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An analytic series method for Laplacian problems with mixed boundary conditions

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Abstract

Mixed boundary value problems are characterised by a combination of Dirichlet and Neumann conditions along at least one boundary. Historically, only a very small subset of these problems could be solved using analytic series methods ("analytic" is taken here to mean a series whose terms are analytic in the complex plane). In the past, series solutions were obtained by using an appropriate choice of axes, or a co-ordinate transformation to suitable axes where the boundaries are parallel to the abscissa and the boundary conditions are separated into pure Dirichlet or Neumann form. In this paper, I will consider the more general problem where the mixed boundary conditions cannot be resolved by a co-ordinate transformation. That is, a Dirichlet condition applies on part of the boundary and a Neumann condition applies along the remaining section. I will present a general method for obtaining analytic series solutions for the classic problem where the boundary is parallel to the abscissa. In addition, I will extend this technique to the general mixed boundary value problem, defined on an arbitrary boundary, where the boundary is *not* parallel to the abscissa. I will demonstrate the efficacy of the method on a well known seepage problem.

MSC: 34B05; 35J25; 42B05; 65M70; 74H05

Keywords: Mixed boundary value problem; Series solution; Analytic solution; Seepage; Pseudo-spectral method

1. Introduction

Laplacian mixed boundary value problems occur in a wide range of engineering and applied mathematics applications [3–5,15]. Applications include classic electrical potential and electric field conditions on a disk [17], stress and strain conditions around a punch pressing on an elastic surface [6] as well as applications in porous media [14,19]. The boundary conditions in these problems range from straightforward Dirichlet/Neumann conditions through mixed Dirichlet–Neumann type problems that cannot be resolved by a suitable choice of axes. As an example, problems of these types regularly occur in the study of transport processes in porous media [1,16]. In particular, infiltration and seepage problems generally involve mixed boundary value problems.

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The general Laplacian mixed boundary value problem for the potential function $\phi(x, y)$ has a boundary condition of the form

$$\alpha(x, y)\phi(x, y) + \beta(x, y)\frac{\partial}{\partial m}\phi(x, y) = \gamma(x, y), \tag{1}$$

along at least one boundary $y = f(x)$ ($\partial/\partial m$ denotes differentiation normal to $y = f(x)$). The classic problem as posed for analytic solution appears somewhat simpler than this. The coefficients α and β are step functions, the boundary $y = f(x)$ reduces to $y = c$, where c is a constant, and the Neumann term reduces to $\partial/\partial y$. The boundary conditions along $y = c$ reduce to

$$\phi(x, y) = g(x), \quad 0 \leqslant x \leqslant a, \quad \frac{\partial}{\partial y}\phi(x, y) = h(x), \quad a < x \leqslant s. \tag{2}$$

There are other arrangements of the intervals where ϕ and $\partial/\partial y$ apply along $0 \leqslant x \leqslant s$. However, the essence of the problem is captured in this representation.

The classic problem appears deceptively simple to solve, given the ease with which solutions are obtained for the pure Dirichlet and Neumann equivalents to Eq. (2) (i.e., $a = s$ or $a = 0$, respectively). In the classic series approach, one of the other boundary conditions is used to produce a set of eigenfunctions that are then used to expand one of $h(x)$ or $g(x)$ over the interval $[0, s]$, and the expansion coefficients are then evaluated using an orthogonality relationship. But in this case, there are two different sets of eigenfunctions; in addition, the expansions are over subintervals of $[0, s]$ so that neither orthogonality relationship applies, and the process breaks down at this point. Another solution approach involves reducing the problem to a pair of integral equations which can then be solved by the use of Jacobi polynomials [17]. (Note that Jacobi polynomials are the class of orthogonal polynomials that include the spectral polynomials.) This method (after several pages of innovative and ingenious mathematics) has been used to determine solutions for infiltration from a shallow pond [20]. However, none of these techniques are simple to apply and do not appear tractable for most problems.

Recently, analytic series solutions have been developed to solve known and free boundary problems (where the location of one of the boundaries is initially unknown), on irregular solution domains [8,10,11,13]. The series solution is obtained by using non-orthogonal basis functions to obtain a set of equations for the series coefficients. The free boundary problem is solved by guessing an initial location for the free boundary and then iteratively improving this initial approximation. At each step, the series solution is obtained by using the (arbitrarily shaped) free boundary estimate to define the problem fully. There is a striking similarity between the mixed boundary value problem and the free boundary value problem—Dirichlet and Neumann conditions must both be satisfied on the same boundary. The method used to solve free boundary problems can be applied to solve the mixed boundary problem [12]. However, the main weakness with this method is that it is iterative and the accuracy of the solution depends on the number of iterations as well as the number of terms in the solution.

