Detecting strange attractors in turbulence.

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1. Introduction.

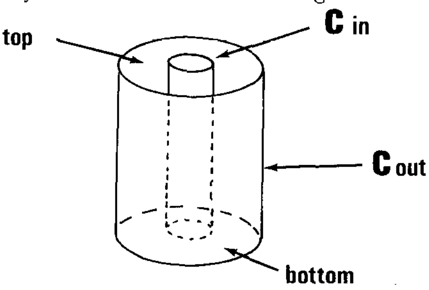
Since [19] was written, much more accurate experiments on the onset of turbulence have been made, especially by Fenstermacher, Swinney, Gollub and Benson [6,8,9,10]. These new experimental data should be interpreted according to [19] in terms of strange attractors, or they should falsify the whole picture given in that paper. For such interpretations one uses in general the so-called power spectrum. It is however not at all clear how to reconstruct the "strange attractors" from a power spectrum (with continuous parts); even worse : how can one see whether a given power spectrum (with continuous parts) might have been "generated" by a strange attractor? In this paper I present procedures to decide whether one may attribute certain experimental data, as in the onset of turbulence, to the presence of strange attractors. These procedures consist of algorithms, to be applied to the experimental data itself and not to the power spectrum; in fact, I doubt whether the power spectrum contains the relevant information.

In order to describe the problems and results, treated in this paper, in more detail, I shall first review the ideas of [19], also comparing them with those exposed by Landau and Lifschitz [13], in relation with the flow between two rotating cylinders. It was this same experiment which was carried out to great precision by Swinney et.al. [6, 8, 10].

It should be noted that the discussion in [19] is not restricted to this situation but should also be applicable to other situations where an orderly dynamic changes to a chaotic one; see [8] for a discussion of some examples. Also, our present discussion should be applicable to these cases.

The Taylor-Couette Experiment.

We consider the region D between two cylinders as indicated in figure 1. In this region we have a fluid. We study its motion when the outer cylinder, the top and bottom are at rest, while the inner cylinder has an angular velocity Ω. p is some fixed point in the interior of D. For a number of values of Ω, one component



of the velocity of the fluid at p is measured as a function of time. In [19] the idea was the following : for each value of Ω the set of all "possible states" is a Hilbert space H_Ω consisting of (divergence free) vector fields on D satisfying the appropriate boundary conditions (these vector fields represent velocity distributions of the fluid). For each Ω there is an evolution semi-flow

$$\{\varphi_t^\Omega : H_\Omega \to H_\Omega\}_{t \in \mathbb{R}_+}, \mathbb{R}_+ = \{t \in \mathbb{R} | t \geq 0\} ,$$

such that if $X \in H_\Omega$ represents the state at time $t = 0$ then $\varphi_{t_0}^\Omega(X)$ represents the state at time t_0. We assume that for all values of Ω under consideration, there is an "attractor" $\Lambda_\Omega \subset H_\Omega$ to which (almost) all evolution curves $\varphi_t^\Omega(X)$ tend as $t \to \infty$. (At this point we don't want to specify the term "attractor".) Λ_Ω and $\varphi_t^\Omega|\Lambda_\Omega$ then describe the asymptotic behaviour of all evolution curves $\varphi_t^\Omega(X)$. Roughly the main assumptions in [19] could be rephrased as : $\varphi_t^\Omega|\Lambda_\Omega$ behaves just as an attractor in a finite dimensional differentiable dynamical system. In more detail, the assumption was that for all values of Ω under consideration there is a smooth finite dimensional manifold $M_\Omega \subset H_\Omega$, smoothly depending on Ω, such that :

(i) M_Ω is invariant in the sense that for $X \in M_\Omega$, $\varphi_t^\Omega(X) \in M_\Omega$;

(ii) M_Ω is attractive in the sense that evolution curves $\varphi_t^\Omega(X)$, starting outside M_Ω tend to M_Ω for $t \to \infty$;

(iii) the flow, induced in M_Ω by φ_t^Ω, is smooth, depends smoothly on Ω and has an attractor Λ_Ω.

Some justification for this assumption was given by Marsden [15, 16]. Apart from this we used genericity assumptions : if Z_Ω denotes the vector field on M_Ω which is the infinitessimal generator of $\varphi_t^\Omega|M_\Omega$, we assume (M_Ω, Z_Ω) to be a generic one-parameter family of vector fields. (If however the physical system under consideration has symmetry, like the case of the Couette flow, then a same type of symmetry must hold for $M_\Omega, \varphi_t^\Omega$, and hence for Z_Ω. In this case genericity should be understood within the class of vector fields having this symmetry; see [18].)

