

# On the $T$ -matrix for scattering by small obstacles

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## Abstract

Acoustic scattering by bounded obstacles is considered, in both two and three dimensions. Relations between the  $T$ -matrix and the far-field pattern are derived, and then used to obtain new approximations for the  $T$ -matrix for small obstacles. The problem of scattering by a pair of small sound-soft circular cylinders is also solved, in the Rayleigh approximation, using bipolar coordinates. © 2006 Elsevier B.V. All rights reserved.

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## 1. Introduction

Consider the scattering of acoustic waves by a bounded, three-dimensional obstacle,  $B$ . Choose an origin  $O$  inside  $B$ , and let  $C$  denote the smallest sphere that is centred at  $O$  and encloses  $B$ . If we know the  $T$ -matrix for  $B$ , we can calculate the scattered field outside  $C$  for any given incident field. Similarly, if we know the far-field pattern,  $f$ , we can also calculate the scattered field outside  $C$ , *but only for the incident field that generated the far-field pattern via the scattering process*:  $f$  depends on the direction of observation and on the choice of incident field.

Evidently, we can calculate the far-field pattern from the  $T$ -matrix. However, we can also calculate the  $T$ -matrix from the far-field pattern, provided we know  $f$  for all directions of observation and for all directions of incidence when the incident field is a plane wave. This simple observation means that we can use known results for low-frequency scattering of plane waves to obtain expressions for the  $T$ -matrix of small scatterers.

The main utility of these results occurs with multiple-scattering problems, where waves interact with two or (many) more obstacles [10]. Such problems are often treated using  $T$ -matrix methods. Note that the basic ideas are not limited to problems of acoustics, but may be generalised to electromagnetic and elastodynamic problems.

Although most of the paper is concerned with three-dimensional problems, we also discuss two-dimensional problems in Section 6. These problems are more complicated because low-frequency approximations are more difficult to obtain. In an appendix, we also obtain such an approximation for a pair of sound-soft circular cylinders, using bipolar coordinates.

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### 2. Formulation

Suppose that the scatterer  $B$  has surface  $S$ . Suppressing a time dependence of  $e^{-i\omega t}$ , the total field  $u$  satisfies the Helmholtz equation,

$$(\nabla^2 + k^2)u = 0,$$

in the unbounded region outside  $S$ , where  $k = \omega/c$  and  $c$  is the constant sound speed. We write  $u = u_{\text{in}} + u_{\text{sc}}$ , where  $u_{\text{in}}$  is the known incident field and  $u_{\text{sc}}$  is the unknown scattered field. We require that  $u_{\text{sc}}$  satisfies the Sommerfeld radiation condition at infinity. Consequently,

$$u_{\text{sc}}(\mathbf{r}) \sim f(\hat{\mathbf{r}})h_0(kr) \quad \text{as } r \rightarrow \infty,$$

where  $r = |\mathbf{r}|$ ,  $\hat{\mathbf{r}} = \mathbf{r}/r$  is a unit vector in the direction of observation (from  $O$  towards  $P$ , the point with position vector  $\mathbf{r}$  with respect to  $O$ ),  $h_n(kr) \equiv h_n^{(1)}(kr)$  is a spherical Hankel function, and  $f(\hat{\mathbf{r}})$  is known as the *far-field pattern*. Note that  $h_0(kr) = e^{ikr}/(ikr)$ .

For direct problems, one is often interested in calculating  $f$ . For inverse problems, one often starts with  $f$  and then tries to say something about the scatterer. It is well known that if one knows  $f(\hat{\mathbf{r}})$  for all  $\hat{\mathbf{r}} \in \Omega$  (the unit sphere), then one can reconstruct  $u_{\text{sc}}(\mathbf{r})$  everywhere outside the escribed sphere  $C$ ; this sphere has radius  $r_c$ . Explicitly, we have the Atkinson–Wilcox theorem,

$$u_{\text{sc}}(\mathbf{r}) = h_0(kr) \sum_{n=0}^{\infty} \frac{f_n(\hat{\mathbf{r}})}{r^n} \quad \text{for } r > r_c, \tag{1}$$

where  $f_0 \equiv f$ . For  $n = 1, 2, \dots$ ,  $f_n$  is obtained by applying a second-order differential operator (essentially, the angular part of the Laplacian) to  $f_{n-1}$ . In principle, Eq. (1) can be used to continue  $u_{\text{sc}}$  from the far field to the near field.

### 3. The $T$ -matrix and the far-field pattern

Outside the described sphere  $C$ , we have the expansion

$$u_{\text{sc}}(\mathbf{r}) = \sum_{n,m} c_n^m h_n(kr) Y_n^m(\hat{\mathbf{r}}), \quad r > r_c, \tag{2}$$

where  $Y_n^m$  is a spherical harmonic and

$$\sum_{n,m} = \sum_{n=-\infty}^{\infty} \sum_{m=-n}^n .$$

We use normalised complex-valued spherical harmonics, so that

$$\overline{Y_n^m} = (-1)^m Y_n^{-m}$$

and

$$\int_{\Omega} Y_n^m \overline{Y_n^\mu} d\Omega = \delta_{n\nu} \delta_{m\mu}, \tag{3}$$

where the overbar denotes complex conjugation. Using  $h_n(x) \sim (-i)^n h_0(x)$  as  $x \rightarrow \infty$ , we have

$$f(\hat{\mathbf{r}}) = \sum_{n,m} (-i)^n c_n^m Y_n^m(\hat{\mathbf{r}}). \tag{4}$$