In this paper I will use the non-orthogonal basis function approach to solve the mixed boundary value problem. There is no need for an iterative procedure—the series coefficients are obtained by solving a set of equations for the series coefficients. These equations are obtained using the principle of eigenfunction expansions, applied to non-orthogonal basis functions. I will provide an efficient method to estimate the expansion coefficients using a collocation (or pseudo-spectral) approach [18]. I will demonstrate the efficacy of the method on a well-known seepage problem for both regular and irregular boundaries.

This paper is organised as follows. In Section 2, a formal mathematical description of the problem is developed. The series solution method is described in Section 3, followed by the test problem1 results in Section 4. Finally, the method and results are discussed in Section 5.

2. Problem definition

In this section, the Laplacian mixed boundary value problem is defined. In the interior of the solution domain, Laplace's equation for $\phi(x, y)$ is satisfied

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0. \tag{3}$$

Without loss of generality, we choose the top boundary $y = f(x)$ as the location of the mixed boundary condition. On $y = f(x)$, the following condition must be satisfied:

$$\phi(x, y) = g(x), \quad 0 \leqslant x \leqslant a, \tag{4}$$

$$\frac{\partial}{\partial m}\phi(x, y) = h(x), \quad a < x \leqslant s, \tag{5}$$

where $\partial/\partial m$ denotes differentiation normal to the boundary. We assume that this boundary condition cannot be broken down into separate Dirichlet and Neumann conditions by a suitable choice of axes or co-ordinate transformation. Note that when $y = c$, where c is constant, this reduces to the classic problem.

We assume that the other boundaries $x = 0$, $x = s$ and $y = 0$ are subject to either Dirichlet or Neumann conditions, so that separation of variables can be applied. At this point we will introduce an application to use as a vehicle to demonstrate the method.

2.1. A specific problem

Consider steady saturated seepage from a dam through an aquifer to a pond. The surface of the aquifer lies at the base of the dam wall and extends through to the pond. The aquifer is of length s and lies on top of a horizontal impermeable aquiclude. Upstream, water has ponded to a height h_2, while downstream the ponded water has height h_1 ($h_1 < h_2$). Along the upstream section of the soil surface, the base of the dam wall is impermeable, while the downstream section acts as a seepage face. Fig. 1 gives a schematic of the soil horizon.

Mathematically, the problem can be formulated as follows. Inside the saturated soil, seepage is governed by Darcy's law. Assuming constant hydraulic conductivity K and invoking the continuity condition, the hydraulic potential ϕ is governed by Laplace's equation, inside the aquifer:

$$\nabla^2\phi(x, y) = 0. \tag{6}$$

Along the vertical boundaries at $x = 0$ and s, the boundary conditions are given by

