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# ON SOLUTIONS OF ELLIPTIC EQUATIONS SATISFYING MIXED BOUNDARY CONDITIONS*

A. AZZAM† AND E. KREYSZIG†

**Abstract.** We consider the mixed boundary value problem for linear second order elliptic equations in a plane domain $\Omega$ whose boundary has corners, and obtain conditions sufficient for the solution to be in $C^{2+\alpha}(\bar{\Omega})$, where $0<\alpha<1$. This result means that under those conditions, solutions are as smooth as they would be in the absence of corners, so that, in this sense, the present result is best possible.

**1. Introduction.** We shall be concerned with the mixed boundary value problem for second order linear elliptic equations in a two-dimensional domain whose boundary has corners. More specifically, we shall study the effect of these corners on the Hölder smoothness of solutions. To motivate this investigation, we first give a general orientation about the development and present situation in this field, beginning with mixed problems in domains with a smooth boundary and then turning to the case of domains with corners on the boundary.

Early results on the regularity of solutions of boundary value problems concern *domains with a smooth boundary*, first for the Laplace and Poisson equations and corresponding Dirichlet and Neumann problems, and later for general second order elliptic equations as well as general boundary conditions. In particular, the *mixed problem* was first considered by Zaremba [54], and is often called *Zaremba's problem.* Further work on the mixed problem in domains with a smooth boundary up to about 1970 is reviewed by Miranda [32] (and a few additional references are given in [18] and [30]), so that it will suffice to mention some of the major contributions during that period and add an outline of some more recent basic results not yet included in any monograph. Of course, we shall be able to select only a small number of articles from the very extensive literature in the field.

Work by Signorini [41], almost contemporary with that of Zaremba, and similar results by Keldysh and Sedov [24] concern the mixed problem for harmonic functions in a half-plane. Slightly earlier than the latter two authors, Giraud [19] proposed a method of solving the mixed problem by first converting it to a Neumann problem on some Riemannian manifold. In 1949, Fichera [14] (cf. also [15]) proved a general existence theorem by transforming the problem into a system of Riesz–Fischer equations; this is known as Picone's method and is also of interest in numerical analysis. Direct methods of the calculus of variations were applied to the mixed problem by Stampacchia [45], whose results are particularly important since they also concern nonlinear equations. The existence of Hölder continuous solutions was proved by Miranda [31], using Schauder type estimates. The method of integral equations was first applied successfully to the mixed problem by Vekua [47]. See also Muskhelishvili [33], whose references reflect the development of that method until about 1955.

Beginning with a paper by Schechter [40], some of the work on (interior and boundary) regularity of solutions of the mixed problem is based on the Sobolev space approach and the use of coercive quadratic forms. For the general idea and setting (which also apply to other "non-Dirichlet problems"), we refer to Agmon [1]. An important contribution specifically devoted to the mixed problem is the thesis by Purmonen [37], which also contains numerous references. Purmonen's work concerns

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† Department of Mathematics, University of Windsor, Windsor, Ontario, Canada N9B 3P4.

rather general mixed problems for linear elliptic equations in $n$ variables, and his results include conditions sufficient for a priori regularity of strong solutions as well as for the existence and some regularity properties of weak solutions. Subsequently, Purmonen [38] studied the well-posedness of the two-dimensional mixed problem in Sobolev spaces. Another approach is the conversion of mixed problems to Wiener–Hopf type problems; this is known as Peetre's method and has recently been extended by Pryde [36].

It is clear that mixed problems in domains with smooth boundary are of great interest in physics, and there also exists an extensive literature on corresponding numerical methods. We cannot go into details, but want to mention that some of the references on domains with corners given below also include the case of domains with a smooth boundary; for further applications we refer to Sneddon [43] and a recent paper by Wendland, Stephan and Hsiao [52] on harmonic functions in two variables, in which two Fredholm equations resulting from the integral equation method are solved constructively, using finite element functions augmented by singular functions, an approach which would be difficult to extend to equations with *variable* coefficients, as is known (cf. Grisvard [20, p. 215] and Kawohl [23]).

