Metrized Graphs, Laplacian Operators, and Electrical Networks

Matthew Baker and Xander Faber

Abstract. A metrized graph is a weighted graph whose edges are viewed as line segments, or alternatively, it is a singular Riemannian 1-manifold. In this expository paper, we study the Laplacian operator on a metrized graph and some important objects related to it, including the "j-function", the effective resistance, and the "canonical measure". We discuss the relationship between metrized graphs and electrical networks, which provides some physical intuition for the concepts being dealt with. We also explain the relation between the Laplacian on a metrized graph and the combinatorial Laplacian matrix. Finally, we obtain a new proof of Foster's network theorem.

1. An informal discussion

The basic idea of a metrized graph is simple: identify each edge of a weighted graph with a line segment and define the distance between two points of the graph to be the length of the shortest path connecting them. We will provide more details on this definition in §2.

Metrized graphs appear in the literature of several areas of science and mathematics. For example: in number theory, they are used to study arithmetic intersection theory on algebraic curves (see [**CR**], [**Zh**]); in algebra, a certain moduli space of metrized graphs is used to study the automorphisms of free groups (see [**Vo**]); in mathematical biology, they are used to study neuron transmission (see [**Ni**]); they are also used in physics, chemistry, and engineering as wave-propagation models (see [**Ku**] and [**QG**]). These objects are called different names depending on the context in which they arise; they seem to have been discovered independently by several disparate groups of researchers. A *metric graph* is the same as a metrized graph. A c^2-*network* is a metrized graph along with a piecewise $\mathcal{C}^2$-embedding into some Euclidean space $\mathbb{R}^m$. A *quantum graph* is a metrized graph together with a self-adjoint differential operator (such as a Laplacian).

2000 *Mathematics Subject Classification.* 94C05, 35J05.

The authors would like to thank Robert Rumely for numerous useful conversations about metrized graphs. They would also like to thank the participants of the Summer 2003 REU program "Analysis on Metrized Graphs" for their dedication and enthusiasm: Maxim Arap, Jake Boggan, Rommel Cortez, Crystal Gordon, Kevin Mills, Kinsey Rowe, and Phil Zeyliger. The first author's work was supported by NSF grant DMS 0070736. The second author's work was supported by the VIGRE grant at Columbia University: DMS 9810750.

In §3 we define a Laplacian operator on a metrized graph which is closely related to the Laplacian matrix (or Kirchhoff matrix) associated to a weighted graph — see §5 for precise statements about this connection. Metrized graphs can be viewed as one-dimensional Riemannian manifolds with singularities, and from this point of view the Laplacian on a metrized graph is a nontrivial but computationally accessible variant of the Laplacian on a higher-dimensional Riemannian manifold. The theory of harmonic analysis on metrized graphs provides an interesting generalization of Fourier analysis on the circle; see [**BR**] for one account.

There is a well-known and useful interplay between the theories of finite graphs and resistive electrical networks (see e.g., [**Bo**, Ch. II,IX]). This relationship extends beautifully to the setting of metrized graphs (cf. §4,6). For example, a theorem of Foster from 1949 (see [**Fo**]) asserts that

$$\sum_{\text{edges } e} \frac{r(e)}{L_e} = \#V - 1,$$

where $r(e)$ is the effective resistance in the electrical network between the endpoints of the edge e, L_e is the resistance along the edge e, and $\#V$ is the number of nodes (vertices) in the network. In §7 we give a proof of Foster's theorem using the "canonical measure" on a metrized graph.

The theory of electrical networks is itself closely related to the theory of random walks on graphs. We will not touch upon the connection with random walks in this paper, but we refer the interested reader to the delightful monograph [**DS**]. There is a nice proof of Foster's theorem using random walks in [**Te**] (see also [**Bo**, Theorem 25, Exercise 23, Chapter IX]).

This article is a follow-up to the 2003 summer REU on metrized graphs held at the University of Georgia and run by the first author and Robert Rumely. The participating students' enthusiasm for the subject convinced us that a broader audience might appreciate a gentle introduction to the ideas involved. Further information about the REU, its organizers and participants, and the research they performed can be found at `http://www.math.uga.edu/~mbaker/REU/REU.html`

In keeping with the spirit of discovery that spawned this article, we have included a number of exercises to clarify the text or extend the ideas presented. We have also strived to keep the exposition as self-contained as possible with the hopes that it will inspire further students toward this subject.

2. Metrized graphs versus weighted graphs

There is a bijective correspondence between metrized graphs and equivalence classes of weighted graphs. In this section we give an overview of this correspondence, leaving many of the details to the reader. See [**BR**] for more detailed proofs of the assertions made in this section.

DEFINITION 1. For the purposes of this paper, we define a *weighted graph* G to be a finite, connected graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$, edge set $E(G) = \{e_1, \ldots, e_m\}$, and a collection of positive weights $\{w_{e_1} \ldots, w_{e_m}\}$ associated to the edges of G. Further, we require that G have no loop edges or multiple edges. The *length* of the edge e is defined to be $L_e = 1/w_e$.

In classical graph theory, one associates weights to the edges of a graph. When studying metrized graphs, it makes more sense to work with lengths, since distance

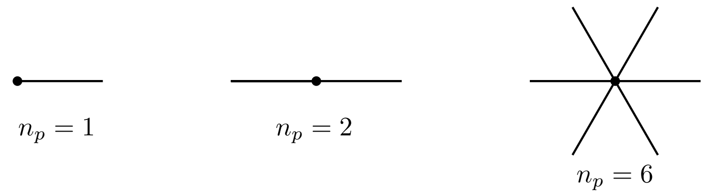


FIGURE 1. Three examples of star-shaped sets and their valences.

is the fundamental notion in a metric space. We will always indicate *lengths* in our figures (e.g., Figure 2).

A weighted graph G gives rise to a metric space Γ in the following way. To each edge e, associate a line segment of length L_e, and identify the ends of distinct line segments if they correspond to the same vertex of G. The points of these line segments are the points of Γ. We call G a *model* for Γ. The distance between two points x and y in Γ is defined to be the length of the shortest path between them, where the length of a path is measured in the obvious way along the line segments traversed. (A path between distinct points always exists because G is connected.)

EXERCISE 1. Show that this notion of distance defines a metric on Γ (which we call the *path metric*).

The space Γ, endowed with the path metric, is called a *metrized graph*. Here is a more abstract definition, taken from [**Zh**]:

DEFINITION 2. A *metrized graph* Γ is a compact, connected metric space such that each $p \in \Gamma$ has a neighborhood U_p isometric to a star-shaped set of valence $n_p \geq 1$, endowed with the path metric (see Figure 1). To be precise, a star-shaped set of valence n_p is a set of the form

$$S(n_p, r_p) \;=\; \{z \in \mathbb{C} : z = te^{k \cdot 2\pi i/n_p} \text{ for some } 0 \leq t < r_p \text{ and some } k \in \mathbb{Z}\}.$$

EXERCISE 2. Check that the metric space Γ arising from a weighted graph G satisfies the abstract definition (Definition 2) of a metrized graph.

The points $p \in \Gamma$ with valence different from 2 are precisely those where Γ fails to look locally like an open interval, and the compactness of Γ ensures that there are only finitely many such points. Let $V(\Gamma)$ be any finite, nonempty subset of Γ such that:

- $V(\Gamma)$ contains all of the points with $n_p \neq 2$. (This implies that $\Gamma \setminus V(\Gamma)$ is a finite, disjoint union of subspaces U_i isometric to open intervals.)
- For each i, the topological closure $\overline{U}_i$ of U_i in Γ is isometric to a line segment (as opposed to a circle). We call $e_i = \overline{U}_i$ a *segment* of Γ.
- For each $i \neq j$, $e_i \cap e_j$ consists of at most one point.