$$\phi(0, y) = h_1, \quad \phi(s, y) = h_2. \tag{7}$$

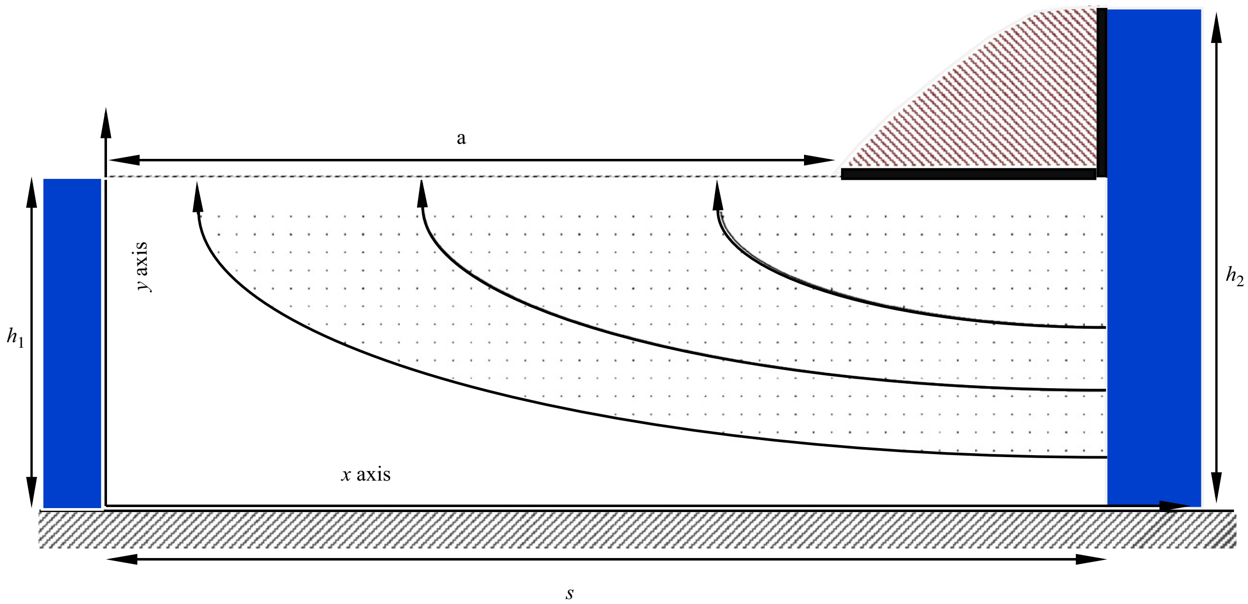

Fig. 1. Schematic of the saturated flow domain.

Along the impermeable aquiclude $y = 0$, the boundary condition becomes

$$K \frac{\partial}{\partial y} \phi(x, 0) = 0. \tag{8}$$

Along the soil surface $y = f(x)$, the mixed boundary condition is

$$\phi(x, f(x)) = f(x), \quad 0 \leqslant x < a, \tag{9}$$

$$-K \frac{\partial}{\partial m} \phi(x, f(x)) = -K R(x) \hat{\mathbf{m}} \cdot \hat{\mathbf{j}}, \quad a \leqslant x \leqslant s, \tag{10}$$

where $R(x)$ is the vertical recharge and $\hat{\mathbf{m}}$ is a unit normal to the boundary. Noting that

$$\hat{\mathbf{m}} = \frac{1}{\sqrt{1 + f'(x)^2}} [-f'(x)\hat{\mathbf{i}} + \hat{\mathbf{j}}] \tag{11}$$

the Neumann condition (10) becomes

$$\frac{\partial}{\partial y} \phi(x, f(x)) - f'(x) \frac{\partial}{\partial x} \phi(x, f(x)) = R(x). \tag{12}$$

This Neumann condition can be further simplified by the introduction of the conjugate stream function $\psi(x, y)$, related to the potential $\phi(x, y)$ by the Cauchy–Riemann equations:

$$\frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}, \quad \frac{\partial \psi}{\partial y} = -\frac{\partial \phi}{\partial x}. \tag{13}$$

Substituting for ϕ in Eq. (12), we obtain

$$-\frac{\partial}{\partial x} \psi(x, f(x)) - \frac{\partial}{\partial y} \psi(x, f(x)) \frac{\mathrm{d}}{\mathrm{d}x} f(x) = -\left[\frac{\partial}{\partial x} \psi(x, y) + \frac{\partial}{\partial y} \psi(x, y) \frac{\mathrm{d}y}{\mathrm{d}x}\right]_{y=f(x)} \tag{14}$$

$$= -\frac{\mathrm{d}\psi}{\mathrm{d}x}. \tag{15}$$

Hence the Neumann condition (12) can be expressed as

$$\psi(x, f(x)) = -\int R(x)\, \mathrm{d}x = r(x). \tag{16}$$

When there is zero flux across the section $a \leqslant x \leqslant s$, then $r(x) = \text{const}$. The mixed boundary condition (Eqs. (9) and (10)) along $y = f(x)$ can be expressed as

$$\phi(x, f(x)) = f(x), \quad 0 \leqslant x < a, \tag{17}$$

$$\psi(x, f(x)) = r(x), \quad a \leqslant x \leqslant s. \tag{18}$$

This representation does not remove the difficulty associated with solving the mixed boundary condition and is not necessary for the solution process that we describe in this paper. However, there are computational advantages in this representation and it is a mathematically more elegant representation of the Neumann condition.

