Marching Schemes for Inverse Acoustic Scattering Problems

Frank Natterer Frank Wübbeling

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Abstract

For the numerical solution of inverse Helmholtz problems the boundary value problem for a Helmholtz equation with spatially variable wave number has to be solved repeatedly. For large wave numbers this is a challenge. In the paper we reformulate the inverse problem as an initial value problem, and describe a marching scheme for the numerical computation that needs only $n^2 \log n$ operations on an $n \times n$ grid. We derive stability and error estimates for the marching scheme. We show that the marching solution is close to the low-pass filtered true solution. We present numerical examples that demonstrate the efficacy of the marching scheme.

1 Introduction

This paper is concerned with the numerical solution of the Helmholtz equation

$$\Delta u + k^2(1+f)u = 0 \tag{1}$$

in $\mathbb{R}^n$. Here, $k > 0$ is the constant wave number and f is a function of compact support in $\mathbb{R}^n$. We are interested in the case where k is large, i.e. the diameter of the computational domain is one hundred (say) wavelengths. The main obstacle in the numerical solution of inverse problems for (1) is the need of solving (1) for k large. This is a challenge even for the direct problem (see e.g. [1]), not to speak of the inverse problem where this has to be done repeatedly. The present paper suggests a highly efficient marching scheme that is derived from stabilized initial value problems for (1). The stabilization is achieved simply by suppressing the evanescent waves. The resulting inversion algorithm is very similar to well-known iterative methods in computerized tomography except that the projection and backprojection steps are replaced by propagation and backpropagation, respectively, that are carried out by marching schemes.

Marching schemes are quite popular for the Helmholtz equation. They are based either on heuristics [5], on parabolic or paraxial approximations [2], or on the Riccati equation [6]. Our approach differs from these works in that it does not make use of first order equations. It rather sticks with the second order equation. The advantage is that it is not restricted to small propagation angles, and it can deal easily with backscatter. In addition our marching scheme admits error and stability estimates whose derivations are the main subject of the present paper. On the other hand our second order marching scheme is restricted to the use in conjunction with inverse problems where the needed Cauchy data can be obtained via the Dirichlet-to-Neumann map; it can't be used in an obvious way for the direct boundary value problem.

The present paper provides the theoretical justification of the marching schemes already used in [8], [10], [12], [14]. In section 2 we derive error and stability estimates for abstract evolution equations that are applied in section 3 to the initial value problem of the Helmholtz equation. In section 4 we give a detailed description of the marching scheme. In section 5 we describe how our marching scheme is applied to an inverse Helmholtz problem that arises in medical ultrasound tomography. In section 6 we give numerical examples for the direct problem. Numerical examples for inverse problems can be found in the above works.

2 Estimates for evolution equations

In the following we derive some simple estimates for the solution of second order evolution equations. These estimates are very much in the spirit of energy methods and are tailored to the needs of section 3. As general reference for the relevant aspects of evolution equations we recommend [13]

Let H be a complex Hilbert space with inner product $(\cdot,\cdot)$ and norm $||\cdot||$. Let $A(t)$, $t \geq 0$, be a densely defined linear operator in H. We assume throughout this section that $A = A_1 + A_2$ with $A_1(t)$ selfadjoint, and that there exist β, $\gamma > 0$ such that

$$\|A_2(t)\| \leq \beta \ , \tag{2}$$

$$(v, A_1'(t)v) \leq \gamma(v,v) \quad \text{for } \ v \in H \tag{3}$$

where A_1' is the derivative of A_1 with respect to t. For $u \in C^1([0,\infty), H)$ we put

$$\|u\|_1^2 = (u', u') + (u, A_1 u) \ .$$

Proposition 1: Let $u \in C^2([0,\infty), H)$ be a solution of

$$u'' + Au = r \tag{4}$$

with some $r \in C([0,\infty), H)$. If there exists $\alpha > 0$ such that

$$(v, A_1(t)v) \geq \alpha^2 (v,v) \ , \quad v \in H \ , \tag{5}$$

then, for any $\varrho > 0$,

$$\|u(t)\|_1^2 \leq \left(\|u(0)\|_1^2 + \frac{1}{\varrho} \int_0^t \|r(\tau)\|^2 d\tau \right) e^{(\varrho + \beta/\alpha + \gamma/\alpha^2)t} \ .$$

