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New Formulation of Acoustic Scattering

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Upon introducing the outgoing spherical (or circular cylinder) partial waves $\{\psi_n\}$ as a basis, the equation $QT=-\mathrm{Re}(Q)$ is obtained for the transition matrix T describing scattering for general incidence on a smooth object of arbitrary shape. Elements of Q involve integrals over the object surface, e.g.

$$Q_{mn}=\pm\left(\frac{i}{2}\right)\delta_{mn}+\left(\frac{k}{8\pi}\right)\int d\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}[\mathrm{Re}(\psi_m)\psi_n].$$

where the $-$, $+$ apply for Dirichlet and Neumann conditions, respectively. For quadric (separable) surfaces, Q is symmetric. Symmetry and unitarity lead to a secular equation defining eigenfunctions for general bodies. Some apparently new closed-form results are obtained in the low frequency limit, and the transition matrix is computed numerically for the infinite strip.

INTRODUCTION

Three methods extensively employed in the literature on scattering and diffraction, especially where explicit numerical results are desired, are separation of variables, variational techniques, and the direct numerical solution of integral equations. The separation-of-variables procedure is, of course, extremely well known,[1–5] and constitutes a formal solution for a class of objects bounded by quadric surfaces. In practice, a good part of the computational effort goes into evaluation of the wavefunctions themselves except for the sphere and the circular cylinder, for which efficient recursion relations are available. The variational method, described by Levine and Schwinger[6] and others,[2,7–11] is equivalent to Galerkin's method, as was shown by Jones.[12,13] For general bodies, the principal effort goes into evaluating matrix elements, which consist of repeated surface or volume integrals with singular kernel, and require, respectively, fourfold and sixfold numerical quadrature.[12] The integral equation method consists of approximating an integral (over the surface or volume of the scattering region) by a discrete sum, then solving the resulting system of equations numerically.[14] In recent years, several applications of this approach have appeared, using the digital computer.[11,15–17]

The purpose of the present work is to describe a new matrix formulation of scattering. In structure, the resulting equations most nearly resemble those of the variational method, with however the computational advantage that, for both surface- and volume-type scattering, elements of the matrix to be inverted are described by a single surface integral with no singularities in the integrand. Essentially the same matrix

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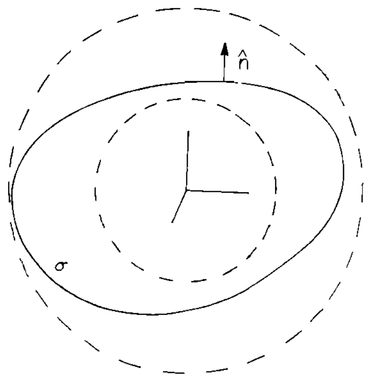

FIG. 1. Geometry of an obstacle bounded by a smooth closed surface σ.

applies for both Dirichlet and Neumann boundary conditions.

In brief, the plan is as follows: In Sec. I, equations are derived for the transition matrix describing the scattering for general incident wave, using a spherical partial wave basis. The derivation is based on the Helmholtz integral formula as applied to both the interior and exterior of the scattering region, and supplemented with analytic continuation arguments. The idea that exterior boundary-value problems can be solved by considerations in the interior is not new, incidentally, and appears to have first been applied in electrostatics by Smythe, in 1956.[18] Symmetry and unitarity are employed in Sec. II to obtain a secular equation for the eigenvalues and eigenvectors of the scattering or transition matrix; and the eigenvectors, in turn, generate eigenfunctions associated with a specified scatterer (including boundary conditions). The matrix elements appropriate to both two- and three-dimensional problems are written out explicitly in Sec. III, and various reductions discussed that depend on the geometry of the scattering region. Finally, in Sec. IV, the transition matrix is computed numerically for the infinite strip. Symmetry and unitarity are verified, and equivalence of the eigenfunctions of Sec. II with the elliptic cylinder functions demonstrated. By specializing to plane wave incidence, the results of earlier workers[19,20] are, in effect, extended to the geometrical optics limit.

It should be emphasized that the method in its present stages is formal in the sense that no rigorous proofs are available dealing with convergence of truncated solutions of the (infinite) matrix equations derived below. It is hoped that the present work may stimulate activity along these lines. In addition to the analytical and numerical results presented here, numerical results have also been obtained for electromagnetic scattering by conducting[21,22] and dielectric obstacles,[23] using a vector formulation of the method. Aside from obvious differences of the vector and scalar cases, the present work goes further in that a unified derivation is given for both surface- and volume-type scattering regions and, what is more important, works directly with the transition matrix, in which setting the rôle of reciprocity and energy conservation is explicitly displayed.

I. DERIVATION OF MATRIX EQUATIONS

Consider the exterior boundary-value problem that consists of finding a solution to the scalar Helmholtz equation

$$\Delta\psi+k^2\psi=0, \tag{1}$$

subject to boundary conditions to be described subsequently on the (two- or three-dimensional) closed surface σ shown in Fig. 1. The surface is assumed smooth in the sense of having continuous turning normal $\hat{n}$, and only simple harmonic time dependence is considered; a factor $\exp(-i\omega t)$ is suppressed in all field quantities.

The total velocity potential ψ consists of the sum of a known incident wave ψ^i, having no sources in the interior of σ, and a scattered wave ψ^s, having the form of outgoing radiation at infinity. Under these conditions the well-known Helmholtz formula asserts that[24,25]

$$\left.\begin{matrix}\psi(\mathbf{r}')\\ 0\end{matrix}\right\} = \psi^i(\mathbf{r}')+\frac{1}{4\pi}\int d\sigma\hat{n}\cdot[\psi_+\nabla g(k|\mathbf{r}-\mathbf{r}'|)-g(k|\mathbf{r}-\mathbf{r}'|)\nabla_+\psi] \quad \text{for } \mathbf{r}' \begin{cases}\text{outside } \sigma\\ \text{inside } \sigma\end{cases}, \tag{2}$$

where ψ_+ and $\hat{n}\cdot\nabla_+\psi$ are the total field and its normal gradient on the surface of the obstacle, approached from the outside, and g is the free space Green's function $ikh_o^{(1)}(kR)=(1/R)\exp(ikR)$ [in two dimensions $i\pi H_o^{(1)}(kR)$, the Hankel function of order zero, of the first kind].

We choose as a basis the set of functions

$$\{\psi_n(\mathbf{r});\, n=1,2,\cdots\}$$

consisting of the outgoing partial wave solutions of Eq. 1 in circular polar or spherical polar coordinates, depending on the dimensionality of the problem. The various indices needed to express parity, and so forth, have been reordered into a single index for simplicity.

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The detailed form of the basis functions, including normalization, is discussed subsequently.

The incident wave is to have no singularities in the neighborhood of the origin, and hence can be expanded in the regular wave functions $\{\mathrm{Re}\psi_n(\mathbf{r})\}$. One writes

$$\psi^i = \sum a_n \,\mathrm{Re}\psi_n, \tag{3}$$

where the expansion coefficients $\{a_n\}$ are assumed to be known. Similarly the free-space Green's function may be expanded in the form[26]

$$g(k|\mathbf{r}-\mathbf{r}'|) = ik \sum \psi_n(k\mathbf{r}_>)\,\mathrm{Re}\psi_n(k\mathbf{r}_<), \tag{4*}$$

where $r_>$ and $r_<$ are respectively the greater and lesser of r, r'.[1] Inserting this expansion in the Helmholtz formula, the scattered wave, which may be identified with the surface integral, is seen to be given for all points outside the circumscribed cylinder (sphere) of Fig. 1 by

$$\psi^s = \sum f_n \psi_n, \tag{5}$$

with expansion coefficients

$$f_n = \frac{ik}{4\pi}\int d\sigma\hat{n}\cdot[\boldsymbol{\nabla}(\mathrm{Re}\psi_n)\psi_+ - (\mathrm{Re}\psi_n)\boldsymbol{\nabla}_+\psi], \qquad n=1,2,\cdots. \tag{6*}$$

On the other hand, for field points inside the inscribed cylinder (sphere), use of Eqs. 3 and 4 will reduce the entire right side of Eq. 2 to an expansion in the complete set of functions $\{\mathrm{Re}\psi_n\}$. This expansion must vanish, and because of orthogonality each coefficient must vanish separately, giving the set of equations

$$\frac{ik}{4\pi}\int d\sigma\hat{n}\cdot[\psi_+\boldsymbol{\nabla}\psi_n - (\boldsymbol{\nabla}_+\psi)\psi_n] = -a_n, \qquad n=1,2,\cdots. \tag{7*}$$

Observe that the right side of Eq. 2 is a regular solution of the differential Eq. 1 of elliptic type throughout the interior of σ. By analytic continuation, it follows that this field will vanish identically not just inside the inscribed volume but throughout the entire interior.