For the incident field, we have the expansion

$$u_{\text{in}}(\mathbf{r}) = \sum_{n,m} d_n^m j_n(kr) Y_n^m(\hat{\mathbf{r}}), \tag{5}$$

where  $j_n$  is a spherical Bessel function. This expansion holds in some ball centred at  $O$ . The coefficients  $d_n^m$  in Eq. (5) are known. In particular, for an incident plane wave,

$$u_{\text{in}}(\mathbf{r}) = \exp(i\mathbf{k}\mathbf{r} \cdot \hat{\boldsymbol{\alpha}}),$$

and then we have

$$d_n^m = 4\pi i^n \overline{Y_n^m(\hat{\boldsymbol{\alpha}})}, \tag{6}$$

here  $\hat{\boldsymbol{\alpha}}$  is the direction of incidence.

The  $T$ -matrix relates the coefficients in Eqs. (2) and (5)

$$c_n^m = \sum_{\nu, \mu} T_{n\nu}^{m\mu} d_\nu^\mu. \tag{7}$$

For properties of the  $T$ -matrix, see [14]. The  $T$ -matrix can be computed in various ways, such as by solving boundary integral equations [9].

For an incident plane wave, with the corresponding far-field pattern denoted by  $f(\hat{\mathbf{r}}; \hat{\boldsymbol{\alpha}})$ , Eqs. (4), (6) and (7) give

$$f(\hat{\mathbf{r}}; \hat{\boldsymbol{\alpha}}) = 4\pi \sum_{n,m} \sum_{\nu, \mu} i^{\nu-n} T_{n\nu}^{m\mu} Y_n^m(\hat{\mathbf{r}}) \overline{Y_\nu^\mu(\hat{\boldsymbol{\alpha}})}. \tag{8}$$

Then, using the orthonormality relation, Eq. (3), twice, we obtain

$$T_{n\nu}^{m\mu} = \frac{i^{n-\nu}}{4\pi} \int_{\Omega} \int_{\Omega} f(\hat{\mathbf{r}}; \hat{\boldsymbol{\alpha}}) \overline{Y_n^m(\hat{\mathbf{r}})} Y_\nu^\mu(\hat{\boldsymbol{\alpha}}) d\Omega(\hat{\mathbf{r}}) d\Omega(\hat{\boldsymbol{\alpha}}). \tag{9}$$

This formula is exact. It can be found in [3]. It may be used to continue  $u_{\text{sc}}$  from the far field to the near field; cf. Eq. (1).

#### 4. Small soft scatterers

As a simple example, consider Rayleigh scattering by a small sound-soft obstacle (so that  $u = 0$  on  $S$ ). Then, it is known that (see, for example, [4])

$$f(\hat{\mathbf{r}}; \hat{\boldsymbol{\alpha}}) = -ik\mathcal{C} + O(k^2) \quad \text{as } k \rightarrow 0,$$

where the constant  $\mathcal{C}$  is the *capacity* of  $S$ ; by definition,

$$\mathcal{C} = -\frac{1}{4\pi} \int_S \frac{\partial \phi}{\partial n} ds,$$

where  $\partial/\partial n$  denotes normal differentiation on  $S$  away from  $B$ , and the potential  $\phi$  solves the following problem:

$$\begin{aligned} \nabla^2 \phi &= 0 && \text{outside } S, \\ \phi &= 1 && \text{on } S, \end{aligned}$$

and

$$\phi = O(r^{-1}) \quad \text{as } r \rightarrow \infty.$$

Then, Eq. (9) gives the corresponding  $T$ -matrix as

$$T_{n\nu}^{m\mu} = -ik\mathcal{C} \overline{y_n^m} y_\nu^\mu + O(k^2) \quad \text{as } k \rightarrow 0,$$

where

$$y_n^m = \frac{(-i)^n}{\sqrt{4\pi}} \int_{\Omega} Y_n^m d\Omega = \delta_{n0} \delta_{m0},$$

using  $Y_0^0 = (4\pi)^{-1/2}$ . Thus, we find that every entry of the  $T$ -matrix is  $O(k^2)$  except that

$$T_{00}^{00} = -ik\mathcal{C} + O(k^2) \quad \text{as } k \rightarrow 0.$$

Consequently, for any incident field,  $u_{\text{in}}(\mathbf{r})$ , we have

$$u_{\text{sc}}(\mathbf{r}) \simeq T_{00}^{00} d_0^0 h_0(kr) Y_0^0,$$

where from Eq. (5),  $d_0^0 Y_0^0 = u_{\text{in}}(\mathbf{0})$ . Hence, we obtain the approximation

$$u_{\text{sc}}(\mathbf{r}) \simeq -ik\mathcal{C} u_{\text{in}}(\mathbf{0}) h_0(kr). \tag{10}$$