In the Landau-Lifschitz picture, one assumes that the limiting motion (or attractor) is quasi-periodic, i.e. of the form

$$\varphi_t^{\Omega}(X) = f^{\Omega}(X, a_1 e^{2\pi i \omega_1 t}, a_2 e^{2\pi i \omega_2 t}, \ldots)$$

where ω_i, and a_i depends on Ω and where for each Ω only a finite number of a_i is non-zero. One can imagine that, as more and more a_i become non-zero, the motion gets more and more turbulent.

Also in this last description we have a smooth finite dimensional manifold as attractor, namely an n-torus, but such attractors do not occur for generic parameter values of generic one-parameter families of vector fields. It should be noted however that for generic one-parameter families of vector fields there may be a set of parameter values with positive measure for which quasi periodic motion occurs; see [11].

This n-torus attractor has topological entropy zero and its dimension is an integer. On the other hand "strange attractors" have in general positive entropy and often non-integral dimension. Hence it would be important to determine entropy and dimension of attractors from "experimental data".

In view of the experiment just described, we have to add one more point to out formal description, namely we have to add the function (observable) from the state space to the reals giving the experimental output (when composed with $\varphi_t^{\Omega}(X)$). In the present example of the Taylor-Couette experiment, this function $y_{\Omega}: H_{\Omega} \to \mathbb{R}$ assigns to each $X \in H_{\Omega}$ the measured component of X(p). As far as the asymptotic behaviour is concerned, we only have to deal with $y_{\Omega}|M_{\Omega}$ (or with $y_{\Omega}|\Lambda_{\Omega}$). Since M_{Ω} depends smoothly on Ω all M_{Ω} are diffeomorphic and so we may drop the Ω.

Summarising, we have a manifold M with a smooth one-parameter family of vector fields Z_{Ω} and a smooth one-parameter family of functions y_{Ω}. For a number of values of Ω the function $y_{\Omega}(\varphi_t^{\Omega}(x))$ is known by measurement (for some x in or near M which may depend on Ω; φ_t^{Ω} denotes here the flow on M generated by Z_{Ω}. The point is to obtain information about the attractor(s) of Z_{Ω} from these measurements, i.e. from the functions $t \mapsto y_{\Omega}(\varphi_t^{\Omega}(x))$. For this we shall allow ourselves to make genericity assumptions on $(M, Z_{\Omega}, y_{\Omega}, x)$.

We shall prove that under suitable genericity assumptions on $(M, Z_{\Omega}, y_{\Omega}, x)$ the positive limit set $L^+(x)$ of x is determined by the function $y_{\Omega}(\varphi_t^{\Omega}(x))$. In our "main theorem" in section 4 we describe algorithms which, when applied to a sequence

$\{a_i = y_\Omega(\varphi^\Omega_{\alpha,i}(x))\}_{i=1}^{\bar{N}}$, $\bar{N}$ sufficiently big, will give an approximation for the dimension of $L^+(x)$, respectively for the topological entropy of $\varphi^\Omega_\alpha | L^+(x)$. This leads in principle to a possibility of testing and comparing the hypothesis made by Landau-Lifschitz [13] and Ruelle-Takens [19]; see the observation at the end of section 4. The author wishes to acknowledge the hospitality of the department of mathematics of Warwick University and the many discussions with participants of the turbulence and dynamical systems symposium there during the preparation of this paper.

2. Dynamical systems with one observable.

Let M be a compact manifold. A dynamical system on M is a diffeomorphism $\varphi: M \to M$ (discrete time) or a vector field X on M (continuous time). In both cases the time evolution corresponding with an initial position $x_0 \in M$ is denoted by $\varphi_t(x_0)$: in the case of discrete time $t \in \mathbb{N}$ and $\varphi_i = (\varphi)^i$; in the case of continuous time $t \in \mathbb{R}$ and $t \mapsto \varphi_t(x_0)$ is the X integral curve through x_0.