Proof: We use the energy method; see e.g. [4]. Multiplying (4) with u' we obtain

$$(u', u'') + (u', Au) = (u', r) \ . \tag{6}$$

Using

$$\begin{aligned} \frac{1}{2}\,\frac{d}{dt}\,(u', u') &= \mathrm{Re}(u', u'') \\ \frac{1}{2}\,\frac{d}{dt}\,(u, A_1 u) &= \mathrm{Re}(u', A_1 u) + \frac{1}{2}(u, A_1' u) \end{aligned}$$

we can rewrite the real part of (6) as

$$\frac{1}{2}\,\frac{d}{dt}\,((u', u') + (u, A_1 u)) = \mathrm{Re}(u', r) + \frac{1}{2}(u, A_1' u) - \mathrm{Re}(u', A_2 u) \ .$$

Integrating we obtain

$$\|u(t)\|_1^2 = \|u(0)\|_1^2 + 2\int_0^t \left\{ \mathrm{Re}(u', r) + \frac{1}{2}(u, A_1' u) - \mathrm{Re}(u', A_2 u) \right\} d\tau \ .$$

Cauchy - Schwarz and (2)-(3) yield

$$\|u(t)\|_1^2 \leq \|u(0)\|_1^2 + \int_0^t \ \left\{ 2\|u'\|\|r\| + \gamma\|u\|^2 + 2\beta\|u'\|\|u\| \right\} d\tau \ .$$

Using the inequality

$$2ab \leq \delta a^2 + \delta^{-1} b^2$$

with $\delta = \varrho$ and $\delta = \alpha^{-1}$ in the first and third term of the integral, respectively, we obtain

$$\|u(t)\|_1^2 \leq \|u(0)\|_1^2 + \frac{1}{\varrho}\int_0^t \|r\|^2 d\tau + \int_0^t \left\{ \left(\frac{\beta}{\alpha} + \varrho\right) \|u'\|^2 + (\gamma + \beta\alpha)\|u\|^2 \right\} d\tau \ .$$

From (5) we see that

$$\|u'\|^2 + \alpha^2\|u\|^2 \leq \|u\|_1^2 \ ,$$

yielding

$$\|u(t)\|_1^2 \leq c_1(t) + c_2 \int_0^t \|u(\tau)\|_1^2 d\tau \ ,$$

$$c_1(t) = \|u(0)\|_1^2 + \frac{1}{\varrho}\int_0^t \|r(\tau)\|^2 d\tau \ , \quad c_2 = \varrho + \frac{\beta}{\alpha} + \frac{\gamma}{\alpha^2} \ .$$

Now the result follows from Gronwall's inequality. We use the following version of Gronwall's inequality: Let $\phi \geq 0$ be a C^1 function and assume that

$$\phi(t) \leq c_1(t) + c_2 \int_0^t \phi(\tau) d\tau.$$

where $c_1' \geq 0, c_2 \geq 0$. Then,

$$\phi(t) \leq c_1(t) e^{c_2 t}.$$

□

We apply Proposition 1 to a situation in which (5) is satisfied only on some subspace $D \subseteq H$ with P: $H \to D$ being the orthogonal projection onto D.

Proposition 2: Let $v \in C^2([0,\infty), H)$ be a solution of

$$v'' + PAv = 0 \ , \quad t > 0 \ ,$$

with $v(0) \in D$. If there exists a constant $\alpha > 0$ such that

$$(v, A_1(t)v) \geq \alpha^2 (v, v) \ , \quad v \in D \ , \tag{7}$$

then

$$\|v(t)\|_1^2 \leq \|v(0)\|_1^2 e^{(\beta/\alpha + \gamma/\alpha^2)t} \ .$$

Proof: It is readily verified that PA as an operator in D satisfies the assumptions (2)-(3) and (5) that A satisfies as an operator in H. The estimate follows by letting $\varrho \to 0$.

□

Proposition 3: Let u, $w \in C^2([0,\infty), H)$ be solutions of

$$u'' + Au = 0$$

and

$$w'' + PAw = 0$$

in $t > 0$, respectively, and let

$$w(0) = Pu(0) \ , \quad w'(0) = Pu'(0) \ .$$

If there exists a constant $\alpha > 0$ such that (7) is satisfied, then, for any $\varrho > 0$,

$$\|(w - Pu)(t)\|_1^2 \leq \frac{1}{\varrho} \int_0^t \|PA(u - Pu)\|^2 d\tau e^{(\varrho + \beta/\alpha + \gamma/\alpha^2)t} \ .$$

Proof: Applying P to the equation for u and subtracting the equation for w we obtain

$$(Pu - w)'' + PA(Pu - w) = PA(Pu - u) \; .$$

The proposition follows by applying Proposition 1 with $r = PA(Pu-u)$ to the operator PA in D.