The procedure from this point will consist of the following: The unknown surface quantities ψ_+, $\hat{n}\cdot\boldsymbol{\nabla}_+\psi$ are expanded in a complete set of functions, utilizing the boundary conditions, so far unspecified, so as to introduce only a single set of independent expansion coefficients, say $\{\alpha_n\}$. Substitution in Eq. 7 will then give a system of linear algebraic equations for computing the surface fields $\{\alpha_n\}$ from the incident wave $\{a_n\}$. In similar fashion, Eq. 6 will give a system of equations to compute the scattered wave $\{f_n\}$ from the surface fields $\{\alpha_n\}$. Our principal concern is with the *transition matrix* T connecting the $\{f_n\}$ with the $\{a_n\}$, and an equation for T may finally be obtained by eliminating the surface fields $\{\alpha_n\}$ between Eqs. 6 and 7.

To proceed with this plan, consider first the homogeneous Dirichlet boundary condition

$$\psi_+ = 0 \text{ on } \sigma. \tag{8}$$

Note that when this condition is inserted in Eq. 2, the remaining kernel, g, is sufficiently well behaved as to produce no jump in value of the integral when crossing the surface. Thus, satisfaction of Eq. 7, which is necessary and sufficient to make the right-hand side of Eq. 2 vanish throughout the interior, also guarantees that ψ will take on the desired boundary value from the exterior. An analogous argument can be made for the Neumann boundary condition discussed below. The choice of expansion functions to represent the unknown surface quantity $\hat{n}\cdot\boldsymbol{\nabla}_+\psi$ is somewhat arbitrary. One useful choice, for reasons that will become clear, is the normal gradients of regular wave functions, i.e., $\{\hat{n}\cdot\boldsymbol{\nabla}\,\mathrm{Re}\psi_n\}$. Thus, assuming these functions are complete[27] on the surface σ described by $r=r(\theta)$ [or, in three dimensions $r=r(\theta,\varphi)$], one writes

$$\hat{n}(\mathbf{r})\cdot\boldsymbol{\nabla}_+\psi(\mathbf{r}) = \sum \alpha_n\hat{n}(\mathbf{r})\cdot\boldsymbol{\nabla}[\mathrm{Re}\psi_n(\mathbf{r})]; \quad \mathbf{r} \text{ on } \sigma. \tag{9}$$

Substitution of this expansion in Eqs. 7 and 6 now gives respectively, in an obvious matrix notation (prime denotes matrix transpose)

$$iQ'\alpha = a, \tag{10}$$

$$f = -i\,\mathrm{Re}(Q')\alpha, \tag{11}$$

where the matrix elements of Q are given by

$$Q_{mn} \equiv \frac{k}{4\pi}\int \mathbf{d\sigma}\cdot\boldsymbol{\nabla}(\mathrm{Re}\psi_m)\psi_n \tag{12*}$$

and may be obtained either analytically or by numerical integration, depending on the complexity of the surface geometry.

Formal elimination of the surface field α between Eqs. 10 and 11 results in a system of equations

$$f = -\mathrm{Re}(Q')(Q')^{-1}a \tag{13}$$

relating the scattered wave directly to the incident wave. The transition matrix T for the Dirichlet problem is defined as just this connecting matrix, which generates the coefficients of the scattered wave by premultiplication on the coefficients of the incident wave. Thus one has (we assume symmetry in order to replace T' by T; see Sec. II)

$$QT = -\mathrm{Re}(Q) \tag{14}$$

for determination of the transition matrix.

For the Neumann problem, on the other hand, one has the boundary condition

$$\hat{n}\cdot\boldsymbol{\nabla}_+\psi = 0 \text{ on } \sigma, \tag{15}$$

[26] This and subsequent equations marked with an asterisk apply to the three-dimensional case. For the two-dimensional case, replace the factor "k" by π.

[27] See Appendix.

and this time the remaining surface field ψ_+ is assumed to be representable[27] in regular wavefunctions $\{\mathrm{Re}\psi_n\}$. The procedure leading to Eq. 14 follows exactly as before, except that Q must be replaced by a new matrix $\hat{Q}$ with elements given by

$$\hat{Q}_{mn}\equiv\frac{k}{4\pi}\int \mathbf{d\sigma}\cdot \mathrm{Re}(\psi_m)\nabla\psi_n.$$

At this point results may be collected in a more symmetric form, as follows: Applying the divergence theorem to $(\hat{Q}-Q)$, using a volume bounded outside by σ, and inside by the inscribed circle (sphere), this difference is readily seen to vanish except for the imaginary parts of diagonal elements, i.e., $\hat{Q}-Q=i\mathbf{1}$ where $\mathbf{1}$ is the identity matrix having elements $\delta_{mn}=1$ for $m=n$, $\delta_{mn}=0$ otherwise. On the other hand, by inspection one sees that the sum $\hat{Q}+Q$ can be written as an integral involving the gradient of the *product* of wavefunctions. Solving these equations for Q and $\hat{Q}$, one has that Eq. 14 is applicable to either boundary condition, with matrix elements given by

$$Q_{mn}=\mp\frac{i}{2}\delta_{mn}+\frac{k}{8\pi}\int \mathbf{d\sigma}\cdot\nabla[\mathrm{Re}(\psi_m)\psi_n], \qquad (16^*)$$

with the minus and plus signs referring to Dirichlet and Neumann conditions, respectively.

This expression has an interesting feature from the numerical point of view. For indices m, $n>ka$ (hence, in the low-frequency limit, all elements) where a equals the maximum radius of the obstacle, the product of radial functions in the integrand can be approximated by the leading term arising from the appropriate power-series expansions. Similarly, the product of angular functions can be expanded in a finite set of angular functions—e.g., in two dimensions

$$\cos m\theta \cos n\theta=\tfrac{1}{2}\cos(m-n)\theta+\tfrac{1}{2}\cos(m+n)\theta.$$

The dominant numerical contribution to the off-diagonal elements generally would be expected to arise from the first and more slowly varying of these terms. Examination of Eq. 16, however, reveals that the contribution in question consists of the surface integral of the normal gradient of a potential function, and hence by the divergence theorem vanishes identically for all off-diagonal elements. Because of the term containing δ_{mn}, this cancellation does not occur with diagonal elements, which might therefore be expected to dominate their off-diagonal neighbors, resulting in a matrix better suited for inversion by numerical techniques. An analogous effect can be seen to occur in Eq. 20 below, and in the three-dimensional case, although the situation is much more complex in the latter because of the additional index attached to the wavefunctions.

For the more general acoustic boundary-value problem, in which fields penetrate the interior of the obstacle, propagation in the interior is described by propagation constant k', in accord with density ρ' and stiffness modulus (reciprocal compressibility) M', all of which may differ from the parameters k, ρ, M of the surrounding medium. Boundary conditions require that the pressure, and the normal component of particle velocity, be continuous across the interface, giving, respectively,

$$\left.\begin{aligned}\psi_+&=(\rho'/\rho)\psi_-\\ \hat{n}\cdot\nabla_+\psi&=\hat{n}\cdot\nabla_-\psi\end{aligned}\right\}. \qquad (17)$$

Observe that, for k' real, the wavefunctions $\{\mathrm{Re}\psi_n(k'\mathbf{r})\}$ form a complete orthonormal set of functions for the total field in the interior, which may hence be expanded in the form

$$\psi(\mathbf{r})=\sum \beta_m\,\mathrm{Re}\psi_m(k'\mathbf{r});\quad \mathbf{r}\text{ inside }\sigma. \qquad (18)$$

Assuming that this expansion and its normal gradient converge on the boundary,[27] the interior surface fields ψ_-, $\hat{n}\cdot\nabla_-\psi$, and hence through the boundary conditions Eq. 17 the exterior surface fields, are all expressible in terms of expansions involving β. Substituting these forms back in Eqs. 6 and 7, and eliminating β as before, one finally obtains

$$\bar{Q}T=-\mathrm{Re}(\bar{Q}), \qquad (19)$$

with matrix elements given by

$$\bar{Q}_{mn}\equiv\frac{k}{4\pi}\int \mathbf{d\sigma}\cdot\left\{\frac{\rho'}{\rho}[\mathrm{Re}\psi_m(k'\mathbf{r})]\nabla\psi_n(k\mathbf{r}) -\nabla[\mathrm{Re}\psi_m(k'\mathbf{r})]\psi_n(k\mathbf{r})\right\}. \qquad (20^*)$$

Notice also that the restriction to nondissipative obstacles is easily removed. The above argument goes through with no essential changes provided one expands the interior field in regular wavefunctions containing the radial functions of the appropriate complex argument $k'r$. The form of Eq. 19 becomes slightly more involved, but the modifications are straightforward.

After solving the appropriate equation for the transition matrix T, the scattering coefficients f are obtainable from Eq. 13 for each desired incident wave. The farfield scattering is then described in the usual manner by introducing the large-argument formulas for the radial functions in Eq. 5. Alternatively, if numerical values of surface field quantities are desired, the expansion coefficients α are obtainable by numerical solution of the system of Eqs. 10. In this event, the scattering coefficients are given directly by Eq. 11.