3. Series solution

The problem formulated in the previous section needs to be transformed slightly [9], so that separation of variables can be used to obtain an analytic series solution. Letting

$$\phi(x, y) = h_1 + \frac{(h_2 - h_1)}{s} x + \varphi, \tag{19}$$

then φ satisfies Laplace's equation

$$\nabla^2\varphi = 0, \tag{20}$$

with homogeneous side boundary conditions

$$\varphi(0, y) = 0, \quad \varphi(s, y) = 0 \tag{21}$$

and homogeneous bottom boundary condition

$$\frac{\partial}{\partial y}\varphi(x, 0) = 0. \tag{22}$$

The classical method of separation of variables can now be applied to (20). Using the homogeneous side (21) and bottom (22) boundary conditions, the analytic series solution is readily shown to be

$$\varphi(x, y) = \sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi y}{s}\right) \sin\left(\frac{n\pi x}{s}\right). \tag{23}$$

The series solution for the original problem becomes

$$\phi(x, y) = h_1 + \frac{(h_2 - h_1)}{s}x + \sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi y}{s}\right) \sin\left(\frac{n\pi x}{s}\right). \tag{24}$$

Using the Cauchy–Riemann equations (13), the stream function is given by

$$\psi(x, y) = A_0 + \frac{(h_2 - h_1)}{s}y + \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi y}{s}\right) \cos\left(\frac{n\pi x}{s}\right). \tag{25}$$

This equation involves an arbitrary constant term A_0 that is the constant of integration in (18) for $r(x)$. In fact, A_0 represents the mass flux through the aquifer at the dam wall (i.e., between $(s, 0$ and $(s, f(s)))$.

Note that this series solution satisfies both the nonhomogeneous side boundary conditions (7) and homogeneous bottom boundary condition (8) of the original problem exactly. The top boundary condition given by Eqs. (17) and (18) is used to evaluate the series coefficients A_n, and so define fully the solution. Unfortunately, the classical approach breaks down at this point, as the boundary condition is not a linear combination of ϕ and $\partial\phi/\partial y$. Consequently, the orthogonality relationship cannot be used (directly!) to determine the A_n. However, we note that the mixed boundary value problem has a unique solution [17], which implies the coefficients A_n are also uniquely defined.

3.1. Series coefficient evaluation

The key to solving the mixed boundary value problem described in the preceding sections is the following representation of the series expansion along the top boundary. Let

$$F(x, y) = \begin{cases} f(x) - h_1 - \dfrac{(h_2 - h_1)}{s}x, & 0 \leqslant x < a, \\ r(x) - \dfrac{(h_2 - h_1)}{s}y, & a \leqslant x \leqslant s, \end{cases} \tag{26}$$

and, for $n = 1, 2, \ldots$

$$U_n(x, y) = \begin{cases} \cosh\left(\dfrac{n\pi y}{s}\right) \sin\left(\dfrac{n\pi x}{s}\right), & 0 \leqslant x < a, \\ \sinh\left(\dfrac{n\pi y}{s}\right) \cos\left(\dfrac{n\pi x}{s}\right), & a \leqslant x \leqslant s, \end{cases} \tag{27}$$

with

$$U_0(x, y) = \begin{cases} 0, & 0 \leqslant x < a, \\ 1, & a \leqslant x \leqslant s. \end{cases} \tag{28}$$

Then the mixed boundary condition given by Eqs. (17) and (18) along $y = f(x)$ can be represented in terms of the potential and stream function solutions given by Eqs. (24) and (25), respectively, as

$$F^t(x) = \sum_{n=0}^{\infty} A_n U_n^t(x), \tag{29}$$

where

$$F^t(x) = F(x, f(x)) \quad \text{and} \quad U_n^t(x) = U_n(x, f(x)). \tag{30}$$

At this point, the series solution is truncated to $N+1$ terms, so that a number of techniques can be employed to evaluate the series coefficients A_n, $n = 0, 1, \ldots, N$.

The most obvious approach is to use least squares. Introducing an inner product notation

$$\langle f, g\rangle = \int_0^s f(x)g(x)\,\mathrm{d}x, \tag{31}$$

the normal equations are readily shown to be

$$\sum_{j=0}^{N} \langle U_i^t, U_j^t\rangle A_j = \langle F^t, U_i^t\rangle, \quad i = 0, 1, \ldots, N. \tag{32}$$

In matrix form, these equations become

$$\mathrm{U}\mathbf{a} = \mathbf{F}, \tag{33}$$

where, for $i, j = 0, 1, \ldots, N$,

$$[\mathrm{U}]_{ij} = \langle U_i^t, U_j^t\rangle, \quad [\mathbf{F}]_i = \langle U_i^t, F^t\rangle, \quad [\mathbf{a}]_i = A_i. \tag{34}$$

Another approach is to use Gram–Schmidt orthogonalisation to generate an orthogonal set of basis functions. Then an orthogonality relationship is used to solve for the coefficients of the series expansion, now defined in terms of the orthogonal basis functions. However, although this approach can be used, there are no intrinsic reasons to use an orthogonal basis—any such representation must be inverted to obtain the original series coefficients A_n in the potential and stream function representations [9]. Also, QR factorisation is a more stable numerical method for solving the least squares system of equations (33) than using the Gram–Schmidt process.

