Change of variable by diffusions: numerical experiments

Idea

let $X = \{x_1, ..., x_N\}$ be $N = 200$ points distributed at random on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. Let f be a function whose samples are known on X, and that one wants to interpolate on a finer grid $Y = \{y_1, ..., y_P\}$ of P=2400 points on $\mathbb{S}^2$. To do so, f is expressed as

$$f = h \circ g$$

where $g : \mathbb{S}^2 \to \mathbb{R}$ is such that its bandwidth is much smaller than that of f.

If such a low-band function g can be computed, then with high probability, the set X is sufficient to interpolate g to Y with high accuracy. Let $\overline{g}(Y) = \{\overline{g}(y_1), ..., \overline{g}(y_N)\}$ be the set of values thus obtained. Then with probability 1, $\{g(x_1), ..., g(x_N)\}$ is made of $N = 200$ distinct numbers at which one knows the value of h. Hence, h can also be interpolated on $\overline{g}(Y)$ with high precision ($\mathcal{O}(N^{-2})$ on average).

Experiments

The coarser set X was made of 200 points uniformly distributed at random in the (φ, θ)-space (these are the spherical angles). The finer grid Y was a grid on $\mathbb{S}^2$ where φ was evenly discretized 60 times, whereas θ was evenly discretized 40 times, leading to a grid of $P = 2400$ points.

We imposed $g(\varphi, \theta) = \sin(\theta) = z$, and we successively tried to interpolate $f(\varphi, \theta) = \sin(2\theta), \sin(4\theta)$,...

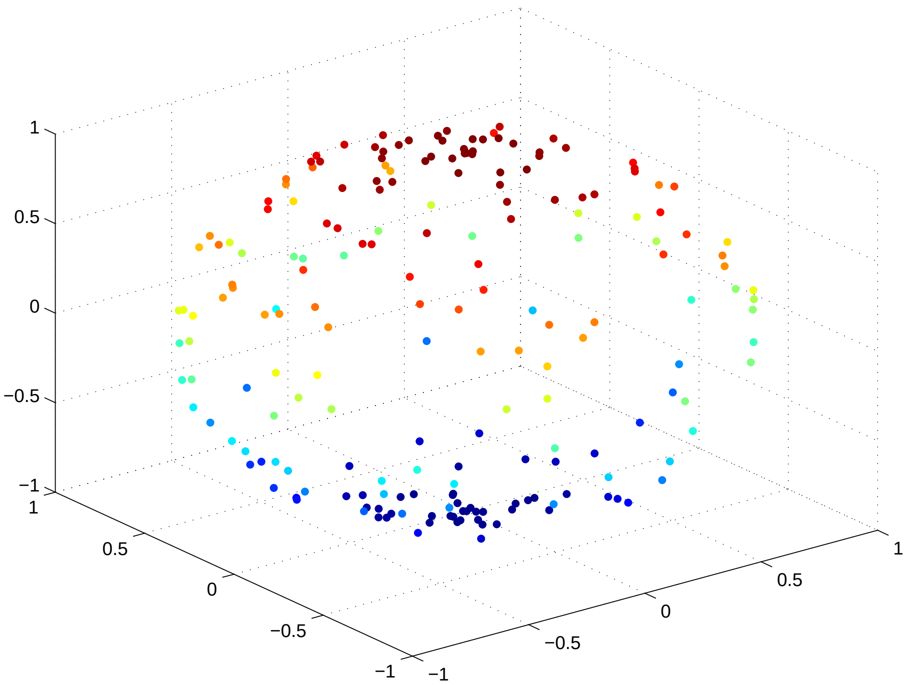
For the interpolation of the function g to the fine grid Y, we used geometric harmonics based on a Gaussian kernel of width equal to 1. We evaluated that the relative L^2 error (on this grid) of this extension with the function it is supposed to approximate, namely $z = \sin(\theta)$, is equal to 8×10^{-3}. Therefore we can conclude that the extension of g is accurate. The function g and its extension are shown on figure 2. The interpolation of h was done using cubic splines.

The performance of the extension of f is shown in the table on figure 1. We computed the relative L^2 error between the interpolation of f on Y and the function it is suppose to approximate. All error estimates were obtained as averages of 20 independent trials. The result was compared with that of the direct geometric harmonic interpolation.

frequency of f	Direct interpolation	Adapted interpolation
2	0.16	0.03
4	0.68	0.03
8	1.03	0.04
16	1.05	0.10

Figure 1: Relative L^2 error of interpolation of f for the direct and adapted interpolations.

The table shows that the adapted method clearly outperforms the direct interpolation.