An observable is a smooth function $y: M \to \mathbb{R}$. The first problem is this : if, for some dynamical system with time evolution φ_t, we know the functions $t \mapsto y(\varphi_t(x))$, $x \in M$, then how can we obtain information about the original dynamical system (and manifold) from this. The next three theorems deal with this problem. (After the research for this paper was completed, the author was informed that this problem, or at least parts of it, was also treated by other authors, see [1, 17]. Since out results are in some sense somewhat more general we still give here a treatment of the problem independent of the results in the above papers.)

Theorem 1. Let M be a compact manifold of dimension m. For pairs (φ, y), $\varphi: M \to M$ a smooth diffeomorphism and $y: M \to \mathbb{R}$ a smooth function, it is a generic property that the map $\Phi_{(\varphi,y)}: M \to \mathbb{R}^{2m+1}$, defined by

$$\Phi_{(\varphi,y)}(x) = (y(x), y(\varphi(x)), \ldots, y(\varphi^{2m}(x))$$

is an embedding; by "smooth" we mean at least C^2.

Proof. We may, and do, assume that if x is a point with period k of φ, $k \le 2m+1$, all eigenvalues of $(d\varphi^k)_x$ are different and different from 1. Also we assume that no two different fixed points of φ are in the same level of y. For $\Phi_{(\varphi,y)}$ to be an immersion near a fixed point x, the co-vectors $(dy)_x, d(y\varphi)_x, \ldots, d(y\varphi^{2m})_x$ must span $T^*_x(M)$. This is the case for generic y if $d\varphi$ satisfies the above condition at each fixed point.

In the same way one proves that $\Phi_{(\varphi, y)}$ is generically an immersion and even an embedding when restricted to the periodic points with period $\leq 2m + 1$. So we may assume that for generic $(\bar{\varphi}, \bar{y})$ we have : $\Phi_{(\bar{\varphi}, \bar{y})}$, restricted to a compact neighbourhood V of the set of points with period $\leq 2m + 1$ is an embedding; for some neighbourhood $\mathfrak{U}$ of $(\bar{\varphi}, \bar{y})$, $\Phi_{(\varphi, y)}|V$ is an embedding whenever $(\varphi, y) \in \mathfrak{U}$. We want to show that for some $(\varphi, y) \in \mathfrak{U}$, arbitrarily near $(\bar{\varphi}, \bar{y})$, $\Phi_{(\varphi, y)}$ is an embedding.

For any point $x \in M$, which is not a point of period $\leq 2m + 1$ for $\bar{\varphi}$, the co-vectors $(d\bar{y})_x, d(\bar{y}\bar{\varphi})_x, d(\bar{y}\bar{\varphi}^2)_x, \ldots, d(\bar{y}\bar{\varphi}^{2m})_x \in T^*(M)$ can be perturbed independently by perturbing $\bar{y}$. Hence arbitrarily near $\bar{y}$ there is $\bar{\bar{y}}$ such that $(\bar{\varphi}, \bar{\bar{y}}) \in \mathfrak{U}$ and such that $\Phi_{(\bar{\varphi}, \bar{\bar{y}})}$ is an immersion. Then there is a positive ε such that whenever $0 < \rho(x, x') \leq \varepsilon$, $\Phi_{(\bar{\varphi}, \bar{\bar{y}})}(x) \neq \Phi_{(\bar{\varphi}, \bar{\bar{y}})}(x')$; ρ is some fixed metric on M. There is even a neighbourhood $\mathfrak{U}' \subset \mathfrak{U}$ of $(\bar{\varphi}, \bar{\bar{y}})$ such that for any $(\varphi, y) \in \mathfrak{U}'$, $\Phi_{(\varphi, y)}$ is an immersion and $\Phi_{(\varphi, y)}(x) \neq \Phi_{(\varphi, y)}(x')$ whenever $x \neq x'$ and $\rho(x, x') \leq \varepsilon$. From now on we also assume that each component of V has diameter smaller than ε.

