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On connections between boundary integral equations and *T*-matrix methods

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Abstract

Acoustic scattering by three-dimensional obstacles is considered, using boundary integral equations, null-field equations and the *T*-matrix. Connections between these techniques are explored. It is shown that solving a boundary integral equation by a particular Petrov–Galerkin method leads to the same algebraic system as obtained from the null-field equations. It is also emphasised that the *T*-matrix can be constructed by solving boundary integral equations rather than by solving the null-field equations.

Keywords: Acoustics; Boundary integral equations; Multiple scattering; Null-field equations; *T*-Matrix

1. Introduction

Readers of this journal will be familiar with boundary integral equation (BIE) methods, and their use in the calculation of acoustic scattering by bounded obstacles. Perhaps less familiar are null-field and *T*-matrix methods, even though these methods have a similar scope. This paper is concerned with some connections between these methods.

For acoustic (and electromagnetic) scattering, the theory of BIEs has been given in detail in the well-known book by Colton and Kress [1]. There are also several books in which the use of boundary elements is described in the context of acoustics; examples are Refs. [2–5].

The null-field and *T*-matrix methods are also used widely for obtaining numerical solutions to various radiation and scattering problems. These methods are related, as we will see. They were first devised by Waterman in 1965 for electromagnetic scattering problems [6]. Later, they were developed for treating problems in acoustics [7–11], elastodynamics [12,13] and hydrodynamics [14,15]. For surveys up to 1979, see Ref. [16]. Many subsequent applications are reviewed in Refs. [17,18] and in chapter 6 of Ref. [19]. The real power of *T*-matrix methods comes when they are used to solve multiple-scattering problems; the key paper is Ref. [20]. Thus, one can construct the *T*-matrix for a group of scatterers from a knowledge of the *T*-matrix for each constituent scatterer in isolation.

We know that BIEs are very effective for solving scattering problems, numerically. They are less effective when used for multiple-scattering problems (where there are several disjoint scatterers), especially when one is interested in the effects of changes in the geometrical configuration of the scatterers.

We also know that the numerical performance of the null-field method (for one scatterer) is strongly dependent on the shape of the scatterer: it tends to degrade as the shape deviates from a sphere.

The null-field method is often used as a way of computing the *T*-matrix for a single scatterer. This matrix satisfies $\mathbf{c} = \mathbf{T}\mathbf{d}$, where $\mathbf{d}$ is a vector of coefficients in an expansion of the incident field in terms of regular spherical wave-functions, and $\mathbf{c}$ is a vector of coefficients in a similar expansion of the scattered field in terms of outgoing spherical wave-functions (see Section 4.5). However, it is important to observe that we are not obliged to use the null-field method.

We can compute the *T*-matrix by first solving a BIE and then calculating the scattered field. This is explained in detail in Section 5.3. The virtue of this approach is that we can use the good numerical properties of BIE methods combined with the useful properties of the *T*-matrix.

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In particular, the T-matrix is a very useful ingredient when one wants to solve multiple-scattering problems.

We limit our discussion to acoustic scattering by a three-dimensional sound-hard obstacle B with a smooth boundary S; the unbounded connected exterior is denoted by B_e. Thus, we consider the following problem.

Sound-hard scattering problem: Find a function $u_{sc}(P)$ for $P \in B_e$, where

$$(\nabla^2 + k^2)u_{sc} = 0 \qquad \text{in } B_e, \tag{1}$$

u_{sc} satisfies the radiation condition at infinity, (2)

and the boundary condition

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } S, \tag{3}$$

where the total field u is defined by

$$u = u_{sc} + u_{in}, \tag{4}$$

and u_{in} is the given incident field. We assume that u_{in} is a regular solution of the Helmholtz equation everywhere, except perhaps at some isolated points in B_e.

