Betti's identity and transition matrix for elastic waves

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A transition matrix relating the coefficients of scattered waves to those of incident waves in elastic solids is derived by applying Betti's third identity and orthogonality conditions for a set of basis functions. The transition matrix for a fluid inclusion, a cavity, a rigid inset, or a solid inclusion in a fluid can all be derived from the general result for an elastic inclusion of arbitrary shape by taking proper limiting values of the general result. This limiting process is illustrated for the case of a spherical inclusion.

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INTRODUCTION

A series of papers has appeared recently to develop the transition matrix for the scattering of elastic waves.[1-5] In 1969, P. Waterman[1] first developed the method for the scattering of acoustic (scalar) waves, starting from the Helmholtz integral formula. The rest of the papers cited dealt with elastic (vector) waves. Because there are two different Helmholtz-type integral formulas for elastic waves, two formulations of the transition matrix for elastic waves have been proposed. Varatharajulu and Pao[2] based their derivation on the integral formula given by Kupradze,[6] which contains traction and displacement as unknown surface sources. Waterman[3] started from the integral formula given by Morse and Feshbach,[7] which involves the normal and tangential components, and the divergence and curl of the displacement vector. Very recently, Waterman has shown that the transition matrix can be derived directly from a new conservation law.[5]

In this paper, we show how the transition matrix given in Ref. 2 can be derived from Betti's third identity for elastic displacements. This identity is analogous to Green's second identity for scalar potentials. Although no new result is added by this approach, the derivation is more elegant and simple than the approach from the Helmholtz-type formulas.[2,3] Furthermore, the results are so compactly stated and clearly identifiable physically that the chance of error is greatly reduced.

The derivations are carried out for the general case of an elastic inclusion of arbitrary shape in a matrix of different material. The case of a cavity is then shown as a special case of the elastic inclusion. The case of a rigid inclusion and a fluid inclusion in an elastic matrix is shown to be derivable from the general case by proper limiting processes. These processes are illustrated by examples of a spherical inclusion and are analogous to those used previously in the eigenfunction expansion solution for a sphere.[8] The case of an elastic inclusion in an inviscid fluid can also be handled by applying the existing process.

Because the tangential components of the displacement are discontinuous at the interface of an elastic solid and an inviscid fluid, some difficulties exist in assigning proper values for the surface sources in the integral formula. We believe the approach of setting up the matrix elements for a general inclusion and then taking the limit as the shear modulus of the solid approaches zero alleviates the difficulty.

Toward the end of this paper, we discuss the pitfall of assuming the zero-displacement vector in the surface integral formula for a "rigid" inclusion. In the Appendix, we supply a proof of the convergence of the series of basis functions representing the waves inside the inclusion and at the boundary. This proof is essential in establishing the transition matrix and was missing in Ref. 2.

I. APPLICATIONS OF BETTI'S IDENTITY

A. Betti's third identity

Let $\mathbf{u}$ and $\mathbf{v}$ be two displacement vectors in an elastic medium, and $\boldsymbol{\tau}(\mathbf{u})$ and $\boldsymbol{\tau}(\mathbf{v})$ be the corresponding stress tensors. At a surface with a unit normal vector $\mathbf{n}$, the traction (stress vector) is related to the stress tensor by

$$\mathbf{t}=\mathbf{n}\cdot\boldsymbol{\tau}\,. \tag{1}$$

For an isotropic material with the Lamé constants λ and μ and density ρ, the generalized Hooke's law is

$$\boldsymbol{\tau}(\mathbf{u})=\mathbf{I}\lambda\nabla\cdot\mathbf{u}+\mu(\nabla\mathbf{u}+\mathbf{u}\nabla) \tag{2}$$

$$=[\lambda\mathbf{I}\,\mathrm{div}+2\mu\,\mathrm{grad}+\mu\mathbf{I}\times\mathrm{curl}]\mathbf{u}\,. \tag{3}$$

In the preceding equations, $\mathbf{I}$ is the identity tensor (idemfactor), $\boldsymbol{\tau}(\mathbf{u})$ is expressed as a linear vector function of $\mathbf{u}$, and the three terms inside the brackets of Eq. (3) constitute a linear dyadic operator. Substitution of Eq. (3) in Eq. (1) yields

$$\mathbf{t}(\mathbf{u})=[\lambda\mathbf{n}\,\mathrm{div}+2\mu\mathbf{n}\cdot\mathrm{grad}+\mu\mathbf{n}\times\mathrm{curl}]\mathbf{u}\,. \tag{4}$$

The divergence of a stress tensor is expressed as another vector function of $\mathbf{u}$,

$$\Delta^*\mathbf{u}\equiv\nabla\cdot\boldsymbol{\tau}(\mathbf{u})=[(\lambda+\mu)\,\mathrm{grad\,div}+\mu\nabla^2]\mathbf{u}\,. \tag{5}$$

Then Betti's third identity may be stated as (Eq. (1.10) of Ref. 6),

$$\int_S[\mathbf{t}(\mathbf{u})\cdot\mathbf{v}-\mathbf{t}(\mathbf{v})\cdot\mathbf{u}]\,dS=\int_V[\Delta^*\mathbf{u}\cdot\mathbf{v}-\Delta^*\mathbf{v}\cdot\mathbf{u}]\,dV, \tag{6}$$

where V is a volume bounded by a closed surface S. This identity can easily be established by noting

$$[\nabla\cdot\boldsymbol{\tau}(\mathbf{u})]\cdot\mathbf{v}=\nabla\cdot[\boldsymbol{\tau}(\mathbf{u})\cdot\mathbf{v}]-\boldsymbol{\tau}(\mathbf{u}):\nabla\mathbf{v}\,, \tag{7}$$

and

$$\boldsymbol{\tau}(\mathbf{u}):\nabla\mathbf{v}=\boldsymbol{\tau}(\mathbf{v}):\nabla\mathbf{u}\,. \tag{8}$$

The Cartesian components of the double scalar product $\boldsymbol{\tau}:\nabla\mathbf{v}$ are $\tau_{ij}\partial_i v_j$; and Eq. (8) is true because of the linear Hooke's law. Subtracting Eq. (7) from a second equation which is obtained by interchanging u and v in Eq. (7), integrating the difference of these two equations over the volume V, and then applying the divergence theorem and Eq. (1) and (5), we obtain Eq. (6). In the application of the divergence theorem, it is assumed that u and v and their first and second derivatives are continuous inside the volume V.

For steady elastic waves with a time factor $\exp(-i\omega t)$, the displacement amplitude satisfies the reduced elastic wave equation

$$\Delta^*(\mathbf{u}) = -\rho\omega^2\mathbf{u}\ . \tag{9}$$

If both u and v satisfy Eq. (9) with the same circular frequency ω, the volume integral in Eq. (6) vanishes. We thus obtain

$$\int_S [\mathbf{t}(\mathbf{u})\cdot\mathbf{v} - \mathbf{t}(\mathbf{v})\cdot\mathbf{u}]\,dS = 0\ . \tag{10}$$

Furthermore, the surface S may be deformed into another surface so long as no new discontinuities are encompassed. This seemingly simple and obvious fact was keenly observed by Waterman from a conservation principle and was applied by him to derive a transition matrix.[5]

B. Basis functions in spherical coordinates

For elastic waves in a three-dimensional medium, the total motion may be decomposed into three parts: **L**, **M**, and **N**. The **L** represents the dilatational (longitudinal) motion propagating with speed c_p and wave number $k=\omega/c_p$; and **M** and **N** are the rotational (transverse) parts, moving with speed c_s and wave number $\kappa=\omega/c_s$. The two wave speeds are given by $c_p^2=(\lambda+2\mu)/\rho$ and $c_s^2=\mu/\rho$.