Finally we have to show that in $\mathfrak{U}'$ we have a pair (φ, y) with $\Phi_{(\varphi, y)}$ injective. For this we need a finite collection $\{U_i\}_{i=1}^N$ of open subsets of M, covering the closure of $M \setminus \{\bigcap_{j=0}^{2m} \varphi^j(V)\}$, and such that :

(i) for each $i = 1, \ldots, N$ and $k = 0, 1 \ldots, 2m$, diameter $(\bar{\varphi}^{-k}(U_i)) < \varepsilon$;

(ii) for each $i, j = 1, \ldots, N$ and $k, l = 0, 1, \ldots, 2m$, $\bar{\varphi}^{-k}(U_i) \cap U_j \neq \emptyset$ and $\bar{\varphi}^{\ell}(U_i) \cap U_j \neq \emptyset$ imply that $k = \ell$;

(iii) for $\bar{\varphi}^j(x) \in M \setminus (\bigcup_i U_i)$, $j = 0, \ldots, 2m, x' \notin V$ and $\rho(x, x') > \varepsilon$, no two points of the sequence $x, \bar{\varphi}(x), \ldots, \bar{\varphi}^{2m}(x), x', \bar{\varphi}(x'), \ldots, \bar{\varphi}^{2m}(x')$ belong to the same U_i.

Note that (ii) implies, but is not implied by

(ii)' no two points of the sequence $x, \bar{\varphi}(x), \ldots, \bar{\varphi}^{2m}(x)$ belong to the same U_i .

We take a corresponding partition $\{\lambda_i\}$ of unity, i.e., λ_i is a non-negative function with support $\bar{U}_i$ and $\sum_{i=1}^N \lambda_i(x) = 1$ for all $x \in \overline{M \setminus V}$. Consider the map

$\Psi: M \times M \times \mathbb{R}^N \to \mathbb{R}^{2m+1} \times \mathbb{R}^{2m+1}$ which is defined in the following way $\Psi(x,x',\varepsilon_1,\ldots,\varepsilon_N) = (\Phi_{(\bar{\varphi},\bar{\bar{y}}_\varepsilon)}(x), \Phi_{(\bar{\varphi},\bar{\bar{y}}_\varepsilon)}(x'))$, where ε stands for $(\varepsilon_1,\ldots,\varepsilon_N)$ and $\bar{\bar{y}}_\varepsilon = \bar{\bar{y}} + \sum_{i=1}^{N} \varepsilon_i\lambda_i$. We define $W \subset M \times M$ as $W = \{(x,x') \in M \times M \mid \rho(x,x') \geq \varepsilon$ and not both x and x' are in $\mathrm{int}(V)\}$. Ψ, restricted to a small neighbourhood of $W\times\{0\}$ in $(M\times M)\times\mathbb{R}^N$, is transverse with respect to the diagonal of $\mathbb{R}^{2m+1} \times \mathbb{R}^{2m+1}$. This transversality follows immediately from all the conditions imposed on the covering $\{U_i\}_{i=1}^N$. From this transversality we conclude that there are arbitrarily small $\bar{\varepsilon} \in \mathbb{R}^N$ such that $\Psi(W\times\{\bar{\varepsilon}\}) \cap \Delta = \emptyset$. If also for such an $\bar{\varepsilon}, (\bar{\varphi},\bar{\bar{y}}_{\bar{\varepsilon}}) \in \mathfrak{U}'$ then $\Phi_{(\bar{\varphi},\bar{\bar{y}}_\varepsilon)}$ is injective and hence an embedding.

This proves that for a dense set of pairs (φ, y), $\Phi_{(\varphi,y)}$ is an embedding. Since the set of all embeddings is open in the set of all mappings, there is an open and dense set of pairs (φ, y), for which $\Phi_{(\varphi,y)}$ is an embedding. This proves the theorem.

Remark. This theorem also works for M non-compact if we restrict our observables to be proper functions.

Theorem 2. Let M be a compact manifold of dimension m. For pairs (X,y), X a smooth (i.e., C^2) vector field and y a smooth function on M, it is a generic property that $\Phi_{X,y}: M \to \mathbb{R}^{2m+1}$, defined by $\Phi_{X,y}(x) = (y(x), y(\varphi_1(x)), \ldots, y(\varphi_{2m}(x))$ is an embedding, where φ_t is the flow of X.

Proof. The proof of this theorem is almost the same as the proof of theorem 1. In this case we impose the following generic properties on X :

(i) if $X(x) = 0$ then all eigenvalues of $(d\varphi_1)_x : T_x(M) \to T_x(M)$ are different and different from 1;

(ii) no periodic integral curve of X has integer period $\leq 2m + 1$.

In this case φ_1 satisfies the same conditions as $\bar{\varphi}$ in the previous proof. The rest of the proof carries over immediately.

The next theorem is only included for the sake of completeness; it will not be used in the sequel of this paper.