We now turn to boundary value problems for *domains with corners on the boundary.* The interest in those problems and in regularity properties of corresponding solutions has several sources. The earliest impetus came from conformal mapping and boundary value problems for harmonic functions; see, for instance, Carleman [11], Kellogg [25] or Warschawski [51]. More recent results of importance, pertaining to the Laplace and Poisson equations in domains with corners, are those by Fufaev [17], Nikol'skiĭ [34] and Volkov [49]. Just as in the case of a smooth boundary, in addition to methods related to Hölder classes, as employed in the present paper, there are other approaches; we mention in particular Sobolev space methods as considered in the reviews by Grisvard [20], [21], then Kondratiev's extension [26] of the Sobolev–Slobodeckii space method by Eskin [13] and Višik [48] (cf. also [3]), furthermore a function theoretic approach by Lewy [29] and his school (see, for instance, Wigley [53]) and, finally, a recent method by Simon [42] based on geometric measure theory.

As a second source for the interest in regularity properties of solutions of boundary value problems in domains with corners we mention physical applications. In fact, it was recognized early that those investigations are important in connection with practical problems in heat conduction, fluid flow and elasticity theory; examples can be found in [12], [44] and other standard monographs. See also [43] and [52].

Thirdly, those problems play a role in numerical analysis, particularly in the study of the accuracy of finite element and finite difference approximations, acceleration of convergence, general convergence analysis, subtraction of singularities and other numerical techniques. Here, in error estimates and other tasks, one often experiences great difficulties caused by the presence of corners, since there may not exist an adequate theory covering such cases. Moreover, in this area there are various traps for the unwary; for a typical example, see [49, p. 157]. For the finite element method, a general characterization of the situation is given by Strang and Fix [46, Chap. 8]. More details are discussed by Fix, Gulati and Wakoff [16] as well as Babuska and Aziz [9, Chaps. 8, 9]; see also Babuska [8] and Babuska and Rosenzweig [10], who use the weighted Sobolev space approach. For the finite difference method, convergence in domains with nonsmooth boundary is studied in basic papers by Laasonen [27], [28]. For a combination of that method with the integral equation method and conformal mapping in the case of the two-dimensional Laplace equation, see Papamichael and Symm [35]. In accelerating convergence, rather natural ideas seem to be the refinement of meshes near corners where convergence becomes poor and the choice of a net that confines slow convergence to small neighborhoods of corners, instead of "polluting" the whole domain; cf. Volkov [50] for finite differences, and the recent work of Schatz and Wahlbin [39] for finite elements in the case of the Poisson equation in the plane. Ref. [39] includes local estimates of convergence rates up to the boundary, estimates of the effect of systematic refinements and calculation procedures for stress intensity factors as well as the location of the maximum error.

Before we start on our actual problem, let us add a few words about the case of a smooth boundary as compared to that of a boundary with corners. The smoothness of solutions depends on that of the coefficients of the equation, of the boundary of the domain and of the boundary data. It is well known that if in a domain $\Omega$ with sufficiently smooth boundary, the regularity properties of the coefficients of the equation and of the boundary data improve, so do the regularity properties on $\bar{\Omega}$ of the solution of the first, second and third boundary value problems. This was first shown for special equations (Laplace and Poisson) and later for general elliptic equations; see Agmon, Douglis and Nirenberg [2]. However, the situation changes drastically in the case of corners at the boundary. Then the smoothness of solutions also depends on the interior angle at the corners. Roughly speaking, small angles are favorable with respect to smoothness of solutions. In addition, there also exist "exceptional angles" for which the smoothness is "exceptionally good", that is, is better than for values of the angle close to those exceptional ones. Our result will be typical in that respect, since it will illustrate this general pattern. We shall find conditions sufficient for the solutions to be as smooth as they would be in the absence of corners, the other conditions remaining the same; *hence our conclusion will be strongest possible.*