The paper is partly in the nature of a review. Thus, we begin with the standard Helmholtz–Sommerfeld formulae in Section 2. Direct BIEs are discussed briefly in Section 3. The null-field and T-matrix methods are developed in Section 4. Three connections between these methods and BIEs are made in Section 5. Thus, connections with modified fundamental solutions and exact Green's functions are reviewed in Section 5.1. In Section 5.2, we show that solving a direct BIE using a particular Petrov–Galerkin method leads to exactly the same system of equations as one obtains from the null-field method. Finally, in Section 5.3, we show explicitly how to compute the elements of the T-matrix from the solution of BIEs. We believe that this should be the preferred approach, especially for obstacles of complicated shape.

2. Integral representations

In order to describe boundary-integral and T-matrix methods for solving the sound-hard scattering problem, we begin with some familiar integral formulae. These are obtained using Green's theorem and a fundamental solution. In three dimensions, the simplest fundamental solution is

$$G(P,Q) = -\exp(\mathrm{i}kR)/(2\pi R), \tag{5}$$

where $R = |\mathbf{r}_P - \mathbf{r}_Q|$ is the distance between the two points P and Q. For each fixed Q, G satisfies the Helmholtz equation with respect to P, for $P \neq Q$. G also satisfies the Sommerfeld radiation condition with respect to P, and is symmetric, $G(P,Q) = G(Q,P)$, so that the roles of P and Q can be interchanged. $G(P,Q)$ is also singular when $P = Q$. Other fundamental solutions can be used instead of G; see Section 5.1.

Let us apply Green's theorem in B_e to u_{sc} and G. Making use of the radiation condition, we obtain

$$\int_S \left\{ G(P,q)\frac{\partial u_{sc}}{\partial n_q} - u_{sc}(q)\frac{\partial}{\partial n_q}G(P,q) \right\} \mathrm{d}s_q = \begin{cases} 2u_{sc}(P), & P \in B_e, \\ 0, & P \in B. \end{cases}$$

Similarly, applying Green's theorem in B to u_{in} and G gives

$$\int_S \left\{ G(P,q)\frac{\partial u_{in}}{\partial n_q} - u_{in}(q)\frac{\partial}{\partial n_q}G(P,q) \right\} \mathrm{d}s_q = \begin{cases} 0, & P \in B_e, \\ -2u_{in}(P), & P \in B, \end{cases}$$

assuming that u_{in} satisfies the Helmholtz equation in B. Adding these equations, making use of Eqs. (3) and (4), gives the integral representation

$$2u_{sc}(P) = -\int_S u(q)\frac{\partial}{\partial n_q}G(P,q)\,\mathrm{d}s_q, \qquad P \in B_e, \tag{6}$$

and the interior integral relation

$$\int_S u(q)\frac{\partial}{\partial n_q}G(P,q)\,\mathrm{d}s_q = 2u_{in}(P), \qquad P \in B. \tag{7}$$

Eq. (6) shows that the field scattered by sound-hard obstacles can always be represented as a double-layer potential, with density $u(q)$.

Eq. (7) can be regarded as an equation for $u(q)$. It is always uniquely solvable and it holds for all $P \in B$. It has various names. For electromagnetic problems, it is called the 'Ewald–Oseen extinction theorem' because it 'expresses the *extinction* of the incident wave…at any point inside…by interference with…the dipole field' [21, p. 102]. Other terminologies are the 'extended boundary condition' [6] and the 'extended integral equation' [9].

3. Boundary integral equations

Letting $P \to p \in S$ in Eq. (6) or (7), we obtain

$$\mathscr{L}u = 2u_{in}, \tag{8}$$

where

$$(\mathscr{L}u)(p) = u(p) + \int_S u(q)\frac{\partial}{\partial n_q}G(p,q)\,\mathrm{d}s_q, \qquad p \in S. \tag{9}$$

Eq. (8) is a Fredholm integral equation of the second kind for the boundary values of the total field u. It is often known as the *Helmholtz integral equation*.

Alternatively, we can evaluate the normal derivative of Eq. (6) on S, giving

$$Nu = 2v_{in}, \tag{10}$$

where $v_{\rm in} \equiv \partial u_{\rm in}/\partial n$ and

$$(Nu)(p) = \frac{\partial}{\partial n_p}\int_S u(q)\frac{\partial}{\partial n_q}G(p,q)\,{\rm d}s_q, \qquad p \in S. \tag{11}$$

N is often called a *hypersingular operator*.