Any finite set $V(\Gamma)$ satisfying these conditions will be called a *vertex set* for Γ, and the elements of $V(\Gamma)$ will be called *vertices* of Γ.

EXERCISE 3.
(a) Prove that a vertex set for Γ always exists.
(b) Let Γ be a circle. Show that any set consisting of only one or two points of Γ cannot be a vertex set.

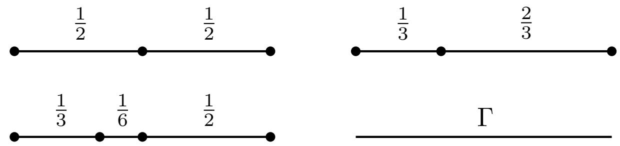


FIGURE 2. Each of the three weighted graphs displayed is a model of the metrized graph Γ, a segment of length 1. These weighted graphs are all distinct, but they lie in the same equivalence class. The lower left weighted graph is a common refinement of the upper left and upper right graphs.

It should be remarked that $V(\Gamma)$ is not unique. For example, if Γ is a circle, then any choice of three distinct points of Γ is a vertex set. The choice of a vertex set $V(\Gamma)$ determines a finite set $\{e_i\}$ of segments of Γ. The endpoints of each segment e_i are vertices of Γ. We emphasize that the segments of Γ depend on our choice of a vertex set.

Given a metrized graph Γ, our next task will be to find a weighted graph G that serves as a model for Γ as above. Pick a vertex set $V(\Gamma)$ for Γ. Define a graph G with vertices indexed by $V(\Gamma)$, and join two distinct vertices p and q of G by an edge if and only if there exists a segment of Γ with endpoints p and q. (So edges of G correspond to segments of Γ.) Define the length of the edge joining p to q to be the length of the segment e. Then G is a weighted graph, with weights given by the reciprocals of the lengths; our definition of $V(\Gamma)$ guarantees that G has no multiple edges or loop edges. Moreover, if we construct the metrized graph associated to G, it is easily seen to be isometric to Γ.

Different choices of a vertex set $V(\Gamma)$ yield distinct weighted graphs in the above construction. Write $G \sim G'$ if the two weighted graphs G, G' admit a *common refinement*, where we *refine* a weighted graph by subdividing its edges in a manner that preserves total length (see Figure 2). This provides an equivalence relation on the collection of weighted graphs, and one can check that two weighted graphs are equivalent if and only if they give rise to isometric metrized graphs.

Having established this correspondence, we are now free to fix a particular model of a metrized graph, without worrying that we've lost some degree of generality in doing so.

3. The Laplacian on a metrized graph

Our goal in this section is to motivate and define the Laplacian of a function on a metrized graph. The Laplacian on a metrized graph is a hybrid between the Laplacian on the real line (i.e., the negative of the second derivative) and the discrete Laplacian matrix studied in graph theory (cf. §5).

Choose a vertex set $V(\Gamma)$ for the metrized graph Γ. Let p be a non-vertex point of Γ, and suppose e is a segment of length L containing p. Parametrize e by an isometry $s_e : [0, L] \to e$ so that we have a real coordinate $t \in [0, L]$ to use for describing points of the segment. We say that f is differentiable at p if the quantity $\frac{d}{dt} f(s_e(t))|_{s_e(t)=p}$ exists. There is precisely one other parametrization of this sort, namely $u_e(t) = s_e(L - t)$. The chain rule shows that

$$\frac{d}{dt} f(u_e(t))\Big|_{u_e(t)=p} = -\frac{d}{dt} f(s_e(t))\Big|_{s_e(t)=p}.$$

Hence the value of the derivative of f at p depends on the parametrization, but only up to a sign. Picking one of the two parametrizations for a segment can be thought of as choosing an *orientation* for the segment, and we will use the two concepts interchangeably.

We can similarly determine if f is n times differentiable at p by looking at the existence of the quantity $\frac{d^n}{dt^n} f(s_e(t))|_{s_e(t)=p}$.

EXERCISE 4. Show that the second derivative $f''(p)$, when it exists, is well-defined independent of the choice of an orientation for the segment containing p.

We also require a notion of differentiability that makes sense at the vertices. The abstract definition of a metrized graph tells us that each point $p \in \Gamma$ has a neighborhood isometric to a star-shaped set with $n_p \geq 1$ arms. Thus there are n_p directions by which a path in Γ can leave p. To each such direction, we associate a formal unit vector $\vec{v}$, and we write $\mathrm{Vec}(p)$ for the collection of all n_p directions at p. We make this convention so that we can write $p + \varepsilon\vec{v}$ for the point of Γ at distance ε from p in the direction $\vec{v}$ for sufficiently small $\varepsilon > 0$.

DEFINITION 3. Given a function $f : \Gamma \to \mathbb{R}$, a point $p \in \Gamma$, and a direction $\vec{v} \in \mathrm{Vec}(p)$, the *derivative of f at p in the direction $\vec{v}$*, written $D_{\vec{v}} f(p)$, is given by

$$D_{\vec{v}} f(p) = \lim_{\varepsilon \to 0^+} \frac{f(p + \varepsilon\vec{v}) - f(p)}{\varepsilon},$$

provided this limit exists. This will also be called a *directional derivative*.

EXERCISE 5. Given a function $f : \Gamma \to \mathbb{R}$ and a point $p \notin V(\Gamma)$ at which f is differentiable, show that the two directional derivatives of f at p exist and sum to zero.

Here is the class of functions on which we intend to apply our Laplacian[1]:

DEFINITION 4. Define $\mathrm{S}(\Gamma)$ to be the class of all continuous functions $f : \Gamma \to \mathbb{R}$ for which there exists a vertex set $V_f(\Gamma)$ (with corresponding segments e_i) such that

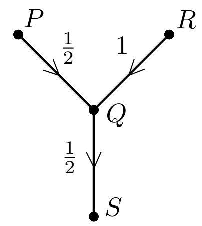
(i) $D_{\vec{v}} f(p)$ exists for each $p \in \Gamma$ and each $\vec{v} \in \mathrm{Vec}(p)$,
(ii) f is twice continuously differentiable on the interior of each segment e_i, and
(iii) f'' is bounded on the interior of each segment e_i.

We call $\mathrm{S}(\Gamma)$ the class of *piecewise smooth functions* on Γ. (This is, of course, a small abuse of terminology as these functions need not be infinitely differentiable away from the vertices.)

EXERCISE 6. Show that hypotheses (ii) and (iii) imply hypothesis (i), and that hypothesis (i) already implies that f is continuous.

We now define the Laplacian operator on a metrized graph. In order to make the Laplacian on a metrized graph compatible with the Laplacian matrix on a weighted graph (see §5), we will define the Laplacian of a function $f \in \mathrm{S}(\Gamma)$ to be a bounded, signed measure rather than a function on Γ.

[1]There exists a much larger class of continuous functions on which the Laplacian can be defined; we have restricted our attention to this particular class for simplicity. See [**BR**, §4,5] for details.



Segment parametrizations:

$$PQ \leftrightarrow [0,1/2]$$
$$QS \leftrightarrow [0,1/2]$$
$$RQ \leftrightarrow [0,1]$$

FIGURE 3. A model of a metrized graph and its segment parametrizations. The arrows in the diagram indicate the directions in which the segment parametrizations increase.