**2. Problem and main result.** We shall consider linear elliptic equations of the form

$$Lu = \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^{2} a_i(x) u_{x_i} + a(x) u = f(x) \tag{2.1}$$

in a plane domain $\Omega$ whose boundary $\partial\Omega$ has corners. Here, $x = (x_1, x_2)$. We assume that $\Omega$ is simply connected and bounded and $L$ is uniformly elliptic in $\Omega$. The boundary conditions are of mixed type; we write them in the form

$$\chi_1(x)u(x) + \chi_2(x)u_n(x) = \chi_1(x)\phi(x) + \chi_2(x)\psi(x) \quad \text{on } \partial\Omega; \tag{2.2}$$

here, the subscript $n$ denotes the outer normal derivative.

The following result in the "regular case" is well known. If $\partial\Omega$ is smooth of class $C^{2+\alpha}$, where $0<\alpha<1$, and if

(A) $a_{ij}$, $a_i$, $a, f \in C^{\alpha}(\bar{\Omega})$, $L$ uniformly elliptic in $\Omega$,

(B) $\chi_1$, $\phi \in C^{2+\alpha}(\partial\Omega)$, $\chi_2$, $\psi \in C^{1+\alpha}(\partial\Omega)$,

then

$$u \in C^{2+\alpha}(\bar{\Omega}). \tag{2.3}$$

See Agmon, Douglis and Nirenberg [2].

We now turn to the case when $\partial\Omega$ is not smooth, a case which we also considered in [5] and [7]. Then [2] implies that in a compact subregion $\Omega_1$ of $\bar{\Omega}$ with positive distance from the corner points, $u$ is smooth as before. More precisely we have the following. Without loss of generality we may assume that $\partial\Omega$ has a single corner, which is located at the origin $x=0$, the interior angle being $\gamma$, $0<\gamma<2\pi$. Let $\Gamma_1$ and $\Gamma_2$ denote the two arcs of $\partial\Omega$ that form the corner at $x=0$. Suppose that $\partial\Omega\backslash\{0\}$ is smooth of class $C^{2+\alpha}$. Let $u$ be a solution of (2.1) satisfying the boundary conditions (2.2). Further, assume that conditions (A) and

$$\text{(B*)} \qquad \begin{aligned} &\chi_1, \phi \in C^{2+\alpha}(\partial\Omega\backslash\{0\}),\\ &\chi_2, \psi \in C^{1+\alpha}(\partial\Omega\backslash\{0\}), \end{aligned}$$

as well as

$$\text{(C)} \qquad \begin{aligned} &\chi_2 = 0, \qquad \phi = 0 \quad \text{on } \Gamma_1,\\ &\chi_1 = 0, \qquad \psi = 0 \quad \text{on } \Gamma_2, \end{aligned}$$

hold true. Then, by [2],

$$u \in C^{2+\alpha}(\Omega_1) \cap C^0(\bar{\Omega}), \tag{2.4}$$

with $\Omega_1$ as indicated before.

To characterize the smoothness of $u$ near the corner point, we introduce

$$\omega = \arctan \frac{[a_{11}(0)a_{22}(0) - a_{12}^2(0)]^{1/2}}{a_{22}(0)\cot\gamma - a_{12}(0)}. \tag{2.5}$$

This is the angle obtained from $\gamma$ in the transformation of the equation

$$\sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij}(0) u_{x_i x_j} = 0$$

to normal form. In [7] we proved that, under assumptions (A), (B*), (C) and $\omega < \pi/2$, we have

$$u \in C^{\nu}(\bar{\Omega}), \qquad \nu = \min\left(\frac{\pi}{2\omega} - \varepsilon, 2\right), \tag{2.6}$$

with arbitrarily small $\varepsilon > 0$. Substantially improving that result, we shall now obtain sufficient conditions in order that even (2.3) be valid; those conditions will concern small angles as well as an exceptional angle ($\pi/4$). Note well that (2.3) refers to the "regular case" of a smooth boundary. Accordingly, despite the presence of corners, *our result to be obtained is as strong as that in the case of the absence of corners*; in that sense, this result is best possible.

