Scattering matrix for elastic waves. I. Theory

Vasundara Varatharajulu and Yih-Hsing Pao

Department of Theoretical and Applied Mechanics, Cornell University, Ithaca, New York 14853
(Received 25 January 1976; revised 1 June 1976)

A matrix theory is developed for investigating the scattering of elastic waves in solids by an obstacle of arbitrary shape. The scattering matrix which depends only on the shape and nature of the obstacle relates the scattered field to any type of harmonic incident field. Expressions are obtained for the elements of the scattering matrix in the form of surface integrals around the boundary of the obstacle, which can be evaluated numerically. Using the principle of reciprocity and the conservation of energy, the scattering matrix is shown to be symmetric and unitary. These properties are essential to assure the accuracy of numerical calculations. Both two- and three-dimensional problems are discussed, and the obstacle may be an elastic inclusion, a fluid inclusion, a cavity, or a rigid inclusion of arbitrary shape.

Subject Classification: [43]20.15, [43]20.30.

INTRODUCTION

A matrix theory for the scattering of acoustic[1] and electromagnetic[2] waves by a single object of arbitrary size was presented by Waterman a few years ago. Comparisons of this theory with other established methods of analysis such as the eigenfunction expansion method, the boundary integral technique and the variational method were discussed in the original paper.[1] Based on extensive numerical calculations for the scattering of acoustic waves by an elliptical or a square scatterer, Bolomey and Wirgin[3] summarized that the matrix method is efficient and is some respects superior to the boundary integral technique (Green's function theory), especially when numerical computations become necessary.

In this paper, we extend Waterman's matrix theory to the scattering of elastic waves. In contrast to the acoustic wave which is governed by a single scalar wave equation and the electromagnetic wave governed by a vector wave equation with a single wave speed, the elastic wave is composed of a longitudinal (p-wave) part and a transverse (s-wave) part. The former satisfies a scalar wave equation and the latter, a vector one, each with a distinct velocity. The coupling of these two types of waves at the boundary of an elastic solid makes the analysis very difficult. For instance, while the scattering of acoustic waves by a hard or soft elliptic cylinder can be conveniently analyzed using Mathieu functions as a basis set, the same cannot be done for elastic waves.

A notable exception is the case of an incident s wave with displacement vector parallel to the axis of an infinitely long prismatic cylinder, propagating in a direction normal to the axis. In this case, the scattered wave can be represented by a two-dimensional scalar wave function and the problem is identical to the scattering of acoustic waves by the same cylinder. In the literature of elastodynamics this is known as the *sh* wave (horizontally polarized shear wave) and is excluded from our discussion.

For a normally incident s wave with any other polarization, the wave scattered by a prismatic cylinder is still two dimensional but it is composed of both p and S waves. This is known as the problem of sv waves (vertically polarized shear waves). The scattering of p- and sv waves normally incident on an infinitely long prismatic cylinder constitute the two dimensional problems discussed in this paper.

As reviewed by Pao and Mow[4] only a circular cylinder (two dimensional) or a sphere (three dimensional) can be analyzed effectively by the eigenfunction expansion method. The application of the boundary integral equation technique has been tried only for two-dimensional problems with limited success.[5] For these reasons, the matrix method as presented in this paper offers a promising alternative to the existing methods of analysis.

For scalar waves, the matrix method begins with the Helmholtz integral formula for the harmonic wave field exterior to the bounding surface S of an obstacle,[6]

$$\int_S \{\psi' \hat{n}' \nabla' g - g \hat{n}' \cdot \nabla' \psi'\} dS' = \begin{cases} \psi^s(\vec{x}), & \vec{x} \text{ outside } S, \\ -\psi^0(\vec{x}), & \vec{x} \text{ inside } S, \end{cases}$$

where ψ^0 is the incident field and $\psi^s = \psi - \psi^0$ is the scattered wave field, $g(\vec{x}|\vec{x}')$ is the Green function of the scalar wave equation and $\hat{n}' \cdot \nabla'$ is the normal gradient at the surface S and primes indicate that they are functions of $\vec{x}'$ which is any point on S. The procedure is to expand ψ^0 in a complete, orthogonal basis set $\psi_n(\vec{x})$ with known coefficients A_n, and the scattered field in terms of the unknown coefficients B_n. The expansion of the unknown surface field introduces an additional coefficient α_n whereas the expansion of g in a circular or spherical basis set is well known. The essence of the matrix method is to eliminate the unknown α_n by using the two cases in the Helmholtz formula and express B_m as $\Sigma T_{mn} A_n$. The infinite square matrix T_{mn} known as the transition or T matrix in quantum mechanics yields the scattered wave coefficient B_n for a given incident wave.

The matrix method has several advantages over other known methods. First the T matrix depends only on the size and shape of the scatterer. Thus the same matrix can be used to analyze the response of an obstacle to any incident wave. Secondly, the principle of reciprocity implies that the T matrix is symmetric and the conservation of energy within any surface surrounding the scatterer implies that the scattering matrix (the S

matrix which is formed by $S = 1 - 2T$ should be unitary. Conforming to these two properties ensures the accuracy in the numerical evaluation of the matrix elements. Thirdly, each element of the matrix involves a line integral (two-dimensional problems) or a surface integral (three-dimensional problems) of the basis functions. Once a computer program is developed for numerical integration, only the equation to the surface of the obstacle need be changed for computing the T matrix for obstacles of different shape. The great efficiency of this method is demonstrated in Ref. 3.

In order to adapt this method for the scattering of waves in elastic solids, we need first a Helmholtz-type integral formula, like the one given above, for elastic waves. In the literature several formulations exist. As discussed by Pao and Varatharajulu,[7] the most convenient formula contains a second-rank Green's displacement dyadic and a third-rank Green's stress tensor in the integral representation. The surface integral then involves only the surface displacements and the surface tractions. These quantities are precisely those prescribed by the boundary conditions of elastodynamics. The success of the matrix method depends crucially on whether the surface fields can be eliminated in an efficient manner. A brief discussion of the convenient integral representation is given in Sec. I.

Next we need three sets of orthogonal basis functions for elastic waves, one for the longitudinal part and two for the transverse part. They are solutions of the vector wave equation and their properties have been well studied.[8] These basis functions have also been used in studying the scattering of elastic waves by a circular cylinder[9] and by a sphere.[10,11] For convenience, they are briefly summarized in Sec. II.

With all preliminary grounds covered, we show in Sec. III how the elements of the T matrix are evaluated for elastic wave scattering. To simplify the discussion, the scattering by an arbitrary shape cavity in two and three dimensions is illustrated in detail. The more general problem of an elastic solid inclusion with the rigid inclusion, the fluid inclusion, and the cavity as limiting cases are discussed in Sec. VI.

Finally, there is the need of a reciprocity theorem that relates the amplitude of the far field scattered along any direction for a given direction of incidence to the amplitude of the far field when the directions of incidence and scattering are interchanged and reversed. Because of mode conversion in elastic waves, p waves may be scattered into s waves and vice versa, reciprocity relations for elastic waves are far more complicated than those for acoustic or electromagnetic waves. A general theorem establishing reciprocity for elastic solid obstacles of all shapes was recently established by Varatharajulu.[12] With this theorem, the symmetry property of the S and T matrix and the unitarity of the S matrix are proved in Sec. IV and V.

Only the general formulation of the matrix theory is presented here. Numerical examples which illustrate the application of this theory will be presented in the second part of this paper.

I. INTEGRAL REPRESENTATION OF THE ELASTIC FIELD

Consider an infinite, homogeneous, isotropic and linearly elastic medium of density ρ and Lamé constants λ and μ, in which is embedded an elastic inclusion of different material constants ρ_1, λ_1, and μ_1. The surface S of the obstacle is assumed to be smooth, with a continuous turning normal.

A monochromatic wave of frequency ω is incident on S. The displacement vector corresponding to the incident wave is denoted by $\vec{u}^0$ and that corresponding to the scattered wave by $\vec{u}^s$. Since the incident wave has time dependence $\exp(-i\omega t)$, all field quantities will have the same time dependence and this time factor is suppressed for notational convenience. $\vec{u}^0$ has no singularity in the region enclosed by S and $\vec{u}^s$ has no sources outside S. The total displacement field outside S is given by

$$\vec{u}(\vec{r}) = \vec{u}^0(\vec{r}) = \vec{u}^s(\vec{r}) , \tag{1}$$

where $\vec{r}$ is the position vector of a field point (observation point).

The equation of motion in the elastic medium is given by

$$\nabla \cdot \overleftrightarrow{\tau} + \rho\omega^2 \vec{u} = 0 , \tag{2}$$

where $\overleftrightarrow{\tau}$ is the stress tensor related to the displacement gradients by

$$\overleftrightarrow{\tau} = \lambda \overleftrightarrow{I} \nabla \cdot \vec{u} + \mu(\nabla\vec{u} + \vec{u}\nabla) \tag{3}$$

and $\overleftrightarrow{I}$ is the idemfactor. The traction vector $\vec{t}$ is defined as

$$\vec{t} = \hat{n} \cdot \overleftrightarrow{\tau} = \overleftrightarrow{\tau} \cdot \hat{n} \tag{4}$$

where $\hat{n}$ is the unit outward normal to the surface S.

Corresonding to Eq. (2), a Green's dyadic $\overleftrightarrow{G}$ and a Green's stress tensor $\overleftrightarrow{\Sigma}$ are defined by

$$\nabla \cdot \overleftrightarrow{\Sigma}(\vec{r}\,|\,\vec{r}') + \rho\omega^2 \overleftrightarrow{G}(\vec{r}\,|\,\vec{r}') = -\overleftrightarrow{I}\delta(\vec{r} - \vec{r}') , \tag{5}$$

where $\delta(\vec{r} - \vec{r}')$ is the three- or two-dimensional delta function and $\vec{r}'$ is a source point. $\overleftrightarrow{\Sigma}$ is related to $\overleftrightarrow{G}$ as $\overleftrightarrow{\tau}$ is to $\vec{u}$ according to

$$\overleftrightarrow{\Sigma} = \lambda \overleftrightarrow{I} \nabla \cdot \overleftrightarrow{G} + \mu(\nabla \overleftrightarrow{G} + \overleftrightarrow{G}\nabla) . \tag{6}$$

In indicial notation Eq. (6) reads

$$\Sigma_{ijk} = \lambda\delta_{ij}\partial_l G_{lk} + \mu(\partial_i G_{jk} + \partial_j G_{ik}) , \tag{7}$$

where $\partial_i = \partial/\partial x_i$ and the subscripts i, j, k, l refer to vector components which take on values 1, 2, 3 in three dimensions (3-D) and 1, 2 in two dimensions (2-D). Note that Σ_{ijk} is symmetric only in the first two indices and the tensor $\overleftrightarrow{G}\nabla$ in Eq. (6) should be interpreted according to the last term in Eq. (7), which is different from the conventional dyadic notation.