Theorem 3. Let M be a compact manifold of dimension m. For pairs (X, y), X a smooth vector field and y a smooth function on M, it is a generic property that the map $\tilde{\Phi}_{X,y}: M \to \mathbb{R}^{2m+1}$, defined by

$$\tilde{\Phi}_{X,y}(x) = (y(x), \frac{d}{dt}(y(\varphi_t(x)))|_{t=0}, \ldots, \frac{d^{2m}}{dt^{2m}}(y(\varphi_t(x)))|_{t=0})$$

is an embedding. Here φ_t again denotes the flow of X; this time, smooth means at least C^{2m+1}.

Proof. Also this proof is quite analogous to that of theorem 1. First we may, and do, assume that a generic vector field X has the property that whenever X(x) = 0, all eigenvalues of $(dX)_x$ are different and different from zero. Sing(X) denotes the set of points where X is zero; this set is finite.

As in the proof of theorem 1, for such a vector field X the set of functions $y: M \to \mathbb{R}$ such that $\tilde{\Phi}_{X,y}$ is an immersion and, when restricted to a small neighbourhood of Sing(X), an embedding, is residual.

Finally, to obtain an embedding for $(X, \bar{y})$, $\bar{y}$ near y, we don't need an open covering in the present case. One can construct directly a map y_v, v in some finite dimensional vector space V, which is the analogue of y_ε, with the following properties :

(i) $y_0 = y$;

(ii) for $x \in$ Sing(X), the 1-jet of y_v is independent of v;

(iii) for $x, x' \notin$ Sing(X), $x \neq x'$ the map

$$j_x^{2m} \times j_{x'}^{2m} : V \to J_x^{2m}(M) \times J_{x'}^{2m}(M)$$

has a surjective derivative for all (x, x') in v = 0; $J_x^{2m}(M)$ is the vector space of 2m-jets of functions on M in x; $j_x^{2m}(v)$ is the 2m-jet of y_v in x.

Using y_v one defines a map

$$\tilde{\Phi}: M \times M \times V \to \mathbb{R}^{2m+1} \times \mathbb{R}^{2m+1} \text{ as before .}$$

The rest of the proof of theorem 1 now carries over to the present situation.

From the last three theorems it is clear how a dynamical system with time evolution φ_t and observable y is determined generically by the set of all functions $t \to y(\varphi_t(x))$. In practice the following situation may occur : we have a dynamical system with continuous time, but the value of the observable y is only determined for a discrete set $\{0, \alpha, 2\alpha, \dots\}$ of values of t; $\alpha > 0$. This happens e.g. in the measurements of the onset of turbulence [6, 8, 9, 10]. Also instead of all sequences of the form $\{y(\varphi_{i\alpha}(x))\}_{i=0}^{\infty}$, $x \in M$, we only know such a sequence for one, or a few values of x (depending on the number of experiments) and these sequences are not known entirely but only for $i = 1, \dots, \bar{N}$ for some finite but big $\bar{N}$ (in [6], $\bar{N} = 8192 = 2^{13}$). In this light we should know whether, under generic assumptions, the topology of, and dynamics in the positive limit set

$$L^+(x) = \{x' \in M \mid \exists t_i \to \infty \text{ with } \varphi_{t_i}(x) \to x'\}$$

of x is determined by the sequence $\{y(\varphi_{i,\alpha}(x))\}_{i=0}^{\infty}$. This question is treated in the next theorem and its corollary; in later sections we come back to the point that these sequences are only known up to some finite $\bar{N}$.

<u>Theorem 4.</u> Let M be a compact manifold, X a vector field on M with flow φ_t and p a point in M. Then there is a residual subset $C_{X,p}$ of positive real numbers such that for $\alpha \in C_{X,p}$, the positive limit sets of p for the flow φ_t of X and for the diffeomorphism φ_α are the same. In other words, for $\alpha \in C_{X,p}$ we have that each point $q \in M$ which is the limit of a sequence $\varphi_{t_i}(p)$, $t_i \in \mathbb{R}$, $t_i \to +\infty$, is the limit of a sequence $\varphi_{n_i.\alpha}(p)$, $n_i \in \mathbb{N}$, $n_i \to \infty$.