In spherical coordinates (r,θ,ϕ), the three elastic wave functions satisfying Eq. (9) are (p. 1865 of Ref. 7)

$$\mathbf{L}^\sigma_{mn} = (1/k)\nabla[h_n(kr)Y^\sigma_{mn}(\theta,\phi)]$$
$$= h_n'(kr)\mathbf{A}^\sigma_{mn} + (n^2+n)^{1/2}[h_n(kr)/kr]\,\mathbf{B}^\sigma_{mn}\ , \tag{11}$$

$$\mathbf{M}^\sigma_{mn} = \nabla\times[\mathbf{r}h_n(\kappa r)Y^\sigma_{mn}(\theta,\phi)]$$
$$= (n^2+n)^{1/2}h_n(\kappa r)\mathbf{C}^\sigma_{mn}\ , \tag{12}$$

$$\mathbf{N}^\sigma_{mn} = (1/\kappa)\nabla\times\mathbf{M}^\sigma_{mn} \tag{13}$$
$$= (n^2+n)\left[\frac{h_n(\kappa r)}{\kappa r}\right]\mathbf{A}^\sigma_{mn} + \left[\frac{(n^2+n)^{1/2}}{\kappa r}\right][\kappa r h_n(\kappa r)]'\mathbf{B}^\sigma_{mn}\ .$$

In the above equations, h_n is the spherical Hankel function of the first kind. It can be replaced also by one of the spherical Bessel functions j_n and y_n and the spherical Hankel functions of the second kind. The Y^σ_{mn} are the spherical surface harmonics, which have been split into an even part ($\sigma=1$) and an odd part ($\sigma=2$),

$$\begin{aligned} Y^1_{mn}(\theta,\phi) &= P^m_n(\cos\theta)\cos m\phi \quad \text{(even)},\\ Y^2_{mn}(\theta,\phi) &= P^m_n(\cos\theta)\sin m\phi \quad \text{(odd)}. \end{aligned} \tag{14}$$

The P^m_n is the associate Legendre polynomials, $m=0, 1, \ldots, n$, and $n=0,1,2,\ldots,\infty$. The three mutually perpendicular vectors **A**, **B**, and **C** are related to the three unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_\phi$ in spherical coordinates:

$$\begin{aligned} \mathbf{A}^\sigma_{mn} &= \mathbf{e}_r\,Y^\sigma_{mn}\ ,\\ \mathbf{B}^\sigma_{mn} &= (n^2+n)^{-1/2}r\nabla Y^\sigma_{mn} = \mathbf{e}_r\times\mathbf{C}^\sigma_{mn}\\ &= (n^2+n)^{-1/2}[\mathbf{e}_\theta\partial/\partial\theta + (\mathbf{e}_\phi/\sin\theta)\partial/\partial\phi]\,Y^\sigma_{mn}\ ,\\ \mathbf{C}^\sigma_{mn} &= (n^2+n)^{-1/2}\nabla\times(\mathbf{r}Y^\sigma_{mn}) = -\mathbf{e}_r\times\mathbf{B}^\sigma_{mn}\ . \end{aligned} \tag{15}$$

The tractions at a surface with unit normal n can be calculated from Eq. (4), or

$$\mathbf{t} = \lambda\mathbf{n}(\nabla\cdot\mathbf{u}) + \mu\mathbf{n}\cdot(\nabla\mathbf{u}+\mathbf{u}\nabla)\ . \tag{16}$$

Hence we have

$$\begin{aligned} \mathbf{t}(\mathbf{L}^\sigma_{mn}) &= \lambda\mathbf{n}\nabla\cdot\mathbf{L}^\sigma_{mn} + \mu\mathbf{n}\cdot(\nabla\mathbf{L}^\sigma_{mn}+\mathbf{L}^\sigma_{mn}\nabla)\ ,\\ \mathbf{t}(\mathbf{M}^\sigma_{mn}) &= \mu\mathbf{n}\cdot(\nabla\mathbf{M}^\sigma_{mn}+\mathbf{M}^\sigma_{mn}\nabla)\ ,\\ \mathbf{t}(\mathbf{N}^\sigma_{mn}) &= \mu\mathbf{n}\cdot(\nabla\mathbf{N}^\sigma_{mn}+\mathbf{N}^\sigma_{mn}\nabla)\ . \end{aligned} \tag{17}$$

In spherical coordinates, the components of a dyadic $(\nabla\mathbf{A}+\mathbf{A}\nabla)$ are listed in Ref. 7 (p. 117). If $\mathbf{n}=\mathbf{e}_r$, as in the case of a spherical surface, the three fractions are

$$\mathbf{t}^r(\mathbf{L}^\sigma_{mn}) = 2\mu k\left\{\left[(n^2+n-\tfrac{1}{2}\kappa^2r^2)\frac{h_n(kr)}{(kr)^2} - 2\frac{h_n'(kr)}{kr}\right]\mathbf{A}^\sigma_{mn} + \left[\frac{h_n(kr)}{kr}\right]'(n^2+n)^{1/2}\mathbf{B}^\sigma_{mn}\right\},$$

$$\mathbf{t}^r(\mathbf{M}^\sigma_{mn}) = \mu\kappa[h_n'(\kappa r) - h_n(\kappa r)/\kappa r](n^2+n)^{1/2}\mathbf{C}^\sigma_{mn}\ , \tag{18}$$

$$\mathbf{t}^r(\mathbf{N}^\sigma_{mn}) = 2\mu\kappa\left\{\left[\frac{h_n(\kappa r)}{\kappa r}\right]'(n^2+n)\mathbf{A}^\sigma_{mn} + \left[(n^2+n-1-\tfrac{1}{2}\kappa^2r^2)\frac{h_n(\kappa r)}{(\kappa r)^2} - \frac{h_n'(\kappa r)}{\kappa r}\right](n^2+n)^{1/2}\mathbf{B}^\sigma_{mn}\right\}.$$

Throughout this paper, a prime over a function indicates the derivative with respect to its argument. Thus $[xh_n(x)]' = xh_n'(x) + h_n(x)$, and $h_n'(x) = dh_n(x)/dx$.

The asymptotic formulae for spherical Bessel functions with large arguments ($z\to\infty$) are

$$\begin{aligned} j_n(z) &\to (1/z)\cos(z-\pi_n)\ ,\\ h_n(z) &\to (1/z)\exp(iz-i\pi_n)\ ,\\ h_n'(z) &\to (i/z)\exp(iz-i\pi_n)\ , \end{aligned} \tag{19}$$

where $\pi_n=(n+1)\pi/2$. Applying them to Eqs. (11)–(13), we find, as $kr\to\infty$ or $\kappa r\to\infty$,

$$\begin{aligned} \mathbf{L}^\sigma_{mn} &\to (i/kr)\exp(ikr-i\pi_n)\mathbf{A}^\sigma_{mn}\ ,\\ \mathbf{M}^\sigma_{mn} &\to (1/\kappa r)\exp(i\kappa r-i\pi_n)(n^2+n)^{1/2}\mathbf{C}^\sigma_{mn}\ ,\\ \mathbf{N}^\sigma_{mn} &\to (i/\kappa r)\exp(i\kappa r-i\pi_n)(n^2+n)^{1/2}\mathbf{B}^\sigma_{mn}\ . \end{aligned} \tag{20}$$

Similarly, the asymptotical values for tractions are

$$\begin{aligned} \mathbf{t}(\mathbf{L}^\sigma_{mn}) &\to [-k(\lambda+2\mu)/kr]\exp(ikr-i\pi_n)\mathbf{A}^\sigma_{mn}\ ,\\ \mathbf{t}(\mathbf{M}^\sigma_{mn}) &\to (i\mu\kappa/\kappa r)\exp(i\kappa r-i\pi_n)(n^2+n)^{1/2}\mathbf{C}^\sigma_{mn}\ ,\\ \mathbf{t}(\mathbf{N}^\sigma_{mn}) &\to (-\mu\kappa/\kappa r)\exp(i\kappa r-i\pi_n)(n^2+n)^{1/2}\mathbf{B}^\sigma_{mn}\ . \end{aligned} \tag{21}$$

From the previous equations, we note that only as $r\to\infty$, is the dilatational wave **L** polarized in the radial direction (for both u and t) and the rotational waves in the direction tangent to the sphere.

C. Orthogonality of the spherical basis functions

The three vector functions **A**, **B**, and **C** satisfy the orthogonality conditions[7]

$$\mathbf{A}^{\sigma}_{mn}\cdot\mathbf{B}^{\sigma}_{mn}=\mathbf{A}^{\sigma}_{mn}\cdot\mathbf{C}^{\sigma}_{mn}=\mathbf{B}^{\sigma}_{mn}\cdot\mathbf{C}^{\sigma}_{mn}=0 \tag{22}$$

and

$$\iint \mathbf{A}^{\sigma}_{mn}\cdot\mathbf{A}^{\nu}_{pq}\,d\Omega=\iint \mathbf{B}^{\sigma}_{mn}\cdot\mathbf{B}^{\nu}_{pq}\,d\Omega=\iint \mathbf{C}^{\sigma}_{mn}\cdot\mathbf{C}^{\nu}_{pq}\,d\Omega$$
$$=(1/\gamma_{mn})\delta_{mp}\delta_{nq}\delta_{\sigma\nu}\;. \tag{23}$$

The integration is over a spherical surface with a solid angle $d\Omega=\sin\theta\,d\theta\,d\phi$. The normalization constant is

$$\gamma_{mn}=\epsilon_m(2n+1)(n-m)!/4\pi(n+m)!\,, \tag{24}$$

where $\epsilon_m=1$ when $m=0$ and $\epsilon_m=2$ when $m>0$.