The solution to Eq. (5) is well known (see for example, Ref. 8, Chap. 13) and is of the form

$$G_{ij}(\vec{r}\,|\,\vec{r}') = \kappa' \{\delta_{ij} k_s^2 g(k_s;\ \vec{r}\,|\,\vec{r}') - \partial_i \partial_j [g(k_p;\ \vec{r}\,|\,\vec{r}') - g(k_s;\ \vec{r}\,|\,\vec{r}')]\} ,$$

where $\kappa' = 1/(4\pi\rho\omega^2)$, $k_s^2 = \rho\omega^2/\mu$, and $k_p^2 = \rho\omega^2/(\lambda + 2\mu)$ are the squares of the shear wave number and compressional wave number, respectively. In Eq. (8), $g(k_p)$ and $g(k_s)$ are the Green's functions to the scalar wave equation given by

$$g(k;\,\vec{r}\,|\,\vec{r}\,') = \begin{cases} i\pi H_0(k|\vec{r}-\vec{r}\,'|)\,; & 2-\mathrm{D} \\ \exp(ik|\vec{r}-\vec{r}\,'|)/|\vec{r}-\vec{r}\,'| & 3-\mathrm{D} \end{cases}, \tag{9}$$

where H_0 is the Hankel function of the first kind and of order zero. To obtain $g(k_p)$ and $g(k_s)$ we simply replace k in Eq. (9) by k_p and k_s, respectively.

From Eqs. (1), (2), and (5), one can obtain the interior and exterior Helmholtz formulae for elastic waves in a straightforward manner.[7] One simply makes use of the fact that $\vec{u}^0$ is regular inside S and that $\vec{u}^s$ is regular outside S, and then applies the divergence theorem to Eq. (2) when it is postmultiplied by $\overleftrightarrow{G}$ and Eq. (5) after it is premultiplied by $\vec{u}$. The formula is

$$\vec{u}^0(\vec{r}) + \int_S \{\vec{u}'[\hat{n}'\cdot\overleftrightarrow{\Sigma}(\vec{r}\,|\,\vec{r}\,')] - \hat{n}'\cdot\overleftrightarrow{\tau}'\cdot\overleftrightarrow{G}(\vec{r}\,|\,\vec{r}\,')\}\,dS' = \begin{cases} \vec{u}(\vec{r})\,; & \vec{r}\text{ outside } S \\ 0\,; & \vec{r}\text{ inside } S \end{cases}. \tag{10}$$

In Eq. (10), the primes indicate that $\vec{u}'$, $\vec{\tau}'$, and $\hat{n}'$ are functions of $\vec{r}\,'$ and dS' is an element of area on S centered at $\vec{r}\,'$. In deriving the above equation, we have assumed that suitable radiation conditions are imposed on the scattered field far from S.[7] For two-dimensional scattering geometries the surface integral in Eq. (10) will be replaced by a contour integral along the circumference of the cylinder.

Equation (10) is the starting point for deriving the transition matrix for the scattering of elastic waves, which will be discussed in the following sections.

II. SPHERICAL AND CIRCULAR VECTOR BASIS FUNCTIONS

The matrix formulation of scattering differs from the eigenfunction expansion technique in that the same basis set may be used for obstacles of any shape. As discussed in the introduction, the vector spherical wave functions and the vector cylindrical wave functions form the basis sets for three- and two-dimensional problems, respectively. These functions are discussed in detail in Ref. 8. Since the elastic wave field is composed of solenoidal (divergence free) and irrotational (curl free) components, it is practical to choose one vector basis that is irrotational and two basis sets that are solenoidal.

In elasticity, the displacement vector can be constructed from three scalar functions P, Q, and S (two in 2-D) as

$$\vec{u}(\vec{r}) = \begin{cases} \nabla P + \nabla\times(\hat{z}Q)\,; & 2\text{-D} \\ \nabla P + k_s\nabla\times(\vec{r}Q) + \nabla\times[\nabla\times(\vec{r}S)]\,; & 3\text{-D} \end{cases}. \tag{11}$$

The potential P is associated with p waves and Q and S with s waves; each function satisfies a scalar wave equation with the appropriate wave number k_p or k_s. The unit vector $\hat{z}$ in Eq. (11) is along the axis of the cylinder and $\vec{r}$ is the radial vector in spherical polar coordinates.

Let $\vec{\phi}_n^\sigma$, $\vec{\psi}_n^\sigma$, and $\vec{\chi}_n^\sigma$ be the vector basis functions ($\vec{\chi}_n^\sigma$ being absent in 2-D). Following Eq. (11), they are defined as follows:

A. Two-dimensional basis functions

Let r, θ be plane polar coordinates perpendicular to the axis of a prismatic cylinder. We define

$$\vec{\phi}_n^\sigma(\vec{r}) = \begin{cases} \nabla[\epsilon_n^{1/2}H_n(k_p r)\cos n\theta]\,; & \sigma=1 \\ \nabla[\epsilon_n^{1/2}H_n(k_p r)\sin n\theta]\,; & \sigma=2 \end{cases} \tag{12a}$$

and

$$\vec{\psi}_n^\sigma(\vec{r}) = \begin{cases} \nabla\times[\hat{z}\epsilon_n^{1/2}H_n(k_s r)\cos n\theta]\,; & \sigma=1 \\ \nabla\times[\hat{z}\epsilon_n^{1/2}H_n(k_s r)\sin n\theta]\,; & \sigma=2 \end{cases} \tag{12b}$$

where $\epsilon_0=1$, and $\epsilon_n=2(n>0)$ is the Neumann factor. $H_n(kr)$ are Hankel functions of the first kind of order n and represent outgoing cylindrical waves for large values of kr. If a basis set that is regular at the origin is needed, we simply construct $\vec{\phi}_n^\sigma$ and $\vec{\psi}_n^\sigma$ with the real part of H_n, which is J_n, the Bessel function of the first kind. In the following sections, this regular set will be denoted by $\mathrm{Re}\vec{\phi}_n^\sigma$, or $\mathrm{Re}\vec{\psi}_n^\sigma$.

As shown in Ref. 8 (Chap. 13) the angular parts of these basis functions satisfy the following orthogonality conditions:

$$\int_0^{2\pi} \vec{\phi}_n^\sigma(\vec{r})\cdot\vec{\phi}_m^\nu(\vec{r})\,d\theta = E_n(k_p r)\delta_{mn}\delta_{\sigma\nu} \tag{13}$$

$$\int_0^{2\pi} \vec{\psi}_n^\sigma(\vec{r})\cdot\vec{\psi}_m^\nu(\vec{r})\,d\theta = F_n(k_s r)\delta_{mn}\delta_{\sigma\nu} \tag{14}$$

$$\int_0^{2\pi} \vec{\phi}_n^\sigma(\vec{r})\cdot\vec{\psi}_m^\nu(\vec{r})\,d\theta = G_n(k_p r,\, k_s r)\delta_{mn}(1-\delta_{\sigma\nu})\,, \tag{15}$$

where E_n, F_n, G_n are functions of the Bessel or Hankel functions, which are not relevant to this paper and δ_{nm} is the Kronecker delta. In the above, n, m are integers which take values 0, 1, 2, ... for σ, $\nu=1$, and 1, 2, 3, ... for σ, $\nu=2$.

B. Three-dimensional basis functions

In three-dimensional problems the polarization of the shear wave can be in any direction in a plane perpendicular to the wave vector. Since the polarization can not be easily resolved along two preferred directions, we need two separate basis functions to describe the shear wave. The three sets of basis functions in spherical coordinates r, θ, ϕ are:

$$\vec{\phi}_n^\sigma(\vec{r}) = \begin{cases} \Lambda^{1/2}\zeta_n\nabla[h_n(k_p r)P_n^m(\cos\theta)\cos m\phi]\,; & \sigma=1 \\ \Lambda^{1/2}\zeta_n\nabla[h_n(k_p r)P_n^m(\cos\theta)\sin m\phi]\,; & \sigma=2 \end{cases} \tag{16a}$$

$$\vec{\psi}_n^\sigma(\vec{r}) = \begin{cases} k_s\eta_n\nabla\times[\vec{r}\,h_n(k_s r)P_n^m(\cos\theta)\cos m\phi]\,; & \sigma=1 \\ k_s\eta_n\nabla\times[\vec{r}\,h_n(k_s r)P_n^m(\cos\theta)\sin m\phi]\,; & \sigma=2 \end{cases} \tag{16b}$$

$$\vec{\chi}_n^\sigma(\vec{r}) = (1/k_s)\nabla\times\vec{\psi}_n^\sigma(\vec{r})\,. \tag{16c}$$

In the above, $h_n(kr)$ are spherical Hankel functions of the third kind, of order n, P_n^m are associated Legendre polynomials, m is an integer that takes values 0, 1, 2, ..., n, and n is an integer that takes values 0, 1, 2, ... for $\sigma=1$ and 1, 2, 3, ... for $\sigma=2$, and ζ_n and η_n are normalization factors given by

$$\zeta_n=\left[\epsilon_n\,\frac{(2n+1)(n-m)!}{4\pi(n+m)!}\right]^{1/2}\;;\quad \eta_n=\left[\frac{\epsilon_n(2n+1)(n-m)!}{4\pi n(n+1)(n+m)!}\right]^{1/2}$$

and the parameter Λ is given by

$$\Lambda=k_p/k_s\ .$$

We note that the normalization of the compressional wave functions is different from the shear wave functions. This was not so in the two-dimensional case. In three-dimensional case this is necessary in order to prove the symmetry and unitarity of the scattering matrix.

To describe field quantities that are regular at the origin, we simply replace h_n by j_n, the spherical Bessel function of the first kind. The set will be denoted by $\mathrm{Re}\vec{\phi}_n^\sigma$, $\mathrm{Re}\vec{\psi}_n^\sigma$, and $\mathrm{Re}\vec{\chi}_n^\sigma$.