Choose a model for the metrized graph Γ and parametrize each segment e of Γ by $s_e : [0, L_e] \to e$. For a function $f : \Gamma \to \mathbb{R}$, we define $f_e : [0, L_e] \to \mathbb{R}$ by $f_e = f \circ s_e$. This notation will be used without comment for the rest of the paper.

For our purposes, a measure on Γ will be an expression of the form

$$\mu = \sum_{\text{segments } e} g_e(t)dt|_e + \sum_{i=1}^{n} c_i \delta_{p_i},$$

where $g_e : (0, L_e) \to \mathbb{R}$ is continuous and bounded, $c_i \in \mathbb{R}$, and $p_1, \ldots, p_n$ are points of Γ. A measure of the form $\sum_e g_e(t)dt|_e$ will be called a *continuous measure*, and a measure of the form $\sum_{i=1}^{n} c_i \delta_{p_i}$ will be called a *discrete measure*. If $g : \Gamma \to \mathbb{R}$ is a function such that $g \circ s_e(t) = g_e(t)$ for all segments e of Γ and all $t \in (0, L_e)$, we will usually write $g(x)dx$ instead of $\sum_e g_e(t)dt|_e$. The *total mass* of a measure μ is defined to be $\int_\Gamma \mathbb{1}(x)d\mu(x)$, where $\mathbb{1}$ denotes the constant function with value 1.

DEFINITION 5. The *Laplacian* of a function $f \in \mathrm{S}(\Gamma)$ is given by the measure

$$\Delta f = -f''(x)dx - \sum_{p \in \Gamma} \sigma_p(f)\delta_p,$$

where $\sigma_p(f) = \sum_{\vec{v} \in \mathrm{Vec}(p)} D_{\vec{v}} f(p)$, dx denotes the Lebesgue measure on Γ, and δ_p is the Dirac measure (unit point mass) at p.

By Exercise 5, the sum $\sum_p \sigma_p(f)$ is actually finite as $\sigma_p(f) = 0$ for any p not in $V_f(\Gamma)$. Also, Δf is independent of segment orientations by Exercise 4.

EXAMPLE. Consider the metrized graph Γ modelled in Figure 3. Define a function on Γ by

$$f_e(t) = \begin{cases} t+1 & \text{if } e = PQ, \\ 3(t+\frac{1}{2}) & \text{if } e = QS, \\ t^2 + \frac{1}{2} & \text{if } e = RQ. \end{cases}$$

Then $\Delta f = -2dx|_{RQ} - \delta_P + 3\delta_S$. Note that Δf has total mass zero; we will see shortly that this is not an accident.

Now we give an alternate formulation of the Laplacian which will be easier to use for computations. For a function $f \in \mathrm{S}(\Gamma)$ write $f'_e(0)$ for the right-hand derivative of f_e at 0 (as the limit only makes sense from one side). Similarly, write $f'_e(L_e)$ for the left-hand derivative at L_e if it exists. If p and q are the endpoints of the segment e, with $s_e(0) = p$ and $s_e(L_e) = q$, we'll say that e *begins* at p and *ends* at q. If $\vec{v} \in \mathrm{Vec}(p)$ and $\vec{w} \in \mathrm{Vec}(q)$ are the directions pointing inward along e, then $D_{\vec{v}}f(p) = f_e'(0)$ and $D_{\vec{w}}f(q) = -f_e'(L_e)$. Observe that $-\sigma_p(f)$ counts $-f_e'(0)$ for each segment e beginning at p, and it counts $f_e'(L_e)$ for each segment e ending at p. Thus Δf can be written

$$(*) \qquad \Delta f = \sum_{\text{segments } e} \left\{ -f_e''(x)\,dx|_e + f_e'(L_e)\delta_{s_e(L_e)} - f_e'(0)\delta_{s_e(0)} \right\}.$$

The contribution of the segment e to the Laplacian is independent of the choice of parametrization of e, but it is necessary to choose a parametrization to write it down.

THEOREM 1 (Symmetry of Δ). *Suppose $f, g \in \mathrm{S}(\Gamma)$. Then*

$$\int_\Gamma f\Delta g = \int_\Gamma g\Delta f = \int_\Gamma f'(x)g'(x)\,dx.$$

PROOF. Choose a model for Γ with vertex set $V(\Gamma) = V_f(\Gamma) \cup V_g(\Gamma)$. Then f'' is continuous on the interior of each segment of Γ, and the directional derivatives of f and g exist for all vertices in $V(\Gamma)$ (see Definition 4). Choose parametrizations for each segment of Γ and define f_e and g_e as before. Using integration by parts, we obtain:

$$\begin{aligned} \int_\Gamma g\Delta f &= \sum_e \left\{ f_e'(L_e)g_e(L_e) - f_e'(0)g_e(0) - \int_0^{L_e} g_e(t)f_e''(t)\,dt \right\} \\ &= \sum_e \int_0^{L_e} f_e'(t)g_e'(t)\,dx = \int_\Gamma f'(x)g'(x)\,dx. \end{aligned}$$

The rest of the result follows by symmetry. □

COROLLARY 1. *If $f \in \mathrm{S}(\Gamma)$, then Δf has total mass 0.*

PROOF. Set $g = \mathbb{1}$ in the statement of Theorem 1. Then

$$\int_\Gamma \mathbb{1} \cdot \Delta f = \int_\Gamma f'(x)\mathbb{1}'(x)\,dx = \int_\Gamma 0\,dx = 0.$$

□

Before stating the next result about the Laplacian, we need to define another useful class of functions:

DEFINITION 6. Define $\mathrm{A}(\Gamma)$ to be the subclass of functions $f \in \mathrm{S}(\Gamma)$ such that for each oriented segment e of Γ, there exist real constants A_e, B_e so that $f_e(t) = A_e t + B_e$ for $t \in [0, L_e]$. A function in $\mathrm{A}(\Gamma)$ is called *piecewise affine*.

EXERCISE 7. Show that a function $f \in \mathrm{A}(\Gamma)$ is completely determined by its values on a vertex set $V_f(\Gamma)$ for f.

EXERCISE 8. Show that if $f \in \mathrm{S}(\Gamma)$, then f is piecewise affine if and only if Δf is a discrete measure.

The next result is a graph-theoretic analogue of the second derivative test from calculus.

THEOREM 2 (The Maximum Principle). *Suppose $f \in \mathrm{A}(\Gamma)$ is nonconstant. Then f achieves its maximum value on Γ at a vertex $p \in V_f(\Gamma)$ for which $\sigma_p(f) < 0$.*

PROOF. It is easy to see that the function f must take on its maximum value at a vertex $p \in V_f(\Gamma)$. Moreover, since f is nonconstant, we may select p so that f decreases along some segment e_0 having p as an endpoint. (This uses the fact that Γ is connected.) Re-parametrize each segment e having p as an endpoint, if necessary, so that $s_e(0) = p$, where $s_e : [0, L_e] \to e$. Then $\sigma_p(f) = \sum f'_e(0)$, where the summation is over all segments e beginning at p. Each of the slopes $f'_e(0)$ must be non-positive; otherwise f would grow along e, violating the fact that f is maximized at p. We know $f'_{e_0}(0) < 0$ since f decreases along e_0. Hence $\sigma_p(f) < 0$, which completes the proof. □

THEOREM 3. *Suppose $f, g \in \mathrm{S}(\Gamma)$. If $\Delta f = \Delta g$ and $f(p) = g(p)$ for some $p \in \Gamma$, then $f \equiv g$.*

PROOF. If $h = f - g$, then $\Delta h = 0$. By Exercise 8, $h \in \mathrm{A}(\Gamma)$. As $\Delta h = 0$, it follows from the Maximum Principle that h is constant. The hypothesis that $f(p) = g(p)$ for some p now implies that $h \equiv 0$, so that $f \equiv g$ as desired. □

Note in particular that if $f \in \mathrm{S}(\Gamma)$ is *harmonic* (i.e., $\Delta f = 0$), then f must be constant.