$\square$

3 The initial value problem for the Helmholtz equation

We apply the estimates of the previous section to the initial value problem

$$\begin{aligned} &\Delta u + k^2(1+f)u = 0 \;\text{ in }\; x_n > 0 \; , \\ &u = g \; , \; \frac{\partial u}{\partial x_n} = h \qquad \text{on }\; x_n = 0 \; . \end{aligned} \tag{8}$$

This can be put into the framework of the preceding section, with x_n playing the role of the parameter t, by the following settings.

We put $H = L_2(\mathbb{R}^{n-1})$, $D = \{v \in H : \hat{v}(\xi) = 0 \text{ for } |\xi| \geq \kappa\}$ where $\hat{v}$ is the $(n-1)$-dimensional Fourier transform of v. The orthogonal projection P of H onto D is given by

$$(Pv)^\wedge(\xi) = \begin{cases} \hat{v}(\xi) & , \quad |\xi| \leq \kappa \; , \\ 0 & , \quad \text{otherwise} \; . \end{cases}$$

We make the following assumptions on f : $f = f_1 + \frac{i}{k} f_2$ with f_1, f_2 real, and there exist constants $m_1 \leq 0$, M_1, M_1' and M_2 such that

$$-1 < m_1 \leq f_1, |f_1| \leq M_1 \; , \; \left| \frac{\partial f_1}{\partial x_n} \right| \leq M_1' \; , \; |f_2| \leq M_2 \; . \tag{9}$$

We put $A = A_1 + A_2$ where

$$\begin{aligned} A(x_n)v &= \sum_{\ell=1}^{n-1} \frac{\partial^2 v}{\partial x_\ell^2} + k^2(1+f)v \\ A_1(x_n)v &= \sum_{\ell=1}^{n-1} \frac{\partial^2 v}{\partial x_\ell^2} + k^2(1+f_1(\cdot, x_n))v \; , \\ A_2(x_n)v &= ikf_2(\cdot, x_n)v \; . \end{aligned}$$

Obviously, (2), (3) are satisfied with

$$\beta = kM_2 \; , \quad \gamma = k^2 M_1' \; .$$

For $v \in D$ we have by Parseval's relation

$$\begin{aligned}
(v, A_1 v) &= \left(v, \sum_{\ell=1}^{n-1} \frac{\partial^2 v}{\partial x_\ell^2} + k^2(1 + f_1(\cdot, x_n))v \right) \\
&= \left(\hat{v}, -\sum_{\ell=1}^{n-1} \xi_\ell^2 \hat{v} \right) + k^2(v, (1 + f_1(\cdot, x_n))v) \\
&\geq -\kappa^2(\hat{v}, \hat{v}) + k^2(v, (1 + f_1(\cdot, x_n))v) \\
&= -\kappa^2(v, v) + k^2(v, (1 + f_1(\cdot, x_n))v) \\
&\geq (k^2(1 + m_1) - \kappa^2)(v, v) \;.
\end{aligned}$$

Thus, if $\kappa = (1 - \varepsilon)k\sqrt{1 + m_1}, 0 < \varepsilon < 1$, (7) is satisfied with

$$\alpha^2 = k^2(1 + m_1) - \kappa^2 = (2\varepsilon - \varepsilon^2)(1 + m_1)k^2. \tag{10}$$

Likewise we have for $v \in H$

$$(v, A_1 v) \leq k^2(1 + M_1)\|v\|^2 \;. \tag{11}$$

Now we come to the main result of the paper. We define an approximate solution u_κ of the initial value problem (8), show that u_κ is stably determined by the initial values and derive an error estimate for u_κ.