It turns out that Eq. (8) is uniquely solvable for u, for any $u_{\rm in}$, except when k^2 coincides with an eigenvalue of the corresponding interior Dirichlet problem; these are called the *irregular values* of k^2, or *irregular frequencies*. Similarly, Eq. (10) is uniquely solvable except when k^2 is an eigenvalue of the interior Neumann problem.

Irregular frequencies can always be eliminated. Indeed, there is an extensive literature on devising integral-equation methods that do not suffer from irregular frequencies; for reviews, see Ref. [1, Section 3.6] or Ref. [22]. One convenient method is the (direct) method of Burton and Miller [23]. Thus, if we add Eq. (8) to a multiple of Eq. (10), we obtain

$$\mathscr{L}_\eta u = 2(u_{\rm in} + {\rm i}\eta v_{\rm in}), \tag{12}$$

where η is a real, non-zero coupling parameter and $\mathscr{L}_\eta = \mathscr{L} + {\rm i}\eta N$. Eq. (12) is a hypersingular BIE for $u(q)$. It is always uniquely solvable [1, Theorem 3.43].

Note that the integral representation for $u_{\rm sc}(P)$ has not been modified: it remains as Eq. (6).

There are many papers in which the method of Burton and Miller has been implemented, numerically; examples are Refs. [24–28].

4. Null-field and *T*-matrix methods

4.1. The null-field equations

Let us seek the function $u(q)$ that satisfies the interior integral relation (7) for all $P \in B$. Waterman [6] noted that we only have to satisfy this equation for all P in an open region $B_- \subset B$; we shall assume that B_- is a ball. Explicitly, choose an origin $O \in B$. Let S_- be the inscribed sphere to S which is centred at O, so that B_- is the largest ball in B (centred at O). Similarly, let S_+ be the smallest escribed sphere to S, centred at O, so that S_+ encloses the obstacle. The region exterior to S_+ is $B_+ \subset B_{\rm e}$. These definitions are convenient because they enable us to use the bilinear expansion of G, namely

$$G(P,Q) = -2{\rm i}k\sum_{n,m}(-1)^m\hat{\psi}_n^m(\mathbf{r}_P)\psi_n^{-m}(\mathbf{r}_Q), \tag{13}$$

where $\mathbf{r}_P$ is the position vector of P with respect to O, $r_P = |\mathbf{r}_P| < |\mathbf{r}_Q| = r_Q$, and we have used the shorthand notation

$$\sum_{n,m} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}. \tag{14}$$

If P and Q are interchanged, we obtain

$$G(P,Q) = -2{\rm i}k\sum_{n,m}(-1)^m\hat{\psi}_n^m(\mathbf{r}_Q)\psi_n^{-m}(\mathbf{r}_P), \qquad r_Q < r_P. \tag{15}$$

The functions $\hat{\psi}_n^m$ and ψ_n^m in Eqs. (13) and (15) are separated solutions of the Helmholtz equation in spherical polar coordinates. Thus, with $\mathbf{r} = r\hat{\mathbf{r}}$,

$$\psi_n^m(\mathbf{r}) = h_n(kr)Y_n^m(\hat{\mathbf{r}}) \quad\text{and}\quad \hat{\psi}_n^m(\mathbf{r}) = j_n(kr)Y_n^m(\hat{\mathbf{r}}),$$

are outgoing and regular spherical wave-functions, respectively. j_n is a spherical Bessel function, $h_n \equiv h_n^{(1)}$ is a spherical Hankel function, and Y_n^m is a normalised spherical harmonic. Note that $\psi_n^m(\mathbf{r})$ satisfies the radiation condition and is singular at the origin, where $\mathbf{r} = \mathbf{0}$.

The formulas (13) and (15) are well known; see, for example, p. 352 of Ref. [29].