4. Metrized graphs versus electrical networks

We now take a moment to give some physical intuition about the Laplacian coming from the theory of electrical networks. (For a more detailed account of the theory of electrical networks, see [**Bo**], [**DS**], and [**CR**].) For our purposes, a *(resistive) electrical network* is a physical model of a metrized graph Γ obtained by viewing the vertices of Γ as nodes of the network and the segments of Γ as branches (wires), each with a resistance given by its length.

Using an external device (such as a battery), one can force *current* to flow through the network; for simplicity, we consider only the case where a quantity $I > 0$ of current enters the circuit at some point a and exits at some point b. At all other points of Γ, we have *Kirchhoff's current law*: The total current flowing into any node equals the current flowing out of any node. Mathematically, current is a function which assigns to each oriented segment e of Γ a real number i_e, the *current flow* across e. *Kirchhoff's node law* says that it is possible to define an *electric potential function* $\phi(x) \in \mathrm{A}(\Gamma)$ such that for every oriented segment e, $\phi'_e(x) = -i_e$. (The minus sign is due to the convention that current flows from high potential to low potential.) In particular, if p is the initial endpoint and q the terminal endpoint of an oriented segment e, then *Ohm's law* $\phi(p) - \phi(q) = i_e L_e$ holds. The potential function $\phi(x)$ is only determined up to an additive constant; one needs to pick a reference voltage at some point of Γ in order to define the potential at other points.

In our language of directional derivatives, if p is a point of Γ and $\vec{v} \in \mathrm{Vec}(p)$ is any direction at p, then the current flowing away from p in the direction $\vec{v}$ is $-D_{\vec{v}}\phi(p)$. Mathematically, Kirchhoff's current law states that for $p \notin \{a, b\}$, we have $-\sigma_p(\phi) = -\sum D_{\vec{v}}\phi(p) = 0$. We have $-\sigma_a(\phi) > 0$, which says that a is a *current source*, and $-\sigma_b(\phi) < 0$, which says that b is a *current sink*. The current entering the network at a is $-\sigma_a(\phi)$; the current exiting the network at b is $\sigma_b(\phi)$; and we have $-\sigma_a(\phi) = \sigma_b(\phi) = I$.

Taken together, Kirchhoff's node and potential laws say that given $I > 0$, there is a function $\phi \in \mathrm{A}(\Gamma)$ such that $\Delta\phi = I \cdot \delta_a - I \cdot \delta_b$. (This will be proved mathematically as a consequence of Corollary 3 in §6.) Note that ϕ is determined up to an additive constant by Theorem 3. Note also that we initially required $-\sigma_a(\phi) = \sigma_b(\phi) = I$ (*conservation of current*), which is demanded mathematically by Corollary 1.

In accordance with physical intuition, the Maximum Principle (Theorem 1) implies that the electric potential in the network is highest at a (where current enters) and lowest at b (where it exits). By convention, one often sets the potential at b to be zero, in which case we say that the node b is *grounded*.

5. The Laplacian on a weighted graph

In this section, we explain some connections between the classical Laplacian matrix on a weighted graph and the Laplacian on a metrized graph.

Suppose G is a weighted graph with vertex set $V(G) = \{v_i\}$, edge set $E(G) = \{e_k\}$, and weights $\{w_{e_k}\}$. If the edge e_k has endpoints v_i and v_j, then we will use the notation $w_{ij} = w_{e_k} = w_{ji}$ to show the dependence of the weights on the vertices. For convenience, we set $w_{ij} = 0$ if v_i and v_j are not connected by an edge. In particular $w_{ii} = 0$ for all i.

Definition 7. The *Laplacian matrix* associated to a weighted graph G is the $n \times n$ matrix Q with entries

$$Q_{ij} = \begin{cases} \sum_k w_{ik} & \text{if } i = j, \\ -w_{ij} & \text{if } i \neq j. \end{cases}$$

We should note that in the literature, our Q is often called the *combinatorial Laplacian* or *Kirchhoff matrix* (see e.g., [**Bo**]).

The Laplacian matrix encodes interesting information about the graph G (see e.g., [**Mo**], [**GR**, §13], [**CDS**], [**CDGT**], [**Ch**], [**CdV**]). For example, zero appears as an eigenvalue of Q with multiplicity equal to the number of connected components of G (so exactly once in our case). Kirchhoff's famous Matrix-Tree Theorem (see [**Bo**, Corollary 13, Chapter II]) equates the weighted number of spanning trees of the graph with the absolute value of the determinant of the matrix obtained by deleting any row and column from Q.

Returning to metrized graphs, we've already noted in Exercise 7 that a function $f \in \mathrm{A}(\Gamma)$ is completely determined by its values on the finite set $V_f(\Gamma)$. Thus, a piecewise affine function on Γ yields a function on the vertices of a certain model for Γ, and conversely, given a model G and a function on $V(G)$, we can linearly interpolate to obtain a piecewise affine function on Γ. Our two notions of Laplacian honor this correspondence:

Theorem 4. *Suppose Γ is a metrized graph, $f \in \mathrm{A}(\Gamma)$, and G is a model of Γ with vertex set $V_f(\Gamma) = \{v_1, \dots, v_n\}$. Let $\vec{f}$ be the $n \times 1$ vector with $\vec{f}_i = f(v_i)$. Then*

$$\Delta f = \sum_i \left[Q\vec{f}\right]_i \delta_{v_i}.$$

Proof. We already know that Δf is discrete if f is piecewise affine, so it suffices to show that $\left[Q\vec{f}\right]_i = -\sigma_{v_i}(f)$ for any vertex v_i. We parametrize each segment e having v_i as an endpoint so that $s_e(0) = v_i$. As f is piecewise affine, the directional derivatives of f at v_i are given by $f_e'(0) = [f_e(L_e) - f_e(0)]/L_e$. Recall that the weight of an edge is the reciprocal of its length. Conclude that

$$\sigma_{v_i}(f) = \sum_{\substack{\text{segments } e \\ \text{adjacent to } v_i}} \frac{f_e(L_e) - f_e(0)}{L_e} = \sum_j w_{ij} \left\{ f(v_j) - f(v_i) \right\}$$

$$= -\left\{ \left(\sum_k w_{ik} \right) f(v_i) - \sum_{j \neq i} w_{ij} f(v_j) \right\} = -\left[Q\vec{f} \right]_i .$$

□

As a bonus, one can use results about Δ to deduce two standard facts about the Laplacian matrix:

Corollary 2. *If G is a weighted graph with $n \times n$ Laplacian matrix Q, then*

(i) *The kernel of Q is 1-dimensional with basis $[1, \ldots, 1]^t$.*
(ii) *If $x \in \mathbb{R}^n$ is a vector, then $\sum_i [Qx]_i = 0$.*

Proof. Identify $\mathbb{R}^n$ with the n-dimensional vector space spanned by the vertices of G. A vector $x \in \mathbb{R}^n$ can be interpreted as a function on the vertices of G, and this function can be linearly interpolated to yield a piecewise affine function f on the associated metrized graph Γ. If $Qx = 0$, then Theorem 4 implies that $\Delta f = 0$. The Maximum Principle shows f must be constant, so $x = [c, \ldots, c]^t$ for some real number c. This proves (i). For (ii), use Corollary 1 and Theorem 4 to get

$$\sum_i [Qx]_i = \int_\Gamma \Delta f = 0.$$

□

Now we know the relationship between the Laplacian operator acting on $\mathrm{A}(\Gamma)$ and the Laplacian matrix. In fact, one can prove that the Laplacian of a piecewise smooth function f is a limit of Laplacians of piecewise affine approximations of f. To state the result, we introduce the following notation. If $f \in \mathrm{S}(\Gamma)$ and G_N is a model of Γ whose vertices contain $V_f(\Gamma)$, define f_N to be the unique piecewise affine function with $f_N(p) = f(p)$ for each vertex p of G_N (restrict f to the vertices of G_N and linearly interpolate).