It is of interest to examine some limiting cases of Eq. 20. First, taking $k'=k$, so that phase velocities are equal within the scattering region and its surroundings (hence the density and stiffness ratios are both arbitrary, but equal, i.e., $\rho'/\rho=M'/M$), one has by inspection

$$\bar{Q}=(\rho'/\rho)\hat{Q}-Q;\quad k'=k. \qquad (20')$$

Using this equation, we see the preceding results to be limiting cases of the present one. Letting $\rho'/\rho = M'/M \rightarrow \infty$ (rigid boundary), as discussed by Rayleigh,[28] Eq. 19 goes over to the Neumann case—i.e., Eq. 14 using $\hat{Q}$. Similarly, for the alternative limit $\rho'/\rho = M'/M \rightarrow 0$ (soft boundary), Eq. 19 yields the Dirichlet case, involving Q.

A second situation of interest arises when the scattering region is a perturbation on its surroundings, i.e., when $\Delta_\rho \equiv (\rho'-\rho)/\rho \ll 1$, $\Delta_M \equiv (M'-M)/M \ll 1$. In this event, $\tilde{Q}$ may be transformed, by separating off the first term in the integrand times a factor $(\rho'-\rho)/\rho$, then applying the divergence theorem. Neglecting terms of first order or higher in the small quantities, one obtains

$$\tilde{Q} \rightarrow i1,$$

so that, from Eq. 19, the transition matrix in this limit is just proportional to $\mathrm{Re}(\tilde{Q})$ and is given by

$$T_{mn} \rightarrow \frac{k^3}{4\pi i}\Delta_M \int d\tau\, \mathrm{Re}\psi_m(k\mathbf{r})\, \mathrm{Re}\psi_n(k\mathbf{r})$$
$$-\frac{k}{4\pi i}\Delta_\rho \int d\tau\, \nabla[\mathrm{Re}\psi_m(k\mathbf{r})]\cdot\nabla[\mathrm{Re}\psi_n(k\mathbf{r})]. \quad (21^*)$$

The scattered wave is obtained according to Eqs. 5 and 13, by multiplying by the incident wave coefficients a_n, the outgoing wavefunctions ψ_m, and summing over both indices (see Eqs. 3 and 4). For example, for incident plane wave $\exp(i\mathbf{k}_0\cdot\mathbf{r})$ one gets, after some straightforward simplifications,

$$\psi^s(k\mathbf{r}) \longrightarrow -\frac{k^2\Delta_M}{4\pi}\int d\tau' g(kR)e^{i\mathbf{k}_0\cdot\mathbf{r}'}$$
$$\mp\frac{\Delta_\rho}{4\pi}\int d\tau'\nabla' g(kR)\cdot\nabla' e^{i\mathbf{k}_0\cdot\mathbf{r}'}$$
$$\xrightarrow[kr\to\infty]{} -\frac{k^2}{4\pi r}e^{ikr}[\Delta_M-(\hat{k}_0\cdot\hat{k})\Delta_\rho]$$
$$\times\int d\tau' e^{i(\mathbf{k}_0-\mathbf{k})\cdot\mathbf{r}'} \quad (22^*)$$

where $\hat{k}_0$ and $\hat{k}$ are unit vectors in the direction of incidence and observation, respectively.

This extension of the first Born approximation (in the usual case $\Delta_\rho = 0$) has recently been found by different techniques by Morse and Ingard.[29] It could alternatively be obtained by iteration of the integral representation given by Gerjuoy and Saxon.[30] The same result was also found by Kleinman.[31] In the low-frequency limit, the last integral above is seen to give precisely the volume of the scattering region, and Eq. 22 agrees with the result originally obtained by Rayleigh.[32]

II. SYMMETRY, UNITARITY, AND EIGENFUNCTIONS

Before proceeding further, it is appropriate to examine the properties of symmetry and unitarity, as they relate to the matrix equations. This is conveniently done in terms of the scattering matrix S defined by

$$S = 1 + 2T. \quad (23a)$$

T serves to compute the expansion coefficients for the outgoing waves due to a given *regular* incident wave, whereas S performs the same computation for an incident field specified by *incoming* waves singular at the origin. More specifically, the total field can be written (outside the circumscribing sphere of Fig. 1)

$$\psi = \sum_{m,n} [a_n \,\mathrm{Re}(\psi_n) + T_{mn}a_m\psi_n]; \quad (23b)$$

after a little manipulation, this same field becomes

$$\psi = (\tfrac{1}{2}) \sum_{m,n} [a_n\psi_n^* + S_{nm}a_m\psi_n]. \quad (23c)$$

The scattering matrix has been discussed by Gerjuoy and Saxon for acoustic problems.[30] Upon introducing the incoming–outgoing partial-wave basis in their results, it is not difficult to show that S must be both symmetric and unitary, i.e.,

$$S' = S \text{ (or } T' = T), \quad (24a)$$

and

$$S'^*S = 1 \text{ (or } T'^*T = -\mathrm{Re}T). \quad (24b)$$

These conditions stem, respectively, from the reciprocity principle and energy conservation requirements.

The matrix equations derived earlier are not independent of the constraints of Eqs. 24, but satisfy them in part, as follows: The basic equation is

$$QS = -Q^*, \quad (25)$$

for which the formal solution is[33]

$$S = -Q^{-1}Q^*. \quad (26)$$

Now forming the product S^*S from Eq. 26, one immediately obtains

$$S^*S = 1. \quad (27)$$

Because of this property, the conditions (Eqs. 24) are no longer separate constraints; that is, if S is symmetric, it will automatically be unitary, and vice versa.

Next, let us consider the eigenvectors of S. The eigenvalues of a unitary matrix lie on the unit circle in

[28] J. W. Strutt Lord Rayleigh, *The Theory of Sound* (Dover Publications, Inc., New York, 1945), Vol. 2, p. 284.

[29] P. M. Morse and K. U. Ingard, *Theoretical Acoustics* (McGraw-Hill Book Co., New York, 1968), p. 413.

[30] E. Gerjuoy and D. S. Saxon, Phys. Rev. 94, 1445–1458 (1954). See Eq. 11 therein.

[31] R. E. Kleinman, private communication.

[32] Reference 28, pp. 149–152.

[33] Note that it follows readily from Eq. 26 that, if S is to be symmetric, the matrix product $\mathrm{Im}(Q)\,\mathrm{Re}(Q)$ must be symmetric.

the complex plane, so one can write

$$Su^{(j)} = e^{i\lambda_j} u^{(j)}, \tag{28}$$

where the jth eigenvector $u^{(j)}$ has components $u_1^{(j)}$, $u_2^{(j)}$, $\cdots$, and the λ_j are real. Because S is in addition symmetric, one can show (premultiply Eq. 28 by S'^*, then employ Eqs. 24) that the eigenvectors constitute a *real* orthonormal set. Operating on $u^{(j)}$ with the matrix equality (Eq. 25), there results

$$e^{i\lambda_j} Q u^{(j)} = -Q^* u^{(j)},$$

which, in view of the fact that $u^{(j)}$ is real, may be rewritten

$$\mathrm{Re}(Q) u^{(j)} = \tan(\lambda_j/2)\, \mathrm{Im}(Q) u^{(j)}. \tag{29a}$$

This is a real homogenous system of equations, from which the eigenvectors may be determined after first solving the secular equation

$$|\mathrm{Re}Q - \tan(\lambda_j/2)\, \mathrm{Im}(Q)| = 0 \tag{29b}$$

for the eigenvalues.[34]

The *eigenfunctions* $\{\varphi_j(\mathbf{r})\}$ can now be constructed using the eigenvectors as expansion coefficients with the basis functions; i.e., by definition

$$\varphi_j(\mathbf{r}) \equiv \sum_n u_n^{(j)} \psi_n(\mathbf{r}), \quad j = 1, 2, \cdots. \tag{30}$$

Just as with the original basis functions, these outgoing fields have as their counterparts the regular eigenfunctions $\{\mathrm{Re}\varphi_j(\mathbf{r})\}$ which are well behaved at the origin. The set $\{\varphi_j(\mathbf{r})\}$ constitute outgoing waves reflected intact except for a phase shift upon incidence of the corresponding incoming wave. That is, introducing the eigenvectors in Eq. 23b, one sees that the linear combinations $\varphi_j^* + \exp(i\lambda_j)\varphi_j$, $j = 1, 2, \cdots$, are fields satisfying the boundary conditions imposed by the presence of the obstacle. Note also that these linear combinations may be written in terms of the regular functions as $\mathrm{Re}\varphi_j + \frac{1}{2}[\exp(i\lambda_j) - 1]\varphi_j$.