Suppose that P is restricted to being in B_- (where $r_P < r_q$). Then, using Eq. (13) in the interior integral relation (7), we obtain

$$u_{\rm in}(P) = -{\rm i}k\sum_{n,m}(-1)^m\hat{\psi}_n^m(\mathbf{r}_P)\int_S u(q)\frac{\partial}{\partial n_q}\psi_n^{-m}(\mathbf{r}_q)\,{\rm d}s_q, \tag{16}$$

$P \in B_-$. But, as $u_{\rm in}(P)$ is assumed to be a regular wave-function in B_-, there exist coefficients d_n^m such that

$$u_{\rm in}(P) = \sum_{n,m} d_n^m\hat{\psi}_n^m(\mathbf{r}_P), \qquad P \in B_-; \tag{17}$$

this is separation of variables inside a ball, using spherical polar coordinates. The coefficients d_n^m can be considered as known. Equating coefficients in Eqs. (16) and (17) gives

$$-{\rm i}k(-1)^m\int_S u(q)\frac{\partial}{\partial n_q}\psi_n^{-m}(\mathbf{r}_q)\,{\rm d}s_q = d_n^m, \tag{18}$$

$$n = 0, 1, 2, \ldots, \quad m = -n, \ldots, n.$$

These are the *null-field equations* for scattering by a sound-hard obstacle. They were first obtained by Waterman [8]. They are uniquely solvable for all real values of k^2.

4.2. The scattered field

Having found $u(q)$ on S, somehow, the scattered field $u_{\rm sc}(P)$ for $P \in B_{\rm e}$ is given by Eq. (6). If P is outside S_+, we can use Eq. (15) to give

$$u_{\rm sc}(P) = \sum_{n,m} c_n^m\psi_n^m(\mathbf{r}_P), \qquad P \in B_+, \tag{19}$$

where the coefficients c_n^m are given by

$$c_n^m = {\rm i}k(-1)^m\int_S u(q)\frac{\partial}{\partial n_q}\hat{\psi}_n^{-m}(\mathbf{r}_q)\,{\rm d}s_q. \tag{20}$$

The formula (19) merely states that an outgoing wave-function can be expanded as an infinite series of outgoing spherical wave-functions, outside S_+. The assumption that Eq. (19) can be used everywhere in $B_{\rm e}$, including in

the region between S_+ and S, is known as the *Rayleigh hypothesis* [30].

4.3. Solving the null-field equations

The most common method for solving Eq. (18) begins by choosing a set of expansion functions, $\{\phi_n^m(q)\}$, which is suitable for representing functions defined on S; thus, suppose that we can write

$$u(q) = \sum_{\nu,\mu} u_\nu^\mu \phi_\nu^\mu(q), \qquad q \in S, \tag{21}$$

where u_ν^μ are coefficients to be found and we have used Eq. (14). Note that we could re-order the summation so that there is a single summation index; this is often done in the literature.

Substituting Eq. (21) in Eq. (18) gives

$$\sum_{\nu,\mu} Q_{n\nu}^{m\mu} u_\nu^\mu = d_n^m, \qquad \begin{matrix} n = 0, 1, 2, \ldots \\ n = -n, \ldots, n, \end{matrix} \tag{22}$$

where

$$Q_{n\nu}^{m\mu} = -\mathrm{i}k(-1)^m \int_S \phi_\nu^\mu \frac{\partial}{\partial n_q} \psi_n^{-m}(\mathbf{r}_q)\, \mathrm{d}s_q. \tag{23}$$

Eq. (22) gives an infinite system of linear algebraic equations for u_ν^μ; truncating this system leads to a numerical method for solving the null-field equations.

Similarly, the scattered field is given in the region outside S_+ by Eq. (19) in terms of the coefficients c_n^m, defined by Eq. (20). Substituting Eq. (21) gives

$$c_n^m = -\sum_{\nu,\mu} \hat{Q}_{n\nu}^{m\mu} u_\nu^\mu, \tag{24}$$

where

$$\hat{Q}_{n\nu}^{m\mu} = -\mathrm{i}k(-1)^m \int_S \phi_\nu^\mu \frac{\partial}{\partial n_q} \hat{\psi}_n^{-m}(\mathbf{r}_q)\, \mathrm{d}s_q. \tag{25}$$

This formula should be compared with Eq. (23).