Our main result can be stated as follows.

THEOREM 1. *Let $u$ be a bounded solution of* (2.1), (2.2) *in* $\Omega$. *Suppose that* (A), (B*), (C) *hold true and* $\omega$ *in* (2.5) *satisfies the condition*

(D$_1$) $\omega < \pi/(4+2\alpha)$

*or the condition*

(D$_2$) $\omega = \pi/4$.

*Then*

$$u \in C^{2+\alpha}(\bar{\Omega}). \tag{2.7}$$

From the statement involving (2.4), we conclude that it suffices to prove Theorem 1 in

$$N = \{x \mid x \in \bar{\Omega}, |x| < r_0\}, \qquad r_0 > 0.$$

Furthermore, by [7] it is sufficient to consider the case of a circular sector and impose the additional condition

$$a_{ij}(0) = \delta_{ij}, \qquad i, j = 1, 2.$$

Indeed, the transition from this special setting to the general case is the same as in [7] (and is relatively simple), so that we need not reproduce it here.

At this point, we should notice that [7] concerns arbitrary $n$, whereas here we take $n=2$ because later (near the end of the paper) we have to use a result by Volkov which is known to hold for $n=2$ only. Actually, we need Volkov's result only in connection with condition $(D_2)$, so that the assertion of Theorem 1 under condition $(D_1)$ could be proved for any $n$ by an argument similar to the present one.

**3. The case of a sector.** Let $r$, $\theta$ be polar coordinates defined by $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ and consider the sector

$$\Omega_{2\sigma} = \{(r, \theta) \mid 0 < r < 2\sigma, 0 < \theta < \omega\},$$

where $\sigma = \text{const} > 0$. Let

$$\Gamma_1: \theta = 0, \quad r < 2\sigma, \qquad \Gamma_2: \theta = \omega, \quad r < 2\sigma.$$

A theorem analogous to Theorem 1 but referring to the present setting can be stated as follows.

THEOREM 2. *Let $u$ be a bounded solution of the mixed boundary value problem for the equation*

$$Lu = f \quad \text{in } \Omega_{2\sigma}, \tag{3.1}$$

*with $L$ as in* (2.1) *and $a_{ij}(0) = \delta_{ij}$ and assume that $u$ satisfies the conditions*

$$\text{(a)}\ u|_{\Gamma_1} = 0, \qquad \text{(b)}\ u_n|_{\Gamma_2} = 0. \tag{3.2}$$

*Suppose that* (A) *with $\Omega$ replaced by $\Omega_{2\sigma}$ and* $(D_1)$ *or* $(D_2)$ *hold. Then*

$$u \in C^{2+\alpha}(\bar{\Omega}_\sigma). \tag{3.3}$$

By what has been said, in order to obtain Theorem 1, it suffices to prove Theorem 2. The proof of the latter theorem will result from two lemmas.

In the first lemma, we obtain bounds for $u$ and its first and second partial derivatives as well as a statement on the Hölder smoothness of $u$. Here, $D^k u$ denotes any $k$th partial derivative of $u$.