We define u_κ to be the solution of

$$\begin{aligned}
\frac{\partial^2 u_\kappa}{\partial x_n^2} + P\left(\sum_{\ell=0}^{n-1} \frac{\partial^2 u_\kappa}{\partial x_\ell^2} + k^2(1 + f)u_\kappa \right) &= 0 \qquad \text{in} \quad x_n > 0 \;, \\
u_\kappa = Pg \;, \quad \frac{\partial u_\kappa}{\partial x_n} &= Ph \qquad \text{on} \quad x_n = 0 \;.
\end{aligned} \tag{12}$$

Theorem. Let u_κ be a solution to (12). Assume that $\kappa = (1 - \varepsilon)k\sqrt{1 + m_1}$. Put

$$\vartheta = \sqrt{(1 + m_1)(2\varepsilon - \varepsilon^2)} \;, \quad \delta = M_2/\vartheta + M_1'/\vartheta^2 \;.$$

Then we have

$$\|u_\kappa'(\cdot, x_n)\|^2 \leq \left(\|h\|^2 + k^2(1 + M_1)\|g\|^2\right) e^{\delta x_n} \;, \tag{13}$$

$$\|(u_\kappa - Pu)'(\cdot, x_n)\|^2 \leq ek^2(kM_1 + M_2)^2 x_n \int_0^{x_n} \|(u - Pu)(\cdot, x_n')\|^2 dx_n' e^{\delta x_n}, \tag{14}$$

$$\|u_\kappa - Pu\|^2 \leq \frac{e}{2}(kM_1 + M_2)^2 x_n \int_0^{x_n} \|(u - Pu)(\cdot, x_n')\|^2 dx_n' e^{\delta x_n}.$$

Proof: (12) can be written as

$$\begin{aligned} & u_\kappa'' + PAu_\kappa = 0 && \text{in } x_n > 0 \ , \\ & u_\kappa = Pg \ , \quad u_\kappa' = Ph && \text{on } x_n = 0 \end{aligned} \tag{15}$$

where the prime denotes differentiation with respect to x_n. From Proposition 2 we get

$$\|u_\kappa(\cdot, x_n)\|_1^2 \le \|u_\kappa(\cdot, 0)\|_1^2 e^{\delta x_n} \ ,$$
$$\delta = \beta/\alpha + \gamma/\alpha^2 = M_2/\vartheta + M_1'/\vartheta^2 \ .$$

From (7), (11) we obtain

$$\|u'(\cdot, x_n)\|^2 \le (\|h\|^2 + k^2(1+M_1)\|g\|^2)e^{\delta x_n} \ .$$

This is (13). For the proof of (14) we rewrite (8) as

$$\begin{aligned} & u'' + Au = 0 \quad \text{in} \quad x_n > 0 \ , \\ & u = g \ , \quad u' = h \quad \text{on} \quad x_n = 0 \ . \end{aligned} \tag{16}$$

Applying Proposition 3 to (16), (15) we obtain

$$\|(u_\kappa - Pu)(\cdot, x_n)\|_1^2 \le \frac{1}{\varrho} \int_0^{x_n} \|PA(u - Pu)(\cdot, x_n')\|^2 dx_n' e^{(\varrho+\delta)x_n} \ .$$

Since $P^2 = P$ we have

$$\begin{aligned} PA(u - Pu) &= P\left(\sum_{\ell=1}^{n-1} \frac{\partial^2}{\partial x_\ell^2} + k^2 + k^2 f\right)(u - Pu) \\ &= \left(\left(\sum_{\ell=1}^{n-1} \frac{\partial^2}{\partial x_\ell^2} + k^2\right) P + k^2 Pf\right)(u - Pu) \\ &= k^2 Pf(u - Pu) \ , \end{aligned}$$

hence

$$\begin{aligned} \|PA(u - Pu)(\cdot, x_n')\| &= \|k^2 Pf(u - Pu)(\cdot, x_n')\| \\ &\le k^2\left(M_1 + \frac{M_2}{k}\right)\|(u - Pu)(\cdot, x_n)\| \ . \end{aligned}$$

Using this and (2.6) yields

$$\|(u_\kappa - Pu)'(\cdot, x_n)\|^2 \le k^4\left(M_1 + \frac{M_2}{k}\right)^2 \frac{1}{\varrho}\int_0^{x_n} \|(u - Pu)(\cdot, x_n')\|^2 dx_n' e^{(\varrho+\delta)x_n} \ .$$

Putting $\varrho = 1/x_n$ yields (3.7).

$\square$

The decisive parameter in the application of Theorem 3.1 is ε. The ideal choice would be $\varepsilon = 0$, leading to the maximal bandwidth $\kappa = \sqrt{1+m_1}k$. In practice we rather think of $\varepsilon = 0.1$, say, yielding a 10% loss in bandwidth while making δ not too big.