Solution of the scattering problem is immediate in terms of the eigenfunctions: First, the incident wave is expanded in regular eigenfunctions to get

$$\psi^i(\mathbf{r}) = \sum_j c_j\, \mathrm{Re}\varphi_j(\mathbf{r}). \tag{31a}$$

The coefficients may be obtained from the observation that the φ_j are orthogonal with respect to integration over the large circular cylinder (or sphere) σ_o at infinity, because of orthogonality in Eq. 30 of both the angular functions appearing in the ψ_n, and the eigenvectors $u^{(j)}$. One thus has

$$c_j = \int d\sigma_o \psi^i\, \mathrm{Re}(\varphi_j) \Big/ \int d\sigma_o [\mathrm{Re}(\varphi_j)]^2. \tag{31b}$$

In view of the comments of the preceding paragraph, the resulting scattered wave is given by (assuming the expansion converges)

$$\psi^s(\mathbf{r}) = \sum_j \tfrac{1}{2}(e^{i\lambda_j} - 1) c_j \varphi_j(\mathbf{r}). \tag{32}$$

For the elementary case of Dirichlet boundary conditions on a circular cylinder of radius $r = a$, using circular cylindrical wavefunctions ψ_n, the Q matrix in Eq. 14 is diagonal, and the φ_j coincide with the ψ_j. The real and imaginary parts of the elements Q_{jj} differ only in containing the factor $J_j(ka)$ or $N_j(ka)$, respectively (Bessel or Neumann functions). From Eq. 29b, one has $\tan(\lambda_j/2) = J_j(ka)/N_j(ka)$, and the factor in Eq. 32 yields the well-known result

$$\tfrac{1}{2}(e^{i\lambda_j} - 1) = -J_j(ka)/H_j(ka)$$

involving the Hankel function of the first kind H_j. Corresponding known results can be seen to obtain with Neumann conditions, Eq. 16, or the penetrable acoustic cylinder, Eq. 19.

Next in order of difficulty would be Dirichlet or Neumann boundary conditions on a cylinder of elliptic cross section. In both cases, the eigenfunctions are the same and are known, from the standard separation of variables procedure, in the form of products of Mathieu functions in elliptic cylinder coordinates. Expansion of the regular eigenfunctions, i.e., the real part of Eq. 30, has been given for example by Stratton,[35] who also gives the expansion of Eq. 31a for an incident plane wave. In problems of this type, where separation of variables is directly applicable, the method of the present section can be reduced to a simpler form, because both real and imaginary parts of Q turn out to be symmetric, as is shown below. In this event[33] it follows that ImQ and ReQ commute and must have common eigenvectors. The generalized eigenvalue problem given in Eqs. 29a,b may consequently be replaced by either one of the two ordinary eigenvalue problems $\mathrm{Re}(Q)u^{(j)} = \alpha_j u^{(j)}$, or $\mathrm{Im}(Q)u^{(j)} = \beta_j u^{(j)}$ (where $\alpha_j\ \beta_j = \tan\lambda_j/2$). These equations are of interest in providing a new method for determination of the elliptic cylinder wavefunctions, not involving elliptic cylinder coordinates. Whether or not the method will turn out to have computational advantages in practice remains to be seen.

If now the boundary conditions be changed, to apply to a penetrable elliptic cylinder, then the eigenfunctions are determined from Eqs. 29 and 30, using the matrix $\bar{Q}$ given in Eq. 20. The separation of variables procedure, on the other hand, does not lead to eigenfunctions. As shown by Yeh for the mathematically equivalent problem of the dielectric cylinder,[3] one can nevertheless solve the problem numerically by expanding in elliptic cylinder wavefunctions, if proper

[34] Eigenvalue problems of this form have been discussed by W. V. Petryshyn, Phil. Trans. Roy. Soc. (London) A262, 413–458 (1968), and references therein.

[35] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Co., New York, 1941), pp. 375–387.

account is taken of the fact that all coefficients are coupled through the boundary conditions.

For this last problem, or the general case of non-separable boundary geometry, the merits of employing eigenfunctions, rather than the original basis functions, in practice have not been established. One criterion, however, consists of the relative numerical difficulty of straight matrix inversion of Q (in truncation) to solve, say, Eq. 14, versus the resolution into eigenvectors and eigenvalues described by Eqs. 29a,b.

III. STRUCTURE OF THE Q MATRIX

In order to understand the matrix equations better, it is helpful to examine the matrix elements Q_{mn} in some detail. For two dimensions, the basis functions are[1]

$$\psi_{(e/o)n}(\mathbf{r})=(\epsilon_n)^{\frac{1}{2}}\begin{matrix}\cos\\ \\ \sin\end{matrix}\, n\theta H_n(kr) \tag{33}$$

in circular cylinder coordinates r, θ. The Neumann factor ϵ_n has the value $\epsilon_0=1$, $\epsilon_n=2$ otherwise. It is notationally convenient to break Q into four blocks according to parity, writing

$$Q=\begin{bmatrix}Q^{ee} & Q^{eo}\\ Q^{oe} & Q^{oo}\end{bmatrix}. \tag{34}$$

A corresponding block notation is then used for the transition matrix in Eq. 14.

The discussion can be simplified slightly by making the restriction that the obstacle have mirror symmetry across the plane $y=r\sin\theta=0$, so that $r(\theta)=r(2\pi-\theta)$, and the integrals involving mixed products of sines and cosines are seen to vanish. Because of the block diagonal nature of Q in Eq. 34, the matrix equation, Eq. 14, is now seen to reduce to the two (single) block equations

$$Q^{ee}T^{ee}=-\mathrm{Re}Q^{ee};\quad Q^{oo}T^{oo}=-\mathrm{Re}Q^{oo}. \tag{35}$$

For the matrices Q^{ee} and T^{ee}, indices run $m, n=0, 1, 2, \cdots$; whereas for Q^{oo} and T^{oo}, one has $m, n=1, 2, \cdots$.

From Eq. 16, it is not difficult to show that

$$Q_{mn}{}^{ee,\,oo}=\mp\frac{i}{2}\delta_{mn}+(\epsilon_m\epsilon_n)^{\frac{1}{2}}\frac{1}{8}\int_0^{2\pi}d\theta\left[\frac{r\partial}{\partial r}-\frac{1}{r}\frac{dr}{d\theta}\frac{\partial}{\partial\theta}\right]J_mH_n\begin{matrix}\cos m\theta\cos n\theta\\ \\ \sin m\theta\sin n\theta\end{matrix}, \tag{36}$$

where r is set equal to $r(\theta)$ after the partial derivatives are taken. The low-frequency limiting form of Q may now be obtained by keeping only the leading terms in the power-series expansions of the Bessel and Hankel functions. It has already been noted, following Eq. 16, that certain numerically dominant terms will integrate to zero for off-diagonal elements. From an analytical point of view, on the other hand, writing $r(\theta)=a\rho(\theta)$, so that a and $\rho(\theta)$ characterize the size and shape, respectively, of the obstacle, Eq. 36 reduces schematically to (for either Q^{ee} or Q^{oo})

$$\left.\begin{aligned}\mathrm{Re}(Q_{mn})&\doteq\epsilon^{m+n} \qquad (37\mathrm{a})\\ \mathrm{Im}(Q_{mn})&\doteq\epsilon^{m-n} \qquad (37\mathrm{b})\end{aligned}\right\}\ m, n=1, 2, \cdots$$

for $\epsilon=ka\ll 1$. For Q^{ee}, one can verify in addition the terms

$$\mathrm{Re}(Q_{0m}{}^{ee})\doteq\mathrm{Re}(Q_{m0}{}^{ee})\doteq\epsilon^{m+2};\quad m=0, 1, \cdots, \tag{37c}$$

$$\mathrm{Im}(Q_{0m}{}^{ee})\doteq\epsilon^{2-m},\ \mathrm{Im}(Q_{m0}{}^{ee})\doteq\epsilon^{m};\ m=1, 2, \cdots, \tag{37d}$$

and finally

$$\mathrm{Im}(Q_{00}{}^{ee})\doteq\begin{cases}\epsilon^2 \text{ (Dirichlet)}\\ 1 \text{ (Neumann)}\end{cases}. \tag{37e}$$

The form of the numerical coefficients appearing on the right-hand side of Eqs. 37, including dependence on $\ln\epsilon$ in some cases, is easily obtained from Eq. 36.

We are interested in the structure of the transition matrix T insofar as dependence on powers of ϵ is concerned. In order to balance out the indicated dependence on the parameter ϵ in Eq. 35, assuming $\mathrm{Im}(Q)$ to be nonsingular, it is necessary that T^{ee} have the form

$$\mathrm{Im}T_{mn}{}^{ee}\doteq\epsilon^{m+n};\quad m, n=0, 1, \cdots \text{ (Dirichlet)}, \tag{38}$$

with $\mathrm{Re}T_{mn}{}^{ee}$ involving higher-order terms in each element. This can be verified by substitution, along with Eqs. 37, in the first of Eqs. 35. One then notes that any larger terms (i.e., lower powers of ϵ) than shown in Eq. 38 could only be accommodated if $\mathrm{Im}(Q)$ were singular, which is contrary to assumption. In the Rayleigh limit, the scattering with Dirichlet boundary conditions is thus isotropic, and described by the leading term $T_{00}{}^{ee}$ which can depend on ϵ only logarithmically. In order to obtain any of the appropriate numerical coefficients suppressed on the right side of Eqs. 38, it is necessary in general to solve the infinite matrix Eq. 35 numerically by a limiting process.