The most efficient and direct approach is to use a pseudo-spectral or collocation approach [18]. The series approximation is made to satisfy the function exactly at $N + 1$ collocation points:

$$F(x_i) = \sum_{j=0}^{N} U_j^t(x_i) A_j, \quad i = 0, 1, \ldots, N. \tag{35}$$

The matrix equation for the pseudo-spectral approach is the same as for the least squares approach (33), with for $i,\ j = 0, 1, \ldots, N$

$$[\mathrm{U}]_{ij} = U_j^t(x_i), \quad [\mathbf{F}]_i = F(x_i), \tag{36}$$

and **a** as before. The collocation points x_i are chosen to minimise the Runge phenomenon [18]. Given the sine/cosine basis functions, equally spaced collocation points appear to be the best choice, although this is not sufficient to stop the Runge phenomenon in all cases. When the Runge Phenomenem persists, there are a number of possible ways to avoid it, including non-uniform spacing. For a Fourier basis, we have found that “over collocation” or discrete least squares works well. Instead of using $N + 1$ collocation points, we use $M \approx 2N \sim 5N$ points, and then use discrete least squares to determine the series coefficients. The system of Equations (33) that need to be solved for the series coefficients becomes

$$\mathrm{U}^{\mathrm{T}}\mathrm{U}\mathbf{a} = \mathrm{U}^{\mathrm{T}}\mathbf{F}. \tag{37}$$

The pseudo-spectral (and discrete least squares) approach has the advantage of not requiring the calculation of $(N+1)^2$ inner products (integrals). This becomes a much more significant consideration when the integrals have to be evaluated numerically, both in terms of accuracy and computational complexity.

4. Examples from seepage in porous media

The accuracy and convergence properties of the series expansion can be determined by examining the boundary errors. This is a consequence of the analytic nature of the series solution—the truncation errors $\phi(x,y)-\phi_N(x,y)$ and $\psi(x,y)-\psi_N(x,y)$ also satisfy Laplace's equation, an elliptic equation, and thus obey a maximum principle [7]. The side and bottom boundary conditions are satisfied exactly, so bounds on the error can be determined by examining the error in the top boundary approximation.

We can calculate the root mean square (rms) error $\varepsilon_\phi(N)$ and $\varepsilon_\psi(N)$ of the potential and stream function approximations $\phi_N(x,y)$ and $\psi_N(x,y)$, respectively, for N terms in the series, by

$$\varepsilon_\phi(N)=\left(\frac{1}{a}\int_0^a(\phi(x,1)-\phi_N(x,1))^2\,\mathrm{d}x\right)^{1/2}, \tag{38}$$

$$\varepsilon_\psi(N)=\left(\frac{1}{s-a}\int_a^s(\psi(x,1)-\psi_N(x,1))^2\,\mathrm{d}x\right)^{1/2}. \tag{39}$$

I will demonstrate the efficacy of the method on the two examples of the problem described in Section 2.1: seepage under a dam wall. In the first example, the soil surface is horizontal and represents the classic mixed boundary value problem. In the second example, the soil surface is functionally dependent on x and extends the classic mixed boundary value problem to irregular shaped solution domains. In both examples, we will assume that the seepage face on the soil surface is from $0\leqslant x\leqslant a$ and that there is no seepage (or recharge) from $a\leqslant x\leqslant s$.

Example 1. Letting $h_1=1$, $h_2=1.5$, $s=10$, $a=5.0$ and $f(x)=1$, the mixed boundary condition in Equations (17) and (18) becomes

$$\phi(x,1)=1,\quad 0\leqslant x<5.0,\quad \psi(x,1)=\text{const.},\quad 5.0\leqslant x\leqslant 10. \tag{40}$$

In this example, the soil surface between $x=5$ and 10 is impermeable so the seepage face is from $x=0$ to 5. Using the collocation approach, the matrix equation (37) is readily calculated and solved. The matrix U generated was very well conditioned for large values of N, with the condition number less than 10 even when $N\sim 1000$. Fig. 2 shows a