Since $P_n^m(\cos\theta)$, $\sin m\phi$ and $\cos m\phi$ form complete orthogonal sets, we can again establish certain orthogonality relations among the basis functions.

$$\int \vec{\phi}_n^\sigma(\vec{r})\cdot\vec{\phi}_l^\nu(\vec{r})\,d\Omega=L_n(k_p r)\delta_{nl}\,\delta_{\sigma\nu}\ , \tag{17}$$

$$\int \vec{\psi}_n^\sigma(\vec{r})\cdot\vec{\psi}_l^\nu(\vec{r})\,d\Omega=M_n(k_s r)\,\delta_{nl}\,\delta_{\sigma\nu}\ , \tag{18}$$

$$\int \vec{\chi}_n^\sigma(\vec{r})\cdot\vec{\chi}_l^\nu(\vec{r})d\Omega=N_n(k_s r)\delta_{nl}\,\delta_{\sigma\nu}\ , \tag{19}$$

$$\int \vec{\phi}_n^\sigma(\vec{r})\cdot\vec{\psi}_l^\nu(\vec{r})d\Omega=0\ , \tag{20}$$

$$\int \vec{\psi}_n^\sigma(\vec{r})\cdot\vec{\chi}_l^\nu(\vec{r})d\Omega=0\ , \tag{21}$$

and

$$\int \vec{\phi}_n^\sigma(\vec{r})\cdot\vec{\chi}_l^\nu(\vec{r})d\Omega=I_n(k_p r,\ k_s r)\delta_{nl}\,\delta_{\sigma\nu}\ . \tag{22}$$

In the above, $d\Omega=\sin\theta\,d\theta\,d\phi$, the range of integration being $0\le\theta\le\pi$ and $0\le\phi\le 2\pi$. The L_n, M_n, I_n, and N_n are functions involving the spherical Bessel or Hankel functions; the exact expressions for them are not relevant to the present discussion.

III. EVALUATION OF THE TRANSITION MATRIX

In this section, we present the transition matrix (the T matrix) for the scattering of elastic waves. All field quantities in the Helmholtz formula [Eq. (10) of Sec. I], the incident, scattered, and surface fields, the Green's displacement and stress tensors, are all expanded in the vector basis functions defined in Sec. II. The unknown expansion coefficients of the scattered displacement field are related to the known coefficients of the incident wave through the transition matrix. The elements of the T matrix involve surface integrals; for obstacles of arbitrary shape, they can only be evaluated numerically.

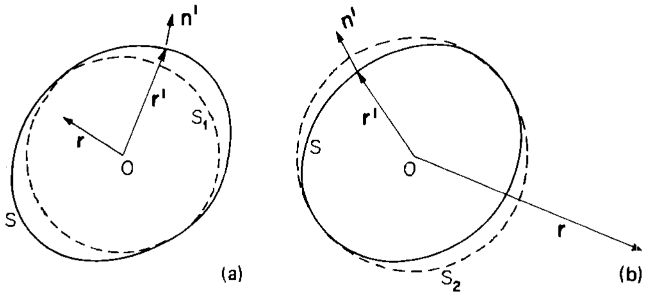

FIG. 1. (a) Geometry for points $\vec{r}$ interior to S. (b) Geometry for points $\vec{r}$ exterior to S.

Since the incident field has no sources in the region occupied by S, we expand $\vec{u}^0(\vec{r})$ in Eq. (1) in terms of basis functions that are regular at the origin of the coordinate system that is situated within S (Fig. 1). Thus,

$$\vec{u}^0(\vec{r})=\sum_{m,\sigma}\left[A_m^\sigma\mathrm{Re}\vec{\phi}_m^\sigma(\vec{r})+B_m^\sigma\mathrm{Re}\vec{\psi}_m^\sigma(\vec{r})+C_m^\sigma\mathrm{Re}\vec{\chi}_m^\sigma(\vec{r})\right]\ , \tag{23}$$

where A, B, C are assumed known [for plane wave incidence A, B, C are given by Eqs. (68), (70), (71)]; Re denotes the real part of the basis functions. The double summation is over $\sigma=1$, 2 and over $m=0$, 1, 2, ..., for $\sigma=1$ and $m=1$, 2, 3, ..., for $\sigma=2$. Actually, there is a third summation over the integral multiples of the azimuth angle ϕ [the integer m in Eq. (16)]. This third summation is omitted in the writing.

The outgoing scattered field is represented by

$$\vec{u}^s(\vec{r})=\sum_{m,\sigma}\left[\alpha_m^\sigma\vec{\phi}_m^\sigma(\vec{r})+\beta_m^\sigma\vec{\psi}_m^\sigma(\vec{r})+\gamma_m^\sigma\vec{\chi}_m^\sigma(\vec{r})\right]\ , \tag{24}$$

where α,β,γ are unknown coefficients to be determined.

The remaining functions to be expanded in the integral formulae, Eq. (10), are the Green's dyadic $\overleftrightarrow{G}$ and the Green's stress tensor $\overleftrightarrow{\Sigma}$. The expansion of $g(k_p)$ and $g(k_s)$ in the scalar basis sets are well known (Ref. 8, p. 827 for 2-D and p. 1874 for 3-D). Substituting these expansions in Eq. (8), we obtain after some amount of reduction

$$\overleftrightarrow{G}(\vec{r}\,|\,\vec{r}\,')=i\kappa\sum_{m,\sigma}\{(k_s)\vec{\phi}_m^\sigma(\vec{r}_>)\mathrm{Re}\vec{\phi}_m^\sigma(\vec{r}_<)$$
$$+(k_s)[\vec{\psi}_m^\sigma(\vec{r}_>)\mathrm{Re}\vec{\psi}_m^\sigma(\vec{r}_<)+\vec{\chi}_m^\sigma(\vec{r}_>)\mathrm{Re}\vec{\chi}_m^\sigma(\vec{r}_<)]\}\ , \tag{25}$$

where $\kappa=(\rho\omega)^{-2}$. In Eq. (25), $\vec{r}_>$ and $\vec{r}_<$ refer to the greater and lesser of $\vec{r}$ and $\vec{r}\,'$, respectively. Note that $\overleftrightarrow{G}(\vec{r}\,|\,\vec{r}\,')=\overleftrightarrow{G}(\vec{r}\,'\,|\,\vec{r})$.

We discuss in the sequel only the three-dimensional problems. For two-dimensional problems, one replaces (k_s) in Eq. (25) by π, and omits all terms involving $\vec{\chi}_m^\sigma$.

Now field quantities in Eq. (10) have all been expanded in the same basis set. The next step is to consider the two cases on the right hand side of the Helmholtz formula, namely, $\vec{r}$ inside S and $\vec{r}$ outside S.

First consider points $\vec{r}$ that lie inside the inscribed sphere, S_1 (circle in 2-D) and let the origin of the coordinate system be situated at the center of the sphere [Fig. 1(a)]. Since now, $|\vec{r}|<|\vec{r}\,'|$, we set $\vec{r}_>=\vec{r}\,'$ and $\vec{r}_<=\vec{r}$. Substituting Eqs. (23) and (25) into Eq. (10) and using the definition of $\overleftrightarrow{\Sigma}$ from Eq. (6), we obtain

$$\sum_{m,\sigma}[A_m^\sigma \mathrm{Re}\vec{\phi}_m^\sigma(\vec{r}) + B_m^\sigma \mathrm{Re}\vec{\psi}_m^\sigma(\vec{r}) + C_m^\sigma \mathrm{Re}\vec{\chi}_m^\sigma(\vec{r})] = -\sum_{n,\nu}\int_S [\vec{u}'\cdot\hat{n}'\cdot \overleftrightarrow{\Sigma}_n^\nu(\vec{r}|\vec{r}') - \hat{n}'\cdot\overleftrightarrow{\tau}'\cdot\overleftrightarrow{G}_n^\nu(\vec{r}|\vec{r}')]dS'\,; \quad \vec{r}\ \text{inside}\ S_1\,, \tag{26}$$

where

$$\overleftrightarrow{G}_n^\nu(\vec{r}|\vec{r}') = i\kappa[(k_s)\vec{\phi}_n^\nu(\vec{r}')\mathrm{Re}\vec{\phi}_n^\nu(\vec{r}) + (k_s)\vec{\psi}_n^\nu(\vec{r}')\mathrm{Re}\vec{\psi}_n^\nu(\vec{r}) + (k_s)\vec{\chi}_n^\nu(\vec{r}')\mathrm{Re}\vec{\chi}_n^\nu(\vec{r})]$$

$$\overleftrightarrow{\Sigma}_n^\nu(\vec{r}|\vec{r}') = i\kappa(k_s)\{\lambda\overleftrightarrow{I}\mathrm{Re}\vec{\phi}_n^\nu(\vec{r})\nabla'\cdot\vec{\phi}_n^\nu(\vec{r}') + \mu[\nabla'\vec{\phi}_n^\nu(\vec{r}') + \vec{\phi}_n^\nu(\vec{r}')\nabla']\mathrm{Re}\vec{\phi}_n^\nu(\vec{r})\}$$
$$+ i\kappa\mu(k_s)\{[\nabla'\vec{\psi}_n^\nu(\vec{r}') + \vec{\psi}_n^\nu(\vec{r}')\nabla']\mathrm{Re}\vec{\psi}_n^\nu(\vec{r}) + [\nabla'\vec{\chi}_n^\nu(\vec{r}') + \vec{\chi}_n^\nu(\vec{r}')\nabla']\mathrm{Re}\vec{\chi}_n^\nu(\vec{r})\}\,.$$

In the above equations $\kappa = 1/(\rho\omega^2)$, and primes on dS', ∇', $\vec{u}'$, $\overleftrightarrow{\tau}'$, $\hat{n}'$ indicate that they are functions of $\vec{r}'$ which is the source coordinate and the variable of integration. Again, for two-dimensional problems, (k_s) should be replaced by π, and the $\vec{\chi}$ term should be dropped. In writing the expressions for $\overleftrightarrow{G}_n^\nu$ and $\overleftrightarrow{\Sigma}_n^\nu$ we have used the symmetry properties of $\overleftrightarrow{G}$ and $\overleftrightarrow{\Sigma}$.

Next, consider points $\vec{r}$ that lie outside the sphere S_2 circumscribed on S and choose the origin of the coordinate system at the center of S_2 [Fig. 1(b)]. Clearly $|\vec{r}| > |\vec{r}'|$, we now obtain from Eqs. 24 and 25,

$$\sum_{m,\sigma}[\alpha_m^\sigma \vec{\phi}_m^\sigma(\vec{r}) + \beta_m^\sigma \vec{\psi}_m^\sigma(\vec{r}) + \gamma_m^\sigma \vec{\chi}_m^\sigma(\vec{r})] = \sum_{n,\nu}\int_S [\vec{u}'\cdot\hat{n}'\cdot \overleftrightarrow{\Sigma}_n^\nu(\vec{r}'|\vec{r}) - \hat{n}'\cdot\overleftrightarrow{\tau}'\cdot\overleftrightarrow{G}_n^\nu(\vec{r}'|\vec{r})]dS'\,, \quad \vec{r}\ \text{outside}\ S_2\,, \tag{27}$$

$$\overleftrightarrow{\Sigma}_n^\nu(\vec{r}'|\vec{r}) = i\kappa(k_s)\{\lambda\overleftrightarrow{I}\nabla'\cdot\mathrm{Re}\vec{\phi}_n^\nu(\vec{r}')\vec{\phi}_n^\nu(\vec{r}) + \mu[\nabla'\mathrm{Re}\vec{\phi}_n^\nu(\vec{r}') + \mathrm{Re}\vec{\phi}_n^\nu(\vec{r}')\nabla']\vec{\phi}_n^\nu(\vec{r})\}$$
$$+ i\kappa\mu(k_s)\{[\nabla'\mathrm{Re}\vec{\psi}_n^\nu(\vec{r}') + \mathrm{Re}\vec{\psi}_n^\nu(\vec{r}')\nabla']\vec{\psi}_n^\nu(\vec{r}) + [\nabla'\mathrm{Re}\vec{\chi}_n^\nu(\vec{r}') + \mathrm{Re}\vec{\chi}_n^\nu(\vec{r}')\nabla']\vec{\chi}_n^\nu(\vec{r})\}\,.$$

The $\overleftrightarrow{G}_n^\nu(\vec{r}'|\vec{r})$ in Eq. (27) is obtained from $\overleftrightarrow{G}_n^\nu(\vec{r}|\vec{r}')$ in Eq. (26) by interchanging $\vec{r}$ and $\vec{r}'$.