4.4. Numerical experience

To proceed further, we must choose a set $\{\phi_n^m\}$. In theory, we can choose any convenient basis. However, in practice, the choice may be crucial: we want the truncated system of equations to be well conditioned and to yield a good approximation to $u(q)$. Most authors have chosen wave-functions for $\phi_n^m(q)$; there are eight obvious choices, namely,

$$\psi_n^m, \quad \partial\psi_n^m/\partial n_q, \quad \hat{\psi}_n^m, \quad \partial\hat{\psi}_n^m/\partial n_q, \tag{26}$$

and their complex conjugates. Those involving ψ_n^m are known to give complete sets, whereas those involving $\hat{\psi}_n^m$ do not. For example, $\{\hat{\psi}_n^m\}$ is not complete whenever k^2 is an eigenvalue of the interior Dirichlet problem for B. Thus, the use of regular wave-functions reintroduces the difficulties at irregular frequencies.

In his 1965 paper on electromagnetic scattering, Waterman [6] used outgoing spherical vector wave-functions for ϕ_n^m; the scalar equivalent would be to use ψ_n^m. In his 1969 paper on acoustic scattering [8], he showed that it is better to use regular spherical wave-functions $\hat{\psi}_n^m$, because this choice leads to a symmetric Q-matrix if S is an ellipsoid. As he emphasised in his 1999 paper, this offers 'a tremendous advantage: in the [low-frequency] Rayleigh limit, for example, what are apparently large numerical values below the diagonal are replaced by their small counterparts above the diagonal. Note that this replacement must be carried out to avoid serious precision problems' [31, p. 2970].

To understand this difficulty, observe that, for fixed w,

$$h_n(w) \sim \left(\frac{2}{w}\right)^n \frac{\Gamma\left(n+\frac{1}{2}\right)}{\mathrm{i}w\sqrt{\pi}} \text{ and } j_n(w) \sim \left(\frac{w}{2}\right)^n \frac{\sqrt{\pi}}{2\Gamma\left(n+\frac{3}{2}\right)},$$

as $n \to \infty$, so that $h_n(w)$ grows rapidly with n whereas $j_n(w)$ decays rapidly. Thus, the matrix $Q_{n\nu}^{m\mu}$, defined by Eq. (23), seems to grow as $n \to \infty$. Also, Waterman's $Q_{n\nu}^{m\mu}$, defined by Eq. (23) with $\phi_n^m(q) = \hat{\psi}_n^m(\mathbf{r}_q)$, seems to grow as $n \to \infty$ but seems to decay as $\nu \to \infty$; however, the growth with n is illusory when S is an ellipsoid because we know then that $Q_{n\nu}^{m\mu} = Q_{\nu m}^{\mu m}$, and this fact can be used to construct a diagonally dominant Q-matrix. This explains why regular wave-functions should be used for ϕ_n^m for ellipsoids of any eccentricity.

If S is not an ellipsoid, other methods may help to alleviate the rapid growth of $h_n(kr)$. One way is to extract a factor that grows with n, writing

$$h_n(w) = \left(\frac{2}{w}\right)^n \frac{\Gamma\left(n+\frac{1}{2}\right)}{\mathrm{i}w\sqrt{\pi}} \mathscr{H}_n(w),$$

say, and then re-writing the null-field equations in terms of $\mathscr{H}_n$; see Ref. [32].