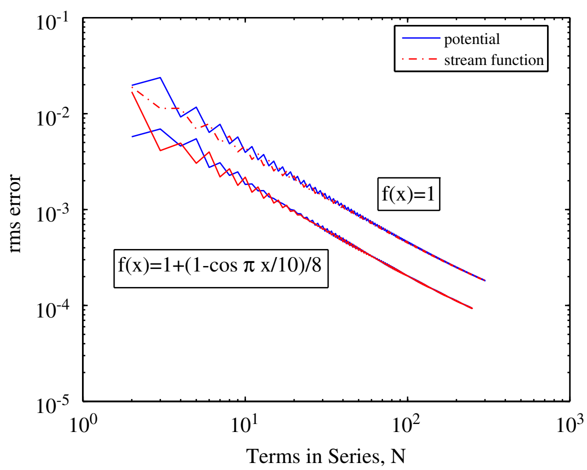

Fig. 2. Rms errors for the top boundary condition.

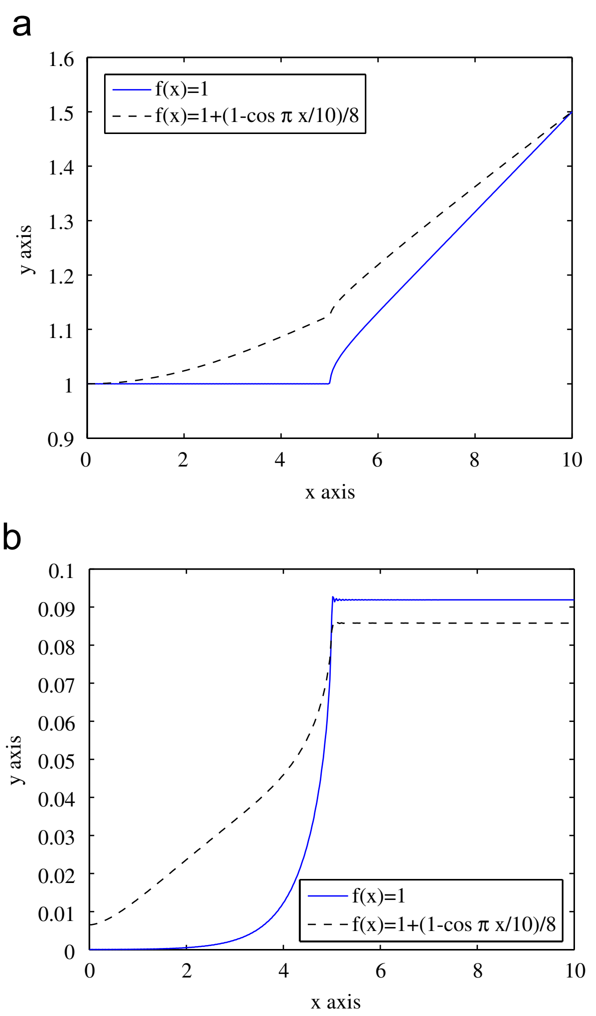

Fig. 3. (a) Potential $\phi(x, f(x))$ and (b) stream function $\psi(x, f(x))$ along the top boundary.

log–log plot of the rms errors $\varepsilon_\phi(N)$ and $\varepsilon_\psi(N)$ for $n = 2, 3, \ldots, 400$ and $M = 2N$ "over collocation" points. As we expect for a Fourier basis and non-periodic boundary conditions, the convergence is algebraic. Both the potential and stream function errors are approximately the same for any given N and decrease at the same rate.

Fig. 3(a) shows a plot of the potential function along the soil surface $y = 1$ ($N = 300$, $M = 2N$). The potential for $5 \leqslant x \leqslant 10$ has been determined as part of the solution, and corresponds to the constant stream function along this section of the soil surface. The potential increases almost linearly to the height of the water $h_2 = 1.5$ at the point (10, 1.5). There appears to be a discontinuity in the first derivative at $x = 5$ and this will limit the rate of convergence of any series method, including pseudo-spectral methods [2].

Fig. 3(b) shows a plot of the stream function along the soil surface. The stream function for $5 \leqslant x \leqslant 10$ is constant as a result of the no flow condition imposed and has been determined as part of the solution process. The value of the stream function is the maximum value $\psi_{\max}$ of $\psi(x, y)$ and represents the mass flux through the aquifer. $\psi_{\max}$ corresponds to the constant term A_0 in Eq. (25) and for this problem is 0.09189. The stream function decreases sharply away from this value as x decreases and is approximately zero when $x \leqslant 2$. As expected, we note that there is a discontinuity in the first derivative of the stream function at $x = 5$.