The situation is somewhat different with Neumann boundary conditions. Because of Eq. 37e, similar analysis leads to (again, $\mathrm{Re}T_{mn}{}^{ee}$ can involve only higher powers of ϵ in each element)

$$\mathrm{Im}T_{mn}{}^{ee}\doteq\begin{cases}\epsilon^{m+n+2} & (m=0 \text{ and/or } n=0)\\ \epsilon^{m+n} & m, n=1, 2, \cdots \text{ (Neumann)}\end{cases} \tag{39}$$

There are thus three leading terms in this case, the isotropic term $T_{00}{}^{ee}$, and the dipole terms $T_{11}{}^{ee}$ and $T_{11}{}^{oo}$ (not shown in Eq. 39), all of order ϵ^2. The coefficient of the isotropic term can be obtained in closed form this time, by noting in Eq. 35 that elements of the top row of Q (i.e., $Q_{0n}{}^{ee}$, $n=0, 1, \cdots$) vary like $i+\epsilon^2$, $i\epsilon+\epsilon^3$, $i+\epsilon^4$, $i/\epsilon+\epsilon^5$, $\cdots$, whereas the first column of T behaves like $i\epsilon^2$, $i\epsilon^3$, $i\epsilon^4$, $\cdots$. Correct to leading terms the product $(Q^{ee}T^{ee})_{00}$ is given by the first term in the sum over row and column, so that $Q_{00}{}^{ee}T_{00}{}^{ee}\approx -\mathrm{Re}Q_{00}{}^{ee}$. From Eq. 36, it is easily seen that

$$Q_{00}{}^{ee}\approx i\frac{k^2}{8}\int_0^{2\pi} d\theta r^2(\theta)=i\frac{k^2A}{4}$$

in terms of the cross-sectional area A of the cylinder, so one finally obtains

$$T_{00}{}^{ee}\approx -(k^2A/4)^2-ik^2A/4 \text{ (Neumann)}. \qquad (40)$$

This result agrees with the classical results for the circular cylinder $[-J_0'(ka)/H_0'(ka)]$ and the elliptic cylinder,[36] obtained by separation of variables.

The analysis for a cylindrical volume of scattering material, starting from Eq. 19, is almost identical, and one obtains for the isotropic term in the Rayleigh limit

$$T_{00}{}^{ee}\approx -\left(\frac{M'-M}{M'}\right)^2\left(\frac{k^2A}{4}\right)^2-i\left(\frac{M'-M}{M'}\right)\frac{k^2A}{4}. \qquad (41)$$

In contrast to the low-frequency results discussed earlier in connection with Eq. 22, where the medium properties were restricted to a perturbation on their surroundings, this equation is valid for general density and stiffness ratios, provided both ka and $k'a\ll 1$, and with the exception of the Dirichlet limit $M'/M=\rho'/\rho\to 0$. Note, for example, that the Neumann limit $M'/M=\rho'/\rho\to\infty$, given in Eq. 40, is obtainable from Eq. 41. It is also of interest to note that for the corresponding boundary-value problems in the electromagnetic case, the dominant terms (i.e., the imaginary parts) of Eqs. 40 and 41 have been obtained by Van Bladel by invoking magnetostatic or electrostatic considerations.[37]

The above discussion changes quite radically when applied to quadric surfaces, for which separation-of-variables techniques are also available. This comes about through the orthogonality of the angular functions, coupled with Wronskian relations for the radial functions in our basis. Consider the elliptic cylinder having semimajor and semiminor axes a, b, respectively, defined by

$$[1/\rho(\theta)]^2=\cos^2\theta+(a/b)^2\sin^2\theta. \qquad (42)$$

Because there is now a second plane of mirror symmetry, the plane $x=0$, one easily sees from Eq. 36 that $Q_{mn}=0$ if $(m+n)$ is odd. For the balance of the elements, the difference $Q_{m(m+2s)}-Q_{(m+2s)m}$ will contain under the integral sign a factor

$$J_m(ka\rho)Y_{m+2s}(ka\rho)-J_{m+2s}(ka\rho)Y_m(ka\rho).$$

Now, by applying the standard recursion formulas to the Wronskian relation $J_m(x)Y_{m+1}(x)-J_{m+1}(x)Y_m(x)=-2/\pi x$, one can show that

$$J_m(x)Y_{m+2s}(x)-J_{m+2s}(x)Y_m(x)\doteq\sum_{p=0}^{s-1}\left(\frac{1}{x}\right)^{2(s-p)}, \qquad (43)$$

where the precise coefficients of the inverse powers of x are not germane to the present discussion and have been omitted.[38] Identifying x with $ka\rho$ and using Eqs. 43 with the defining Eq. 36 for Q_{mn}, the second group of terms in the integrand can be integrated by parts to remove the factor $dr/d\theta$. At this point, from Eq. 42 one sees that the radial functions in the integrand contribute a finite number of trigonometric functions $\{\cos 2s_1\theta\}$, with $0\le s_1\le s$. On the other hand, the angular functions contribute only the two terms $\cos 2s\theta$, $\cos 2(s+m)\theta$. By orthogonality, all of these integrals vanish except one involving $(\cos 2s\theta)^2$, and the latter is precisely the term that vanishes for general shapes, as discussed following Eq. 16. It follows that the Q matrix is exactly symmetric, and given by Eq. 36 using

$$J_mH_n=J_{m>}H_{m<} \qquad (44)$$

for the product of Bessel functions, where $m_>$, $m_<$ are, respectively, the greater and lesser of m, n.

For the more general case of volume scattering by an elliptic cylinder, $\bar{Q}$ is no longer symmetric. Observe, however, that symmetry of Q implies that, after introducing low-frequency expansions for both J_m and Y_n in Eq. 12 and regrouping terms according to ascending powers of kr, all terms involving inverse powers of kr vanish upon integration. Comparison of Eq. 12 with the first term in the integrand of Eq. 20 for $\bar{Q}$ reveals that precisely the same terms will vanish in the latter. Similar comments apply to the second term in the integral of Eq. 20 by analogy with $\hat{Q}$. Thus, $\bar{Q}$ for scattering from an elliptic cylindrical volume is given by Eq. 20 with all singular terms (as described above) from the radial function expansion simply discarded.

In dealing with quadrics, it is of interest to observe that an alternative choice of expansion functions for the surface fields will also lead to a symmetric Q matrix. Thus, instead of the functions of Eq. 9, or the corresponding expansion for Neumann boundary conditions, one can essentially reverse these choices and employ instead the functions[27]

$$w(\mathbf{r})\,\mathrm{Re}\psi_n(\mathbf{r});\quad \mathbf{r}\text{ on }\sigma\text{ (Dirichlet)}$$
$$[w(\mathbf{r})]^{-1}\hat{n}\cdot\nabla[\mathrm{Re}\psi_n(\mathbf{r})];\quad \mathbf{r}\text{ on }\sigma\text{ (Neumann)} \qquad (9')$$

where the weight function $w(\mathbf{r})=k^2r[1+(r'/r)^2]^{-\frac{1}{2}}$ serves to remove a complicating factor appearing in the integrands. For example, using the first of Eqs. 9′ in

[36] J. E. Burke and V. Twersky, J. Opt. Soc. Amer. 54, 732–744 (1964).

[37] J. Van Bladel, Appl. Sci. Res. B (Netherlands) 10, 195–202 (1963); *Electromagnetic Fields* (McGraw-Hill Book Co., New York, 1964), pp. 393–397.

[38] The coefficients can readily be obtained by comparison with results given by Watson: G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, Cambridge, England, 1962), 2nd ed., pp. 145–150.

Eq. 7* gives a new Q matrix with elements

$$Q_{mn}{}'' {}^{\sigma\sigma}=(\tfrac{1}{4})(\epsilon_m\epsilon_n)^{\frac{1}{2}}\int_0^{2\pi} d\vartheta(kr)^2 \times J_m(kr)H_n(kr) \begin{matrix}\cos m\vartheta \cos n\vartheta \\ \sin m\vartheta \sin n\vartheta\end{matrix}. \quad (36')$$

Now essentially by inspection, using Eq. 42, it may be seen again that all terms on the right-hand side of Eq. 43 will drop out in the course of the integration, so that this alternate version of Q is also symmetric, with Eq. 44 applicable.

For the special boundary considered, Eq. 36′ is apparently slightly preferable to Eq. 36 because of the somewhat less involved integrands of the former. Notice, however, that one pays for this advantage in loss of generality, i.e., the earlier Eq. 36 was applicable to both Dirichlet and Neumann conditions. Numerical results have been obtained with Eq. 36′, and are described subsequently.

The simplifications that occur in Q, $\hat{Q}$, and $\bar{Q}$ for boundaries of elliptical cross section are of interest from both theoretical and practical viewpoints. Note first that it is possible to obtain systematically as many terms as desired in the low frequency expansion for the transition matrix. From a practical point of view, the matrices are probably very well behaved as regards truncation and numerical inversion (in this connection, see the following section). Finally, the advantage of Q being symmetric in the eigenfunction computation has been discussed earlier.