Theorem 5. *Suppose $f \in \mathrm{S}(\Gamma)$. There exists a sequence of models $\{G_N\}$ for Γ such that for all continuous functions g on Γ, we have*

$$\int_\Gamma g \; \Delta f_N \longrightarrow \int_\Gamma g \Delta f \qquad \text{as } N \to \infty.$$

That is, the sequence of measures $\{\Delta f_N\}$ converges weakly to Δf on Γ. By Theorem 4 the discrete measures Δf_N can be computed using the Laplacian matrix.

A complete proof can be found in [**Fa**]. We mention the theorem in order to display the very close connection between the Laplacian matrix on a weighted graph and the Laplacian operator on a metrized graph.

6. The j-function

In this section, we introduce a three-variable function $j_z(x,y)$ on the metrized graph Γ which allows us, in a sense to be made precise, to invert the Laplacian operator. Let $\mathrm{Meas}_0(\Gamma)$ denote the space of measures of total mass zero on Γ. We know from Corollary 1 that if $f \in \mathrm{S}(\Gamma)$ then $\Delta f \in \mathrm{Meas}_0(\Gamma)$. The following result is a partial converse to this fact.

THEOREM 6. *Let $\nu = \sum c_i \delta_{p_i} \in \mathrm{Meas}_0(\Gamma)$ be a discrete measure. Then there exists a piecewise affine function f on Γ such that $\Delta f = \nu$.*

PROOF. Let $S = \{p_1, \ldots, p_k\}$, and fix a model G for Γ with vertex set $V(G)$ containing S. Let $n = \#V(G)$, and let W be the n-dimensional real vector space spanned by the vertices of G, which we identify with $\mathbb{R}^n$. If Q is the Laplacian matrix associated to G, then we know $\mathrm{Ker}(Q)$ is 1-dimensional by Corollary 2(i). The rank-nullity theorem implies that $\mathrm{Im}(Q)$ is $(n-1)$-dimensional.

By Theorem 4, solving $\Delta f = \nu$ is equivalent to finding a vector $x \in W$ with $Qx = [c_1, \ldots, c_n]^t$. Let W_0 be the $(n-1)$-dimensional subspace of W consisting of vectors $[a_1, \ldots, a_n]^t$ such that $\sum a_i = 0$. Corollary 2(ii) shows $\mathrm{Im}(Q)$ is contained in W_0. As these two spaces have the same dimension, they must be equal. The condition $\nu \in \mathrm{Meas}_0(\Gamma)$ says $\sum c_i = 0$, so $[c_1, \ldots, c_n]^t$ lies in the image of Q. □

We now single out a special case of this result which is of particular interest. In what follows, we write Δ_x instead of Δ if we wish to emphasize that we are taking the Laplacian with respect to the variable x.

COROLLARY 3. *For fixed $y, z \in \Gamma$, there exists a unique piecewise affine function $j(x) = j_z(x,y)$ satisfying*

$$\Delta_x j_z(x,y) = \delta_y(x) - \delta_z(x), \qquad j_z(z,y) = 0.$$

PROOF. The existence of $j(x)$ follows from Theorem 6, and uniqueness follows from Theorem 3. □

We now justify our assertion that the j-function allows us to "invert the Laplacian" on the space $\mathrm{Meas}_0(\Gamma)$. Recall from Theorem 6 that given a discrete measure $\nu \in \mathrm{Meas}_0(\Gamma)$, there exists a function $f \in \mathrm{A}(\Gamma)$ (unique up to an additive constant) that satisfies the differential equation $\Delta f = \nu$. The next result shows that we can explicitly describe such a function f using the j-function:

THEOREM 7. *Let $\nu = \sum c_i \delta_{p_i} \in \mathrm{Meas}_0(\Gamma)$ be a discrete measure. Then for any fixed $z \in \Gamma$, the function*

$$f(x) = \int_\Gamma j_z(x,y)\, d\nu(y) = \sum_i c_i j_z(x, p_i)$$

is piecewise affine and satisfies the equation $\Delta f = \nu$.

PROOF. The condition $\nu \in \mathrm{Meas}_0(\Gamma)$ means that $\sum c_i = 0$. Therefore

$$\Delta f = \sum_i c_i \left(\delta_{p_i} - \delta_z\right) = \nu.$$

Since the j-function is piecewise affine, f is as well. □

We mention (see [**BR**] for a proof) that Theorem 7 admits the following generalization to arbitrary (not necessarily discrete) measures $\nu \in \mathrm{Meas}_0(\Gamma)$: For fixed $z \in \Gamma$, the function $f(x) = \int_\Gamma j_z(x,y)d\nu(y)$ is in $\mathrm{S}(\Gamma)$ and satisfies the equation $\Delta f = \nu$. In particular, if ν is a measure on Γ, then we can solve the differential equation $\Delta f = \nu$ *if and only if* $\nu \in \mathrm{Meas}_0(\Gamma)$.

The function $j_z(x,y)$ has an interpretation in terms of electrical networks. Recalling our description of the electrical network associated to a metrized graph given in §4, the function $j_z(x,y)$ is the electric potential at x if one unit of current enters the network at y and exits at z, and the node z is grounded.

Exercise 9. Physical intuition suggests that the j-function should be nonnegative; prove more precisely that

$$0 \le j_z(x,y) \le j_z(y,y)$$

for all $x, y, z \in \Gamma$. [**Hint:** Apply the Maximum Principle.]

The three-variable function $j_z(x,y)$ satisfies a magical four-term identity, which will be used in various guises throughout this section and the next.

Theorem 8 (Magical Identity). *For all $x, y, z, w \in \Gamma$, we have the identity*

$$j_z(x,y) - j_z(w,y) = j_w(y,x) - j_w(z,x).$$

Proof. Fix $x, y, z, w \in \Gamma$. On one hand, we have

$$\int_\Gamma j_z(u,y)\Delta_u\left(j_w(u,x)\right) = \int_\Gamma j_z(u,y)\left\{\delta_x(u) - \delta_w(u)\right\} = j_z(x,y) - j_z(w,y).$$

By Theorem 1, this is equal to

$$\int_\Gamma j_w(u,x)\Delta_u\left(j_z(u,y)\right) = \int_\Gamma j_w(u,x)\left\{\delta_y(u) - \delta_z(u)\right\} = j_w(y,x) - j_w(z,x).$$

□

The Magical Identity allows us to prove two useful symmetries for the j-function.