Another way is to replace the spherical wave-function ψ_n^m in Eq. (23) by a *spheroidal* wave-function. This method was introduced by Bates and Wall [9]. They implemented it for two-dimensional problems, using elliptical wave-functions (Mathieu functions). Subsequently, spheroidal wave-functions were used by Hackman [33]. The use of elliptical or spheroidal wave-functions seems to give a Q-matrix that is better conditioned when S is elongated, and this leads to a better numerical scheme. The precise reason for this has not been investigated, but, intuitively, it is because the region in which the interior integral relation (7) is imposed 'explicitly' (rather than by analytic continuation) has been enlarged from the inscribed ball B_- to an inscribed spheroid. A third way to improve the conditioning of the Q-matrix is simply to compute with higher precision, thus reducing the rounding errors by carrying more digits in the floating-point arithmetic. This 'brute-force' method was shown to be very effective by Mishchenko and Travis [34],

who used complex arithmetic with approximately 31 decimal digits; see also Refs. [18: p. 544,19: p. 160].

4.5. The T-matrix

Recall the two expansions Eqs. (17) and (19). As the underlying problem is linear, we must be able to write

$$c_n^m = \sum_{\nu,\mu} T_{n\nu}^{m\mu} d_\nu^\mu, \tag{27}$$

for some (infinite) matrix $\mathbf{T}$ with entries $T_{n\nu}^{m\mu}$. This is usually called the *T-matrix*.

We may use the null-field method to calculate $\mathbf{T}$. Let us use an obvious matrix notation and write Eqs. (22) and (24) as

$$\mathbf{Qu} = \mathbf{d} \quad \text{and} \quad \mathbf{c} = -\hat{\mathbf{Q}}\mathbf{u},$$

respectively. Eliminating $\mathbf{u}$ gives Eq. (27), which we write as

$$\mathbf{c} = \mathbf{Td} \quad \text{with} \quad \mathbf{T} = -\hat{\mathbf{Q}}\mathbf{Q}^{-1}. \tag{28}$$

The first of these equations is the *definition* of the T-matrix, whereas the second gives a prescription for calculating the T-matrix.

The T-matrix characterises the scattering properties of the obstacle: it depends on the scatterer's shape and other properties (boundary condition or internal properties) and on the frequency, but not on the incident field (which is represented by the coefficients d_n^m in $\mathbf{d}$). This makes it useful as a 'building block' for multiple-scattering problems.

4.6. Properties of the T-matrix

It is clear that the unique-solvability of the underlying boundary-value problem implies that the T-matrix exists and is unique. This in turn implies that the T-matrix must be independent of the choice of $\{\phi_n^m(q)\}$. However, this choice may be important in numerical calculations, when $\mathbf{T}$ must necessarily be truncated.

The T-matrix must also satisfy additional constraints. These follow from considerations of reciprocity and energy. With our definitions and normalisations, we have

$$T_{n\nu}^{m\mu} = (-1)^{m+\mu} T_{\nu n}^{-\mu,-m}, \tag{29}$$

and

$$T_{n\nu}^{m\mu} + \overline{T_{\nu n}^{\mu m}} + 2\sum_{L,M} \overline{T_{Ln}^{Mm}} T_{L\nu}^{M\mu} = 0, \tag{30}$$

where the overbar denotes complex conjugation. These relations can be used as a check on computations of $\mathbf{T}$, or they can be incorporated into schemes for computing $\mathbf{T}$ [35–37]; see Ref. [17] for further references.

5. Some connections

We have described two methods for solving problems of acoustic scattering. One uses BIEs; these can be solved in various ways, such as by using boundary elements. This method is not very convenient for multiple-scattering problems, where there are several scatterers: good methods should separate the scattering properties of each scatterer from the spacing and geometrical arrangement of the scatterers. This separation can be made explicit when the T-matrix for each scatterer (in isolation) is known. We have seen that these T-matrices may be calculated using the null-field method, via the second formula in Eq. (28), or one of its variants. However, we have also seen that (some) null-field methods can be sensitive to numerical errors and ill-conditioning.

Thus, it is natural to consider ways of computing the T-matrix using BIEs. Before pursuing this idea, let us review some other connections that have been made.