Turning now to the three-dimensional case, the wavefunctions for the basis are chosen to be[1]

$$\psi_{\sigma mn}(k\mathbf{r})=(\gamma_{mn})^{\frac{1}{2}}h_n(kr)Y_{mn}{}^{\sigma}(\theta,\phi), \quad (45a)$$

in terms of the spherical Hankel functions of the first kind h_n, and the spherical harmonics

$$Y_{mn}{}^{\sigma}(\theta,\varphi)=Y_{mn}{}^{e,o}= \begin{matrix}\cos \\ \sin\end{matrix}\, m\varphi P_n{}^m(\cos\theta). \quad (45b)$$

The normalizing constants in Eq. 45a are given by

$$\gamma_{mn}=\epsilon_m(2n+1)(n-m)!/(n+m)!. \quad (45c)$$

From Eq. 16, the general matrix element becomes

$$Q_{\sigma mn\sigma' m' n'}=\mp\frac{i}{2}\delta_{\sigma\sigma'}\delta_{mm'}\delta_{nn'}+\frac{k}{8\pi}\int_0^{2\pi}\int_0^{\pi} d\theta d\varphi r^2\sin\theta \times\left[\frac{\partial}{\partial r}-\frac{r_\theta}{r^2}\frac{\partial}{\partial\theta}-\frac{r_\varphi}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi}\right]\mathrm{Re}(\psi_{\sigma mn})\psi_{\sigma' m' n'} \quad (46)$$

where $r_\theta=\partial r(\theta,\varphi)/\partial\theta$, $r_\varphi\equiv\partial r(\theta,\varphi)/\partial\varphi$, and r is set equal to $r(\theta,\varphi)$ in the wavefunctions after the additional partial derivatives have been taken.

Various reductions of the Q matrix are possible, depending on the symmetry of the problem. If the obstacle has a plane of mirror symmetry normal to the polar axis—i.e., the plane $\theta=\pi/2$, then from the parity of the associated Legendre functions one has

$$Q_{\sigma mn\sigma' m' n'}=0; \quad (m+n+m'+n') \text{ odd}. \quad (47a)$$

On the other hand, for a plane of mirror symmetry containing the polar axis, e.g., the plane of azimuth $\varphi=0$, or the plane $\varphi=\pi/2$, one has

$$Q_{\sigma mn\sigma' m' n'}=0; \quad \sigma\neq\sigma', \quad (47b)$$

and if *both* the latter symmetry planes are present then in addition to Eq. 47b

$$Q_{\sigma mn\sigma m' n'}=0; \quad m+m' \text{ odd}. \quad (47c)$$

If the body possesses an axis of rotational symmetry, so that $r=r(\theta)$, then in addition to Eq. 47b, there is no coupling of the different azimuthal modes ($m'\neq m$), and one can write

$$Q_{\sigma mn\sigma' m' n'}=\delta_{\sigma\sigma'}\,\delta_{mm'}\,Q_{nn'}{}^{\sigma m}. \quad (48)$$

There are thus two families of matrices, $Q^{em}(m=0,1,\cdots)$ and $Q^{om}(m=1,2,\cdots)$, each member of which may be treated independently. From examination of Eq. 46, one can furthermore see that the families are identical, i.e., $Q^{em}=Q^{om}$ (except for Q^{o0}, which does not exist).

In view of the governing matrix equation (Eq. 14), it follows that each of the above reduction must apply also to the transition matrix. Note also that the full transition matrix may not be required for a particular problem. For example, a rotationally symmetric incident wave contains only the modes $m=0$. If the scattering surface also possesses rotational symmetry about the same axis, then it is only necessary to invert Q^{e0} and compute T^{e0}, in order to obtain a complete description of the scattering.

Finally, consider the ellipsoid

$$(x/a)^2+(y/b)^2+(z/c)^2=1,$$

which is the most general quadric surface having the three symmetry planes of Eqs. 47a, b, c. In spherical coordinates one can verify without difficulty that the ellipsoid is given by

$$[1/r(\theta,\varphi)]^2 \doteq Y_{00}{}^e+Y_{02}{}^e+Y_{22}{}^e \quad (49)$$

to within constant coefficients depending on a, b, c.

In close analogy with Eq. 43 the spherical Bessel functions may be seen to satisfy[38]

$$xj_n(x)n_{n+2s}(x)-xj_{n+2s}(x)n_n(x)\doteq\sum_{p=0}^{s-1}\left(\frac{1}{x}\right)^{2(s-p)}. \quad (50)$$

For the products of spherical harmonics, one has an expansion theorem of the form[39]

$$Y_{mn}{}^{\sigma}Y_{m'\nu'}{}^{\sigma'}\doteq\sum_{\mu\nu}Y_{\mu\nu}{}^{\sigma''} \quad (51)$$

where $\mu=|m'-m|$, $m'+m$, $|n'-n|\leq\nu\leq n'+n$. Introducing Eq. 50 into the difference (with $m+m'$ even, $n+n'$ even, $\sigma=\sigma'$ in view of symmetries)

$$Q_{\sigma mn\sigma m'n'}-Q_{\sigma m'n'\sigma mn}$$

formed from Eq. 48, one can integrate by parts to remove the terms r_θ and r_φ. From this point, the procedure is closely analogous to that leading to Eq. 44. The inverse powers of $(kr)^2$ contribute a finite sum of spherical harmonics, as can be seen from Eqs. 49 and 51. The angular functions give a second finite sum of spherical harmonics, using Eq. 51. Where the indices are distinct, the corresponding integrals vanish by orthogonality. For the one case where indices coincide, the integral contains the normal gradient of a potential function, and must be identically zero by the divergence theorem. Thus for the ellipsoid Q is symmetric in the sense

$$Q_{\sigma mn\sigma m'n'}=Q_{\sigma m'n'\sigma mn}, \tag{52}$$

so that $j_{n>}h_{n<}$ can be employed for the product of radial functions.

By similar analysis, one can establish that this symmetry also obtains for quadrics of rotational symmetry, but not the mirror symmetry of Eq. 47a, i.e., the surface $1/r(\theta)=1-B\cos\theta$, which constitutes a prolate spheroid $(0\leq B<1)$ or a paraboloid of revolution $(B=1)$. Using the notation of Eq. 48, one has in this case

$$Q_{nn'}{}^{\sigma m}=Q_{n'n}{}^{\sigma m}. \tag{53}$$

Returning to the general equation (Eq. 46) it appears that, in contrast to the notation adopted in Eq. 48 for rotationally symmetric bodies, the general computation is more conveniently organized in terms of a supermatrix $\mathcal{Q}$, each element of which is a matrix having just the number of degrees of freedom required to handle all azimuthal indices and parities associated with the values n, n'. Thus the element $\mathcal{Q}_{nn'}$ is a matrix of $(2n+1)$ rows by $(2n'+1)$ columns. A corresponding notation is employed for the transition matrix $\mathcal{T}$. Carrying out an analysis exactly paralleling that of Eqs. 37–40, this time in terms of elements $\mathcal{Q}_{nn'}$ of the supermatrix, it again turns out that the isotropic term can be obtained in closed form for Neumann conditions from the single equation $\mathcal{Q}_{00}\mathcal{T}_{00}=-\mathrm{Re}\mathcal{Q}_{00}$. Keeping leading powers of kr in Eq. 46 one has

$$\mathcal{Q}_{00}\rightarrow -\frac{k^3}{4\pi}\int\int d\theta d\varphi \sin\theta\frac{r^3}{3} + \frac{i}{2}\left(\mp1+1+\frac{k^2}{6\pi}\int\int d\theta d\varphi \sin\theta r^2\right). \tag{54}$$

The first integral is just the volume V of the obstacle so that, correct to leading terms in real part and imaginary part separately,

$$\mathcal{T}_{00}\approx-(k^3V/4\pi)^2-ik^3V/4\pi \text{ (Neumann)}. \tag{55}$$

Note that this result is in accord with the energy requirement of Eq. 24b. The dominant imaginary term in Eq. 55 agrees with results independently obtained by Van Bladel[40]; agreement is also obtained with the spheroid results given by Senior[41] and Burke.[42]

For the ellipsoid, $\mathcal{Q}$ is symmetric (see Eq. 52) and in the low-frequency limit "diagonal" correct to lowest-order terms in ka, i.e., the isotropic scattering can be obtained from the 1×1 matrix equation preceding Eq. 54 in the text, the dipole terms $\mathcal{T}_{11}$ from a 3×3 matrix equation, and so forth. The Neumann result is of course as given in Eq. 55. For the Dirichlet case, using the minus sign in Eq. 54, with the ellipsoid surface defined by

$$(abc)^2/r^2(\theta,\varphi)=(c\sin\theta)^2(b^2\cos^2\varphi+a^2\sin^2\varphi)+(ab\cos\theta)^2,$$

the integral involving r^2 is recognized as an inverse elliptic function,[43] i.e., (for $b\leq c$),

$$\int\int d\theta d\varphi \sin\theta r^2=\frac{4\pi ab}{\zeta}\int_0^\zeta du[(1-u^2)(1-k^2u^2)]^{-\frac{1}{2}} = \frac{4\pi ab}{\zeta}sn^{-1}\zeta, \tag{56}$$

with argument $\zeta=[1-(b/c)^2]^{\frac{1}{2}}$, and modulus given by $k^2=(c^2-a^2)/(c^2-b^2)$. The isotropic scattering is then

$$\mathcal{T}_{00}\approx-(kc\zeta/sn^{-1}\zeta)^2-ikc\zeta/sn^{-1}\zeta \text{ (Dirichlet)}. \tag{57}$$