Corollary 4. *For $x, y, z \in \Gamma$, the j-function satisfies*

(i) $j_z(x,y) = j_z(y,x)$
(ii) $j_z(x,x) = j_x(z,z)$

Proof. For (i), if we set $w = z$ in the Magical Identity, we obtain

$$j_z(x,y) - j_z(z,y) = j_z(y,x) - j_z(z,x).$$

Since $j_z(z,x) = j_z(z,y) = 0$, the result follows.

For (ii), substitute $x = z, y = w$ into the Magical Identity to get

$$j_z(z,w) - j_z(w,w) = j_w(w,z) - j_w(z,z).$$

Since $j_z(z,w) = j_w(w,z) = 0$, the result follows by swapping w for x. □

In passing, we mention that $j_z(x,y)$ has a very strong continuity property: it is jointly continuous in x, y, and z. That is, the value of the j-function varies continuously if we make small variations to x, y, and z simultaneously. Our electrical network interpretation makes this statement quite plausible: the value on our voltmeter should vary continuously when we move the battery terminals and the point at which we're reading the voltage. A mathematical proof is outlined in the next exercise (see [**CR**] for a different approach).

EXERCISE 10.

(a) Let I, I' be closed intervals in $\mathbb{R}$. Suppose $f : I \times I' \to \mathbb{R}$ has the property that $f(x,y)$ is affine in x and y separately. Then $f(x,y) = c_1 + c_2x + c_3y + c_4xy$ for some $c_1, \ldots, c_4 \in \mathbb{R}$.

(b) Use (a) to show that for fixed $z \in \Gamma$, $j_z(x,y)$ is jointly continuous as a function of x and y.

(c) Use Theorem 3 to prove the five-term identity

$$j_z(x,y) = j_w(x,y) - j_w(x,z) - j_w(z,y) + j_w(z,z).$$

(d) Deduce from (b) and (c) that $j_z(x,y)$ is jointly continuous in x, y, and z.

We now define another useful function motivated by the theory of electrical networks:

DEFINITION 8. The *effective resistance* between two points x, y of a metrized graph is given by

$$r(x,y) = j_y(x,x) = j_x(y,y).$$

The fact that $j_y(x,x) = j_x(y,y)$ is just a restatement of the second symmetry of the j-function in Corollary 4. In terms of electrical networks, the effective resistance between two nodes x and y is the absolute value of the potential difference between x and y when a unit current enters the network at x and exits at y.

We now introduce some useful techniques for calculating the j-function and the effective resistance function. Rules (ii) and (iii) in Theorem 9 below are essentially the familiar series and parallel transforms from circuit theory. The proofs of Theorems 9 and 10 are adapted from [**Zh**].

A *subgraph* of the metrized graph Γ is a subspace of Γ which is a metrized graph in its own right. In the statement of Proposition 9, Γ_1 and Γ_2 will always denote subgraphs of Γ. We let $j_z(x,y)$ (resp. $j_{z,1}(x,y), j_{z,2}(x,y)$) denote the j-function on Γ (resp. on Γ_1, Γ_2), and similarly we let $r(x,y)$ (resp. $r_1(x,y), r_2(x,y)$) denote the effective resistance function on Γ (resp. on Γ_1, Γ_2).

THEOREM 9. *Let Γ be a metrized graph, and let Γ_1 and Γ_2 be subgraphs.*

(i) *Suppose e is a segment in Γ of length L with endpoints x, y, and assume that $\Gamma = \Gamma_1 \cup \Gamma_2 \cup e$ with $\Gamma_1 \cap e = \{x\}$, $\Gamma_2 \cap e = \{y\}$, and $\Gamma_1 \cap \Gamma_2 = \emptyset$. (Compare Figure 4(i).) Then $r(x,y) = L$.*

(ii) *Suppose $\Gamma = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 \cap \Gamma_2 = \{z\}$. (Compare Figure 4(ii).) Then for all $x \in \Gamma_1$ and $y \in \Gamma_2$, we have $r(x,y) = r_1(x,z) + r_2(z,y)$.*

(iii) *Suppose $\Gamma = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 \cap \Gamma_2 = \{x, y\}$. (Compare Figure 4(iii).) Then*

$$\frac{1}{r(x,y)} = \frac{1}{r_1(x,y)} + \frac{1}{r_2(x,y)}.$$

PROOF. For (i), we pick a parametrization $s_e : [0,L] \to e$ such that $s_e(0) = x$ and $s_e(L) = y$. Let $t : e \to [0,L]$ be the inverse of s_e. We claim that

$$j_x(z,y) = \begin{cases} 0 & \text{if } z \in \Gamma_1, \\ t(z) & \text{if } z \in e, \\ L & \text{if } z \in \Gamma_2. \end{cases}$$

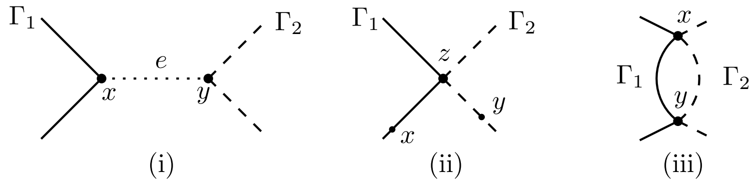


FIGURE 4. These three figures illustrate the three parts of Proposition 9. The solid lines (resp. dashed lines) indicate the segments of the diagram belonging to Γ_1 (resp. to Γ_2).

Indeed, it is easily verified that the Laplacian of the right-hand side with respect to z is $\delta_y - \delta_x$, and that the two sides agree when $z = x$. The claim therefore follows from Theorem 3, and the desired result follows by setting $z = y$.

To prove (ii), we claim that

$$j_x(w, y) = \begin{cases} j_{x,1}(w, z) & \text{if } w \in \Gamma_1, \\ r_1(x, z) + j_{z,2}(w, y) & \text{if } w \in \Gamma_2. \end{cases}$$

For the right-hand side is continuous at $w = z$, has Laplacian equal to $(\delta_z - \delta_x) + (\delta_y - \delta_z) = \delta_y - \delta_x$, and is zero when $w = x$. The result follows by setting $w = y$.

The proof of (iii) proceeds in the same way by verifying the identity

$$j_x(z, y) = \begin{cases} \frac{r_2(x,y)}{r_1(x,y)+r_2(x,y)} j_{x,1}(z, y) & \text{if } z \in \Gamma_1, \\ \frac{r_1(x,y)}{r_1(x,y)+r_2(x,y)} j_{x,2}(z, y) & \text{if } z \in \Gamma_2. \end{cases}$$

We leave the details to the reader. □

EXERCISE 11. Show that the function $r(x, y)$ is jointly continuous in x and y, and that for fixed $y \in \Gamma$, $r(x, y)$ is continuous and piecewise quadratic in x. [**Hint:** Use Exercise 10.]

Using Theorem 9, we can derive an explicit description of the function $r(x, y)$ when x varies along a single segment of Γ having y as an endpoint. To state the result, we define a quantity R_e associated to a segment e of Γ as follows. Let e° denote the interior of the segment e, and let Γ_e be the complement of e° in Γ. If Γ_e is connected, then Γ_e is a subgraph of Γ, and we define R_e to be the effective resistance $r(y, z)$ between the endpoints y and z of e, computed on Γ_e. If Γ_e is not connected (i.e., if e is not part of a cycle), we define R_e to be ∞. Loosely speaking, R_e is the effective resistance between the endpoints of e in the subgraph obtained by deleting e.

The next result is motivated by the following intuition: to calculate $r(x, y)$ on the segment e, we can think of e and its complement Γ_e as being connected in parallel, and x splits e into two segments connected in series. We can then use the parallel and series transforms to calculate $r(x, y)$.