Thus, as is generally known, small soft obstacles scatter isotropically (there is no dependence on  $\hat{\mathbf{r}}$ ), with amplitude proportional to the value of the incident field at the scatterer’s ‘centre’,  $\mathbf{r} = \mathbf{0}$ . This was the starting point for Foldy’s famous study on multiple scattering [5]. In fact, Foldy wrote

$$u_{\text{sc}}(\mathbf{r}) \simeq g u_{\text{in}}(\mathbf{0}) h_0(kr), \tag{11}$$

where  $g$  is a ‘scattering coefficient’. Our asymptotic analysis gives

$$g = -ik\mathcal{C}. \tag{12}$$

However, energy considerations show that  $g$  must satisfy

$$|g|^2 + \text{Re}(g) = 0, \tag{13}$$

so that a better choice for  $g$  is

$$g = -ik\mathcal{C}/(1 + ik\mathcal{C}), \tag{14}$$

this choice satisfies Eq. (13) and agrees with Eq. (12) as  $k \rightarrow 0$ .

It is worth noting that these results all hold for two or more soft scatterers. For example, the capacitance of a pair of spheres can be calculated exactly, using bispherical coordinates. For details, see [2].

### 5. Small hard scatterers

For a sound-hard obstacle, we have  $\partial u / \partial n = 0$  on  $S$ . From [4], we have

$$f(\hat{\mathbf{r}}; \hat{\boldsymbol{\alpha}}) = \frac{ik^3}{4\pi} \left\{ V_B(\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\alpha}} - 1) - \int_S (\hat{\mathbf{r}} \cdot \mathbf{n})(\hat{\boldsymbol{\alpha}} \cdot \boldsymbol{\Psi}) \, ds \right\}$$

as  $k \rightarrow 0$ , with an error that is  $O(k^4)$ . In this formula,  $V_B$  is the volume of  $B$ ,  $\mathbf{n}(q)$  is the unit normal vector at  $q \in S$  pointing away from  $B$ , and the vector field  $\boldsymbol{\Psi}$  solves the following problem (see [4, Eq. (5.20)])

$$\nabla^2 \boldsymbol{\Psi} = \mathbf{0} \quad \text{outside } S, \tag{15}$$

$$\partial \boldsymbol{\Psi} / \partial n = \mathbf{n} \quad \text{on } S, \tag{16}$$

and

$$\boldsymbol{\Psi} = O(r^{-2}) \quad \text{as } r \rightarrow \infty.$$

Now, following Dassios and Kleinman [4, p. 166], we define the *virtual mass tensor*  $\mathbf{W}$  by

$$W_{ij} = - \int_S n_i \Psi_j \, ds = W_{ji}, \tag{17}$$

and the *magnetic polarizability tensor*  $\mathbf{M}$  by

$$M_{ij} = W_{ij} + V_B \delta_{ij} = M_{ji}. \tag{18}$$

(For the special case of a sphere,  $M_{ij} = \frac{3}{2} V_B \delta_{ij}$ .) Then, we can express the far-field pattern concisely by

$$f(\hat{\mathbf{r}}; \hat{\boldsymbol{\alpha}}) = \frac{ik^3}{4\pi} \{\hat{\mathbf{r}} \cdot \mathbf{M} \cdot \hat{\boldsymbol{\alpha}} - V_B\} + O(k^4) \quad \text{as } k \rightarrow 0. \tag{19}$$

Thus, the far field of a small hard scatterer depends linearly on both the observation direction and the incident direction, and it is much smaller than the far field of a small soft scatterer. Of course, this result was known to Lord Rayleigh [12].

We can use Eq. (19) to calculate the  $T$ -matrix for a small sound-hard scatterer. Substituting in Eq. (9), we find after some calculation that the  $T$ -matrix has 10 entries that are  $O(k^3)$  as  $k \rightarrow 0$ :

$$\begin{aligned} T_{00}^{00} &= -ik^3 V_B / (4\pi), \\ T_{11}^{00} &= ik^3 M_{33} / (12\pi), \\ T_{11}^{01} &= -ik^3 (M_{31} + i M_{32}) / (12\pi\sqrt{2}) = -T_{11}^{-1,0}, \\ T_{11}^{10} &= -ik^3 (M_{31} - i M_{32}) / (12\pi\sqrt{2}) = -T_{11}^{0,-1}, \\ T_{11}^{11} &= ik^3 (M_{11} + M_{22}) / (24\pi) = T_{11}^{-1,-1}, \\ T_{11}^{1,-1} &= ik^3 (M_{22} - M_{11} + 2i M_{12}) / (24\pi), \\ T_{11}^{-1,1} &= ik^3 (M_{22} - M_{11} - 2i M_{12}) / (24\pi). \end{aligned}$$

Let us calculate the scattered field for any incident field,  $u_{\text{in}}(\mathbf{r})$ . We introduce a vector  $\mathbf{U}$  with components

$$U_j = \frac{1}{k} \frac{\partial u_{\text{in}}}{\partial x_j} \quad \text{evaluated at } \mathbf{r} = \mathbf{0}. \tag{20}$$

Then, we find that  $d_1^0 = \sqrt{12\pi} U_3$ ,  $d_1^1 = -\sqrt{6\pi}(U_1 - iU_2)$  and  $d_1^{-1} = \sqrt{6\pi}(U_1 + iU_2)$ . Also, as before,  $d_0^0 = \sqrt{4\pi} u_{\text{in}}(\mathbf{0})$ . We then calculate  $c_n^m$ , using Eq. (7) and the approximations to the  $T$ -matrix given above. Eventually, we obtain

$$u_{\text{sc}}(\mathbf{r}) \simeq \frac{ik^3}{4\pi} \{\hat{\mathbf{r}} \cdot \mathbf{M} \cdot \mathbf{U} h_1(kr) - V_B u_{\text{in}}(\mathbf{0}) h_0(kr)\}. \tag{21}$$

This can be used to generalise Foldy’s method to collections of small hard scatterers.

