'} O_{n'n}b_{n'} = \sum_{n'}(R_{nn'}d_{n'} - R_{n'n}c_{n'}), \tag{B5}$$

where

$$R_{nn'} = \oint_s ds\, \mathbf{t}(\operatorname{Re}\psi_n^0)\cdot\psi_n^0 \tag{B6}$$

and where $\hat{R}$ is obtained from R by replacing ψ_n^0 with $\operatorname{Re}\psi_n^0$. It follows from our lemma, specifically Eq. (44), that $\hat{R}$ must be symmetric. Thus

$$\sum_{n'}\hat{R}_{nn'}(d_{n'} - c_{n'}) = 0 \tag{B7}$$

and either

$$d_n = c_n \tag{B8}$$

or

$$\det(\hat{R}) = 0. \tag{B9}$$

As noted by Waterman,[3] Eq. (B9) constitutes the secular equation for the interior resonant cavity modes of a rigid scatterer (see Sec. VIII). In general the determinant of $\hat{R}$ is nonvanishing and it follows that the expansion for $\mathbf{u}_-$ must be differentiable. That this expansion [Eq. (B2)] is identical to Eq. (B1) then follows from Eqs. (46) and (B5).

This establishes the necessary condition for the theorem. That the principle is sufficient follows from assuming Eq. (B1) to be uniformly convergent and then directly substituting into Eq. (65).

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[15]In Appendix B, we demonstrate formally that if Eq. (56) converges in any domain (interior to the scatterer), it must converge both throughout the interior of the scatterer and on the surface of that scatterer, as approached from its interior. Further, this series must also be differentiable in that domain.
[16]This depends upon the properties of O.
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