There exist no such orthogonality conditions for wave functions $\mathbf{L}^{\sigma}_{mn}$, $\mathbf{M}^{\sigma}_{mn}$, and $\mathbf{N}^{\sigma}_{mn}$, even when r is a constant. We shall now show that a combination of **u** and **t**(**u**) according to Eq. (6) satisfies another form of orthogonality relation.

As a compromise of notations used in Refs. 2 and 3, we define three basis functions for elastic waves, $(\psi^{\nu})^{\sigma}_{mn}$ where $\nu=1,2,3$, and

$$(\psi^1)^{\sigma}_{mn}=(\gamma_{mn}k^3/\kappa^3)^{1/2}\mathbf{L}^{\sigma}_{mn}\,,$$
$$(\psi^2)^{\sigma}_{mn}=[\gamma_{mn}/(n^2+n)]^{1/2}\mathbf{M}^{\sigma}_{mn}\,, \tag{25}$$
$$(\psi^3)^{\sigma}_{mn}=[\gamma_{mn}/(n^2+n)]^{1/2}\mathbf{N}^{\sigma}_{mn}\,.$$

The scalar factors are introduced by Waterman[3] so that within a common factor, each ψ^{ν} carries unit energy flux out of any closed surface containing the origin. Our factors differ from his in that the factor $1/\pi$ in Eq. (24) is omitted by him (Eq. (5d) of Ref. 3). The function with $\mathbf{L}^{\sigma}_{mn}$ is designated as ψ_3, etc., in his paper. In Ref. 2, the three basis functions are $(1/\kappa)(\psi^{\nu})^{\sigma}_{mn}$, and they have the dimension of displacement. In the sequal, we abbreviate $(\psi^{\nu})^{\sigma}_{mn}$ as ψ^{ν} ($\nu=1,2,3$) unless the indices σ, m, n must be specified explicitly.

For waves which are nonsingular in a region enclosing the origin of r, we use three basis functions,

$$(\hat{\psi}^{\nu})^{\sigma}_{mn}\equiv\mathrm{Re}\,(\psi^{\nu})^{\sigma}_{mn}\,, \tag{26}$$

where Re stands for the "regular part of." For real wave numbers k and κ, as in the case of waves without damping, Re also means the "real part of." Thus $\hat{\psi}^{\nu}$ is obtained from ψ^{ν} by substituting h_n in **L**, **M**, and **N**, with j_n.

An infinite region which is divided by a surface S of arbitrary shape is shown in Fig. 1. We have added two spherical surfaces s_+ and s_-, exterior and interior to S, respectively, and a large spherical surface s_∞ with radius r_∞. Recall that $\hat{\psi}^{\nu}$ ($\nu=1,2,3$) are regular inside s_∞, and ψ^{ν} are regular outside s_-. If we choose V in Eq. (6) to be the region inside s_+, and let $\mathbf{u}=\hat{\psi}^{\nu}$ and $\mathbf{v}=\hat{\psi}^{\sigma}$ ($\sigma=1,2,3$), then the volume integral vanishes, and Eq. (6) reduces to ($\nu,\sigma=1,2,3$)

$$\int_{s_+}[\mathbf{t}(\hat{\psi}^{\nu})\cdot\hat{\psi}^{\sigma}-\mathbf{t}(\hat{\psi}^{\sigma})\cdot\hat{\psi}^{\nu}]dS=0\,. \tag{27}$$

Equation (27), which is a special case of Eq. (10), is a form of orthogonality conditions for $\hat{\psi}^{\nu}$. Note that the s_+ can be replaced by any surface inside s_+, such as S or s_-.

Next we take $\mathbf{u}=\psi^{\nu}$ and $\mathbf{v}=\psi^{\sigma}$ and choose V to be the region bounded internally by s_+ and externally by s_∞. Since all ψ^{ν} ($\nu=1,2,3$) are regular in this region, we obtain from Eq. (6)

$$\left(\int_{s_\infty}-\int_{s_+}\right)[\mathbf{t}(\psi^{\nu})\cdot\psi^{\sigma}-\mathbf{t}(\psi^{\sigma})\cdot\psi^{\nu}]dS=0\,. \tag{28}$$

The minus sign of the second integral is used because the outer normal at s_+ in this case is $-\mathbf{e}_r$, and we have chosen $\mathbf{n}=+\mathbf{e}_r$ for **t** on all surfaces. From Eqs. (20) and (21), we note that as $r=r_\infty\to\infty$ $t(\psi^{\nu})\cdot\psi^{\sigma}=0$ when $\nu\neq\sigma$. When $\nu=\sigma$, the quantity inside the brackets is identically zero. Thus the integral over s_∞ vanishes, and we have ($\nu,\sigma=1,2,3$)

$$\int_{s_+}[\mathbf{t}(\psi^{\nu})\cdot\psi^{\sigma}-\mathbf{t}(\psi^{\sigma})\cdot\psi^{\nu}]dS=0\,. \tag{29}$$

Finally, we let $\mathbf{u}=\psi^{\nu}$ and $\mathbf{v}=\hat{\psi}^{\sigma}$ ($\nu,\sigma=1,2,3$). Both functions are regular in the region V bounded by s_+ internally and s_∞ at the outside. As in the previous case, by letting $r\to\infty$, we obtain

$$\int_{s_+}[\mathbf{t}(\psi^{\nu})\cdot\hat{\psi}^{\sigma}-\mathbf{t}(\hat{\psi}^{\sigma})\cdot\psi^{\nu}]dS=(i\mu/\kappa)\delta_{\nu\sigma}\,. \tag{30}$$

The constant $(i\mu/\kappa)$ is calculated by substituting Eqs. (20) and (21) into the integral over s_∞ and then by making use of the relations in Eq. (23).

Equations (27), (29), and (30) are the "orthogonality conditions" for the basis functions, which are needed in deriving the transition matrix.

II. TRANSITION MATRIX FOR AN ELASTIC INCLUSION

A. Incident, refracted, and scattered waves

Let the region inside S as shown in Fig. 1 be filled with a material different from the surrounding one, and let all material constants inside S be designated with a subscript 0 (λ_0, μ_0, ρ_0, k_0, and κ_0). An incident wave $\mathbf{u}^{(i)}(\mathbf{r})$ impinging on this inclusion is refracted into the

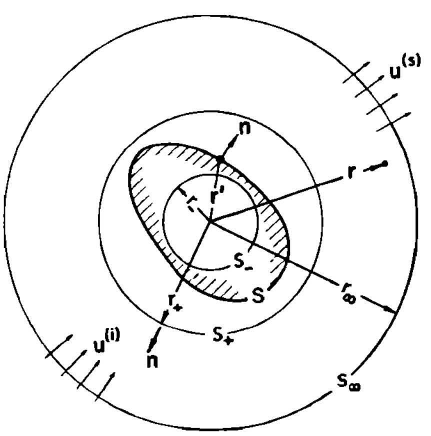

FIG. 1. Waves scattered by an inclusion bounded by the surfaces S.

inclusion material as $\mathbf{u}^{(f)}$ and scattered into the surrounding medium as $\mathbf{u}^{(s)}$. Each of the three waves can be represented by a series of the basis functions within a specific region:

$$\mathbf{u}^{(i)}(\mathbf{r})=\sum a_\nu\hat{\psi}^\nu(\mathbf{r}), \qquad r<r_\infty, \tag{31}$$

$$\mathbf{u}^{(s)}(\mathbf{r})=\sum c_\nu\psi^\nu(\mathbf{r}), \qquad r\geq r_+, \tag{32}$$

$$\mathbf{u}^{(f)}(\mathbf{r})=\sum f_\nu\hat{\psi}_0^\nu(\mathbf{r}), \qquad r\leq r_-. \tag{33}$$

Since ψ^ν is an abbreviation for $(\psi^\nu)^\sigma_{mn}$, the coefficients a_ν, c_ν, and f_ν also carry the indices σ, m, n. The symbol $\sum$ is then an abbreviation for four summations: ν from 1 to 3, σ from 1 (even) to 2 (odd), n from 0 to infinity, and m from 0 to n. The subscript 0 of $\hat{\psi}_0^\nu$ means that the same subscript should be attached to all wave numbers and material constants contained in these regular basis functions.