### 6. Two dimensions

The analysis for two-dimensional problems is more involved. To fix ideas, let us consider scattering by a sound-soft cylindrical obstacle with boundary  $S$ . Let  $B$  denote the interior of  $S$  and let  $B_c$  denote the (unbounded) exterior of  $S$ . The scattered field is given by

$$u_{\text{sc}}(\mathbf{r}) \sim \sqrt{\frac{2}{\pi}} e^{-i\pi/4} \frac{e^{ikr}}{\sqrt{kr}} f(\hat{\mathbf{r}}) \quad \text{as } r \rightarrow \infty, \tag{22}$$

where the far-field pattern is

$$f(\hat{\mathbf{r}}) = -\frac{i}{4} \int_S \frac{\partial u}{\partial n_q} \exp(-ik\hat{\mathbf{r}} \cdot \mathbf{r}_q) ds_q. \tag{23}$$

As  $\hat{\mathbf{r}} = (\cos \theta, \sin \theta)$ , we can write  $f(\hat{\mathbf{r}}) = f(\theta)$ .

For low frequencies, we expand  $u_{\text{in}}$  as

$$u_{\text{in}}(\mathbf{r}) = \sum_{n=0}^{\infty} (ik)^n u_{\text{in}}^n(\mathbf{r}).$$

Then, near  $S$ , we have

$$u(\mathbf{r}) = u_{\text{in}}^0(\mathbf{r}) + v_s(\mathbf{r}) + O(|\log k|^{-1}) \quad \text{as } k \rightarrow 0, \quad (24)$$

where  $v_s$  solves the following Dirichlet problem:

$$\begin{aligned} \nabla^2 v_s &= 0 \quad \text{in } B_e, \\ v_s &= -u_{\text{in}}^0 \quad \text{on } S, \end{aligned}$$

and

$$v_s = O(1) \quad \text{as } r \rightarrow \infty.$$

The unique solution of this problem can be found by solving a boundary integral equation over  $S$ .

The result Eq. (24) is given in [8], for example. However, for an incident plane wave,

$$u_{\text{in}} = e^{ikr \cos(\theta - \alpha)}, \quad (25)$$

we have  $u_{\text{in}}^0 = 1$  and  $v_s = -1$ , so that the leading term in Eq. (24) gives a zero contribution to the integral in Eq. (23): therefore, we have to determine the term of order  $|\log k|^{-1}$ . We do this next, using the results of Kleinman and Vainberg [7].

### 6.1. Kleinman–Vainberg theory

Kleinman and Vainberg [7] have given a systematic method for obtaining low-frequency asymptotic expansions in two dimensions for the following problem:

$$(\nabla^2 + k^2)v = F \quad \text{in } B_e, \quad \mathcal{B}v = 0 \quad \text{on } S$$

and  $v$  satisfies the Sommerfeld radiation condition as  $r \rightarrow \infty$ .

Here,  $\mathcal{B}v$  denotes  $v$  or  $\partial v / \partial n$ , and  $F$  has compact support. We shall refer to this problem as the *K–V problem*.

Our scattering problems can be formulated as

$$(\nabla^2 + k^2)u_{\text{sc}} = 0 \quad \text{in } B_e, \quad \mathcal{B}u_{\text{sc}} = -\mathcal{B}u_{\text{in}} \quad \text{on } S$$

and  $u_{\text{sc}}$  satisfies the Sommerfeld radiation condition as  $r \rightarrow \infty$ .

Let us convert this problem into the K–V problem. Choose  $c$  so that  $B$  is contained inside the disc  $r < c$ . Let  $\chi$  be a  $C^\infty$  cut-off function, so that  $\chi = 0$  for  $r > 2c$  and  $\chi = 1$  for  $0 \leq r < c$ . Put  $u_{\text{sc}} = -u_{\text{in}}\chi + v$ . Then,  $v$  solves the K–V problem with

$$F = (\nabla^2 + k^2)(u_{\text{in}}\chi) = u_{\text{in}}\nabla^2\chi + 2 \text{grad } \chi \cdot \text{grad } u_{\text{in}}.$$

Note that the total field

$$u = u_{\text{sc}} + u_{\text{in}} = (1 - \chi)u_{\text{in}} + v,$$

so that  $v$  is the total field near  $S$  ( $r < c$ ) but  $v$  is the scattered field far from  $S$  ( $r > 2c$ ).