Note that in both Eqs. (26) and (27), the $\vec{u}'$ and $\hat{n}'\cdot\overleftrightarrow{\tau}'$ which are displacement and traction at the surface S respectively, remain unspecified at this stage. To find how A, B, C from Eq. (26) and α, β, γ from Eq. (27) are related to these unspecified surface source quantities, we take the scalar product of Eq. (26) successively with $\mathrm{Re}\vec{\phi}_m^\sigma(\vec{r})$, $\mathrm{Re}\vec{\psi}_m^\sigma(\vec{r})$, and $\mathrm{Re}\vec{\chi}_m^\sigma(\vec{r})$ and the scalar product of Eq. (27) successively with $\vec{\phi}_m^\sigma(\vec{r})$, $\vec{\psi}_m^\sigma(\vec{r})$, and $\vec{\chi}_m^\sigma(\vec{r})$. We then integrate on all possible orientations of $\vec{r}$ keeping $|\vec{r}|$ fixed and apply the orthogonality conditions, Eqs. (17)–(22). We then obtain a set of simultaneous equations for A_m^σ, B_m^σ, $C_m^\sigma(\sigma = 1, 2)$ which can be solved to yield

$$A_m^\sigma = -i\kappa(k_s)\int_S \{\vec{u}'\cdot\hat{n}'\cdot[\overleftrightarrow{I}\lambda\nabla'\cdot\vec{\phi}_m^\sigma(\vec{r}') + \mu\nabla'\vec{\phi}_m^\sigma(\vec{r}') + \mu\vec{\phi}_m^\sigma(\vec{r}')\nabla'] - \hat{n}'\cdot\overleftrightarrow{\tau}'\cdot\vec{\phi}_m^\sigma(\vec{r}')\}dS'\,, \tag{28}$$

$$B_m^\sigma = -i\kappa(k_s)\int_S \{\vec{u}'\cdot\hat{n}'\cdot[\nabla'\vec{\psi}_m^\sigma(\vec{r}') + \vec{\psi}_m^\sigma(\vec{r}')\nabla']\mu - \hat{n}'\cdot\overleftrightarrow{\tau}'\cdot\vec{\psi}_m^\sigma(\vec{r}')\}dS'\,, \tag{29}$$

$$C_m^\sigma = -i\kappa(k_s)\int_S \{\vec{u}'\cdot\hat{n}'\cdot[\nabla'\vec{\chi}_m^\sigma(\vec{r}') + \vec{\chi}_m^\sigma(\vec{r}')\nabla']\mu - \hat{n}'\cdot\overleftrightarrow{\tau}'\cdot\vec{\chi}_m^\sigma(\vec{r}')\}dS'\,; \tag{30}$$

and from Eq. (27)

$$\alpha_m^\sigma = i\kappa(k_s)\int_S \{\vec{u}'\cdot\hat{n}'\cdot[\overleftrightarrow{I}\lambda\nabla'\cdot\mathrm{Re}\vec{\phi}_m^\sigma(\vec{r}') + \mu\nabla'\mathrm{Re}\vec{\phi}_m^\sigma(\vec{r}') + \mu\mathrm{Re}\vec{\phi}_m^\sigma(\vec{r}')\nabla'] - \hat{n}'\cdot\overleftrightarrow{\tau}'\cdot\mathrm{Re}\vec{\phi}_m^\sigma(\vec{r}')\}\,dS'\,, \tag{31}$$

$$\beta_m^\sigma = i\kappa(k_s)\int_S \{\vec{u}'\cdot\hat{n}'\cdot[\nabla'\mathrm{Re}\vec{\psi}_m^\sigma(\vec{r}') + \mathrm{Re}\vec{\psi}_m^\sigma(\vec{r}')\nabla']\mu - \hat{n}'\cdot\overleftrightarrow{\tau}'\cdot\mathrm{Re}\vec{\psi}_m^\sigma(\vec{r}')\}dS'\,, \tag{32}$$

and

$$\gamma_m^\sigma = i\kappa(k_s)\int_S \{\vec{u}'\cdot\hat{n}'\cdot[\nabla'\mathrm{Re}\vec{\chi}_m^\sigma(\vec{r}') + \mathrm{Re}\vec{\chi}_m^\sigma(\vec{r}')\nabla']\mu - \hat{n}'\cdot\overleftrightarrow{\tau}'\cdot\mathrm{Re}\vec{\chi}_m^\sigma(\vec{r}')\}\,dS'\,. \tag{33}$$

At this stage, we must be specific about the boundary conditions at the surface S. Consider, for example, the obstacle in the form of a cavity. The surface of a cavity is stress free, so on the right-hand side of Eqs. (28)–(33) we set

$$\hat{n}'\cdot\overleftrightarrow{\tau}' = \hat{n}(\vec{r}')\cdot\overleftrightarrow{\tau}(\vec{r}') = 0\,, \quad \vec{r}'\ \text{on}\ S\,. \tag{34}$$

The still unknown surface displacement is expanded in the vector basis set,

$$\vec{u}(\vec{r}') = \sum_{n,\nu}[a_n^\nu \mathrm{Re}\vec{\phi}_n^\nu(\vec{r}') + b_n^\nu \mathrm{Re}\vec{\psi}_n^\nu(\vec{r}') + c_n^\nu \mathrm{Re}\vec{\chi}_n^\nu(\vec{r}')]\,, \quad \vec{r}'\ \text{on}\ S \tag{35}$$

where a, b, c are new unknown coefficients.

Substituting Eqs. (34) and (35) in Eqs. (28)–(30), we obtain, in matrix notation,

$$\begin{bmatrix} A_m^\sigma \\ B_m^\sigma \\ C_m^\sigma \end{bmatrix} = -i \begin{bmatrix} (Q^{11})_{mn}^{\sigma\nu} & (Q^{12})_{mn}^{\sigma\nu} & (Q^{13})_{mn}^{\sigma\nu} \\ (Q^{21})_{mn}^{\sigma\nu} & (Q^{22})_{mn}^{\sigma\nu} & (Q^{23})_{mn}^{\sigma\nu} \\ (Q^{31})_{mn}^{\sigma\nu} & (Q^{32})_{mn}^{\sigma\nu} & (Q^{33})_{mn}^{\sigma\nu} \end{bmatrix} \begin{bmatrix} a_n^\nu \\ b_n^\nu \\ c_n^\nu \end{bmatrix}\,. \tag{36}$$

Similarly, from Eqs. (31)–(33), we obtain

$$\begin{bmatrix} \alpha_m^\sigma \\ \beta_m^\sigma \\ \gamma_m^\sigma \end{bmatrix} = i \begin{bmatrix} \mathrm{Re}(Q^{11})_{mn}^{\sigma\nu} & \mathrm{Re}(Q^{12})_{mn}^{\sigma\nu} & \mathrm{Re}(Q^{13})_{mn}^{\sigma\nu} \\ \mathrm{Re}(Q^{21})_{mn}^{\sigma\nu} & \mathrm{Re}(Q^{22})_{mn}^{\sigma\nu} & \mathrm{Re}(Q^{23})_{mn}^{\sigma\nu} \\ \mathrm{Re}(Q^{31})_{mn}^{\sigma\nu} & \mathrm{Re}(Q^{32})_{mn}^{\sigma\nu} & \mathrm{Re}(Q^{33})_{mn}^{\sigma\nu} \end{bmatrix} \begin{bmatrix} a_n^\nu \\ b_n^\nu \\ c_n^\nu \end{bmatrix}\,. \tag{37}$$

In the above, the surface integrals that were present on the right-hand side of Eqs. (28)–(33) have been defined compactly in terms of nine Q matrices, infinite in size.

The expressions for the Q matrices in the case of a cavity are given below:

$$(Q^{11})^{\sigma\nu}_{mn} = \kappa(k_s)\int_S \mathrm{Re}\vec{\phi}^{\nu}_n(\vec{r}')\cdot\hat{n}'\cdot[\lambda\overleftrightarrow{I}\nabla'\cdot\vec{\phi}^{\sigma}_m(\vec{r}') + \mu\nabla'\vec{\phi}^{\sigma}_m(\vec{r}') + \mu\vec{\phi}^{\sigma}_m(\vec{r}')\nabla']\,dS' . \quad (38)$$

The expressions for $(Q^{12})^{\sigma\nu}_{mn}$ and $(Q^{13})^{\sigma\nu}_{mn}$ can be obtained by replacing $\mathrm{Re}\vec{\phi}^{\nu}_n(\vec{r}')$ in the integrand of Eq. (38) by $\mathrm{Re}\vec{\psi}^{\nu}_n(\vec{r}')$ and $\mathrm{Re}\vec{\chi}^{\nu}_n(\vec{r}')$, respectively.

$$(Q^{21})^{\sigma\nu}_{mn} = \kappa(k_s)\mu\int_S \mathrm{Re}\vec{\phi}^{\nu}_n(\vec{r}')\cdot\hat{n}'[\nabla'\vec{\psi}^{\sigma}_m(\vec{r}') + \vec{\psi}^{\sigma}_m(\vec{r}')\nabla']\,dS' \quad (39)$$

and Q^{22} and Q^{23} are obtained from Eq. 39 when $\mathrm{Re}\vec{\phi}^{\nu}_n$ are replaced by $\mathrm{Re}\vec{\psi}^{\nu}_n$ and $\mathrm{Re}\vec{\chi}^{\nu}_n$, respectively. Similarly,

$$(Q^{31})^{\sigma\nu}_{mn} = \kappa\mu(k_s)\int_S \mathrm{Re}\vec{\phi}^{\nu}_n(\vec{r}')\cdot\hat{n}'\cdot[\nabla'\vec{\chi}^{\sigma}_m(\vec{r}') + \vec{\chi}^{\sigma}_m(\vec{r}')\nabla']\,dS' \quad (40)$$

and Q^{32} and Q^{33} are obtained by replacing $\mathrm{Re}\vec{\phi}^{\nu}_n$ in the integrand by $\mathrm{Re}\vec{\psi}^{\nu}_n$ and $\mathrm{Re}\vec{\chi}^{\nu}_n$, respectively.