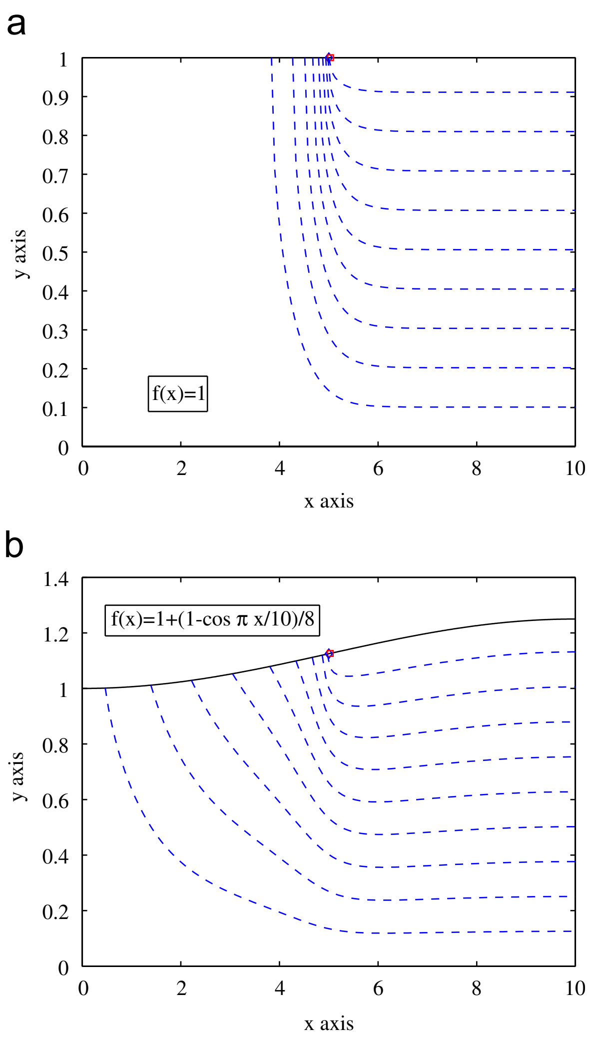

Fig. 4. Stream lines in 10% increments of the mass flux.

Fig. 4(a) shows a stream function plot of the flow domain, with streamlines corresponding to 10% increments in the mass flux (i.e., $\delta\psi = \psi_{\max}/10$). The streamlines have been calculated by solving the implicit equation

$$\alpha = \frac{\psi(x, y)}{\psi_{\max}}, \tag{41}$$

for $\alpha = 0.1, 0.2, \ldots, 0.9$. Note that no smoothing has been used to obtain the streamlines—they are accurate to machine precision. The streamline plot reflects the features seen in the stream function along the surface: parallel flow from $5 \leqslant x \leqslant 10$ with approximately 90% of the seepage out of the aquifer occurring from $4 \leqslant x \leqslant 5$.

Example 2. Let the equation for the soil surface be

$$f(x) = 1 + \frac{1 - \cos(\pi x/10)}{8}, \tag{42}$$

with, as for Example 1, $h_1 = 1$, $h_2 = 1.5$, $s = 10$, $a = 5.0$. The mixed boundary condition in Eqs. (17) and (18) becomes

$$\phi(x, f(x)) = f(x), \quad 0 \leqslant x < 5.0, \quad \psi(x, f(x)) = \text{const.}, \quad 5.0 \leqslant x \leqslant 10. \tag{43}$$

This example is very similar to Example 1, with the soil increasing in elevation with increasing x. Note that matrix equation (37) for the series coefficients are obtained in exactly the same way as for the horizontal soil case. The only apparent difference is that $f(x) \neq 1$. A less obvious difference is that the basis vectors are now not orthogonal over the interval [0, 10]. However, we could not use this property for the mixed boundary condition even if the basis was orthogonal.

In Eq. (29), we assume that $F(x) \in \text{SPAN}\{U_0^t(x),\ U_1^t(x),\ \ldots\}$ and that the convergence properties are the same as the sine/cosine basis. The cosh and sinh terms eventually lead to ill-conditioning in the matrix equations and for this example occurs when $N \sim 250$. However, up until this number of terms in the series the basis functions behave analogously to the sine basis. Fig. 2 shows a log–log plot of the rms errors $\varepsilon_\phi(N)$ and $\varepsilon_\psi(N)$ for $n = 2, 3, \ldots, 250$ and $M = 2N$. The errors follow the same pattern as for the previous example with the same convergence rate, although the errors for given N are slightly lower.