THEOREM 10. *Let e be a closed segment of Γ of length L_e, let y, z be the endpoints of e, and parametrize e by $s_e : [0, L_e] \to e$ with $s_e(0) = y$ and $s_e(L_e) = z$. Suppose $t : e \to [0, L_e]$ is the inverse of s_e. Then for $x \in e$, we have*

$$r(x, y) = t(x) - \frac{1}{L_e + R_e} t(x)^2,$$

where $\frac{1}{L_e+R_e} = 0$ if $R_e = \infty$.

PROOF. If Γ_e is not connected, then $\Gamma = \Gamma_1 \cup \Gamma_2 \cup e$, with $\Gamma_1 \cap e = \{y\}$, $\Gamma_2 \cap e = \{z\}$, and $\Gamma_1 \cap \Gamma_2 = \emptyset$. Then $r(x,y) = t(x)$ by part (i) of Proposition 9.

Now suppose that Γ_e is connected. Then x breaks e into two closed segments $\Gamma_1 = [t(y), t(x)]$ and $\Gamma_2 = [t(x), t(z)]$, and $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_e$. Letting $\Gamma_3 = \Gamma_2 \cup \Gamma_e$, we have (with the obvious notation):

$$\begin{aligned}\frac{1}{r(x,y)} &= \frac{1}{r_1(x,y)} + \frac{1}{r_3(x,y)} && \text{by Prop. 9(iii),}\\ &= \frac{1}{r_1(x,y)} + \frac{1}{r_2(x,z)+R_e} && \text{by Prop. 9(ii),}\\ &= \frac{1}{t(x)} + \frac{1}{L_e - t(x) + R_e} && \text{by Prop. 9(i).}\end{aligned}$$

The desired formula now follows because of the simplification

$$\left(\frac{1}{t(x)} + \frac{1}{L_e - t(x) + R_e}\right)^{-1} = t(x) - \frac{1}{L_e + R_e} t(x)^2.$$

□

EXERCISE 12. Let Γ be a metrized graph of total length L. Fix $a, b \in \Gamma$, and choose a vertex set $V(\Gamma)$ containing a and b. Let e be an oriented segment of Γ beginning at p and ending at q. For $x \in \Gamma$, let $\phi(x) = j_b(x,a)$, and define $i_e = \frac{\phi(p)-\phi(q)}{L_e} = -\phi'_e$. (In terms of electrical networks, i_e is the current flowing across e when a unit current enters the network at a and exits at b.) Also, define $r(e) = r(p,q)$.

(a) Show that $r(e) \le L_e$. [**Hint:** Use Prop. 9.]
(b) Show that $r(x,y)$ is a metric on Γ. [**Hint:** For the triangle inequality, use Exercise 10.]
(c) Deduce that $r(a,b)$ is bounded above by the length of any path from a to b, and conclude that $0 \le r(x,y) \le L$ for all $x, y \in \Gamma$.

7. The canonical measure and Foster's Theorem

Calculating the Laplacian of the effective resistance function $r(x,y)$ for fixed y is not so easy just from the definitions, but our explicit description in Theorems 10 and 11 below will allow us to do it indirectly. The first half of this section will be devoted to figuring out $\Delta_x r(x,y)$, and in the second half we reap the benefits of this calculation by proving some interesting results from graph theory, including Foster's theorem. The method presented here is a simplified version of §2 of [**CR**].

EXAMPLE. If $\Gamma = [0,1]$, then Theorem 9(i) shows that $r(x,y) = |x-y|$, and a simple calculation shows that $\Delta_x r(x,y) = \delta_0(x) + \delta_1(x) - 2\delta_y(x)$. Interestingly, we see that $\Delta_x r(x,y) + 2\delta_y(x)$ is independent of y. This simple example actually illustrates a general phenomenon.

THEOREM 11. *For any metrized graph, $\Delta_x r(x,y) + 2\delta_y(x)$ is a measure which is independent of y.*

PROOF. Let $z, w \in \Gamma$ be arbitrary. Set $x = y$ in the Magical Identity of §6 to get

$$j_z(y,y) - j_z(w,y) = j_w(y,y) - j_w(z,y).$$

Applying Corollary 4, we obtain

$$r(y,z) - j_z(y,w) = r(y,w) - j_w(y,z).$$

Taking the Laplacian of both sides with respect to y and recalling that $\Delta_y j_z(y,x) = \delta_x - \delta_z$, we get

$$\Delta_y r(y,z) - \delta_w + \delta_z = \Delta_y r(y,w) - \delta_z + \delta_w.$$

Rearranging, we see that $\Delta_y r(y,z) + 2\delta_z = \Delta_y r(y,w) + 2\delta_w$. As w and z were arbitrary, the result follows. ☐

DEFINITION 9. The *canonical measure* on a metrized graph Γ is given by

$$\mu_{\mathrm{can}} = \frac{1}{2}\Delta_x r(x,y) + \delta_y(x),$$

where $y \in \Gamma$ is arbitrary. Theorem 11 shows that μ_{can} is independent of the choice of y.

Recall from Corollary 1 that $\Delta_x r(x,y)$ is a measure of total mass zero, so we see from Definition 9 that μ_{can} has total mass 1.

EXERCISE 13. Use Theorem 1 to show that the quantity

$$\tau(\Gamma) = \frac{1}{2}\int_\Gamma r(x,y)d\mu_{\mathrm{can}}(x)$$

is a positive real number which is independent of the choice of $y \in \Gamma$.[2]

We now give an explicit description of the measure μ_{can}.

THEOREM 12. *Let n_p denote the valence of a vertex $p \in V(\Gamma)$. Then*

$$\mu_{\mathrm{can}} = \sum_{\text{vertices } p} \left(1 - \frac{1}{2}n_p\right)\delta_p + \sum_{\text{segments } e} \frac{1}{R_e + L_e} dx|_e.$$

PROOF. We compute the discrete and continuous parts of μ_{can} separately.

Continuous part: Let e be an oriented segment of Γ which begins at y and ends at z. If x lies on e, then we're in the situation of Theorem 10, and we calculate that

$$(\dagger) \qquad \Delta_x \left\{ r(x,y)|_e \right\} = \frac{2}{L_e + R_e} dx|_e + \delta_z - \delta_y.$$

Since $\mu_{\mathrm{can}} = \frac{1}{2}\Delta_x r(x,y) + \delta_y$ is independent of our choice of y, (†) shows that the continuous part of μ_{can} along e must be $\frac{1}{L_e+R_e}dx|_e$.

Discrete part: If y is an endpoint of a segment e, then $r(x,y)$ is quadratic along the interior of e by Theorem 10. It follows that the discrete part of μ_{can} is supported on $V(\Gamma)$. Let $p \in V(\Gamma)$ be a vertex. Using $(*)$ in §3, we calculate from Equation (†) that $\frac{1}{2}\Delta_x r(x,p)$ contributes $-\frac{1}{2}\delta_p$ to the discrete part of μ_{can} at p for each segment e beginning at p. Recalling that $\mu_{\mathrm{can}} = \frac{1}{2}\Delta_x r(x,p) + \delta_p$, the coefficient of δ_p in μ_{can} must therefore be $1 - \frac{1}{2}n_p$.

☐

EXAMPLE. If Γ is a circle of length 1, then every vertex has valence 2, so μ_{can} has no discrete part. For the continuous part, divide the circle into three segments e_1, e_2, e_3, each of length 1/3. Then we get $\mu_{\mathrm{can}} = dx\,|_{e_1} + dx\,|_{e_2} + dx\,|_{e_3} = dx$.