The incident wave is given and the series in Eq. (31) is uniformly convergent for $r<r_\infty$. If the incident wave is generated by a point or line source, the radius r_∞ should pass through this source. For plane incident waves, r_∞ is infinite.

The basis functions for $\mathbf{u}^{(s)}$ are regular outside the inclusion, and the series in Eq. (32) converges uniformly at and outside a sphere of radius r_+. The actual value for r_+ is unknown at this stage, but it can be ascertained when the unknown coefficients c_ν are found. Similarly, the series for $\mathbf{u}^{(f)}$ is uniformly convergent at least at and inside a sphere of radius r_-. Again the radius of convergence will be ascertained after the determination of the unknown coefficients f_ν. The two spheres s_+ and s_- need not be concentric.

To calculate c_ν, we consider the region V bounded by s_+ and S, and let $\mathbf{u}(r)$ be a solution of Eq. (9) and $\mathbf{v}=\psi^\nu(r)$ in Eq. (6). The volume integral in Eq. (6) then vanishes, and

$$\int_S[\mathbf{t}(\mathbf{u})\cdot\psi^\nu-\mathbf{t}(\psi^\nu)\cdot\mathbf{u}]dS=\int_{s_+}[\mathbf{t}(\mathbf{u})\cdot\psi^\nu-\mathbf{t}(\psi^\nu)\cdot\mathbf{u}]dS. \tag{34}$$

As in Eq. (28), all $\mathbf{n}$'s for $\mathbf{t}$ at S and at s_+ are in the direction of $\mathbf{e}_r$.

At the surface S, both $\mathbf{u}$ and $\mathbf{t}(\mathbf{u})$ are unknown, and they will be denoted by $\mathbf{u}_+$ and $\mathbf{t}_+$, respectively. However, at s_+, $\mathbf{u}$ can be represented by the series in Eqs. (31) and (32).

$$\mathbf{u}=\mathbf{u}^{(i)}+\mathbf{u}^{(s)}=\sum a_\sigma\hat{\psi}^\sigma+\sum c_\sigma\psi^\sigma \qquad \text{on } s_+. \tag{35}$$

Again the summation is over σ ($\sigma=1,2,3$) and three more indices. Since both series are uniformly convergent, we can apply the vector differential operator $\mathbf{t}$ as defined by Eq. (4) to the series in Eq. (35) and obtain

$$\mathbf{t}(\mathbf{u})=\sum a_\sigma\mathbf{t}(\hat{\psi}^\sigma)+\sum c_\sigma\mathbf{t}(\psi^\sigma) \qquad \text{on } s_+. \tag{36}$$

Substituting Eqs. (35) and (36) to the right-hand side of Eq. (34), we find

$$\begin{aligned}\int_S[\mathbf{t}_+\cdot\psi^\nu-\mathbf{t}(\psi^\nu)\cdot\mathbf{u}_+]dS &=\sum_\sigma\left\{a_\sigma\int_{s_+}[\mathbf{t}(\hat{\psi}^\sigma)\cdot\psi^\nu-\mathbf{t}(\psi^\nu)\cdot\hat{\psi}^\sigma]dS\right\}\\ &+\sum_\sigma\left\{c_\sigma\int_{s_+}[\mathbf{t}(\psi^\sigma)\cdot\psi^\nu-\mathbf{t}(\psi^\nu)\cdot\psi^\sigma]dS\right\}.\end{aligned} \tag{37}$$

The integral associated with c_σ vanishes because of Eq. (29), and that with a_σ equals $(i\mu/\kappa)\delta_{\nu\sigma}$ according to Eq. (30). Hence we obtain the following identity:

$$a_\nu=\frac{i\kappa}{\mu}\int_S[\mathbf{t}_+\cdot\psi^\nu-\mathbf{t}(\psi^\nu)\cdot\mathbf{u}_+]dS \qquad (\nu=1,2,3). \tag{38}$$

The subscript + means approaching S from the $+\mathbf{n}$ side. Since a_ν are known, the preceding equation implies that the unknown surface traction $\mathbf{t}_+$ and surface displacement $\mathbf{u}_+$ are not independent of each other.

Similarly, we let $\mathbf{v}=\hat{\psi}^\nu(\mathbf{r})$ in Eq. (6) and obtain

$$c_\nu=-\frac{i\kappa}{\mu}\int_S[\mathbf{t}_+\cdot\hat{\psi}^\nu-\mathbf{t}(\hat{\psi}^\nu)\cdot\mathbf{u}_+]dS \qquad (\nu=1,2,3). \tag{39}$$

This equation shows clearly that the coefficients of the scattered waves are determined by the dynamic sources $\mathbf{t}_+$ and $\mathbf{u}_+$ on the surface S. Equations (38) and (39) are a manifestation of Huygens' principle for elastic waves.[9]

To determine f_ν, we consider the region V inside the inclusion bounded by s_- and S. Again we let in Eq. (6), $\mathbf{t}(\mathbf{u})=\mathbf{t}_-$ and $\mathbf{u}=\mathbf{u}_-$ at S, and

$$\mathbf{u}=\mathbf{u}^{(f)}=\sum f_\sigma\hat{\psi}_0^\sigma, \tag{40}$$

on s_-.

$$\mathbf{t}(\mathbf{u})=\sum f_\sigma\mathbf{t}(\hat{\psi}_0^\sigma), \tag{41}$$

By setting $\mathbf{v}=\hat{\psi}_0^\nu$ and ψ_0^ν in successive order, we obtain

$$0=\int_S[\mathbf{t}_-\cdot\hat{\psi}_0^\nu-\mathbf{t}(\hat{\psi}_0^\nu)\cdot\mathbf{u}_-]dS, \tag{42}$$

$\nu=1,2,3.$

$$f_\nu=\frac{i\kappa_0}{\mu_0}\int_S[\mathbf{t}_-\cdot\psi_0^\nu-\mathbf{t}(\psi_0^\nu)\cdot\mathbf{u}_-]dS, \tag{43}$$

The subscript − means approaching S from inside ($-\mathbf{n}$ direction). In the derivation of the above results, we have made use of Eqs. (27) and (30), respectively. Equations (42) and (43) are a mathematical representation of Huygens' principle for waves inside the inclusion.

B. Transition matrix for an elastic inclusion

If an elastic inclusion is perfectly welded to the surrounding medium, traction and displacement must be continuous at the interface S:

$$\left.\begin{aligned}\mathbf{t}_-&=\mathbf{t}_+,\\ \mathbf{u}_-&=\mathbf{u}_+,\end{aligned}\right\} \quad \text{on } S. \tag{44}$$

Thus the $\mathbf{u}_+$ and $\mathbf{t}_+$ in Eqs. (38) and (39) can be replaced by $\mathbf{u}_-$ and $\mathbf{t}_-$, respectively. Furthermore, we prove in the Appendix that the series in Eq. (33) is not just convergent within $r\leq r_-$, it is also convergent and differentiable throughout the interior volume of the inclusion, including the surface approached from inside. Therefore, we extend the series representation of $\mathbf{u}^{(f)}$ to S:

$$\mathbf{u}_- = \sum f_\sigma \hat{\psi}_0^\sigma , \qquad \text{on } S . \qquad (45)$$
$$\mathbf{t}_- = \mathbf{t}\Big(\sum f_\sigma \hat{\psi}_0^\sigma\Big) = \sum f_\sigma \mathbf{t}(\hat{\psi}_0^\sigma),$$

Substituting the above series into Eq. (38) for $\mathbf{u}_+$ and $\mathbf{t}_+$ because of the continuity condition, we find

$$a_\nu = -i\sum_\sigma Q^{\nu\sigma} f_\sigma , \qquad (46)$$

where

$$Q^{\nu\sigma} = \frac{\kappa}{\mu}\int_S [\mathbf{t}(\psi^\nu)\cdot\hat{\psi}_0^\sigma - \mathbf{t}(\hat{\psi}_0^\sigma)\cdot\psi^\nu]\,dS \quad \text{(elastic)}. \qquad (47)$$

Similarly, a substitution of Eq. (45) into Eq. (39) gives rise to

$$c_\nu = i\sum_\sigma \hat{Q}^{\nu\sigma} f_\sigma , \qquad (48)$$

where

$$\hat{Q}^{\nu\sigma} = \frac{\kappa}{\mu}\int_S [\mathbf{t}(\hat{\psi}^\nu)\cdot\hat{\psi}_0^\sigma - \mathbf{t}(\hat{\psi}_0^\sigma)\cdot\hat{\psi}^\nu]\,dS . \qquad (49)$$

Note that $\hat{Q}^{\nu\sigma}$ is the regular part of $Q^{\nu\sigma}$.