The ellipsoid has been considered by Sleeman, who constructed the formal solution using separation of variables,[44] and subsequently obtained explicit results[45] using the low-frequency iterative procedure developed by Kleinman.[46] Equation 57 is in precise agreement with Sleeman's results [which also included terms of order $(kc\zeta)^3$]. Note that Eq. 57 contains as special cases the elliptic disk $(a=0)$,[47] prolate $(a=b<c)$ and oblate $(a<b\rightarrow c)$ spheroids considered by Senior[41] and Burke,[48] and of course the circular disk $(a=0;\, b\rightarrow c)$.[2]

Finally, in the event one is dealing with a volume scattering region of general shape, the analysis leading to Eq. 40 can again be applied with the result that

$$\mathcal{T}_{00}\approx-\left[\frac{(M'-M)}{M'}\right]^2\left(\frac{k^3V}{4\pi}\right)^2-i\left[\frac{(M'-M)}{M'}\right]\frac{k^3V}{4\pi} \tag{58}$$

in the low-frequency limit, with arbitrary disparities

[39] A. Messiah, *Quantum Mechanics*, J. Potter, Transl. (John Wiley & Sons, Inc., New York, 1965), p. 1057.
[40] J. Van Bladel, J. Acoust. Soc. Amer. **44**, 1069–1073 (1968).
[41] T. B. A. Senior, Can. J. Phys. **38**, 1632–1641 (1960).
[42] J. E. Burke, J. Acoust. Soc. Amer. **40**, 325–330 (1966).
[43] Reference 1, p. 432.
[44] B. D. Sleeman, J. Inst. Math. Its Appl. **3**, 4–15 (1967).
[45] B. D. Sleeman, J. Inst. Math. Its Appl. **3**, 291–312 (1967).
[46] R. E. Kleinman, Arch. Rat. Mech. Anal. **18**, 205–229 (1965).
[47] B. D. Sleeman, Proc. Cambridge Phil. Soc. **63**, 1273–1280 (1967).
[48] J. E. Burke, J. Acoust. Soc. Amer. **39**, 826–831 (1966).

in compressibility and density. This apparently new result is verified in one instance by comparison with Burke, who obtained all terms up to order k^6 for penetrable spheroids, using low-frequency expansions of the spheroidal wavefunctions.[5]

IV. NUMERICAL RESULTS AND DISCUSSION

As an example showing the usefulness of the present techniques in practice, consider the two dimensional problem of scattering by the strip $y=0$, $-a\leq x\leq a$ with Dirichlet boundary conditions. This problem has been considered by many authors.[2,25] Numerical results were obtained by Morse and Rubenstein, using separation of variables to carry out the analysis in terms of Mathieu functions.[19] Among subsequent extensions, the work of Skavlem for a slit (equivalent by Babinet's principle) is particularly useful for present purposes in that tables of numerical results were included.[20]

In applying the present method to the strip, two aspects of theoretical interest can be anticipated. *First*, the singularities of the outgoing wave functions ψ_n fall on the path of integration for matrix elements Q_{mn}, and must be dealt with. *Second*, the edge condition requires that the unknown surface field behave like $[1-(x/a)^2]^{-\frac{1}{2}}$ times an analytic function of x.[25] Both of these aspects are handled without difficulty by considering the strip as the limit of the elliptic cylinder, Eq. 42, when $b\rightarrow 0$. For the Q matrix we employ Eq. 36′ with the auxiliary symmetry condition of Eq. 44. The variable of integration can be changed over to Cartesian coordinates by observing that

$$d\theta r^2=dxr^2/x'(\theta)=-bdx[1-(x/a)^2]^{-\frac{1}{2}}. \tag{59}$$

Thus the nonanalytic behavior required by the edge condition enters naturally in the Wronskian of the transformation from polar to Cartesian integration variable.

Using Eq. 59 in Eq. 36′, the limit $b\rightarrow 0$ is readily taken. All elements of Q^{oo} vanish identically, and for Q^{ee} one obtains[49]

$$Q_{mn}=\begin{cases}(\epsilon_m\epsilon_n)^{\frac{1}{2}}\int_0^1 dx[1-x^2]^{-\frac{1}{2}}J_{m>}(kax)H_{m<}(kax); & \\ & (m+n)_{\text{even}} \\ 0 \quad \text{otherwise.} & \end{cases} \tag{60}$$

The transition matrix T^{ee} is then determined from

$$QT=-\mathrm{Re}(Q), \tag{61}$$

with $T_{mn}=0$ for $(m+n)$ odd.

The elements of Q may be computed either by numerical quadrature, or analytically by expansion of the product $J_{m>}H_{m<}$ in powers of kax (including terms in $\ln kax$).[38] We employ the latter method, which is somewhat more convenient for small to moderate values of ka. For large ka, precision difficulties would presumably be encountered because of strong cancellations among numerically large terms in the series (as occurs with the expansion of $\sin x$, for example). Term-by-term integration of the series is straightforward in terms of tabulated integrals,[50] and we proceed directly to results.

TABLE I. Complex even-index elements T_{mn} of the transition matrix for $ka=0.2$. The exponentiation factor is shown in parentheses, e.g., $1.483(-3)=1.483\times10^{-3}$.

T_{mn} m \ n	0	2	4
0	$-2.966(-1)$ $-i4.567(-1)$	$-1.483(-3)$ $-i2.283(-3)$	$-9.271(-7)$ $-i1.427(-6)$
2	$-1.483(-3)$ $-i2.283(-3)$	$-7.417(-6)$ $-i5.068(-5)$	$-4.636(-9)$ $-i3.986(-8)$
4	$-9.271(-7)$ $-i1.427(-6)$	$-4.636(-9)$ $-i3.986(-8)$	$-2.897(-12)$ $-i3.513(-11)$

Equations 61 were programmed for solution on the Philco 2000 computer by successive elimination of all off-diagonal elements of Q on the left-hand side, proceeding column by column. For the low-frequency case, $ka=0.2$, Table I shows the even-index elements of the transition matrix that resulted by truncating Eq. 61 at 3×3 matrices (i.e., m, $n=0$, 2, 4). Observe that T is exactly symmetric to the four significant figures shown. One can easily verify that the energy requirement $T'^*T=-\mathrm{Re}T$ of Eq. 24b is also satisfied to three or four significant figures.

No detailed numerical study of the alternative eigenfunction formulation of Sec. II has been performed as yet. One can, however, verify in part the approprpiate relationships using Table I. In elliptic cylinder coordinates, the expansion coefficients for the eigenfunctions in circular wave functions coincide with those for expanding the radial Mathieu functions in Bessel functions.[35] Numerical values can be obtained from results of Barakat and co-workers[51]; for the case at hand, one gets for the first Mathieu function Je_0 the (normalized) coefficients

$$u^{(0)}=\begin{bmatrix}1.000\\ 5.000\times10^{-3}\\ 3.124\times10^{-6}\end{bmatrix}. \tag{62}$$

This column array should be an eigenvector of the transition matrix, and indeed is. Using Table I, one

[49] We have dropped a factor k^2ab (common to both sides of Eq. 61) in front of the integral sign and introduced a dimensionless integration variable by letting $x\rightarrow ax$.

[50] W. Gröbner and N. Hofreiter, *Integraltafel, Part Two, Definite Integrals* (Springer-Verlag, Vienna, 1950), pp. 38, 79.

[51] R. Barakat, A. Houston, and E. Levin, J. Math. and Phys. 42, 200–247 (1963).