[2] An interesting and difficult problem is to determine whether or not $\tau(\Gamma)$ can be arbitrarily small for a metrized graph Γ of total length 1. (See §14 of [**BR**] for a discussion of this problem.)

EXAMPLE. Let Γ be the star of Figure 3. Then μ_{can} has no continuous part because R_e is infinite for all edges. Therefore $\mu_{\mathrm{can}} = \frac{1}{2}\delta_P - \frac{1}{2}\delta_Q + \frac{1}{2}\delta_R + \frac{1}{2}\delta_S$.

Theorem 12 has some interesting consequences for weighted graphs. For example, we have the following result from [**CR**]:

COROLLARY 5. *Let G be a weighted graph with vertex set $V(G)$ and edge set $E(G)$. Then*

$$\sum_{\text{edges } e} \frac{L_e}{R_e + L_e} = 1 + \#E(G) - \#V(G).$$

PROOF. Integrating both sides of the formula in Theorem 12 over Γ, we obtain:

$$1 = \sum_{\text{vertices } p} \left(1 - \frac{1}{2}n_p\right) + \sum_{\text{edges } e} \frac{L_e}{R_e + L_e}.$$

As each edge in G connects exactly 2 vertices, we have $\sum_{p \in V(G)} n_p = 2\{\#E(G)\}$. Therefore

$$1 = \#V(G) - \#E(G) + \sum_{\text{edges } e} \frac{L_e}{R_e + L_e},$$

which is equivalent to the desired formula. □

It is a well-known fact from graph theory that $1 + \#E(G) - \#V(G)$ is the number of linearly independent cycles on G (see [**Bo**, Theorem 9, Chapter II]). This topological invariant depends only on the associated metrized graph Γ.

The above corollary was first proved in [**Fo**] as a consequence of "Kirchhoff's Rule" (a relation similar to the Matrix-Tree Theorem). The next statement, which was also proved in [**Fo**], has become known more widely as Foster's Theorem.

COROLLARY 6 (Foster's Theorem). *For an edge e in a weighted graph G, let $r(e)$ denote the effective resistance $r(x, y)$ between the endpoints x and y of e on the associated metrized graph Γ. Then*

$$\sum_{\text{edges } e} \frac{r(e)}{L_e} = \#V(G) - 1.$$

PROOF. If $R_e = \infty$, then $r(e) = L_e$ by Proposition 9(i). Otherwise, by Theorem 10, we have

$$r(e) = L_e - \frac{L_e^2}{L_e + R_e} = \frac{L_e R_e}{L_e + R_e}.$$

Combining these observations, we see that

$$\begin{aligned} \sum_{\text{edges } e} \frac{r(e)}{L_e} &= \sum_{\substack{\text{edges } e \\ \text{with } R_e = \infty}} 1 + \sum_{\substack{\text{edges } e \\ \text{with } R_e \neq \infty}} \frac{R_e}{L_e + R_e} \\ &= \#E(G) + \sum_{\substack{\text{edges } e \\ \text{with } R_e \neq \infty}} \left\{ \frac{R_e}{L_e + R_e} - 1 \right\} \\ &= \#E(G) - \sum_{\text{edges } e} \frac{L_e}{L_e + R_e}. \end{aligned}$$

The result follows immediately from Corollary 5. □

Example. If G is a tree, then we have $r(e) = L_e$ for all e by Proposition 9(i), and $\#E(G) = \#V(G) - 1$. Therefore $\sum_e \frac{r(e)}{L_e} = \#E(G) = \#V(G) - 1$ as predicted by Foster's theorem.

More generally, for arbitrary G it follows from Exercise 12(a) that $0 \leq \frac{r(e)}{L_e} \leq 1$ for each edge e, so that *a priori* we have $\sum_e \frac{r(e)}{L_e} \leq \#E(G)$. Foster's theorem is equivalent to the assertion that the difference $\#E(G) - \sum_e \frac{r(e)}{L_e}$ is equal to the number of independent cycles in G.

Example. Foster's theorem can be a useful tool for calculating effective resistances, especially in the presence of symmetry. For example, let $G = K_n$ be the complete graph on $n \geq 2$ vertices, with all edge weights equal to 1. By symmetry, the effective resistance $r(x, y)$ between distinct points $x, y \in V(G)$ is independent of x and y; let r denote the common value. Foster's theorem gives

$$\sum_{\text{edges } e} w_e \, r(e) = \binom{n}{2} \cdot r = n - 1,$$

so that $r = 2/n$.

References

[BR] Matthew Baker and Robert Rumely, Harmonic analysis on metrized graphs, to appear in *Canad. J. Math.*

[Bo] Béla Bollobás, *Modern Graph Theory*, Springer-Verlag, New York, 1998.

[CdV] Yves Colin de Verdière, *Spectres de Graphes*, Soc. math. Fr., Paris, 1998.

[Ch] Fan R.K. Chung, *Spectral Graph Theory*, Amer. Math. Soc., Providence, R.I., 1997.

[CDGT] Dragoš Cvetković, Michael Doob, Ivan Gutman and Aleksandar Torgašev, *Recent Results in the Theory of Graph Spectra*, North-Holland Publishing Co., Amsterdam, 1988.

[CDS] Dragoš Cvetković, Michael Doob and Horst Sachs, *Spectra of Graphs – Theory and Application*, third edition, Johann Ambrosius Barth, Heidelberg, 1995.

[CR] Ted Chinburg and Robert Rumely, The capacity pairing, *J. reine angew. Math.* **434** (1993), 1–44.

[DS] Peter G. Doyle and J. Laurie Snell, *Random Walks and Electric Networks*, Math. Assoc. Amer., Washington, D.C., 1984.

[Fa] Xander W.C. Faber, Spectral convergence of the discrete Laplacian on models of a metrized graph, to appear in *New York J. Math.*

[Fo] Ronald M. Foster, The average impedance of an electrical network, *Contributions to Applied Mechanics. (Reissner Anniversary Volume)* (1949) Ed. J.W. Edwards, Ann Arbor, Michigan, 333–340.

[GR] Chris Godsil and Gordon Royle, *Algebraic Graph Theory*, Springer-Verlag, New York, 2001.

[Ku] Peter Kuchment, Graph models for waves in thin structures, *Waves Random Media* **12** (2002), no. 4, R1–R24.

[Mo] Bojan Mohar, The Laplacian spectrum of graphs, *Graph Theory, Combinatorics, and Applications.* **2**, Ed. Y. Alavi, G. Chartran, O.R. Oellermann, A. J. Schwenk, Wiley, 1991, 871–898.

[Ni] Serge Nicaise, Some results on spectral theory over networks, applied to nerve impulse transmission, *Lecture Notes in Mathematics* **1171**, Springer-Verlag, Berlin and New York, 1985, 532–541.

[QG] Quantum Graphs and Their Applications, special issue of *Waves Random Media* **14** (2004), no. 1.

[Te] Prasad Tetali, Random walks and the effective resistance of networks, *J. Theoret. Prob.* **4** (1991), no. 1, 101–109.

[Vo] Karen Vogtmann, Automorphisms of Free Groups and Outer Space, *Geom. Dedicata* **94** (2002), 1–31.

[Zh] Shou-wu Zhang, Admissible pairing on a curve, *Invent. math.* **112** (1993), 171–193.

Matthew Baker, School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA

E-mail address: `mbaker@math.gatech.edu`

Xander Faber, Department of Mathematics, Columbia University, New York, NY 10027, USA

E-mail address: `xander@math.columbia.edu`