In matrix notation, Eqs. (46) and (48) are

$$\mathbf{a} = -i\mathbf{Q}\mathbf{f}, \qquad (50)$$

$$\mathbf{c} = i\hat{\mathbf{Q}}\mathbf{f} , \qquad (51)$$

where $\mathbf{a}$, $\mathbf{c}$, $\mathbf{f}$ are column matrices, and $\mathbf{Q}$, $\hat{\mathbf{Q}}$ are square matrices. Recall that each of ψ^ν and $\hat{\psi}^\nu$ carries three more indices in the form of $(\psi^\nu)^\mu_{mn}$, and the $Q^{\nu\sigma}$ and $\hat{Q}^{\nu\sigma}$ should also carry the same indices as $(Q^{\nu\sigma})^\mu_{mn}$. Thus the $\sum$ in Eqs. (48) and (49) means a quadruple summation over four indices, and $\mathbf{Q}$ and $\hat{\mathbf{Q}}$ are infinite matrix.

Denoting the inverse of the matrix $\mathbf{Q}$ by $\mathbf{Q}^{-1}$, we obtain

$$\mathbf{f} = i\mathbf{Q}^{-1}\mathbf{a}, \qquad (52)$$

$$\mathbf{c} = -(\hat{\mathbf{Q}}\mathbf{Q}^{-1})\mathbf{a} = \mathbf{T}\mathbf{a} . \qquad (53)$$

The product $\hat{\mathbf{Q}}\mathbf{Q}^{-1}$ is called the *transition matrix* $\mathbf{T}$,

$$\mathbf{T} \equiv -\hat{\mathbf{Q}}\mathbf{Q}^{-1} \quad \text{or} \quad -\hat{\mathbf{Q}} = \mathbf{T}\mathbf{Q} . \qquad (54)$$

It relates the unknown scattering coefficients $\mathbf{c}$ to the given incident coefficients $\mathbf{a}$. The refracting coefficients $\mathbf{f}$ are related to $\mathbf{a}$ through the $\mathbf{Q}^{-1}$ matrix. Based on conservation laws, the T matrix has been shown to be symmetric.[2,3]

The elements of $\mathbf{Q}$ depend on the integrals of the basis functions over the surface of a scatterer. In Eqs. (47) and (49), the symbols ψ^ν ($\nu = 1, 2, 3$) stand for $\mathbf{L}^\sigma_{mn}$, $\mathbf{M}^\sigma_{mn}$, and $\mathbf{N}^\sigma_{mn}$ as defined in Eq. (25), and $\mathbf{L}^\sigma_{mn}$, $\mathbf{t}(\mathbf{L}^\sigma_{mn})$, etc., are listed in Eqs. (11) and (17), respectively. Since we have chosen the spherical elastic wave functions as the basis functions, these integrals can be evaluated analytically if S is spherical. Otherwise, $Q^{\nu\sigma}$ and $\hat{Q}^{\nu\sigma}$ can only be evaluated numerically.

C. Transition matrix for a cavity and a rigid inclusion

In Refs. 2 and 3, where a transition matrix for a fixed and rigid insert, a cavity, and a fluid inclusion are given, each is derived from a distinct set of boundary conditions. In principle, the results for these special cases should be derivable from those for the elastic inclusion. We shall show that this is indeed the case.

The problem of a cavity is particularly simple, as we can set in Eq. (44)

$$\mathbf{t}_- = 0 \quad \text{on } S , \qquad (55)$$

and leave $\mathbf{u}_-$ unspecified. Thus $\mathbf{t}_+ = 0$ in Eqs. (38) and (30). By assuming

$$\mathbf{u}_+ = \sum b_\sigma \hat{\psi}^\sigma \quad \text{on } S , \qquad (56)$$

we obtain

$$a_\nu = -i\sum_\sigma Q^{\nu\sigma} b_\sigma , \qquad (57)$$

$$c_\nu = +i\sum_\sigma \hat{Q}^{\nu\sigma} b_\sigma , \qquad (58)$$

where

$$Q^{\nu\sigma} = \frac{\kappa}{\mu}\int_S [\mathbf{t}(\psi^\nu)\cdot\hat{\psi}^\sigma]\,dS \quad \text{(cavity)} . \qquad (59)$$

Explicit expressions for $Q^{\nu\sigma}$ are given in Ref. 2.

Note that Eq. (59) can be deduced directly from Eq. (47) by setting $\mathbf{t}(\hat{\psi}_0^\nu) = 0$ because of Eq. (55), and by changing $\hat{\psi}_0^\nu$ to $\hat{\psi}^\nu$ because the unspecified displacements should be continuous at S.

The boundary conditions for a rigid inclusion are often assumed as[5,10]

$$\mathbf{u}_- = 0 . \qquad (60)$$

The traction $\mathbf{t}_-$ is left unspecified. From Eq. (47), we can derive the $Q^{\nu\sigma}$ for a rigid inclusion by setting $\hat{\psi}_0^\sigma = 0$ because of Eq. (60) and by changing the unspecified $\mathbf{t}(\hat{\psi}_0^\sigma)$ to $\mathbf{t}(\hat{\psi}^\sigma)$ at S. The answer is

$$Q^{\nu\sigma} = -\frac{\kappa}{\mu}\int_S [\psi^\nu\cdot\mathbf{t}(\hat{\psi}^\sigma)]\,dS \quad \text{(rigid)}. \qquad (61)$$

The reader should be cautioned that Eqs. (60) and (61) will yield a result for the rate of scattering energy which contradicts Rayleigh's law of inverse fourth power of wave length. Details are discussed at the end of the next section.

For a fluid inclusion, the tangential traction $\mathbf{n}\times\mathbf{t}(\hat{\psi}_0^\sigma)$ vanishes at S, but the tangential displacement vector does not vanish and is discontinuous at S. Thus it is difficult to derive the $Q^{\nu\sigma}$ directly from Eq. (47). We believe, however, the elements of $Q^{\nu\sigma}$ for a fluid inclusion can be properly derived by applying a limiting process to all radial functions (j_n, h_n, etc.) inside the integrand. This process is illustrated with examples for a spherical inclusion in Sec. III.

III. TRANSITION MATRIX FOR A SPHERICAL INCLUSION

A. Elastic sphere in a solid

When S is the surface of a sphere with radius a, the $\mathbf{t}(\psi^\nu)$, $\mathbf{t}(\psi_0^\nu)$, etc., in the integrals for $Q^{\nu\mu}$ and $\hat{Q}^{\nu\mu}$ are replaced by $\mathbf{t}^r(\mathbf{L})$, $\mathbf{t}^r(\mathbf{M})$, and $\mathbf{t}^r(\mathbf{N})$ in Eq. (18), each multiplied by a constant as defined in Eq. (25). The resulting integrals can all be evaluated in closed forms by applying Eq. (23). As in Ref. 3, we adopt the notation that the argument of a spherical Bessel function is omitted whenever it is multiplied or divided by the same argument, that is, $(kah_n)\equiv kah_n(ka)$, $(h_n/\kappa a)\equiv(1/\kappa a)\times h_n(\kappa a)$, $(kah_n')\equiv ka\, d[h_n(ka)]/d(ka)$, etc. The elements for **Q** as found from Eq. (47), are