LEMMA 1. *Under the assumptions of Theorem* 2 *we have in* $\bar{\Omega}_\sigma$

$$\begin{aligned} &\text{a)} && |Du(x)| \leqq Mr^\nu, \\ (3.4)\quad &\text{b)} && |D^k u(x)| \leqq Mr^{\nu-k}, \qquad k = 1, 2, \\ &\text{c)} && u \in C^\nu(\bar{\Omega}_\sigma), \end{aligned}$$

*where*

$$\nu = \min\left(\frac{\pi}{2\omega} - \varepsilon, 2 + \alpha\right).$$

*Proof.* a) We consider in $\bar{\Omega}_{2\sigma}$ the function

$$w(x) = Mr^\nu \cos\lambda(\omega - \theta),$$

with $\nu$ defined as in the lemma, $\lambda = (\pi - 2\delta)/2\omega$ and $\delta > 0$ so small that $\lambda > \nu$. Using the method developed in [7], one can show that $w$ may serve as a barrier function for $u$, provided $M$ is taken sufficiently large. In this way we obtain (3.4a).

b), c) From [5] it can be seen that in the case of the *Dirichlet problem*, the proof of the statements corresponding to our present (3.4b) and (3.4c) depends mainly on the analog of our present (3.4a) and on a Schauder estimate of the form

$$\|u\|_{2+\alpha}^{\Omega^*} \leqq \kappa[\|u\|_0^{\Omega} + \|f\|_\alpha^{\Omega} + \|\phi\|_{2+\alpha}^{\Gamma}],$$

where $\bar{\Omega}^* \subset \bar{\Omega}$ and $\Gamma = \partial\Omega \cap \partial\Omega^*$ is of class $C^{2+\alpha}$. Such an estimate also holds for the *mixed* boundary value problem, the only difference being the absence of the last term. In this way, following the general idea in [5], we obtain (3.4b) and (3.4c). This completes the proof.

From (3.4c) it follows that Theorem 2 with condition $(D_1)$ holds. Finally, we must prove Theorem 2 under condition $(D_2)$. If $(D_2)$ holds, then (3.4b) yields

$$|D^2 u(x)| \leqq M r^{-\varepsilon} \quad \text{in } \bar{\Omega}_\sigma$$

and (3.4c) gives

$$u \in C^{2-\varepsilon}(\bar{\Omega}_\sigma).$$

To prove Theorem 2 in the present case, we first investigate the nature of singular behavior of the second derivatives of $u$ near the corner point.

LEMMA 2. *Let $v$ be a solution of* (3.1) *in $\Omega_{2\sigma}$ satisfying* (3.2), *and suppose that the assumptions of Theorem* 2 *hold true. Suppose further that in $\bar{\Omega}_\sigma$*

$$|D^2 v(x)| \leqq M_1 r^{-\eta}, \qquad 0 \leqq \eta < 1.$$

*Let $h \in C^\tau(\bar{\Omega}_\sigma)$, where $1 > \tau \geqq \eta$ and $h(0) = 0$. Then*

$$hD^2 v \in C^\mu(\bar{\Omega}_\sigma), \qquad \mu = \min(\alpha, \tau - \eta). \tag{3.5}$$

*Proof.* In $\Omega_\sigma$ consider any two points $P_j: (r_j, \theta_j)$, $j = 1, 2$. By abuse of notation, we write $h(P_j)$ for $h(r_j, \theta_j)$ and so on. We must show that there exists a constant $H > 0$ such that

$$d(P_1, P_2)^{-\mu} |h(P_1)D^2 v(P_1) - h(P_2)D^2 v(P_2)| \leqq H. \tag{3.6}$$

Let $0 \leqq r_2 \leqq r_1 \leqq \sigma$, without restriction. If $r_2 \leqq \frac{1}{2} r_1$, then $d(P_1, P_2) \geqq \frac{1}{2} r_1$, and from

$$|h(P_j)| \leqq M_2 r_j^\tau, \qquad j = 1, 2,$$

we can obtain (3.6).