The results of [7] assume that  $F$  does not depend on  $k$ , so we write

$$F(\mathbf{r}) = \sum_{n=0}^{\infty} (ik)^n F_n(\mathbf{r}),$$

where

$$F_n = u_{\text{in}}^n \nabla^2 \chi + 2 \text{grad } \chi \cdot \text{grad } u_{\text{in}}^n. \quad (26)$$

We write  $v_n$  for the solution of the K–V problem with  $F = F_n$ , so that

$$v(\mathbf{r}) = \sum_{n=0}^{\infty} (ik)^n v_n(\mathbf{r}). \tag{27}$$

We are going to approximate  $v_n$  near  $S$  using solutions of related static problems. The results are different depending on the boundary condition on  $S$ . Therefore, we now consider soft and hard cylinders separately.

6.2. Small sound-soft cylinders

Consider the following uniquely solvable problems,

$$\nabla^2 u_0^n = F_n \quad \text{in } B_e,$$

$$u_0^n = 0 \quad \text{on } S,$$

and

$$u_0^n = O(1) \quad \text{as } r \rightarrow \infty$$

and

$$\nabla^2 u_1 = 0 \quad \text{in } B_e,$$

$$u_1 = 0 \quad \text{on } S,$$

and

$$u_1 = \log(r/\ell) + o(1) \quad \text{as } r \rightarrow \infty. \tag{28}$$

Here, the constant  $\ell$  is unknown, but it is to be determined by solving the problem for  $u_1$  with the specified logarithmic growth at infinity. (In [7], there is a parameter  $\lambda_0 = -\log \ell$ .) Also, let  $\beta$  be the complex constant occurring in the asymptotic approximation

$$H_0^{(1)}(w) = (2i/\pi)(\log w - \beta) + O(w^2 \log w) \quad \text{as } w \rightarrow 0,$$

where  $H_n^{(1)}$  is a Hankel function; thus,  $\beta = \log 2 - \gamma + i\pi/2$  where  $\gamma = 0.5772\dots$  is Euler’s constant. Then [7, Theorem 1], near  $S$ ,

$$v_n(\mathbf{r}) = u_0^n(\mathbf{r}) + \frac{C_0^n u_1(\mathbf{r})}{\log k\ell - \beta} + O(k^2 \log k) \quad \text{as } k \rightarrow 0,$$

where  $C_0^n = \lim_{r \rightarrow \infty} u_0^n(\mathbf{r})$ . It follows from Eq. (27) that we only need  $v_0$  here. Then, the far-field pattern is given by Eq. (23) as

$$f(\theta) = -\frac{i}{4} \int_S \frac{\partial v_0}{\partial n_q} ds_q + O(k) \quad \text{as } k \rightarrow 0. \tag{29}$$

For a plane wave, given by Eq. (25),  $u_{in}^0 \equiv 1$  and  $F_0 = \nabla^2 \chi$ . Thus  $u_0^0(\mathbf{r}) = \chi(\mathbf{r}) - 1$ , which is constant (in fact, zero) near  $S$ , and so (as expected) the leading term does not contribute to Eq. (29). The next term then gives

$$f(\theta; \alpha) = -\frac{i}{4} \frac{C_0^0}{\log k\ell - \beta} \int_S \frac{\partial u_1}{\partial n_q} ds_q + O(k) \quad \text{as } k \rightarrow 0. \tag{30}$$

However, we have  $C_0^0 = \lim_{r \rightarrow \infty} u_0^0 = -1$  and Green’s theorem gives

$$\int_S \frac{\partial u_1}{\partial n_q} ds_q = \lim_{r \rightarrow \infty} \int_0^{2\pi} \frac{\partial u_1}{\partial r} r d\theta = 2\pi.$$

Hence, Eq. (30) reduces to

$$f(\theta; \alpha) = \frac{1}{2}\pi i (\log k\ell - \beta)^{-1} + O(k) \quad \text{as } k \rightarrow 0. \tag{31}$$

The parameter  $\ell$  can be obtained explicitly for simple shapes. For a circle of radius  $a$ ,  $u_1 = \log(r/a)$  and so  $\ell = a$ . For an ellipse, we can use elliptic cylindrical coordinates,  $(\xi, \eta)$ , defined by  $x = c_0 \cosh \xi \cos \eta$  and  $y = c_0 \sinh \xi \sin \eta$ , giving  $u_1 = \xi - \xi_0$  (where  $\xi = \xi_0$  defines  $S$ ); then it follows that

$$\ell = \frac{1}{2}c_0 e^{\xi_0} = \frac{1}{2}(a + b).$$

The corresponding low-frequency approximation for scattering by a soft elliptic cylinder can be found in [11, Eq. (48a)] and [1, Section 3.2.1.2].

We can also calculate  $\ell$  for a pair of sound-soft circular cylinders, using bipolar coordinates; see Appendix A for details.

We can use Eq. (31) to calculate the  $T$ -matrix. In two dimensions, the analogues of Eqs. (8) and (9) are

$$f(\theta; \alpha) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (-i)^n i^m T_{nm} e^{in\theta} e^{-im\alpha}$$

and

$$T_{nm} = \frac{i^n (-i)^m}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(\theta; \alpha) e^{-in\theta} e^{im\alpha} d\theta d\alpha, \tag{32}$$

respectively. Substituting in Eq. (32), we find that

$$T_{00} = \frac{1}{2}i\pi(\log k\ell - \beta)^{-1} + O(k) \quad \text{as } k \rightarrow 0;$$

all other entries  $T_{nm}$  are asymptotically smaller. Consequently, for any incident field  $u_{\text{in}}(\mathbf{r})$ , we obtain

$$u_{\text{sc}}(\mathbf{r}) \simeq \frac{1}{2}i\pi(\log k\ell - \beta)^{-1} u_{\text{in}}(\mathbf{0}) H_0^{(1)}(kr). \tag{33}$$

Again, we see that small soft cylinders scatter isotropically.