$$\begin{aligned}
&\left(\frac{kk_0^3}{\kappa\kappa_0^3}\right)^{-1/2} Q^{11}=\left[-\kappa^2a^2 j_n'(k_0a)+2\left(1-\frac{\mu_0}{\mu}\right) n(n+1)k_0a\left(\frac{j_n}{k_0a}\right)'\right]h_n(ka)+\left\{2\left(\frac{\mu_0}{\mu}-1\right)\left[2j_n'(k_0a)-n(n+1)\frac{j_n}{k_0a}\right]+\frac{\rho_0\kappa^2}{\rho k_0^2}k_0aj_n\right\}(kah_n'),\\
&(\kappa a)^{-3}Q^{22}=j_n(k_0a)\left(\frac{h_n}{\kappa a}\right)'-\left(\frac{\rho_0}{\rho}\right)\left(\frac{j_n}{\kappa_0a}\right)' h_n(\kappa a),\\
&Q^{33}=\left[-\frac{\kappa}{\kappa_0}(\kappa_0aj_n)'+2n(n+1)\left(1-\frac{\mu_0}{\mu}\right)\frac{\kappa_0}{\kappa}\left(\frac{j_n}{\kappa_0a}\right)'\right]\kappa ah_n+\left[\frac{\mu_0}{\mu}\kappa_0aj_n-2\left(1-\frac{\mu_0}{\mu}\right)j_n'(\kappa_0a)-(n^2+n-1)\frac{j_n}{\kappa_0a}\right](\kappa ah_n)',\\
&\left[\frac{n(n+1)k}{\kappa}\right]^{-1/2}Q^{13}=2\left(1-\frac{\mu_0}{\mu}\right)\kappa_0a\left(\frac{j_n}{\kappa_0a}\right)'(ka)^2\left(\frac{hn}{ka}\right)'+\left[\left(\frac{\rho_0}{\rho}-1\right)(\kappa a)^2+2\left(1-\frac{\mu_0}{\mu}\right)(n^2+n-2)\right]j_n(\kappa_0a)h_n(ka),\\
&\left[\frac{n(n+1)k_0^3}{\kappa_0^3}\right]^{-1/2}Q^{31}=2\left(1-\frac{\mu_0}{\mu}\right)k_0a\left(\frac{j_n}{k_0a}\right)'(\kappa a)^2\left(\frac{h_n}{\kappa a}\right)'+\left[\left(\frac{\rho_0}{\rho}-1\right)(\kappa a)^2+2\left(1-\frac{\mu_0}{\mu}\right)(n^2+n-2)\right]\frac{j_n}{k_0a}h_n(\kappa a),\\
&Q^{12}=Q^{21}=Q^{23}=Q^{32}=0.
\end{aligned}\tag{62}$$

Each element carries with it three more indices in the form $(Q^{\nu\sigma})^{\mu}_{mn}$. These expressions are identical to those in Ref. 3 [Eq. (64)].

The elements of the T matrix are as follows:

$$\begin{aligned}
\Delta T^{11}&=\hat{Q}^{13}Q^{31}-\hat{Q}^{11}Q^{33},\\
T^{22}&=-\hat{Q}^{22}/Q^{22},\\
\Delta T^{33}&=\hat{Q}^{31}Q^{13}-\hat{Q}^{33}Q^{11},\\
\Delta T^{13}&=\hat{Q}^{11}Q^{13}-\hat{Q}^{13}Q^{11},\\
\Delta T^{31}&=\hat{Q}^{33}Q^{31}-\hat{Q}^{31}Q^{33},\\
T^{12}&=T^{21}=T^{23}=T^{32}=0,
\end{aligned}\tag{63}$$

and

$$\Delta=Q^{11}Q^{33}-Q^{31}Q^{13}.\tag{64}$$

Substitution of the above results in Eq. (53) and then in Eq. (32) completes the solution for the waves scattered by a sphere, which has been investigated by many authors[3,8,10,11]:

$$\begin{aligned}
c_1&=T^{11}a_1+T^{13}a_3,\\
c_2&=T^{22}a_2,\\
c_3&=T^{31}a_1+T^{33}a_3.
\end{aligned}\tag{65}$$

B. Fluid sphere in a solid

In the case of a fluid inclusion, μ_0 approaches 0 and ρ_0 and λ_0 are finite. Thus κ_0 approaches ∞, but k_0 remains finite. To compare orders of magnitude, we use $o(\epsilon)$ to denote the "order of infinitesimal ϵ," and $O(1/\epsilon)$ to denote the order of infinity $(1/\epsilon)$. For convenience, we let

$$\mu_0/\mu=o(\epsilon^2),\qquad \kappa_0a=O(1/\epsilon).\tag{66}$$

From Eq. (19), we find as $\kappa_0a\to\infty$,

$$j_n(\kappa_0a)=o(\epsilon),\qquad j_n'(\kappa_0a)=o(\epsilon).\tag{67}$$

Applying these limiting values to $Q^{\nu\sigma}$ in Eq. (62), and neglecting terms of higher order of ϵ in each element, we obtain,

$$\begin{aligned}
Q^{11}&\to\{(1/\kappa_0a)^{3/2}\}(k/\kappa)^{1/2}(k_0a)^{1/2}.\\
&\quad\{[(2n^2+2n-\kappa^2a^2)(k_0aj_n')-2n(n+1)j_n(k_0a)]h_n(ka)\\
&\quad+[(2n^2+2n+\rho_0\kappa^2a^2/\rho)j_n(k_0a)-4(k_0aj_n')](kah_n')\},\\
Q^{22}&\to\{j_n(\kappa_0a)\}(\kappa a)^3(h_n/\kappa a)',\\
Q^{33}&\to\{j_n'(\kappa_0a)\}[(2n^2+2n-\kappa^2a^2)h_n(\kappa a)-2(\kappa ah_n)'],\\
Q^{13}&\to\{j_n'(\kappa_0a)\}[k(n^2+n)/\kappa]^{1/2}2(ka)^2(h_n/ka)',\\
Q^{31}&\to\{(1/\kappa_0a)^{3/2}\}[(n^2+n)k_0a]^{1/2}\\
&\quad\times\{(k_0aj_n')-j_n(k_0a)]2(\kappa a)^2(h_n/\kappa a)'\\
&\quad+j_n(k_0a)[(\rho_0/\rho-1)(\kappa a)^2+2(n^2+n-2)]h_n(\kappa a)\}.
\end{aligned}\tag{68}$$

In the preceding expressions, we have grouped all functions that involve κ_0a inside curly brackets as the first factor of each element.

Note that in the second form of Eq. (54), a common factor which appears in the same column of both matrices $\hat{Q}$ and Q can be cancelled without altering the T matrix. We thus drop the first factor (inside the curly brackets) of all elements to obtain the $Q^{\nu\sigma}$ for a fluid spherical inclusion. The results so obtained disagree with those given in Ref. 3 [Eq. (52)], but they can be reduced to the latter by multiplying the Q matrix in Eq. (68) with a real matrix.

C. Spherical cavity in a solid

The case of a cavity is discussed in the previous section. It can also be considered as a special case of a fluid inclusion with $\lambda_0\to0$ and $\rho_0\to0$. Since the dilatational wave speed $(\lambda_0/\rho_0)^{1/2}$ should be also approaching zero, we have $k_0\to\infty$.

To calculate the limiting values, we let, in addition to Eq. (66),

$$\lambda_0/\lambda=o(\epsilon^2),\qquad \rho_0/\rho=o(\epsilon),\qquad k_0a=O(\epsilon^{-1/2}).\tag{69}$$

The results thus derived from Eq. (68) are

$$\begin{aligned}
Q^{11} &\to \{(k_0a)^{3/2}j_n'(k_0a)\}(k/\kappa)^{1/2} \\
&\quad \times[(2n^2+2n-\kappa^2a^2)h_n(ka)-4(kah_n')], \\
Q^{22} &\to (\kappa a)^3(h_n/\kappa a)', \\
Q^{33} &\to (2n^2+2n-\kappa^2a^2)h_n(\kappa a)-2(\kappa ah_n)', \\
Q^{13} &\to 2[n(n+1)k/\kappa]^{1/2}(ka)^2(h_n/ka)', \\
Q^{31} &\to \{(k_0a)^{3/2}j_n'(k_0a)\}2(n^2+n)^{1/2}(\kappa a)^2(h_n/\kappa a)'.
\end{aligned} \tag{70}$$

By dropping the common factor that involves $\kappa_0 a$ (inside curly brackets) of Q^{11} and Q^{31}, we obtain the $Q^{\nu\sigma}$ for a spherical cavity. These answers are the same as those derived directly from Eq. (59), and they also agree with those given in Ref. 3 [Eq. (44b)].