The interpretation of Theorem 3.1 is as follows. (3.6) states that the approximate solution u_κ is stably determined by the initial values. (3.7) states that the approximate solution u_κ is close to the low-pass filtered version Pu of the true solution provided u is. This is the case since the Helmholtz equation acts as a low pass filter. More precisely the Fourier transform of u along lines not meeting the support of f decays exponentially beyond frequency k; see [10].

4 The marching scheme

The marching scheme for the solution of (8) is derived by discretizing (12) on a cartesian grid. For ease of exposition we restrict ourselves to the case $n = 2$. We work on the grid $(mh_1, \ell h_2)$, $m = -q, \ldots, q$, $\ell = 0, 1, \ldots$ with h_1, $h_2 > 0$ being the step sizes in directions x_1 and x_2, respectively. We denote by $u_{m,\ell}$ the approximation to $u(mh_1, \ell h_2)$ and by u_ℓ the $2q$ vector with components $u_{-q,\ell}, \ldots, u_{q-1,\ell}$.

The marching step $\ell \to \ell + 1$ in the direction x_2 is as follows. First compute a preliminary vector $u^*_{\ell+1}$ by

$$\begin{aligned} \frac{u^*_{m,\ell+1} - 2u_{m,\ell} + u_{m,\ell-1}}{h_2^2} &+ \frac{u_{m+1,\ell} - 2u_{m,\ell} + u_{m-1,\ell}}{h_1^2} \\ &+ k^2(1 + f_{m,\ell})u_{m,\ell} = 0\ , \quad |m| < q\ . \end{aligned} \tag{17}$$

Then, compute the final vector $u_{\ell+1}$ by

$$u_{\ell+1} = Pu^*_{\ell+1}\ .$$

Here, P is a discrete version of the projection P in (12). It is conveniently implemented by the discrete Fourier transform F of length $2q$ on vectors v with components $v_{-q}, \ldots, v_{q-1}$, i.e.

$$(Fv)_j = \sum_{m=-q}^{q-1} e^{-\pi i m j/q} v_m\ ,$$

the inverse being

$$(F^{-1}v)_j = \frac{1}{2q} \sum_{\ell=-q}^{q-1} e^{\pi i \ell j/q} v_\ell\ .$$

With the truncation operator

$$(T_\kappa v)_j = \begin{cases} v_j & , \quad |j| \leq \frac{h_1 q}{\pi}\kappa \\ 0 & , \quad \text{otherwise} \end{cases}$$

we have

$$P = F^{-1} T_\kappa F \ . \tag{18}$$

With the help of the fast Fourier transform P can be applied with O $(q \log q)$ operations. Thus the marching scheme needs only O $(q \log q)$ operations in each step $\ell \to \ell + 1$.

The choice of the parameter κ in (18) is crucial. To ensure stability κ has to satisfy the condition in Theorem (8), i.e.

$$\kappa = ck \quad , \quad c = (1 - \varepsilon)\sqrt{1 + m_1} < \sqrt{1 + m_1} \ . \tag{19}$$

Let $\lambda = 2\pi/k$ be the wavelength in the surrounding medium. Then,

$$\lambda_1 = 2\pi/\kappa = \lambda/c \tag{20}$$

is the smallest wavelength in the direction x_1 perpendicular to the marching direction x_2, that is treated accurately. The wavelength in direction x_1 of a plane wave $e^{ikx\cdot\theta}$ that makes an angle β with the marching direction x_2, i.e. $\theta = (\sin\beta, \cos\beta)^T$ is $\lambda/\sin\beta$. Hence we must have $\sin\beta \leq c$. For c close to 1 this condition is much weaker than the one for parabolic approximations. The superiority of our marching scheme is further illuminated by considering the circular wave $H_0(k|x|)$ with wavelength $\lambda = 2\pi/k$ far from the source. The wavelength in x_1 direction on the ray through the source making an angle β with the marching direction is approximately $\lambda/\sin\beta$. Again we can expect accuracy in the angular range $\sin\beta \leq c$ around the marching direction. We note that the theoretical limit for the angle β is $\sin\beta = \sqrt{1 + m_1}$. Thus, for $m_1 = 0$ (i.e.$f_1 \geq 0$) we get good accuracy in all directions.