5.1. Modified fundamental solutions

BIEs can be derived using a different fundamental solution instead of G. These usually have the form

$$G_1(P,Q) = G(P,Q) - 2ik\sum_{n,m} (-1)^m a_n^m \psi_n^m(\mathbf{r}_P)\psi_n^{-m}(\mathbf{r}_Q), \tag{31}$$

where the coefficients a_n^m are prescribed. Fundamental solutions of this type can be used to eliminate irregular frequencies [1, Section 3.6], [38].

In Ref. [39], the authors generalise Eq. (31), and consider

$$G_1(P,Q) = G(P,Q) - 2ik\sum_{n,m}\sum_{\nu,\mu} (-1)^m \alpha_{n\nu}^{m\mu} \psi_n^m(\mathbf{r}_P)\psi_\nu^{-\mu}(\mathbf{r}_Q),$$

where $\alpha_{n\nu}^{m\mu}$ are constants. They show that

$$\int_{S_-}\int_S \left|\frac{\partial}{\partial n_q} G_1(p,q)\right|^2 \mathrm{d}s_q\, \mathrm{d}s_p$$

will be minimised if $\alpha_{n\nu}^{m\mu} = T_{n\nu}^{m\mu}$, which means that G_1 will then be the best L^2-approximation to the exact Green's function for the exterior Neumann problem (see Ref. [40] for more information on exact Green's functions). Ahner [41] has discussed further connections with exact Green's functions.

5.2. The Petrov–Galerkin method

Let us suppose that k^2 is not an eigenvalue of the interior Dirichlet problem, so that Eq. (8) is uniquely solvable. A general method for solving such equations is the *Petrov–Galerkin method* [42, Section 3.1.2] [43, Section 3.2]; this is also called the *method of moments*. Thus, truncate Eq. (21)

and write

$$u(q) \simeq \sum_{\nu=0}^{N} \sum_{\mu=-\nu}^{\nu} u_\nu^\mu \phi_\nu^\mu(q) = u_N(q),$$

say. Then, determine the coefficients u_ν^μ by solving

$$(\mathscr{L} u_N - 2u_{\rm in}, \chi_n^m) = 0, \quad n = 0,1,2,\ldots,N, \ m = -n,\ldots,n,$$

where the test functions $\{\chi_n^m(q)\}$ are chosen and we have used the inner product in $L^2(S)$

$$(u,v) = \int_S u(q)\overline{v(q)}\,{\rm d}s_q.$$

(The choice $\chi_n^m = \phi_n^m$ gives Galerkin's method). Explicitly, we have the system

$$\sum_{\nu=0}^{N} \sum_{\mu=-\nu}^{\nu} A_{n\nu}^{m\mu} u_\nu^\mu = b_n^m, \qquad \begin{matrix} n = 0,1,2,\ldots,N \\ m = -n,\ldots,n, \end{matrix}$$

where

$$A_{n\nu}^{m\mu} = (\mathscr{L}\phi_\nu^\mu, \chi_n^m) \text{ and } b_n^m = 2(u_{\rm in}, \chi_n^m). \tag{32}$$

The Petrov–Galerkin method is a well-studied projection method. For example, necessary and sufficient conditions on ϕ_n^m and χ_n^m can be found which guarantee that $u_N \to u$ as $N \to \infty$; see Ref. [44, Section 16.2].

Now, we are going to make a specific choice for χ_n^m [45]. Write

$$\psi_n^{-m}(\mathbf{r}_P) = \frac{\rm i}{2k}(-1)^m \int_S \tau_n^m(q) G(P,q)\,{\rm d}s_q, \quad P \in B_{\rm e}, \tag{33}$$

where τ_n^m is a source density. As k^2 is not an eigenvalue of the interior Dirichlet problem, we could determine τ_n^m by solving the integral equation

$$-2{\rm i}k(-1)^m \frac{\partial}{\partial n_p} \psi_n^{-m}(\mathbf{r}_p)$$

$$= r_n^m(p) + \int_S \tau_n^m(q) \frac{\partial}{\partial n_p} G(p,q)\,{\rm d}s_q, \ p \in S, \tag{34}$$

which is obtained by calculating the normal derivative of Eq. (33) at $p \in S$. Also, use of Eq. (13) in Eq. (33) for $P \in B_+$ shows that

$$(\hat{\psi}_\nu^\mu, \overline{\tau_n^m}) = \delta_{n\nu}\delta_{m\mu}. \tag{35}$$