Note that although the values of the Q-matrix elements depends on the shape of S, the integrand does not depend explicitly on the shape.

The matrix equations (36) may be formally inverted to solve for a, b, c in terms of A, B, C and substituted into Eq. (37) to yield

$$\begin{bmatrix}\alpha^{\sigma}_m\\ \beta^{\sigma}_m\\ \gamma^{\sigma}_m\end{bmatrix} = -\begin{bmatrix}(T^{11})^{\sigma\nu}_{mn} & (T^{12})^{\sigma\nu}_{mn} & (T^{13})^{\sigma\nu}_{mn}\\ (T^{21})^{\sigma\nu}_{mn} & (T^{22})^{\sigma\nu}_{mn} & (T^{23})^{\sigma\nu}_{mn}\\ (T^{31})^{\sigma\nu}_{mn} & (T^{32})^{\sigma\nu}_{mn} & (T^{33})^{\sigma\nu}_{mn}\end{bmatrix}\begin{bmatrix}A^{\nu}_n\\ B^{\nu}_n\\ C^{\nu}_n\end{bmatrix}, \quad (41)$$

where we have introduced the notation

$$T \equiv \begin{bmatrix}T^{11} & T^{12} & T^{13}\\ T^{21} & T^{22} & T^{23}\\ T^{31} & T^{32} & T^{33}\end{bmatrix} \equiv \begin{bmatrix}\mathrm{Re}\,Q^{11} & \mathrm{Re}\,Q^{12} & \mathrm{Re}\,Q^{13}\\ \mathrm{Re}\,Q^{21} & \mathrm{Re}\,Q^{22} & \mathrm{Re}\,Q^{23}\\ \mathrm{Re}\,Q^{31} & \mathrm{Re}\,Q^{32} & \mathrm{Re}\,Q^{33}\end{bmatrix}\begin{bmatrix}Q^{11} & Q^{12} & Q^{13}\\ Q^{21} & Q^{22} & Q^{23}\\ Q^{31} & Q^{32} & Q^{33}\end{bmatrix}^{-1}. \quad (42)$$

The double infinite matrix T is the desired transition matrix.

In Eq. (42), we have suppressed the indices mn, $\sigma\nu$ on the Q and T matrices for brevity. In two-dimensional problems both the Q and T matrices have a 2×2 structure rather than a 3×3 structure as in three-dimensional cases.

Equation (41) conveys the interesting result that the waves scattered due to any type of harmonic incident wave represented by the coefficients A, B, C is completely characterized by the T matrix. The T matrix depends only on the nature and shape of the obstacle and is independent of the exciting field. The symmetry properties due to the geometry of an obstacle are reflected in simplified expressions for the Q matrix elements. For instance, the sphere and a right ciruclar cylinder are trivial cases in the matrix formulation, as the Q and T matrices are diagonal if the 3×3 structure is taken as a unit element. Even for elliptic cylinders and ellipsoids, we can use symmetry arguments to set many of the elements of the Q matrix to zero.

Before presenting the T matrix for an elastic obstacle, we discuss first two global properties of the T matrix, which are a consequence of the conservation of energy in elastic solids and the principle of reciprocity.

IV. SYMMETRY OF THE TRANSITION MATRIX

Just as in acoustic, electromagnetic, and quantum mechanical scattering processes, certain reciprocity relations are satisfied in elastic wave scattering. These have been proved for elastic waves by Varatharajulu.[12] Reciprocity refers to the equality of the farfield amplitude for two processes which are obtained by interchanging the *position* of source and observer and reversing the sign of all momenta. In this sense it is not the same as time reversal in which the source for one scattering process becomes the receiver for the second process and vice versa, in addition to reversing all momenta.

For elastic waves, we get an additional interesting reciprocity relation which states that the amplitudes of two waves which have undergone mode conversion in the opposite sense are proportional to each other.[12] For acoustic and electromagnetic waves, the reciprocity relation can be used to prove the symmetry of the T matrix. In this section, the symmetry of the T matrix is proved for elastic wave scattering. Although reciprocity relations have been derived only for plane wave incidence, the results will be true for the Fourier components of other types of incident waves.

Since the proof is somewhat involved, the two- and three-dimensional cases will be treated separately.

A. Two-dimensional geometry

A plane p or s wave of frequency ω propagating along $\hat{k}$ is incident on an obstacle. The expansion of a plane wave in circular cylinder functions is given by (Ref. 8, p. 828)

$$\exp(i\vec{k}\cdot\vec{r}) = \sum_n \epsilon_n i^n J_n(kr)\cos n(\theta-\theta_k) , \tag{43}$$

where k, θ_k are the polar coordinates of $\vec{k}$ and r, θ those of $\vec{r}$.

An incident compressional wave is given by

$$\vec{u}^0(\vec{r}) = \nabla \exp(i\vec{k}_p\cdot\vec{r}) = \sum_{n,\sigma} A_n^\sigma(\hat{k})\mathrm{Re}\vec{\phi}_n^\sigma(\vec{r}) , \tag{44}$$

where

$$A_n^\sigma(\hat{k}) = \begin{cases} \epsilon_n^{1/2} i^n \cos n\theta_k , & \sigma=1 , \\ \epsilon_n^{1/2} i^n \sin n\theta_k , & \sigma=2 . \end{cases} \tag{45}$$

Similarly an incident shear wave is given by

$$\vec{u}^0(\vec{r}) = \nabla x[\hat{z}\exp(i\vec{k}_s\cdot\vec{r})] = \sum_{n,\sigma} B_n^\sigma(\hat{k})\mathrm{Re}\vec{\psi}_n^\sigma(\vec{r}) , \tag{46}$$

where

$$B_n^\sigma(\hat{k}) = \begin{cases} \epsilon_n^{1/2} i^n \cos n\theta_k , & \sigma=1 , \\ \epsilon_n^{1/2} i^n \sin n\theta_k , & \sigma=2 . \end{cases} \tag{47}$$

Thus for plane wave incidence $A_n^\sigma = B_n^\sigma$.

When $k_s r$ is large, the scattered displacement in Eq. (24) may be written as

$$\vec{u}^s(\vec{r}) \underset{r\to\infty}{\longrightarrow} \hat{r} f_p(\hat{k}, \hat{r})(2/i\pi k_p r)^{1/2}\exp(ik_p r) + \hat{\theta} f_s(\hat{k}, \hat{r})(2/i\pi k_s r)^{1/2}\exp(ik_s r) . \tag{48}$$

$\vec{u}^s$ consists of two outgoing cylindrical waves propagating along $\vec{r}$, the direction of observation are polarized parallel and perpendicular to $\hat{r}$, respectively. The wave polarized along $\hat{r}$ is the p wave and its amplitude f_p depends only on θ and the direction of incidence, namely $\hat{k}$. Similarly f_s is the amplitude of the scattered s wave. Expressions for f_p and f_s may be obtained from Eq. (24) by substituting the asymptotic form of the radial part of the basis functions,

$$f_p(\hat{k}, \hat{r}) = ik_p \sum_{n,\sigma} [A_n^\sigma(\hat{r})]^* \alpha_n^\sigma(\hat{k}) \tag{49}$$

and

$$f_s(\hat{k}, \hat{r}) = -ik_s \sum_{n,\sigma} [A_n^\sigma(\hat{r})]^* \beta_n^\sigma(\hat{k}) , \tag{50}$$

where the asterisk indicates complex conjugate, and $A(\hat{r})$ is given in Eq. (45) with θ_k replaced by θ.

Given below are the reciprocity relations derived in Ref. 12:

$$f_{pp}(\hat{k}, \hat{r}) = f_{pp}(-\hat{r}, -\hat{k}) , \tag{51}$$

$$f_{ss}(\hat{k}, \hat{r}) = f_{ss}(-\hat{r}, -\hat{k}), \tag{52}$$

and

$$k_p f_{sp}(\hat{k}, \hat{r}) = -k_s f_{ps}(-\hat{r}, -\hat{k}) . \tag{53}$$

An additional subscript (the second) is assigned to denote the polarization of the incident wave. Thus, $f_{sp}(\hat{k}, \hat{r})$ is the amplitude of an s wave scattered along $\hat{r}$ when a p wave is incident along $\hat{k}$.

From Eq. (45), it may be observed that $A_n^\sigma(-\hat{r}) = [A_n^\sigma(\hat{r})]^*$. For a p wave, $B=0$. Using the solution of the scattered wave coefficients given in Eq. (41), we obtain from Eq. (49),

$$\begin{aligned} f_{pp}(\hat{k}, \hat{r}) &= -ik_p \overline{A^*}(\hat{r}) T^{11} A(\hat{k}) \\ &= -ik_p \overline{A}(\hat{k}) \overline{T^{11}} A^*(\hat{r}) . \end{aligned} \tag{54}$$

In the above we have suppressed all indices and summation signs, $A(\hat{r})$ denotes the column matrix formed by A_n^σ and the overhead bar indicates transposition. For example $\overline{A}T^{11}$ indicates the matrix product of the row vector formed by $\overline{A}$ and the matrix T^{11}. The second equality in Eq. (54) is derived by taking the transpose of the equation. Since f_{pp} is just a scalar, $\overline{f_{pp}} = f_{pp}$.

Similarly

$$f_{pp}(-\hat{r}, -\hat{k}) = -ik_p \overline{A}(\hat{k}) T^{11} A^*(\hat{r}) . \tag{55}$$

Substituting Eqs. (54)–(55) in Eq. (51), we find

$$T^{11} = \overline{T^{11}} . \tag{56}$$

Next note that

$$f_{ss}(\hat{k}, \hat{r}) = ik_s \overline{A^*}(\hat{r}) T^{22} A(\hat{k}) = ik_s A(\hat{k}) \overline{T^{22}} A^*(\hat{r}) \tag{57}$$

and

$$f_{ss}(-\hat{r}, -\hat{k}) = ik_s \overline{A}(\hat{k}) T^{22} A^*(\hat{r}) . \tag{58}$$

The second reciprocity relation, Eq. (32) then implies

$$T^{22} = \overline{T^{22}} . \tag{59}$$

Finally, note that

$$f_{sp}(\hat{k}, \hat{r}) = ik_s \overline{A^*}(\hat{r}) T^{21} A(\hat{k}) \tag{60}$$

and

$$f_{ps}(-\hat{r}, -\hat{k}) = -ik_p \overline{A}(\hat{k}) \overline{T^{12}} A^*(\hat{r}) = -ik_p \overline{A^*}(\hat{r}) \overline{T^{12}} A(\hat{k}) , \tag{61}$$

and the third reciprocity relation implies

$$T^{21} = \overline{T^{12}} . \tag{62}$$

From Eqs. (56), (59), and (62)

$$T = \begin{bmatrix} T^{11} & T^{12} \\ T^{21} & T^{22} \end{bmatrix} = \begin{bmatrix} \overline{T^{11}} & \overline{T^{21}} \\ \overline{T^{12}} & \overline{T^{22}} \end{bmatrix} = \overline{T} . \tag{63}$$

B. Three-dimensional scattering geometry

Again assume that a plane wave of frequency ω propagating along $\hat{k}$ is incident on the obstacle. If the incident wave is a shear wave, the polarization vector of the incident wave can lie in any direction in a plane perpendicular to $\hat{k}$, unlike the two-dimensional case where it is uniquely fixed.