The recursion (17) is initiated by

$$u_0 = Pg \quad , \quad u_1 - u_{-1} = 2h_2 Ph$$

where g, h are discrete versions of the functions g, h in (8). The boundary values, i.e. $u_{\ell,m}$ for $|m| = q$, are assumed to be known.

5 Application to an inverse problem

In the following we describe how our marching scheme is applied to an inverse problem for the Helmholtz equation. Our method is an adaption of the PBP algorithm [8]. We consider only the 2D case, the extension to 3D being obvious.

Let $\Omega = (0, L) \times (0, D) \subseteq \mathbb{R}^2$, and let $y \in \partial\Omega$. Let u be the solution of

$$\begin{aligned} \Delta u + k^2(1 + f)u &= 0 \qquad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} &= -\delta_y \qquad \text{on } \partial\Omega \end{aligned} \tag{21}$$

where δ_y is a δ-like function on $\partial\Omega$ that peaks at y and ν is the exterior normal on $\partial\Omega$. Let

$$g_y = u|_{\partial\Omega} \ .$$

The problem is to recover f from g_y, $y \in \partial\Omega$.

We rewrite the boundary value problem (21) as an initial value problem. To fix ideas let $y = (y_1, 0)$ be on the top part Γ^+ of $\partial\Omega$ (the x_2 axis pointing downwards), Γ^- the bottom part, and Γ the vertical parts. Define u to be the solution of the initial value problem

$$\begin{aligned} \Delta u + k^2(1+f)u = 0 \qquad & \text{in } \Omega \ , \\ \frac{\partial u}{\partial \nu} = -\delta_y \ , \ u = g_y \qquad & \text{on } \Gamma^+ \ , \\ u = g_y \qquad & \text{on } \Gamma \ . \end{aligned} \tag{22}$$

Define the operator $R_y : L_2(\Omega) \to L_2(\Gamma^-)$ by

$$R_y(f) = u\big|_{\Gamma^-} \ .$$

Strictly speaking u is the solution of a stabilized initial value problem according to (12). Then the inverse problem is equivalent to the nonlinear system

$$R_y(f) = g_y^- \ , \quad y \in \partial\Omega$$

where $g_y^- = g_y\big|_{\Gamma^-}$. Often y varies only in a part of $\partial\Omega$, and g_y will be measured only in a part of $\partial\Omega$. The necessary modifications of our procedure are obvious.

The algorithm we are suggesting is a nonlinear version of the Kaczmarz procedure [11]. The update is done by

$$f \leftarrow f - \omega R'^*(f) C_y^{-1}(R_y(f) - g_y^-) \ . \tag{23}$$

Here, R'_y is the Fréchet derivative of R_y and $R'^*_y : L_2(\Gamma^-) \to L_2(\Omega)$ its adjoint. ω is a relaxation parameter. C_y is a positive definite matrix, which is supposed to mimic the action of $R'_y(f)R'^*(f)$, and which usually is chosen as a multiple of the identity.

The evaluation of (23) requires two steps. First we have to compute $R_y(f)$. This can be done by our marching scheme. Then we have to compute $R'^*_y(f)r$ with r the weighted residual $C_y^{-1}(R_y(f) - g_y^-)$. This can be done in the following way [9]: Solve the initial (or final) value problem

$$\begin{aligned} \Delta z + k^2(1+f)z = 0 \qquad & \text{in } \Omega \\ z = 0 \ , \quad \frac{\partial z}{\partial \nu} = \overline{r} \qquad & \text{on } \Gamma^- \ , \\ z = 0 \qquad & \text{on } \Gamma \end{aligned} \tag{24}$$

and put

$$R'^*_y(f)r = k^2\overline{uz}$$

with u the solution of (22). The computation of z again can be done by our marching scheme.

Note that (24) is nothing but time reversal [3] in Fourier space, i.e. the phase conjugation $r \to \overline{r}$, followed by the backpropagation $r \to z$.

Reconstruction methods based on the principles outlined in this section have been tested numerically in [8], [12] and [14].