We consider the case $r_2 > \frac{1}{2} r_1$. Let

$$x = \xi y, \quad \xi = \frac{2r_1}{\sigma}, \quad y = (y_1, y_2).$$

This transformation maps

$$\Omega_0 = \{(r, \theta) | \tfrac{1}{2} r_1 \leqq r \leqq r_1, 0 < \theta < \tfrac{1}{4}\pi\}$$

onto

$$\Omega_1 = \{(\rho, \theta) | \tfrac{1}{4}\sigma \leqq \rho \leqq \tfrac{1}{2}\sigma, 0 < \theta < \tfrac{1}{4}\pi\},$$

where $\rho = r/\xi$. As in [6], it can be shown that in $\Omega_1$ the function $V(y) = v(\xi y)$ satisfies

$$\|V\|_{2+\alpha}^{\Omega_1} \leqq M_3 r_1^{2-\eta}.$$

Now, for any $\mu \leqq \alpha$,

$$\xi^{2+\mu} H_\mu^{\Omega_0}(D^2 v) = H_\mu^{\Omega_1}(\tilde{D}^2 V) \leqq M_4 r_1^{2-\eta},$$

where $\tilde{D}^2 V$ denotes the partial derivative corresponding to $D^2 v$ and $H_\mu^{\Omega_0}$ is the Hölder coefficient. Hence,

$$H_\mu^{\Omega_0}(D^2 v) \leqq M_5 r_1^{-\mu-\eta}.$$

We now obtain (3.6) in the case $r_2 > \frac{1}{2} r_1$ as follows, writing $\delta = d(P_1, P_2)$:

$$\begin{aligned}
&|h(P_1)D^2v(P_1) - h(P_2)D^2v(P_2)|\delta^{-\mu} \\
&\leqq |h(P_1)||D^2v(P_1) - D^2v(P_2)|\delta^{-\mu} \\
&\quad + |D^2v(P_2)|\{|h(P_1) - h(P_2)|\delta^{-\tau}\}^{\mu/\tau}|h(P_1) - h(P_2)|^{1-\mu/\tau} \\
&\leqq M_2 r_1^{\tau} M_5 r_1^{-\mu-\eta} + M_1 r_2^{-\eta} M_6 (2M_2 r_1^{\tau})^{1-\mu/\tau} \\
&\leqq H.
\end{aligned}$$

This proves Lemma 2.

We can now prove Theorem 2 under assumption ($D_2$). We remember that by Lemma 1, under the assumptions of the theorem [with ($D_1$) or ($D_2$)] we have

$$u \in C^{2-\varepsilon}(\bar{\Omega}_\sigma)$$

and in $\bar{\Omega}_\sigma$

$$|D^2 u(x)| \leqq M r^{-\varepsilon},$$

as was stated above. Equation (3.1) can be written

$$\Delta u = f_1 = f - au - \sum_{i=1}^{2} a_i u_{x_i} - \sum_{i=1}^{2} \sum_{j=1}^{2} (a_{ij} - \delta_{ij}) u_{x_i x_j}. \tag{3.7}$$

Since $f$, $a$, $a_i \in C^\alpha(\bar{\Omega}_\sigma)$ and $u \in C^{2-\varepsilon}(\bar{\Omega}_\sigma)$, the first three expressions on the right-hand side of (3.7) are of class $C^\alpha(\bar{\Omega}_\sigma)$. Using Lemma 2 with

$$h = a_{ij} - \delta_{ij}, \quad \tau = \alpha, \quad \eta = \varepsilon,$$

we have

$$(a_{ij} - \delta_{ij}) u_{x_i x_j} \in C^{\alpha-\varepsilon}(\bar{\Omega}_\sigma).$$

Hence, $f_1 \in C^{\alpha-\varepsilon}(\bar{\Omega}_\sigma)$. From this and [49, p. 128], it follows that $u \in C^{2+\alpha-\varepsilon}(\bar{\Omega}_\sigma)$. Using this in the last term of (3.7) and applying again Lemma 2, with $\tau = \alpha$ and $\eta = 0$, we obtain (3.3). This completes the proof of Theorem 2.

From Theorem 2, our main result (Theorem 1) follows as indicated in § 2.

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