D. Elastic sphere in a fluid

The same limiting process can be applied to the case when the surrounding medium is fluid. We let $\mu\to 0$ in Eq. (62), and assume

$$\mu/\mu_0=o(\epsilon^2), \qquad \kappa a=O(1/\epsilon). \tag{71}$$

The $Q^{\nu\sigma}$ are reduced to the following expressions when higher order terms are neglected:

$$\begin{aligned}
Q^{11} &\to \{(\mu^2\kappa)^{-1/2}\}(kk_0^3/\kappa_0^3)^{1/2} \\
&\quad \mu_1\{[-\kappa_0^2a^2j_n'(k_0a)-2n(n+1)k_0a(j_n/k_0a)']h_n(ka) \\
&\quad +[(\kappa_0/k_0)^2(k_0aj_n)+4j_n'(k_0a)-2n(n+1)(j_n/k_0a)](kah_n')\}, \\
Q^{22} &\to \{(\kappa a)^{-3}h_n(\kappa a)\}(-\rho_0/\rho)(j_n/\kappa_0a)', \\
Q^{33} &\to \{(\kappa ah_n')/\mu\}(2\mu_0/\kappa_0a) \\
&\quad \times[(\kappa_0aj_n')-(n^2+n-1-\tfrac{1}{2}\kappa_0^2a^2)j_n(\kappa_0a)], \\
Q^{13} &\to \{(\mu\kappa)^{-1/2}\}2\mu_0[n(n+1)k]^{1/2} \\
&\quad \times\{-(\kappa_0a)(j_n/\kappa_0a)'(ka)^2(h_n/ka)', \\
&\quad -[(n^2+n-2)(j_n/\kappa_0a)+(\rho/\rho_0-1)(\kappa_0aj_n)/2]h_n(ka)\}, \\
Q^{31} &\to \{(\kappa ah_n')/\mu\}2\mu_0[n(n+1)k_0^3/\kappa_0^3]^{1/2}[(j_n/k_0a)-j_n'(k_0a)].
\end{aligned} \tag{72}$$

As in the previous cases, we have grouped all quantities involving μ and κa as the first factor of each element (inside the curly brackets). Note that in this case, there are no common factors in the corresponding column of Q and $\hat{Q}$ matrices. However, we can still substitute Eq. (72) into Eq. (63) and then compare the order of magnitudes of $T^{\nu\sigma}$. After substitution, we find that T^{11} does not contain any function which involves κa. The remaining elements which involve $j_n(\kappa a)$ or $h_n(\kappa a)$ are of the following orders as $\kappa a\to\infty$:

$$\begin{aligned}
T^{22}&=o(\epsilon^0), \qquad T^{33}=o(\epsilon^0), \\
T^{13}&=O(\epsilon^{1/2}), \qquad T^{31}=O(\epsilon^{-1/2}).
\end{aligned} \tag{73}$$

For an incident wave in a fluid medium, $a_2=a_3=0$ in Eqs. (31) and (65). Thus we find $c_1=T^{11}a_1$ and $c_3=T^{31}a_1$. The c_3 factor is troublesome as T^{31} is approaching infinity in the limit. However, we note from Eq. (20), $|\psi^3(r)|=o(\epsilon)$ as $\kappa a\to\infty$. Therefore, we can drop the product $c_3\psi^3(\mathbf{r})$ in the scattered waves, Eq. (32),

$$\begin{aligned}
\mathbf{u}^{(s)}(\mathbf{r})&=c_1\psi^1(\mathbf{r}) \\
&=\sum_n\sum_m\sum_\sigma (T^{11})^\sigma_{mn}(a_1)^\sigma_{mn}(\psi^1)^\sigma_{mn},
\end{aligned} \tag{74}$$

where

$$T^{11}=-(\hat{q}^{11}q^{33}-\hat{q}^{13}q^{31})/(q^{11}q^{33}-q^{31}q^{13}). \tag{75}$$

The $q^{\nu\sigma}$ in the above quotient are obtained from Eq. (72) by dropping the factor $(\mu^2\kappa)^{-1/2}$ or $(\kappa ah_n')/\mu$ in $Q^{\nu\sigma}$. The result in Eq. (74) can be compared with that in Ref. 12.

E. Rigid sphere in a solid

The problem of a rigid sphere in solids has often been treated with the boundary condition Eq. (60).[3,10] However, solutions so derived by the eigenfunction expansion method lead to the conclusion that the scattering cross section of the rigid sphere in the Rayleight limit is independent of the wave length of the incoming wave.[10] This conclusion contradicts Rayleigh's law that, at long wavelength, the rate of energy scattering is inversely proportional to the fourth power of wavelength. As pointed out later, this contradiction is a result of the stringent boundary condition $\mathbf{u}_-=0$.[8]

For if an inclusion made of very rigid material is inbedded in an elastic matrix material, its deformation might be small, but it can still translate and rotate with the surrounding matrix. By setting $\mathbf{u}_-=0$ at the boundary, one also has eliminated the rigid body motion. Thus the inclusion is not only *rigid*, but also *fixed* inside the elastic matrix. The condition of fixation is very difficult to secure in reality.

In the case of a rigid circular cylinder inside an elastic matrix, it was shown that the solution derived from the rigid and fixed boundary conditions $\mathbf{u}_-=0$ is a limiting case of a rigid inclusion with an infinite mass density.[13] We shall now examine the limiting values of $Q^{\nu\sigma}$ in Eq. (62) as $\lambda_0\to\infty$, $\mu_0\to\infty$, and $\rho_0\to\infty$.

Note that $k_0=\omega/c_{p0}$, $\kappa_0=\omega/c_{s0}$ and $c_{p0}=[(\lambda_0+2\mu_0)/\rho_0]^{1/2}$, $c_{s0}=(\mu_0/\rho_0)^{1/2}$. Thus the values for c_{p0} and c_{s0} are indeterminate when λ_0, μ_0, and ρ_0 approach infinity. The wave speeds inside the inclusion could be larger, about equal, or much smaller than the corresponding speeds of the surrounding medium. If ϵ denotes an infinitesimal, these three conditions may be stated as follows:

(1) c_{p0}/c_p and $c_{s0}/c_s=O(\epsilon^{-1/2})$. This implies

$$\mu_0/\mu,\ (\lambda_0+2\mu_0)/(\lambda+2\mu)=O(1/\epsilon^2);\ \ \rho_0/\rho=O(1/\epsilon). \tag{76}$$

(2) c_{p0}/c_p and $c_{s0}/c_s=o(\epsilon^0)$. This implies,

$$\mu_0/\mu,\ (\lambda_0+2\mu_0)/(\lambda+2\mu),\ \rho_0/\rho=O(1/\epsilon^2). \tag{77}$$

(3) c_{p0}/c_p and $c_{s0}/c_s=o(\epsilon^{1/2})$. This implies,

$$\mu_0/\mu,\ (\lambda_0+2\mu_0)/(\lambda+2\mu)=O(1/\epsilon^2);\ \ \rho_0/\rho=O(1/\epsilon^3). \tag{78}$$

Case (1) means that $k_0a=o(\epsilon^{1/2})$ and $\kappa_0a=o(\epsilon^{1/2})$. The expressions of $Q^{\nu\sigma}$ can be simplified somewhat by applying the asymptotic formula of $j_n(z)$ as $z\to 0$. Case (2) leads to no simplification of $Q^{\nu\sigma}$. Case (3) gives rise to a simplified result.