### 6.3. Small sound-hard cylinders

For scattering by a sound-hard cylinder, the far field is given again by Eq. (22) with the far-field pattern now given by

$$f(\hat{\mathbf{r}}) = \frac{k}{4} \int_S u(q) [\hat{\mathbf{r}} \cdot \mathbf{n}(q)] \exp(-ik\hat{\mathbf{r}} \cdot \mathbf{r}_q) ds_q. \tag{34}$$

Then, for low-frequency approximations, we are led to consider the following uniquely solvable problems,

$$\nabla^2 w_m = F_m \quad \text{in } B_e,$$

$$\frac{\partial w_m}{\partial n} = 0 \quad \text{on } S,$$

and

$$w_m = \alpha_m (\log r - \beta) + o(1) \quad \text{as } r \rightarrow \infty,$$

where

$$\alpha_m = \frac{1}{2\pi} \int_{B_e} F_m dV.$$



Then [7, Theorem 2], near  $S$ ,

$$v_n(\mathbf{r}) = \alpha_n \log k + w_n(\mathbf{r}) + O(k^2 \log^3 k) \quad \text{as } k \rightarrow 0.$$

For a plane wave,  $u_{\text{in}}^0 \equiv 1$  and  $u_{\text{in}}^1(\mathbf{r}) = \mathbf{r} \cdot \hat{\boldsymbol{\alpha}}$ . Then, Eq. (26) gives  $F_0 = \nabla^2 \chi$  and  $F_1 = \nabla^2(\chi \mathbf{r} \cdot \hat{\boldsymbol{\alpha}})$ . Hence,

$$2\pi\alpha_0 = \int_{B_e} \nabla^2 \chi \, dV = - \int_S \frac{\partial \chi}{\partial n} \, ds = 0$$

and

$$2\pi\alpha_1 = - \int_S \frac{\partial}{\partial n} \{\chi \mathbf{r} \cdot \hat{\boldsymbol{\alpha}}\} \, ds = - \int_S \mathbf{n} \cdot \hat{\boldsymbol{\alpha}} \, ds = 0,$$

where the last equality follows from an application of the divergence theorem in  $B$ , and we have used  $\chi \equiv 1$  near  $S$ . Thus,

$$v = \sum_{n=0}^{\infty} (ik)^n v_n = w_0 + ikw_1 + O(k^2 \log^3 k) \quad \text{as } k \rightarrow 0.$$

Then, as  $u \equiv v$  near  $S$ , Eq. (34) gives

$$f(\hat{\mathbf{r}}; \hat{\boldsymbol{\alpha}}) = \frac{k}{4} \int_S [\hat{\mathbf{r}} \cdot \mathbf{n}(q)](w_0 + ikf_1) \, ds_q + O(k^3 \log^3 k) \tag{35}$$

as  $k \rightarrow 0$ , where  $f_1(q) = w_1 - w_0 \hat{\mathbf{r}} \cdot \mathbf{r}_q$ .

As  $\alpha_0 = 0$  and  $F_0 = \nabla^2 \chi$ , we obtain  $w_0 = \chi$ . Then, as  $\int_S \hat{\mathbf{r}} \cdot \mathbf{n} \, ds = 0$ , we see that the leading contribution to Eq. (35) vanishes. As  $\alpha_1 = 0$  and  $F_1 = \nabla^2(\chi \mathbf{r} \cdot \hat{\boldsymbol{\alpha}})$ , we obtain

$$w_1(\mathbf{r}) = \chi \mathbf{r} \cdot \hat{\boldsymbol{\alpha}} - \hat{\boldsymbol{\alpha}} \cdot \boldsymbol{\Psi},$$

where the two-dimensional vector field  $\boldsymbol{\Psi}$  is defined by Eqs. (15) and (16) and  $\boldsymbol{\Psi} = O(r^{-1})$  as  $r \rightarrow \infty$ . It follows that  $f_1(q)$  is given by

$$f_1(q) = \mathbf{r}_q \cdot (\hat{\boldsymbol{\alpha}} - \hat{\mathbf{r}}) - \hat{\boldsymbol{\alpha}} \cdot \boldsymbol{\Psi}(q)$$

and then

$$f(\hat{\mathbf{r}}; \hat{\boldsymbol{\alpha}}) = \frac{1}{4} ik^2 \{\hat{\mathbf{r}} \cdot \mathbf{M} \cdot \hat{\boldsymbol{\alpha}} - A_B\} + O(k^3 \log^3 k) \quad \text{as } k \rightarrow 0, \tag{36}$$

where  $A_B$  is the area of  $B$ ,

$$M_{ij} = W_{ij} + A_B \delta_{ij} = M_{ji}$$

and  $W_{ij}$  is defined by Eq. (17). Thus, the far-field pattern is given by a formula that is similar to the analogous formula in three dimensions, namely, Eq. (19). A formula similar to Eq. (36) can be found in [6].