In order to write the expansion of a plane wave in spherical polar coordinates, we first define the vector spherical harmonics which form a complete orthogonal set (p. 1898 of Ref. 8),

$$\vec{A}_m^\sigma(\hat{r}) = \begin{cases} \zeta_m^\sigma \hat{r}[P_m^l(\theta)\cos l\phi] , & \sigma=1 , \\ \zeta_m^\sigma \hat{r}[P_m^l(\theta)\sin l\phi] , & \sigma=2 , \end{cases} \tag{64}$$

$$\vec{B}_m^\sigma(\hat{r}) = \begin{cases} \eta_m^\sigma \hat{r}\times\nabla\times[\vec{r}P_m^l(\theta)\cos l\phi] , & \sigma=1 , \\ \eta_m^\sigma \hat{r}\times\nabla\times[\vec{r}P_m^l(\theta)\sin l\phi] , & \sigma=2 , \end{cases} \tag{65}$$

and

$$\vec{C}_m^\sigma(\hat{r}) = \begin{cases} \eta_m^\sigma \nabla \times [\vec{r} P_m^l(\theta) \cos l\phi], & \sigma = 1, \\ \eta_m^\sigma \nabla \times [\vec{r} P_m^l(\theta) \sin l\phi], & \sigma = 2, \end{cases} \tag{66}$$

where ζ_m^σ and η_m^σ have been defined in Eq. (16).

The expansion coefficients of a p wave propagating along $\hat{k}(\theta_k, \phi_k)$ can be obtained from Ref. 8, p. 1866 as

$$\vec{u}^0(\vec{r}) = \sum_{n,\sigma} \Lambda^{-1/2} i^{n-1} [\hat{k} \cdot \vec{A}_n^\sigma(\hat{k})] \operatorname{Re}\vec{\phi}_n^\sigma(\vec{r}) . \tag{67}$$

Thus the first coefficient in Eq. (23) is

$$A_n^\sigma(\hat{k}) = \Lambda^{-1/2} i^{n-1} \hat{k} \cdot \vec{A}_n^\sigma(\hat{k}) . \tag{68}$$

A plane shear wave with polarization vector $\hat{v}$ and propagation vector $\hat{k}$ is written as

$$\vec{u}^0(\vec{r}) = \sum_{n,\sigma} i^{n-1} \hat{v} \cdot [i \vec{B}_n^\sigma(\hat{k}) \operatorname{Re}\vec{\psi}_n^\sigma(\vec{r}) + \vec{C}_n^\sigma(\hat{k}) \operatorname{Re}\vec{\chi}_n^\sigma(\vec{r})] . \tag{69}$$

Thus, in Eq. (23),

$$B_n^\sigma(\hat{k}) = i^n \hat{v} \cdot \vec{B}_n^\sigma(\hat{k}) \tag{70}$$

and

$$C_n^\sigma(\hat{k}) = i^{n-1} \hat{v} \cdot \vec{C}_n^\sigma(\hat{k}) . \tag{71}$$

The scattered field far from S can be obtained by using the asymptotic forms for the spatial part of the basis functions in Eq. (24), with

$$\vec{u}^s(\vec{r}) \xrightarrow[r\to\infty]{} \hat{r} f_p(\hat{k}, \hat{r}) \exp(ik_p r)/r + [\hat{\theta} f_{s1}(\hat{k}, \hat{r}) + \hat{\phi} f_{s2}(\hat{k}, \hat{r})] \exp(ik_s r)/r . \tag{72}$$

The scattered field again consists of two outgoing spherical waves, polarized parallel and perpendicular to the direction of propagation $\hat{r}$. The polarization of the shear wave is further resolved in two orthogonal directions $\hat{\theta}$ and $\hat{\phi}$ with amplitudes f_{s1} and f_{s2}. Expressions for the scattered amplitudes can be written as

$$f_p(\hat{k}, \hat{r}) = -i\Lambda \sum_{n,\sigma} [A_n^\sigma(\hat{r})]^* \alpha_n^\sigma(\hat{k}) \tag{73}$$

and

$$\hat{u} \cdot [\hat{\theta} f_{s1}(\hat{k}, \hat{r}) + \hat{\phi} f_{s2}(\hat{k}, \hat{r})] = -i \sum_{n,\sigma} \{[B_n^\sigma(\hat{r})]^* \beta_n^\sigma(\hat{k}) + [C_n^\sigma(\hat{r})]^* \gamma_n^\sigma(\hat{k})\} , \tag{74}$$

where $\hat{u}$ is the polarization vector of the outgoing, spherical, shear wave.

The reciprocity relations obtained for three-dimensional scattering are quoted below from Ref. 12,

$$f_{pp}(\hat{k}, \hat{r}) = f_{pp}(-\hat{r}, -\hat{k}) , \tag{75}$$

$$\hat{u} \cdot [\hat{\theta} f_{s1s}(\hat{k}, \hat{r}) + \hat{\phi} f_{s2s}(\hat{k}, \hat{r})] = -\hat{v} \cdot [\hat{\theta}_{-k} f_{s1s}(-\hat{r}, -\hat{k}) + \hat{\phi}_{-k} f_{s2s}(-\hat{r}, -\hat{k})] , \tag{76}$$

and

$$f_{ps}(\hat{k}, \hat{r}) = \Lambda(-\hat{v}) \cdot [\hat{\theta}_{-k} f_{s1p}(-\hat{r}, -\hat{k}) + \hat{\phi}_{-k} f_{s2p}(-\hat{r}, -\hat{k})] . \tag{77}$$

In Eqs. (76) and (77), $\hat{v}$ is the polarization vector of the wave propagating along $\hat{k}$ and $\hat{u}$, that of the wave propagating along $\hat{r}$. $\hat{\theta}_{-k}$ and $\hat{\phi}_{-k}$ are the unit vectors corresponding to $-\hat{k}$ in spherical polar coordinates.

Under the transformation $\hat{r} \to -\hat{r}$,

$$A_n^\sigma(-\hat{r}), B_n^\sigma(-\hat{r}), C_n^\sigma(-\hat{r}) \to [A_n^\sigma(\hat{r})]^*, [B_n^\sigma(\hat{r})]^*, [C_n^\sigma(\hat{r})]^* .$$

Substituting the expressions for the scattered amplitudes from Eqs. (73) and (74) and making use of the expression for α, β, γ in terms of the T matrix and the incident wave coefficients A, B, C from Eq. (41), we obtain

$$f_{pp}(\hat{k}, \hat{r}) = i\Lambda \overline{A^*(\hat{r})} T^{11} A(\hat{k}) = i\Lambda \overline{A(\hat{k})}\, \overline{T^{11}} A^*(\hat{r}) \tag{78}$$

and

$$f_{pp}(-\hat{r}, -\hat{k}) = i\Lambda \overline{A(\hat{k})} T^{11} A^*(\hat{r}) . \tag{79}$$

From the first reciprocity relation we obtain

$$T^{11} = \overline{T^{11}} . \tag{80}$$

In the above we have made use of the fact that $B, C = 0$ for an incident p wave.

For an incident s wave, $A = 0$,

$$\hat{u} \cdot [\hat{\theta} f_{s1s}(\hat{k}, \hat{r}) + \hat{\phi} f_{s2s}(\hat{k}, \hat{r})] = i \begin{bmatrix} B^*(\hat{r}) \\ C^*(\hat{r}) \end{bmatrix}^t \begin{bmatrix} T^{22} & T^{23} \\ T^{32} & T^{33} \end{bmatrix} \begin{bmatrix} B(\hat{k}) \\ C(\hat{k}) \end{bmatrix} = i \begin{bmatrix} B(\hat{k}) \\ C(\hat{k}) \end{bmatrix}^t \begin{bmatrix} T^{22} & T^{23} \\ T^{32} & T^{33} \end{bmatrix}^t \begin{bmatrix} B^*(\hat{r}) \\ C^*(\hat{r}) \end{bmatrix} , \tag{81}$$

and

$$-\hat{v} \cdot [\hat{\theta}_{-k} f_{s1s}(-\hat{r}, -\hat{k}) + \hat{\phi}_{-k} f_{s2s}(-\hat{r}, -\hat{k})] = i \begin{bmatrix} B(\hat{k}) \\ C(\hat{k}) \end{bmatrix}^t \begin{bmatrix} T^{22} & T^{23} \\ T^{32} & T^{33} \end{bmatrix} \begin{bmatrix} B^*(\hat{r}) \\ C^*(\hat{r}) \end{bmatrix} , \tag{82}$$

where a superscript t also indicates transpose of the matrix. The second reciprocity relation implies that

$$\begin{bmatrix} T^{22} & T^{23} \\ T^{32} & T^{33} \end{bmatrix} = \begin{bmatrix} \overline{T^{22}} & \overline{T^{32}} \\ \overline{T^{23}} & \overline{T^{33}} \end{bmatrix} . \tag{83}$$

For the mode conversion result,

$$f_{ps}(\hat{k}, \hat{r}) = i\Lambda \overline{A^*(\hat{r})} (T^{12} T^{13}) \begin{bmatrix} B(\hat{k}) \\ C(\hat{k}) \end{bmatrix} = i\Lambda \begin{bmatrix} B(\hat{k}) \\ C(\hat{k}) \end{bmatrix}^t (T^{12} T^{13})^t A^*(\hat{r}) \tag{84}$$

and

$$-\hat{v} \cdot [\hat{\theta}_{-k} f_{s1p}(-\hat{r}, -\hat{k}) + \hat{\phi}_{-k} f_{s2p}(-\hat{r}, -\hat{k})] = i \begin{bmatrix} B(\hat{k}) \\ C(\hat{k}) \end{bmatrix}^t \begin{bmatrix} T^{21} \\ T^{31} \end{bmatrix} A^*(\hat{r}) , \tag{85}$$

and the third reciprocity relation implies that

$$\overline{T^{12}} = T^{21} \text{ and } \overline{T^{13}} = T^{31} . \tag{86}$$

combining Eqs. (80), (83), and (86) yields the required result that

$$T = \overline{T} . \tag{87}$$

V. UNITARITY AND ENERGY CONSERVATION

We now proceed to explore some further properties of the T matrix which are due to the fact that no energy is dissipated in an elastic solid containing an elastic or an ideal fluid inclusion. Another way of stating this is that averaged over a period of time, the rate of energy flux across a closed surface surrounding the obstacle is zero. In Ref. 12, the total energy flux carried by the scattered field has been related to the amplitude of the scattered wave in the forward direction, in what are referred to as forward amplitude theorems.