In the third case, k_0a equals $o(\epsilon^{-1/2})$, and κ_0a equals $o(\epsilon^{-1/2})$. Applying Eq. (19) to all $j_n(\kappa_0a)$ and $j_n(k_0a)$ in Eq. (62), and keeping only terms of largest magnitude according to Eq. (78), we obtain

$$
\begin{aligned}
&Q^{11} \sim \{(k_0/\kappa_0)^{3/2}(\rho_0/\rho)(j_n/k_0a)\}(k/\kappa)^{1/2}(\kappa a)^2(kah_n'),\\
&Q^{22} \sim \{(-\rho_0/\rho)(j_n/\kappa_0 a)'\}(\kappa a)^2(\kappa a h_n),\\
&Q^{33} \sim \{(\mu_0/\mu)(\kappa_0 a j_n)\}(\kappa a h_n)' \qquad (79)\\
&Q^{13} \sim \{(\rho_0/\rho)(\kappa a)^2(j_n/\kappa_0 a)\}[n(n+1)k/\kappa]^{1/2}h_n(ka),\\
&Q^{31} \sim \{(k_0/\kappa_0)^{3/2}(\rho_0/\rho)(j_n/k_0 a)\}[n(n+1)]^{1/2}(\kappa a)^2 h_n(\kappa a).
\end{aligned}
$$

All factors involving $\kappa_0 a$, $k_0 a$, μ_0, and ρ_0 are grouped inside the curly brackets. Note that $\rho_0\kappa^2/\rho\kappa_0^2=\mu_0/\mu$. Thus the same columns of $Q^{\nu\sigma}$ and $\hat{Q}^{\nu\sigma}$ contain a common real factor (inside the curly brackets) which can be canceled while calculating $T^{\nu\sigma}$. Hence we finally obtain

$$
\begin{aligned}
&Q^{11} \sim (k/\kappa)^{1/2}(\kappa a)^2(kah_n') \sim (kah_n'),\\
&Q^{22} \sim (\kappa a)^2(\kappa a h_n) \sim h_n(\kappa a),\\
&Q^{33} \sim (\kappa a h_n)', \qquad (80)\\
&Q^{13} \sim [n(n+1)k/\kappa]^{1/2}h_n(ka),\\
&Q^{31} \sim [n(n+1)]^{1/2}(\kappa a)^2 h_n(\kappa a)\\
&\qquad \sim [n(n+1)\kappa/k]^{1/2}h_n(\kappa a).
\end{aligned}
$$

The last expressions, which are derived by additional cancellations of factors in the same column, agree with those given in Refs. 3 and 10. The latter was derived by applying the boundary condition $\mathbf{u}_-=0$ on S.

Note that the conditions in Eq. (78) imply the following inequalities:

$$
\begin{aligned}
&\rho_0/\rho > (\rho_0\mu_0/\rho\mu)^{1/2} > \mu_0/\mu,\\
&\rho_0/\rho > [\rho_0(\lambda_0+2\mu_0)/\rho(\lambda+2\mu)]^{1/2} > (\lambda_0+2\mu_0)/(\lambda+2\mu).
\end{aligned} \qquad (81)
$$

Consulting the table in "Acoustic Properties of Solid Materials,"[14] which lists 26 materials, ranging from heavy metals like platinum and tungsten to light metals like beryllium and light-weight polymers like polyethylene (specific density $=0.90$), we find that only a tungsten inclusion inside aluminum matrix can satisfy all conditions in Eq. (81). However, for the tungsten–aluminum combination, $\mu_0/\mu=13.4/2.4=5.58$, and $\rho_0/\rho=19.0/2.69=7.06$. Both ratios are of the same order, and Eq. (78) is not satisfied.

In conclusion, we find that although Eq. (80) is a valid approximation mathematically for a heavy and rigid inclusion, it is not valid for real materials. Adoption of Eq. (61), or Eq. (80) will lead to a result which contradicts Rayleigh's inverse fourth power law.

IV. CONCLUSION

Starting from the Betti's identity in the theory of elasticity [Eq. (6)], we have shown that the basis functions of elastic waves satisfy the orthogonality conditions, Eqs. (27), (29), and (30). The transition matrix which relates the unknown scattering coefficients to the coefficients of incident waves can then be derived directly from the identity and the orthogonality conditions. This derivation circumvents the use of Green's dyadics for elastic waves and can be adopted easily for any other set of basis functions. The case of two-dimensional problems, using the circular cylinder wave functions as the basis function, along with the derivation of transition matrix for scattering of scalar waves based upon Green's second identity for scalar potentials is discussed in a separate report.[15]

The transition matrix is first derived for an elastic inclusion of arbitrary shape, Eqs. (47), (49), and (54). It is then shown that the transition matrices for other types of inclusions, including a cavity, a rigid inclusion, a fluid inclusion, and a solid in fluid, can all be obtained from the general case by a limiting process.

This limiting process is illustrated by examples of a spherical inclusion. Hence the results for an elastic sphere, given explicitly in Eqs. (62)–(65), encompass all known solutions for the scattering of elastic (acoustic) waves by a sphere. Explicit expressions for the elements of the transition matrix enable us to show that the solutions derived from the zero-displacement boundary condition are invalid for most of the solid materials.

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APPENDIX

In applying a series of spherical basis functions to a boundary value problem of nonspherical geometry, it is important to establish, whenever possible, the region of convergence of the assumed series solution.[16] In Eq. (33), we have assumed first a series solution for $\mathbf{u}^{(f)}$ which is convergent within and at the sphere $r=r_-$. Later we extended the region of convergence of the same series to the boundary of a nonspherical inclusion and differentiated the series to obtain $\mathbf{t}_-$ on S in Eq. (45). Such an extension is valid on the basis of a theorem which may be stated as follows.

Huygens' principle [Eqs. (42) and (43)] is a necessary and sufficient condition for the convergence and differentiability of the expansion Eq. (33) throughout the interior volume of the inclusion, including the surface approached from the inside.

An analogous theorem was given by Waterman.[3] Following his logic, we furnish the following proof of the theorem.

That the principle is sufficient can easily be established by assuming the series is uniformly convergent within the volume bounded by S, and on S,

$$\mathbf{u}^{(f)}=\sum f_\sigma \hat{\psi}_0^\sigma(\mathbf{r}), \qquad \mathbf{r}\ \text{inside, and on}\ S \qquad (A1)$$

(Fig. 1). Since the series can be differentiated to obtain $\mathbf{t}^{(f)}$, we find, on the surface S,

$$
\left.
\begin{aligned}
&\mathbf{u}_-=\mathbf{u}^{(f)},\\
&\mathbf{t}_-=\mathbf{t}(\mathbf{u}^{(f)})=\sum f_\sigma \mathbf{t}(\hat{\psi}_0^\sigma),
\end{aligned}
\right\}\ \text{on}\ S. \qquad (A2)
$$

Substitution of (A2) into Eqs. (42) and (43) shows that they both are identically satisfied.

The necessary condition is established by assuming a series for $\mathbf{u}_-$, and another for the derivatives of $\mathbf{u}_-$ with coefficients f' and f'',

$$\mathbf{u}_- = \sum f'_\sigma \hat{\psi}^\sigma_0(\mathbf{r}), \quad \mathbf{r} \text{ on } S, \tag{A3}$$

$$\left.\begin{aligned} \nabla\cdot\mathbf{u}_- &= \sum f''_\sigma \nabla\cdot\hat{\psi}^\sigma_0(\mathbf{r}), \\ \nabla\mathbf{u}_- + \mathbf{u}_-\nabla &= \sum f''_\sigma[\nabla\hat{\psi}^\sigma_0(\mathbf{r}) + \hat{\psi}^\sigma_0(\mathbf{r})\nabla], \end{aligned}\right\} \quad \mathbf{r} \text{ on } S. \tag{A4}$$

Substituting Eq. (A4) into Eq. (2) or Eq. (16), we find

$$\mathbf{t}_- = \sum f''_\sigma \mathbf{t}(\hat{\psi}^\sigma_0), \quad \mathbf{r} \text{ on } S. \tag{A5}$$

Substituting (A3) and (A5) into Eq. (42) and (43) and rewriting $f''_\nu = f'_\nu + (f''_\nu - f'_\nu)$, we obtain the following results:

$$0 = \sum_\sigma (f'_\sigma - f''_\sigma) Q_0^{\nu\sigma}, \tag{A6}$$

$$-if_\nu(\mu_0/\kappa_0) = \sum_\sigma (f'_\sigma - f''_\sigma) Q_0^{\nu\sigma} - if'_\nu(\mu_0/\kappa_0), \tag{A7}$$

where

$$Q_0^{\nu\sigma} = \int_S [\psi^\nu_0 \cdot \mathbf{t}(\psi^\sigma_0)]\, dS. \tag{A8}$$

In the previous derivations, use has been made of Eqs. (27) and (29) when s_+ is shrunken to S.

Since $Q_0^{\nu\sigma} \neq 0$ in general, Eq. (A6) is satisfied by

$$f'_\sigma = f''_\sigma, \quad \sigma = 1, 2, 3. \tag{A9}$$

From Eq. (A7), it then follows that

$$f_\nu = f'_\nu, \quad \nu = 1, 2, 3. \tag{A10}$$

The special case of $Q_0^{\nu\sigma} = 0$ which corresponds to a condition of resonance is discussed in Ref. 3.

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