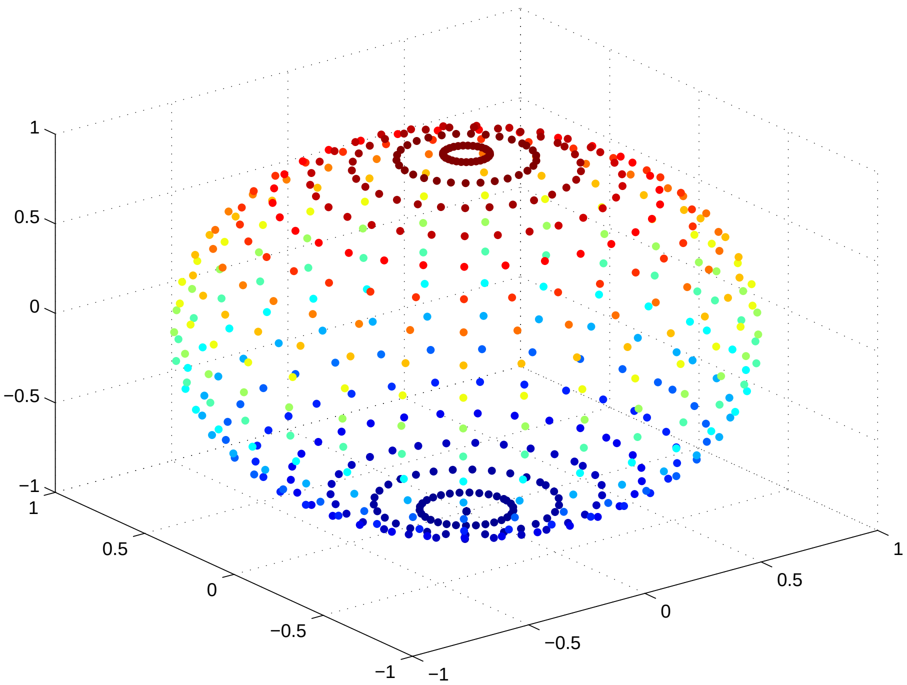


Figure 2: Top: the function g on the original set X of scattered points. Bottom: The extension of g to the fine grid Y using Gaussian geometric harmonics. The relative L^2 error with the actual function $z = \sin(\theta)$ is equal to 8×10^{-3}.

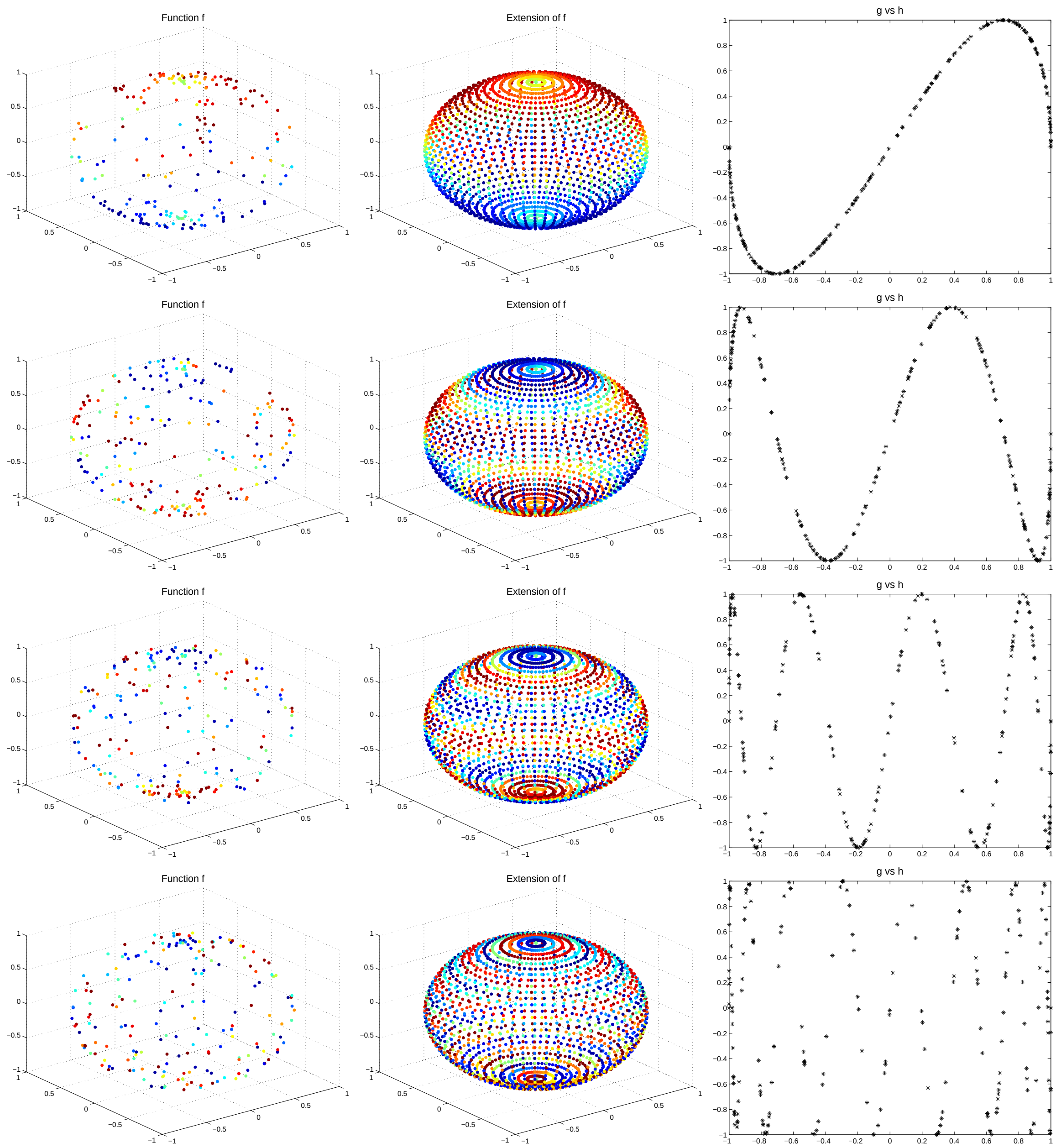


Figure 3: Function f on the set coarse X (left column), its extension to the fine grid Y (middle) and the corresponding plot of h vs g (right column). The rows correspond to f being respectively $\sin(2\theta)$, $\sin(4\theta)$, $\sin(8\theta)$ and $\sin(16\theta)$