Next, we make the choice $\chi_n^m = \overline{\tau_n^m}$ in Eq. (32). Changing the order of integration, using Eqs. (9) and (34), and the symmetry of G gives $A_{n\nu}^{m\mu} = 2Q_{n\nu}^{m\mu}$, where $Q_{n\nu}^{m\mu}$ is defined by Eq. (23). Combining Eqs. (17) and (35) gives $b_n^m = 2d_n^m$. Hence, solving the Helmholtz integral equation using a particular Petrov–Galerkin method leads to precisely the equations $\mathbf{Qu} = \mathbf{d}$, Eq. (22), obtained when we solved the null-field equations.

This connection does not have any practical value, but it may lead to a numerical analysis of the null-field method. (There is a convergence analysis for the null-field method [46,47], but it assumes that the set $\{\partial\psi_n^m(\mathbf{r}_q)/\partial n_q\}$ forms a 'Riesz basis' of $L^2(S)$, and this is now known to be false; see Refs. [48,49: chapter V]).

5.3. Using a boundary integral equation to calculate the T-matrix

It is obvious that we can compute the field scattered by an object in (at least) two ways. We could solve a BIE, such as Eq. (8) or (12), for $u(q)$, and then use Eq. (6) to calculate $u_{\rm sc}(P)$. Alternatively, we could use the T-matrix, combining Eqs. (17), (19) and the first of Eq. (28). As the underlying boundary-value problem has exactly one solution, both ways must give the same result.

The idea of using an integral equation to obtain the T-matrix is explicit in a short note of Gurel and Chew [50]. In the context of electromagnetic scattering by thin strips, they 'show how to obtain a TMM [T-matrix method] solution from an IE [integral-equation] solution, i.e.

$$\begin{aligned} &\text{IE} \Rightarrow \text{[Surface] Current Distribution} \\ &\quad \Rightarrow T \text{ Matrix } (\Rightarrow \text{Scattered Field}). \end{aligned} \tag{36}$$

…Although both of the methods have long histories, the idea of bridging the two has not attracted much attention…The use of Eq. (36) is best appreciated when the individual T matrix of a particular…scatterer is difficult, complicated, or simply impossible to compute using a conventional TMM' [based on a null-field approach] [50, p. 1624].

Let us explain how to realise this idea. First, we make a specific choice for the incident field. Thus, we suppose that

$$d_n^m = \delta_{nN}\delta_{mM},$$

in Eq. (17), so that

$$u_{\rm in}(P) = \hat{\psi}_N^M(\mathbf{r}_P) \text{ and } c_n^m = T_{nN}^{mM}.$$

Denote the corresponding solution of Eq. (8) by $U_N^M(q)$, so that

$$\mathscr{L} U_N^M = 2\hat{\psi}_N^M. \tag{37}$$

More generally, we should solve Eq. (12) if we want to avoid irregular frequencies. Then, having found U_N^M, the scattered field is given by Eqs. (19) and (20). In particular, we can calculate c_n^m, whence

$$T_{nN}^{mM} = {\rm i}k(-1)^m \int_S U_N^M(q) \frac{\partial}{\partial n_q} \hat{\psi}_n^{-m}(\mathbf{r}_q)\,{\rm d}s_q. \tag{38}$$

This is a formula for the entries in the T-matrix.

Notice that the integral equations to be solved, Eq. (37), have the same $\mathscr{L}$ but many different right-hand sides; their solutions, U_N^M, can be obtained efficiently.

The formula (38) is simple and explicit. It bypasses all the problems associated with the ill-conditioning of the Q-matrix, and it permits the use of sophisticated boundary-element methods in the computation of $U_N^M(q)$ for $q \in S$. As far as we know, this connection between

BIEs and T-matrix methods has not been exploited: it should be!

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