For a circle of radius  $a$ ,  $M_{ij} = 2A_B \delta_{ij}$  and then Eq. (36) reduces to

$$f(\theta; \alpha) \simeq \frac{1}{4} i\pi (ka)^2 \{2 \cos(\theta - \alpha) - 1\},$$

in agreement with the known result [1, Eq. (2.43)].

Substitution of Eq. (36) in Eq. (32) gives the  $T$ -matrix for a small sound-hard cylinder. We obtain

$$T_{nm} = \frac{1}{4} ik^2 \{\overline{\mathbf{z}}_n \cdot \mathbf{M} \cdot \mathbf{z}_m - A_B \delta_{n0} \delta_{m0}\} + O(k^3 \log^3 k) \quad \text{as } k \rightarrow 0,$$

where

$$\mathbf{z}_n = \frac{(-i)^n}{2\pi} \int_{-\pi}^{\pi} \hat{\mathbf{r}} e^{in\theta} \, d\theta.$$

Thus,

$$\mathbf{z}_1 = \frac{1}{2}(-i, 1) = \overline{\mathbf{z}_{-1}} \quad \text{and} \quad \mathbf{z}_n = \mathbf{0} \quad \text{for } n \neq \pm 1.$$

Hence, the  $T$ -matrix has five entries that are  $O(k^2)$  as  $k \rightarrow 0$ :

$$T_{00} = -\frac{1}{4}ik^2 A_B,$$

$$T_{11} = \frac{1}{16}ik^2(M_{11} + M_{22}) = T_{-1,-1},$$

$$T_{1,-1} = \frac{1}{16}ik^2(M_{22} - M_{11} + 2iM_{12}),$$

$$T_{-1,1} = \frac{1}{16}ik^2(M_{22} - M_{11} - 2iM_{12}).$$

Next, we calculate the scattered field for any incident field,  $u_{\text{in}}(\mathbf{r})$ . We have

$$d_0 = u_{\text{in}}(\mathbf{0}), \quad d_1 = U_1 - iU_2 \quad \text{and} \quad d_{-1} = -U_1 - iU_2,$$

where  $\mathbf{U} = (U_1, U_2)$  is defined by Eq. (20) with  $x_1 = x$  and  $x_2 = y$ , and

$$u_{\text{in}}(\mathbf{r}) = \sum_{n=-\infty}^{\infty} d_n J_n(kr) e^{in\theta},$$

where  $J_n$  is a Bessel function. For the scattered field, we calculate  $c_n = \sum_m T_{nm} d_m$  and find

$$c_0 = -\frac{1}{4}ik^2 A_B u_{\text{in}}(\mathbf{0}),$$

$$c_1 = \frac{1}{8}ik^2(M_{1j} - iM_{2j})U_j,$$

$$c_{-1} = -\frac{1}{8}ik^2(M_{1j} + iM_{2j})U_j,$$

where summation over  $j = 1$  and  $j = 2$  is implied. All the other coefficients  $c_n$  are asymptotically smaller. Hence, the scattered field is

$$u_{\text{sc}} \simeq \frac{1}{4}ik^2 \left\{ \hat{\mathbf{r}} \cdot \mathbf{M} \cdot \mathbf{U} H_1^{(1)}(kr) - A_B u_{\text{in}}(\mathbf{0}) H_0^{(1)}(kr) \right\}. \quad (37)$$

## 7. Conclusions

We have described a systematic method for obtaining approximations to the  $T$ -matrix, valid for small scatterers of any shape. (The only other related results known to us are for elliptical cylinders and for spheroids in [13].) The method generalises to penetrable scatterers and to other physical situations.

## Appendix A. Scattering by a pair of small soft circular cylinders

The low-frequency asymptotic theory for the scattering of long acoustic waves by a pair of sound-soft circular cylinders leads to the following problem: solve  $\nabla^2 u_1 = 0$  in  $B_e$ , with  $u_1 = 0$  on each cylinder, and the condition Eq. (28) as  $r \rightarrow \infty$ ; the problem is to calculate the constant  $\ell$ .

To solve  $\nabla^2 u_1 = 0$ , we use bipolar coordinates,  $\xi$  and  $\vartheta$ , defined by

$$x = \frac{c \sinh \xi}{\cosh \xi - \cos \vartheta} \quad \text{and} \quad y = \frac{c \sin \vartheta}{\cosh \xi - \cos \vartheta},$$

where  $c$  is a positive constant. Hence

$$r^2 = x^2 + y^2 = c^2 \frac{\cosh \xi + \cos \vartheta}{\cosh \xi - \cos \vartheta}. \quad (\text{A.1})$$

We also have

$$x - iy = c \coth(\zeta/2) \quad \text{with } \zeta = \xi + i\vartheta. \tag{A.2}$$

The line  $\xi = \xi_0 > 0$ ,  $|\vartheta| < \pi$ , corresponds to the circle  $(x - b_0)^2 + y^2 = a_0^2$  with  $b_0 = c \coth \xi_0$  and  $a_0 = c / \sinh \xi_0$ . Similarly, the line  $\xi = -\xi_1$ , with  $\xi_1 > 0$  and  $|\vartheta| < \pi$ , corresponds to the circle  $(x + b_1)^2 + y^2 = a_1^2$  with  $b_1 = c \coth \xi_1$  and  $a_1 = c / \sinh \xi_1$ . Let  $B_e$  denote the region exterior to these two circles in the  $xy$ -plane. This region corresponds to the rectangle

$$-\xi_1 < \xi < \xi_0, \quad -\pi \leq \vartheta < \pi,$$

in the  $\xi\vartheta$ -plane. Note that the origin in this plane corresponds to the point at infinity in the  $xy$ -plane:  $|\zeta| \simeq 2c/r$  near  $\zeta = 0$ .