The average energy flux flowing through a closed surface S' surrounding the obstacle is given by

$$(\omega/2)\,\mathrm{Im}\int_{S'} \vec{u}(\vec{r})\cdot\overleftrightarrow{\tau}^*(\vec{r})\cdot\hat{r}\,dS=0, \tag{88}$$

where S' is the surface of a sphere (circle in 2-D) with radius $\vec{r}$, large compared to the dimensions of S. In Eq. (88) and the sequel, Im denotes the imaginary part. Since $\vec{u}=\vec{u}^0+\vec{u}^s$ and $\overleftrightarrow{\tau}=\overleftrightarrow{\tau}^0+\overleftrightarrow{\tau}^s$, Eq. (88) can be rewritten as

$$\mathrm{Im}\int_{S'}\vec{u}^s\cdot\overleftrightarrow{\tau}^{s*}\cdot\hat{r}\,dS'=-\mathrm{Im}\int_{S'}[\vec{u}^0\cdot\overleftrightarrow{\tau}^{s*}+\vec{u}^s\cdot\overleftrightarrow{\tau}^{0*}]\cdot\hat{r}\,dS. \tag{89}$$

The right-hand side of the above equation has been evaluated in 2-D and 3-D for both p and s wave incidence.

Consider first the case of p wave incidence. The $B(\hat{k})$ and $C(\hat{k})$ in Eq. (23) vanish. The right-hand side of Eq. (89) is found to be $-(2\pi/k_p)\,\mathrm{Im}f_{pp}(\hat{k},\hat{k})$ for 2-D; and $-\mathrm{Re}f_{pp}(\hat{k},\hat{k})$ for 3-D.[12] On the left-hand side, since S' is a sphere of very large radius, the asymptotic form of $\vec{u}^s$ in Eq. (72) may be used, and the $\overleftrightarrow{\tau}^{s*}$ is derived from $\vec{u}^s$ by using Eq. (3). After substitutions, the final results are

$$\int_0^{2\pi}\left[\frac{1}{k_p^2}|f_p(\vec{k},\vec{r})|^2+\frac{1}{k_s^2}|f_s(\hat{k},\hat{r})|^2\right]d\theta =\begin{cases}-(2\pi/k_p)\,\mathrm{Im}f_p(\hat{k},\hat{k}), & p \text{ wave incidence},\\ -(2\pi/k_s)\,\mathrm{Im}f_s(\hat{k},\hat{k}), & s \text{ wave incidence},\end{cases} \tag{90}$$

$$\int[|f_p(\hat{k},\hat{r})|^2+\Lambda\{|f_{s1}(\hat{k},\hat{r})|^2+|f_{s2}(\hat{k},\hat{r})|^2\}]d\Omega =\begin{cases}-\mathrm{Re}f_p(\hat{k},\hat{k}), & p \text{ wave incidence},\\ -\mathrm{Re}f_s(\hat{k},\hat{k}), & s \text{ wave incidence}.\end{cases} \tag{91}$$

In the above, Eq. (90) is for the case of 2-D, and Eq. (91) for 3-D where $d\Omega$ is the solid angle subtended by the element of area dS'.

To prove the unitarity of the scattering matrix, we need to consider the general case when p and s waves are incident simultaneously. Since the right-hand side of Eq. (89) is linear in the incident field, we can superpose the results obtained in Eqs. (90) and (91) for p and s waves incident separately.

The scattered wave amplitudes can be written in terms of the plane wave expansion coefficients and the T matrix, calculations are performed for the three-dimensional case as the two-dimensional case is simpler.

From Eqs. (73), (74), and (41), we note that

$$f_p(\hat{k},\hat{r})=i\Lambda\,\overline{A(\vec{r})^*}(T^{11}T^{12}T^{13})\begin{pmatrix}A(\hat{k})\\ B(\hat{k})\\ C(\hat{k})\end{pmatrix}, \tag{92}$$

$$f_{s1}(\hat{k},\hat{r})=i\overline{B(\hat{r})^*}(T^{21}T^{22}T^{23})\begin{pmatrix}A(\hat{k})\\ B(\hat{k})\\ C(\hat{k})\end{pmatrix}, \tag{93}$$

and

$$f_{s2}(\hat{k},\hat{r})=i\overline{C(\hat{r})^*}(T^{31}T^{32}T^{33})\begin{pmatrix}A(\hat{k})\\ B(\hat{k})\\ C(\hat{k})\end{pmatrix}. \tag{94}$$

From Eq. (92)

$$\int|f_p|^2d\Omega=\Lambda^2\begin{pmatrix}A(\hat{k})\\ B(\hat{k})\\ C(\hat{k})\end{pmatrix}^t\left\{\int A^*(\hat{r})\overline{A(\hat{r})}d\Omega\right\}\times(T^{11*}T^{12*}T^{13*})\begin{pmatrix}A^*(\hat{k})\\ B^*(\hat{k})\\ C^*(\hat{k})\end{pmatrix}. \tag{95}$$

The integral in Eq. (95) can be evaluated using the relation between the column vector A and the vector spherical harmonic $\vec{A}$. Thus

$$\int A_n^{\sigma*}(\hat{r})A_m^{\nu}(\hat{r})d\Omega=i^{-(n-1)}i^{m-1}\Lambda^{-1}\int \vec{A}_n^{\sigma}(\hat{r})\cdot\vec{A}_m^{\nu}(\hat{r})d\Omega =\Lambda^{-1}\delta_{mn}\delta_{\sigma\nu}, \tag{96}$$

where we have used the orthogonality of the vector spherical harmonics (Ref. 8, p. 1900). Similarly,

$$\int B_n^{\sigma*}(\hat{r})B_m^{\nu}(\hat{r})d\Omega=\int C_n^{\sigma*}(\hat{r})C_m^{\nu}(\hat{r})d\Omega=\delta_{mn}\delta_{\sigma\nu}. \tag{97}$$

Writing all terms on the left-hand side of Eq. (91) in a form like Eq. (95) and using Eqs. (96) and (97) to evaluate the angular integrals, adding the three matrix products on the left-hand side, and substituting for the amplitudes on the right-hand side of Eq. (91), we obtain

$$(\overline{A}(\hat{k})\overline{B}(\hat{k})\overline{C}(\hat{k}))T^tT^*\begin{pmatrix}A^*(\hat{k})\\ B^*(\hat{k})\\ C^*(\hat{k})\end{pmatrix} =-\mathrm{Re}\left[(\overline{A}(\hat{k})\overline{B}(\hat{k})\overline{C}(\hat{k}))T\begin{pmatrix}A^*(\hat{k})\\ B^*(\hat{k})\\ C^*(\hat{k})\end{pmatrix}\right]. \tag{98}$$

The right-hand side of Eq. (98) may be written in the form

$$\mathrm{Re}(\overline{x}Tx^*)=\mathrm{Re}\overline{x}\,\mathrm{Im}T\,\mathrm{Im}x-\mathrm{Im}\overline{x}\,\mathrm{Im}T\,\mathrm{Re}x +\mathrm{Im}\overline{x}\,\mathrm{Re}T\,\mathrm{Im}x+\mathrm{Re}\overline{x}\,\mathrm{Re}T\,\mathrm{Re}x,$$

where we have defined $\overline{x}=[\overline{A}(\hat{k})\overline{B}(\hat{k})\overline{C}(\hat{k})]$ for brevity. The first two terms on the right-hand side of the above equation cancel on taking the transpose of the first term and recalling that T is symmetric. The last two terms can be combined to yield

$$\mathrm{Re}[\overline{x}Tx^*]=\overline{x}\,\mathrm{Re}\,Tx^*. \tag{99}$$

Substituting Eq. (99) on the right-hand side of Eq. (98), we obtain

$$TT^*=-\mathrm{Re}T. \tag{100}$$

Exactly the same property results for the two-dimensional case also.

Equation (100) does not imply that the T matrix is unitary, but in the quantum mechanical literature, one defines a scattering matrix S as

$$S = 1 - 2T \quad \text{and thus} \quad S = \bar{S} . \tag{101}$$

Using Eqs. (100) and (101),

$$S\bar{S}^* = 1, \tag{102}$$

thus proving that the S matrix is unitary.

VI. SCATTERING BY AN ELASTIC OBSTACLE

In deriving an expression for the T matrix, the values of the displacement and traction on the surface of the obstacle must be specified in Eqs. (26) and (27). To preserve the continuity of the discussion in Sec. III we completed the derivation for the simplest case, that of a cavity. We should note however, that up to Eq. (34), no boundary conditions have been applied and the results are general. In this section, we treat the general case of an elastic scatterer along with a discussion on the boundary conditions for four types of obstacles.

A. Cavity

As discussed before, the boundary conditions are

$$\hat{n}' \cdot \overleftrightarrow{\tau}' = \hat{n}(\vec{r}') \cdot \overleftrightarrow{\tau}(\vec{r}') = 0, \quad \vec{r}' \text{ on } S . \tag{103}$$

The surface displacement can be expanded in terms of the vector basis set

$$\vec{u}(\vec{r}') = \sum_{n,\sigma} [a_n^\sigma \operatorname{Re}\vec{\phi}_n^\sigma + b_n^\sigma \operatorname{Re}\vec{\psi}_n^\sigma + c_n^\sigma \operatorname{Re}\vec{\chi}_n^\sigma], \quad \vec{r}' \text{ on } S. \tag{104}$$

Determination of the unknown coefficients a, b, c, and the T matrix have been discussed in Sec. III.

B. Rigid obstacle

A rigid obstacle is regarded as the limiting case of a very hard obstacle (λ_1, μ_1 large) embedded in a soft matrix material (λ, μ small). The commonly assumed boundary conditions at the surface S of the obstacle are

$$\vec{u}' \equiv \vec{u}(\vec{r}') = 0, \quad \vec{r}' \text{ on } S. \tag{105}$$

If these conditions were substituted in Eqs. (26) and (27), together with an assumed basis function expansion for the unknown traction at the surface S,

$$\hat{n}(\vec{r}') \cdot \overleftrightarrow{\tau}(\vec{r}') = \sum_{n,\sigma} [a_n^\sigma \operatorname{Re}\vec{\phi}_n^\sigma + b_n^\sigma \operatorname{Re}\vec{\psi}_n^\sigma + c_n^\sigma \operatorname{Re}\vec{\chi}_n^\sigma] , \quad \vec{r}' \text{ on } S \tag{106}$$

one would be able to determine a, b, c, and the T matrix for a rigid obstacle just as in the case of a cavity. However, it is known that the boundary condition (105) leads to an unusual result of scattering, which contradicts the inverse fourth-power-wavelength law in the Rayleigh limit.[10]

As pointed out by Pao and Mow,[13] this unusual result is caused by the unreasonable boundary conditions assumed. Eq. (105), implies that the obstacle is not only rigid, but also fixed in space. In the absence of external agents to restrain it, the obstacle moves as a rigid body under the excitation of an incident wave. Thus a correct formulation of the boundary conditions for a rigid obstacle should allow it to translate and rotate,[13] and the solution based on Eq. (105) is only of academic interest.