Fig. 3(a) shows a plot of the potential function along the soil surface $y = f(x)$ $(N = 200,\ M = 2N)$. The same basic features that are evident in the first example are also observed here. In particular, there is the same discontinuity at $x = 5$ and the overall shape of the two potential functions are similar. The potential for this example has a smoother transition over the discontinuity and this explains the smaller rms errors for given N. Although the rate of decrease of the rms errors is the same for both examples, the constant associated with the rate is smaller for this example. Fig. 3(b) shows a plot of the stream function along the surface. Although the same basic features from Example 1 are observed here (i.e., a discontinuity at $x = 5$ and sharply decreasing with decreasing x just before the discontinuity), there are also some significant differences.

The most important difference is that the mass flux through the aquifer, ψ_{max}, is significantly smaller. For this example, $\psi_{\text{max}} = 0.08578$, 7% smaller than for the first example. The next most important difference is that the discharge out of the aquifer is spread over the seepage face (i.e., $0 \leqslant x \leqslant 5$), with some subsurface discharge ($\sim 7\%$) into the downstream pond. Fig. 4(b) shows a stream function plot of the flow domain, with streamlines corresponding to 10% increments of the mass flux. As we would expect, this plot is very similar to the stream function plot for Example 1, when $5 \leqslant x \leqslant 10$. However, below the seepage face the stream lines are much more evenly distributed, reinforcing the differences seen in the stream function along the soil surface.

5. Discussion

In the previous sections I have detailed a series solution method for the classic mixed boundary value problem defined on both regular and irregular boundary geometry. By representing the mixed boundary conditions suitably, the problem reduces to finding the series expansion of a function using non-orthogonal basis functions. This entails solving an $N \times N$ matrix equation for the N series coefficients. The most straightforward and computationally efficient method of generating the matrix equation uses a pseudo-spectral approach. This approach was combined with discrete least squares (or over collocation) when the Runge phenomenon became a problem.

The classic problem where the boundary is horizontal (i.e., $y = 1$) can be solved with an almost arbitrarily large number of terms in the series as the matrices generated are very well conditioned. The series solution approached the true solution with algebraic convergence of the series coefficients. This convergence rate is the same that we would expect for a Fourier basis on a Dirichlet (or Neumann) boundary value problem without periodic boundary conditions. Due to a discontinuity in the potential (and stream) function at the midpoint of the boundary that was not known a priori to the solution, the optimal convergence rate for any series method on this problem is algebraic.

The irregular boundary value problem where the boundary is functionally dependent on x (i.e., $y = f(x)$) was solved in exactly the same way as the classic problem. The convergence rate of the series was the same as for the classic problem and overall the computational results obtained were very similar. The most significant difference is an upper limit on the number of terms in the series due to ill-conditioning in the matrix equation. This is a result of the sinh/cosh terms in the basis functions which depend on N and become exponentially large as N increases. However, in this example, accurate solutions were obtained before ill-conditioning became a problem.

Both the solved problems are very similar geometrically and computationally and it is somewhat surprising that the results were significantly different in some regions of the flow domain. The slight elevation in the soil profile in the second example resulted in a significantly different flow pattern in the seepage face region, even though the upstream and downstream boundary conditions were the same. The total mass flux through the aquifer was less for this example and more water discharged directly into the pond below the soil surface. From a practical viewpoint, the first model would probably be used in an engineering context to quantify the flow parameters for the second problem. This would result in a significant over-estimation of the mass flux through the aquifer and the seepage face, as well as under estimating the subsurface discharge into the pond.

These simple examples highlight the need for efficient analytic methods for solving this type of problem as well as many others and justifies the amount of time spent by researchers such as Robin Wooding [20]. Clearly this method can be applied to any mixed boundary value problem for any elliptic equation that allows separation of variables (the use of the stream function is a convenience rather than a necessity). Even when analytic techniques such as the method presented in this paper are not applicable to a real problem, they are still inherently valuable as they provide excellent means to calibrate finite element and finite difference solvers on more realistic test problems. This paper also suggests a number of further research topics which I will be pursuing. They include ways to avoid the ill-conditioning problem for the non-orthogonal basis and minimising or removing the effects of the discontinuity in the potential and stream functions at $x = a$.

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