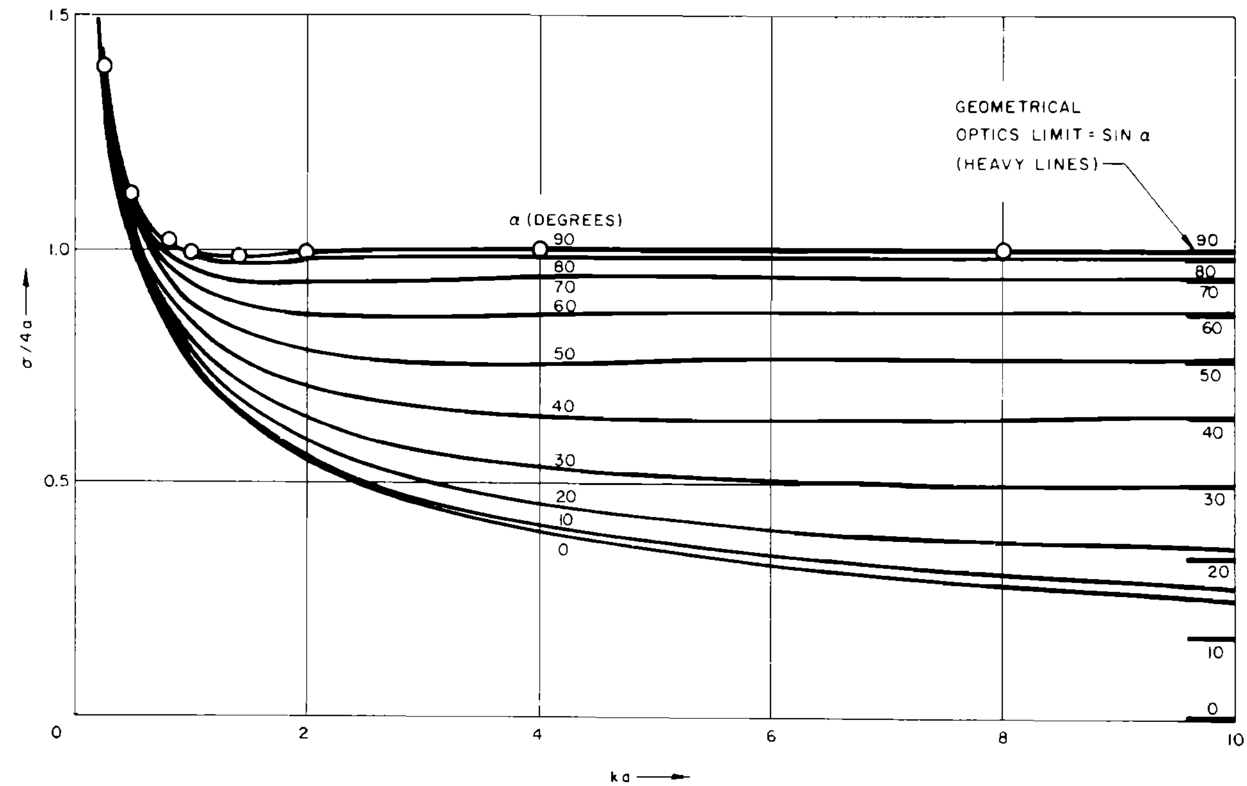

FIG. 2. Normalized scattering width versus ka, for the infinite strip with Dirichlet boundary conditions. Points shown by small circles were computed by Skavlem for normal incidence.[20]

verfies that

$$\mathrm{Re}(T)u^{(0)} = -0.2966\times\begin{bmatrix}1.000\\ 5.000\times10^{-3}\\ 3.125\times10^{-6}\end{bmatrix},$$
$$\mathrm{Im}(T)u^{(0)} = -0.4567\times\begin{bmatrix}1.000\\ 4.999\times10^{-3}\\ 3.124\times10^{-6}\end{bmatrix}. \tag{63}$$

The complex amplitude coefficient arising from Eqs. 63 is also seen to agree well with the appropriate quotient of Mathieu functions

$$-Je_0/He_0 = -0.2967 - i0.4568 \tag{64}$$

obtained from the tables.[31]

Numerical behavior of the solutions of Eq. 61 versus truncation is excellent. For example, for the case of Table I, keeping only one equation in one unknown, the isotropic term T_{00} is obtained correct to five significant figures, as judged by comparison with the larger systems of $N\times N$ equations with $N=2, 3, 4, 5$. For $N\geq 2$, T_{00} is found to remain constant to nine significant figures (computer precision is about 10 figures). T_{24} and T_{32}, obtained as in Table I, agree to seven figures.

At higher frequencies, more elements of the transition matrix are required for an accurate description of scattering. It is found that the elements remain roughly in the range $0.1<|T_{mn}|<1$ until one, or both, of the indices m, n exceed the numerical value of ka. Thus for $ka=10$, the largest value considered, somewhat more than 50 elements are required (recall that the T_{mn} vanish unless indices are both even, or both odd).

Once the transition matrix has been obtained, the scattering coefficients $f=Ta$ are easily computed for any incident wave of the form Eq. 3, and the scattered wave is given, for $r>a$, by $\psi^s=f\cdot\psi$ where ψ is regarded as a column vector made up from the basis functions of Eq. 33. These computations have been performed for plane waves with direction of incidence forming an angle α with the positive x axis (plane of the strip). In this event the scattered wave may be written in full

$$\psi^s(\mathbf{r}) = \sum_{m,n=0}^{N} (i)^m(\epsilon_m\epsilon_n)^{\frac{1}{2}}T_{mn}\cos m\alpha\,\cos n\vartheta H_n(kr);\qquad kr>ka$$
$$\sim f(\alpha,\vartheta)(2/i\pi kr)^{\frac{1}{2}}\exp(ikr);\quad kr\gg1 \tag{65}$$

with farfield amplitude given by

$$f(\alpha,\theta) = \sum_{m,n=0}^{N} (i)^{m-n}(\epsilon_m\epsilon_n)^{\frac{1}{2}}T_{mn}\cos m\alpha\,\cos n\vartheta. \tag{66}$$

The scattering width $\sigma(\alpha)$ may be computed, using the forward amplitude theorem, from the expression

$$\sigma(\alpha) = -(4/k)\,\mathrm{Re}f(\alpha,\alpha). \tag{67}$$

Results of this computation are shown in Fig. 2, in which $\sigma(\alpha)$ (normalized by twice the strip width $4a$) is plotted versus frequency up to $ka=10$, for directions of incidence ranging from grazing ($\alpha=0$) to normal ($\alpha=90°$). The curves appear in good qualitative agreement with those given for a smaller range in ka by Morse and Rubenstein.[19] The circled points shown for normal incidence are those of Skavlem,[20] and agree numerically to the precision given (five or six significant figures) with present results at all ka values common to both computations (0.8, 1, 2, 4, 8). Finally, the geometrical optics limiting values $\sigma(\alpha)/4a \longrightarrow \sin\alpha$ are shown at the right margin. Observe that for angles of incidence at least 30° from grazing, this limit is substantially achieved at $ka=10$.

One concludes that the present computation offers an interesting and practical alternative to the separation of variables procedure for scattering by a strip. Without making an exhaustive comparison, the amount of actual numerical computation in the two methods appears comparable; the additional complexity of matrix inversion in the present method is offset by the advantage of working with circular rather than Mathieu functions, particularly if the latter must be generated in the course of the computation.

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Appendix A. Completeness of the Regular Wavefunctions

The expansion of Eq. 9 employs normal gradients of regular wave functions, restricted to the smooth closed surface σ, to represent an unknown surface field. This expansion is convergent in the mean provided completeness can be established,[A1] and we assert that, considered as a function of k, the functions in question are complete with the exception of those discrete frequencies at which interior resonances (solutions of the homogeneous Neumann problem) occur.

To show this, the interior counterparts of Eqs. 5–7 are first obtained, starting from the alternate form of the Helmholtz formula.[24,25] In this manner, the total field (no incident wave present) in the interior volume is found in the form

$$\psi=\sum d_n \operatorname{Re}\psi_n, \qquad (A1)$$

with coefficients given by[26]

$$d_n=\frac{ik}{4\pi}\int d\sigma\hat{n}\cdot[(\nabla\psi_n)\psi_- - \psi_n\nabla_-\psi], \qquad (A2^*)$$

$$n=1,2,\cdots,$$

where $\hat{n}$ in Fig. 1 now points into the interior. The surface fields themselves are specified by the equations

$$\frac{ik}{4\pi}\int d\sigma\hat{n}\cdot[\nabla(\operatorname{Re}\psi_n)\psi_- - (\operatorname{Re}\psi_n)\nabla_-\psi]=0, \qquad (A3^*)$$

$$n=1,2,\cdots,$$

augmented with boundary conditions.

In particular, for the homogeneous Neumann case, the equations

$$\int d\sigma\hat{n}\cdot\nabla(\operatorname{Re}\psi_n)\psi_-=0,\ \ n=1,2,\cdots \qquad (A4)$$

are necessary and sufficient conditions for determining ψ_-. Furthermore, this problem is known to have only the trivial solution $\psi_-=0$, provided that k does not coincide with any of the discrete resonance frequencies (eigenvalues of the interior).[A2,A3] In this event, by definition, the gradient functions appearing in Eq. A4 form a *closed*, and hence complete, set.[A1] By a similar argument, the wavefunctions $\{\operatorname{Re}\psi_n\}$ used with Eq. 15 are complete except at eigenvalues of the interior *Dirichlet* problem. The above discussion applies also to the "reverse" choice, Eqs. 9′, upon absorbing the weight function in the surface field.

Difficulties with the exterior problem at eigenvalues of the interior are not new, incidentally[25]; methods of treating them have been discussed by Werner[A4] and Schenck.[A5] These difficulties do not appear to be fundamental in the present context. All complicatons arising at an interior resonance are eliminated upon choosing an alternate set of functions for the expansion of Eq. 9 that is complete without exception, e.g., the spherical harmonics. The latter are not to be preferred in general, however, in that they fail to possess the convenient analytical and computational properties shown to exist with the regular wavefunctions.

[A1] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, Inc., New York, 1953), Vol. 1, pp. 51 ff., pp. 110–111.

[A2] F. B. Hildebrand, *Finite-Difference Equations and Simulations* (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1968), pp. 278–285.

[A3] Reference 37, p. 295, pp. 306–312.

[A4] P. Werner, J. Math. Anal. Appl. 7, 348–395 (1963).

[A5] H. A. Schenck, J. Acoust. Soc. Amer. 44, 41–58 (1968).