C. Elastic inclusion

Consider an obstacle composed of an elastic material with material constants ρ_1, λ_1, and μ_1. If the obstacle is completely welded to the matrix material, the six boundary conditions at the interface (four in two-dimensional problems) S are

$$\vec{u}(\vec{r}') = \vec{u}_1(\vec{r}'), \qquad \vec{r}' \text{ on } S , \tag{105}$$

$$\hat{n}' \cdot \overleftrightarrow{\tau}(\vec{r}') = \hat{n}' \cdot \overleftrightarrow{\tau}_1(\vec{r}') , \quad \vec{r}' \text{ on } S . \tag{106}$$

In the above equations and the sequel, the subscript 1 indicates a quantity pertaining to the inclusion and the unsubscripted variables pertain to the surrounding material.

Inside the inclusion, there is a standing wave refracted from the interface. It can be represented as

$$\vec{u}_1(\vec{r}) = \sum_{n,\sigma} [a_n^\sigma \operatorname{Re}\vec{\phi}_{1n}^\sigma(\vec{r}) + b_n^\sigma \operatorname{Re}\vec{\psi}_{1n}^\sigma(\vec{r}) + c_n^\sigma \operatorname{Re}\vec{\chi}_{1n}^\sigma(\vec{r})], \quad \vec{r} \text{ inside } S , \tag{107}$$

where a, b, c are unknown coefficients. The subscript 1 indicates that the wave numbers $k_{p1}^2 = \rho_1\omega^2/(\lambda_1 + 2\mu_1)$ and $k_{s1}^2 = \rho_1\omega^2/\mu_1$ should be used with the basis functions $\vec{\phi}$, $\vec{\psi}$, and $\vec{\chi}$.

Since $\vec{u}_1(\vec{r})$ are regular functions and continuous inside the surface S, we can complete the spatial differentiations and determine $\overleftrightarrow{\tau}_1(\vec{r})$ from Hooke's law [Eq. (3), with constants λ_1 and μ_1] inside S. Both $\vec{u}_1(\vec{r})$ and $\overleftrightarrow{\tau}_1(\vec{r})$ thus assumed are also valid for $\vec{r}$ at the boundary S. These surface quantities are continuous across S according to Eqs. (105) and (106). Hence both $\vec{u}(\vec{r}')$ and $\hat{n}' \cdot \overleftrightarrow{\tau}(\vec{r}')$ for $\vec{r}'$ at S are obtained. Substituting them into Eqs. (26) and (27) for the exterior region, we have expressed the unknown surface sources $\vec{u}'$ and $\hat{n}' \cdot \overleftrightarrow{\tau}'$ in these integral formulae in terms of the basis functions $\operatorname{Re}\vec{\phi}_{1n}^\sigma$, $\operatorname{Re}\vec{\psi}_{1n}^\sigma$ $\operatorname{Re}\vec{\chi}_{1n}^\sigma$ and three unknown coefficients a, b, c. The procedure for determining a, b, c, and the transition matrix is same as in the case of a cavity.

D. Fluid inclusion

If instead of a solid material, the inclusion is filled with an inviscid fluid, it can not sustain shear waves. Thus instead of three continuity conditions for displacements [Eq. (105)], only the normal component is continuous across S,

$$\hat{n}' \cdot \vec{u}(\vec{r}') = \hat{n}' \cdot \vec{u}_1(\vec{r}'), \quad \vec{r}' \text{ on } S . \tag{108}$$

Because of the inviscid assumption, the tangential displacements are discontinuous across S. Furthermore, of the three stresses, only the normal component is continuous.

$$\hat{n}' \cdot [\overleftrightarrow{\tau}(\vec{r}') \cdot \hat{n}'] = \hat{n}' \cdot [\overleftrightarrow{\tau}_1(\vec{r}') \cdot \hat{n}'], \quad \vec{r}' \text{ on } S. \tag{109}$$

The other two tangential stress components of the surrounding medium vanish at S,

$$\hat{n}'\times[\overleftrightarrow{\tau}(\vec{r}')\cdot\hat{n}']=0, \quad \vec{r}' \text{ on } S. \tag{110}$$

The number of boundary conditions are reduced from six to four (three in two-dimensional problems).

It was shown that in the eigenfunction expansion method, the solution for a fluid inclusion can be derived from that of an elastic solid by letting $\mu_1 \to 0$ (Chap. 6 of Ref. 4).[13] In the matrix method, it is easier to assume, instead of Eq. (107),

$$\vec{u}_1(\vec{r}) = \sum_{m,\sigma} a^\sigma_m \operatorname{Re}\vec{\phi}^\sigma_{1m}(\vec{r}); \quad \vec{r} \text{ inside } S. \tag{111}$$

The stress tensor in the fluid is related to $\vec{u}_1$ by

$$\overleftrightarrow{\tau}_1(\vec{r}) = \lambda_1 \overleftrightarrow{I}\nabla\cdot\vec{u}_1(\vec{r}); \quad \vec{r} \text{ inside } S. \tag{112}$$

These two expressions are also valid when $\vec{r}$ is at the surface S. By applying Eqs. (108) and (109), one thus specifies the normal components of the surface displacement $\vec{u}'$ and the traction $\hat{n}'\cdot\overleftrightarrow{\tau}'$ in Eqs. (26) and (27), which are expressed in terms of $\operatorname{Re}\phi^\sigma_{1m}$ and the unknown coefficients a^σ_m. From Eq. (110), we set the remaining two tangential components of the traction in Eqs. (26) and (27) to zero.

The remaining unspecified surface sources in these integrals are the two tangential components of the displacement at the surface S, approached from the exterior. We assume

$$(\overleftrightarrow{I}-\hat{n}\hat{n})\cdot\vec{u}(\vec{r}') = \sum_{m,\sigma}(\overleftrightarrow{I}-\hat{n}\hat{n})\cdot[b^\sigma_m \operatorname{Re}\vec{\psi}^\sigma_m(\vec{r}') + c^\sigma_m \operatorname{Re}\vec{\chi}^\sigma_m(\vec{r}')], \quad \vec{r}' \text{ on } S. \tag{113}$$

The two basic functions should be those pertaining to the surrounding solid matrix material. Note that the dyadic $\overleftrightarrow{I}-\hat{n}\hat{n}$ is perpendicular to $\hat{n}$ as $\hat{n}\cdot(\overleftrightarrow{I}-\hat{n}\hat{n})=0$.

With the surface field completely specified, the integral formulas, Eqs. (26) and (27) again contain three sets of unknown coefficients a, b, c, and the T matrix can be determined as in the previous cases.

In all four cases, the unknown surface sources are expressed in terms of three sets of expansion coefficients a, b, c, and the procedure for determining the T matrix is the same. However, the elements of the Q matrices will be different in each case.

VII. CONCLUDING REMARKS

The matrix formulation of elastic wave scattering as presented in this paper should complement existing methods of analysis. As discussed in the introduction, the chief advantage of the matrix method is that it is applicable to obstacles of arbitrary shape, using the same set of basis functions. This eliminates the need for calculating and tabulating a special set of wavefunctions for each type of geometry. The only special functions needed are the Bessel functions, the spherical Bessel functions and the spherical harmonics.

This paper was confined to the formulation of the method and the general structure and properties of the transition matrix without reference to any particular geometry. The properties of symmetry and unitary of the T and S matrix, respectively are particularly important since they are indispensable for checking the accuracy of the numerical calculations.

The major steps involved in the numerical calculation of the T matrix are (1) evaluation of the Q-matrix elements which consist of surface integrals involving the vector basis functions, (2) inversion of the Q matrix, and (3) computation of the T matrix as the product $\operatorname{Re} Q\, Q^{-1}$.

The integration in step 1 can be performed efficiently by any of the existing algorithms. The second step is somewhat involved since Q is an infinite matrix. Furthermore, to evaluate the T matrix, Q^{-1} and Q have to be computed to the same order of accuracy. The desired accuracy is better attained if all Q matrices are orthogonalized by applying the Schmidt process of orthogonalization. Once Q^{-1} are calculated, the last step is straightforward.

Finally we note that only a single obstacle is discussed in this paper. In multiple scattering of elastic waves, a matrix theory can be formulated analogous to that for acoustic waves.[14] The task of computing the transition matrix for problems involving more than two obstacles would, however, be formidable.

ACKNOWLEDGMENT

The authors wish to thank Dr. P. C. Waterman for reading the entire manuscript and suggesting improvements and corrections.

*This research was supported by a grant from the National Science Foundation and by the Material Science Center at Cornell University.

[1] P. C. Waterman, J. Acoust. Soc. Am. 45, 1417–1429 (1969).
[2] P. C. Waterman, Phys. Rev. D. 3, 825–839 (1971).
[3] J. C. Bolomey and A. Wirgin, Proc. IEE 121, 794–804 (1974).
[4] Y. H. Pao and C. C. Mow, *The diffraction of elastic waves and dynamic stress concentrations* (Crane, Russak, New York, 1973).
[5] R. P. Banaugh and W. Goldsmith, J. Appl. Mech. 30, 589–597 (1963).
[6] B. B. Baker and E. T. Copson, *The Mathematical Theory of Huygens' Principle* (Clarendon, Oxford, 1950).
[7] Y. H. Pao and V. Varatharajulu, J. Acoust. Soc. Am. 59, 1361–1371 (1976).
[8] P. M. Morse and H. Feshbach, *Methods of Mathematical Physics, Parts* I *and* II (Mcgraw–Hill, New York, 1953).
[9] R. M. White, J. Acoust. Soc. Am. 30, 771–785 (1958).
[10] C. F. Ying and R. Truell, J. Appl. Phys. 27, 1086–1097 (1956).
[11] N. G. Einspruch, E. J. Witterholt, and R. Truell, J. Appl. Phys. 31, 806–818 (1960).
[12] V. Varatharajulu, "Reciprocity relations and forward amplitude theorems for the scattering of elastic waves from obstacles of arbitrary shape," Report No. 2568, Material Science Center, Cornell University, Ithaca, NY (Nov. 1975) J. Math. Phys. (to be published).
[13] Y. H. Pao and C. C. Mow, J. Appl. Phys. 34, 493–499 (1963).
[14] B. Peterson and S. Ström, J. Acoust. Soc. Am. 56, 771–780 (1974).