Laplace’s equation is not affected by changing to bipolar coordinates:

$$\frac{\partial^2 u_1}{\partial \xi^2} + \frac{\partial^2 u_1}{\partial \vartheta^2} = 0. \tag{A.3}$$

As we want to solve  $\nabla^2 u_1 = 0$  in  $B_e$ , we can admit only those solutions of Eq. (A.3) that are  $2\pi$ -periodic in  $\vartheta$ . Separated solutions with this property are

$$1, \xi, \cosh n\xi \cos n\vartheta, \sinh n\xi \cos n\vartheta, \tag{A.4}$$

$\cosh n\xi \sin n\vartheta$  and  $\sinh n\xi \sin n\vartheta$ , where  $n$  is a positive integer.

There are also useful non-separated solutions of Eq. (A.3). One is

$$-\frac{1}{2} \log(\cosh \xi - \cos \vartheta) = L(\xi, \vartheta),$$

say. This solution is  $2\pi$ -periodic in  $\vartheta$  and has a logarithmic singularity at  $\zeta = 0$ , corresponding to logarithmic growth as  $r \rightarrow \infty$ ; explicitly, from Eq. (A.1), we have

$$\begin{aligned} L &= \log(r/c) - \frac{1}{2} \log(\cosh \xi + \cos \vartheta) \\ &\sim \log r - \log(c\sqrt{2}) + o(1) \quad \text{as } |\zeta| \rightarrow 0. \end{aligned} \tag{A.5}$$

$L(\xi, \vartheta)$  has a Fourier series in  $\vartheta$ . As

$$|1 - e^{-\zeta}| = 2 e^{-\xi} (\cosh \xi - \cos \vartheta),$$

we obtain

$$\begin{aligned} L(\xi, \vartheta) &= -\frac{1}{2} \log |1 - e^{-\zeta}| + \frac{1}{2} \log(2 e^{-\xi}) \\ &= -\frac{\xi}{2} + \log \sqrt{2} + \sum_{n=1}^{\infty} \frac{e^{-n\xi}}{2n} \cos n\vartheta \end{aligned} \tag{A.6}$$

for  $\xi > 0$ , where we have used the real part of the formula

$$\log(1 - e^{-\zeta}) = -\sum_{n=1}^{\infty} \frac{e^{-n\zeta}}{n}$$

(obtained by integrating the geometric series), which is valid for  $\xi > 0$ . As  $L(\xi, \vartheta)$  is an even function of  $\xi$ , we can replace  $\xi$  by  $|\xi|$  on the right-hand side of Eq. (A.6). Note that, as expected, the series for  $L(0, 0)$  is divergent.

Now, return to the calculation of  $u_1$ , and write

$$u_1 = L + v, \tag{A.7}$$

so that  $v$  solves Eq. (A.3),  $v$  is bounded as  $r \rightarrow \infty$ , and  $v = -L$  on each cylinder. Once we have found  $v(\xi, \theta)$ , Eqs. (A.5), (28) and (A.7) give

$$\log \ell = \log(c\sqrt{2}) - v(0, 0).$$

To solve for  $v$ , we use the solutions in Eq. (A.4); they are all even in  $\vartheta$ , just like  $L$ . Then, write

$$v(\xi, \theta) = A_0 + B_0 \xi + \sum_{n=1}^{\infty} (A_n \cosh n\xi + B_n \sinh n\xi) \cos n\vartheta, \quad (\text{A.8})$$

so that

$$\log \ell = \log(c\sqrt{2}) - \sum_{n=0}^{\infty} A_n.$$

To find  $A_n$ , we apply the boundary conditions using Eqs. (A.6) and (A.8); this gives

$$A_0 = -\log \sqrt{2} + \xi_0 \xi_1 / (\xi_0 + \xi_1)$$

and

$$A_n = -\frac{e^{-n\xi_0} \sinh n\xi_1 + e^{-n\xi_1} \sinh n\xi_0}{2n \sinh n(\xi_0 + \xi_1)}, \quad n \geq 1.$$

In particular, for identical cylinders, we have  $\xi_0 = \xi_1$ , and obtain

$$\log \ell = \log(2c) - \frac{\xi_0}{2} + \sum_{n=1}^{\infty} \frac{e^{-n\xi_0}}{2n \cosh n\xi_0}.$$

Here, both cylinders have radius  $a_0$ , and their centres are distance  $2b_0$  apart; the parameters  $c$  and  $\xi_0$  are defined by

$$b_0/a_0 = \cosh \xi_0 \quad \text{and} \quad c = \sqrt{b_0^2 - a_0^2}.$$

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