6 Numerical examples

In a first numerical example we chose

$$f(x) = \begin{cases} 0.1 & , \quad |x| \leq 0.4 \ , \\ 0 & , \quad \text{otherwise} \ . \end{cases}$$

We assume $u = u_i + u_s$ with $u_i = e^{ikx_2}$ the incoming plane wave and u_s the scattered wave that satisfies the Sommerfeld radiation condition at infinity.
We computed u analytically in the central square with side length 2 and used the values of u and its normal derivative on $x_2 = -1$ as initial values for our marching scheme. For $k = 50$ and stepsize $h = 1/512$ the marching solution on $x_2 = 1$ was virtually indistinguishable from the low pass filtered version (with $\varepsilon = 0.1$, i. e. cut-off $\kappa = (1-\varepsilon)k = 45$) of the exact solution. This confirms the error estimates of Theorem 3.1.
For the second example we chose f to be the "Luneberg lense" [7]

$$f(x) = \begin{cases} 1 - 4|x|^2 & , \quad |x| \leq 0.5 \ , \\ 0 & , \quad \text{otherwise} \ . \end{cases}$$

When illuminated by a plane wave the Luneberg lense generates a focal point at its rim. Therefore we consider solving (1) with this f for large k as a challenge for an initial value solver. We chose $k = 200$ and did the computation in the square centered at the origin with side length 2.

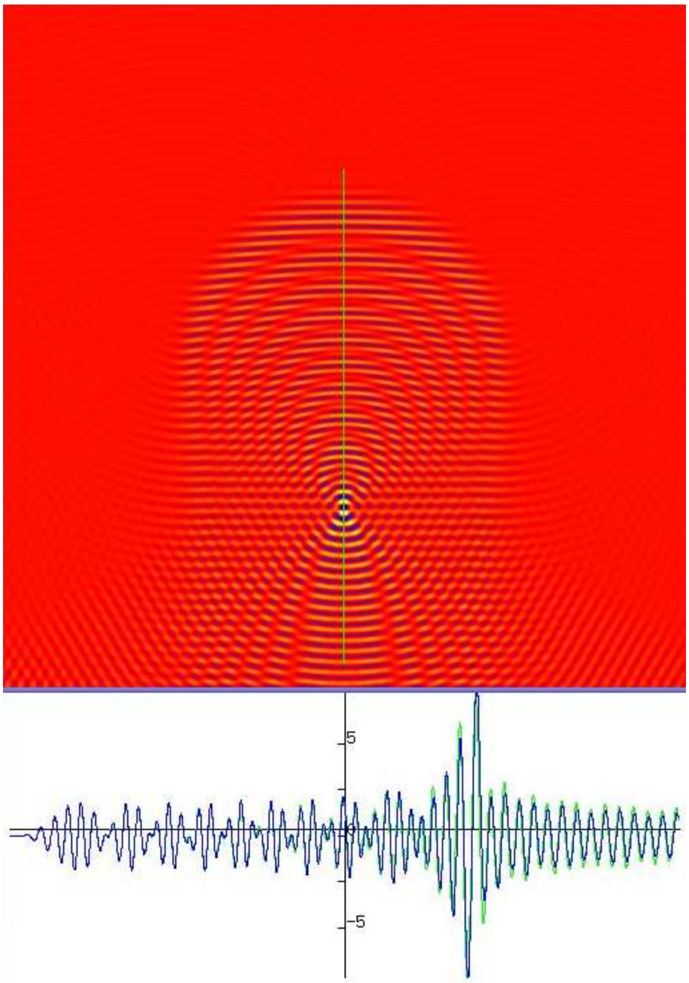

Fig. 1: Scattered field for Luneberg lense.

Top:	Real part of field computed by the marching scheme.
Bottom:	Cross sections through the real part of the exact and approximate field along vertical line.

In Fig. 1 we display the approximate field computed by the marching scheme (top) and a vertical cross section through the exact and the approximate fields (bottom). The irradiating plane wave is falling in from top and the marching is done from top to bottom. (Reversing the direction of the marching produced results that are virtually identical). The exact field was computed by a finite difference time domain method, followed by a Fourier transform. The focal point is clearly visible. The exact field is not shown since it is indistinguishable from the approximate field as computed by the marching scheme. The vertical cross section reveals small differences in the vicinity of the focal point that are mainly due to the bandlimiting in the marching scheme. This illustrates the fact that the marching scheme approximates a low-pass filtered version of the true solution. The spatial step size used in both computations is $h = 1/512$. We remark that for this example $m_1 = 0, M_1 = 1, M_1' = 4, M_2 = 0$, and we chose $\varepsilon = 0.1$ i.e. we worked with the bandwidth $\kappa = 190$.

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