<u>Proof.</u> Take $q \in L^+(p)$. For ε a (small) positive real number define $C_{\varepsilon,q} = \{\alpha>0 \mid \exists\, n \in \mathbb{N}, \text{ such that } \rho(\varphi_{n.\alpha}(p),q) < \varepsilon\}$, ρ is some fixed metric on M. Clearly $C_{\varepsilon,q}$ is open; it is also dense. To prove this last statement we observe that for any $\bar{\alpha} > 0$ and $\bar{\varepsilon} > 0$, there is a point of $C_{\varepsilon,q}$ in $(\bar{\alpha}, \bar{\alpha} + \bar{\varepsilon})$ if and only if there is a $t \in (n.\bar{\alpha}, n.(\bar{\alpha}+\bar{\varepsilon}))$ with $\rho(\varphi_t(p),q) < \varepsilon$ for some integer n. The existence of such t follows from the fact that for big n the intervals $(n.\bar{\alpha}, n.(\bar{\alpha} + \bar{\varepsilon}))$ overlap (in the sense that for big n, $n.(\bar{\alpha} + \bar{\varepsilon}) > (n+1).\bar{\alpha}$ and the fact that there are arbitrary big values of t with $\rho(\varphi_t(p),q) < \varepsilon$.

Since $C_{\varepsilon,q}$ is open and dense we can take for $C_{X,p} \subset \mathbb{R}_{++}$ the following residual set $C_{X,p} = \bigcap_{i,j=1}^{\infty} C_{\frac{1}{i},q_j}$ where $\{q_j\}$ is a countable dense sequence in $L^+(p)$.

Corollary 5. Let M be a compact manifold of dimension m. We consider quadruples, consisting of a vector field X, a function y, a point p, and a positive real number α. For generic such (X, y, p, α) (more precisely : for generic (X, y) and α satisfying generic conditions depending on X and p), the positive limit set $L^+(p)$ is "diffeomorphic" with the set of limit points of the following sequence in $\mathbb{R}^{2m+1}$:

$$\{(y(\varphi_{k,\alpha}(p)), y(\varphi_{(k+1).\alpha}(p)), \ldots, y(\varphi_{(k+2m).\alpha}(p)))\}_{k=0}^{\infty} .$$

The meaning of "diffeomorphic" should be clear here : it means that there is a smooth embedding of M into $\mathbb{R}^{2m+1}$ mapping $L^+(p)$ bijectively to this set of limit points.

For further reference we remark that the metric properties of $\overline{\{\varphi_{i.\alpha}(p)\}_{i=0}^{\infty}} \subset M$, with $\{\varphi_{i.\alpha}(p)\}$ as a sequence of distinguished points are the same as $\overline{\{b_i\}_{i=1}^{\infty}} \subset \mathbb{R}^{2m+1}$ with $\{b_i\}$ as a sequence of distinguished points :

$$b_i = (y(\varphi_{i.\alpha}(p)), \ldots, y(\varphi_{(1+2m).\alpha}(p))) \in \mathbb{R}^{2m+1} .$$

These metric properties are the same in the sense that distances in M and the corresponding distances in $\mathbb{R}^{2m+1}$ have a quotient which is uniformly bounded and bounded away from zero.

3. Limit capacity and dimension.

There are several ways to define the notion of dimension for compact metric spaces. The definition which we use here gives the so-called limit capacity. Some information on this notion can be found in [14]. Since this limit capacity is not well known we treat here some of its basic properties.

Let (S, ρ) be a compact metric space. For $\varepsilon > 0$ we make the following definitions

- $s(S,\varepsilon)$ is the maximal cardinality of a subset of S such that no two points have distance less than ε; such a set is called a maximal ε-separated set;
- $r(S,\varepsilon)$ is the minimal cardinality of a subset of S such that S is the union of all the ε-neighbourhoods of its points; such a set is also called a minimal ε-spanning set.

Note that

$$r(S,\varepsilon) \leq s(S,\varepsilon) \leq r(S,\tfrac{\varepsilon}{2}) \; \ldots\ldots\ldots\ldots \; (1)$$

The first inequality follows from the fact that a maximal ε-separated set is ε-spanning. The second inequality follows from the fact that in an $\frac{\varepsilon}{2}$-neighbourhood of any point (of a minimal $\frac{\varepsilon}{2}$-spanning set) there can be at most one point of an ε-separated set.

Next we define the limiting capacity D(S) of S as

$$D(S) = \liminf_{\varepsilon\to 0} \frac{\ln (r(S,\varepsilon))}{-\ln \varepsilon} = \liminf_{\varepsilon\to 0} \frac{\ln (s(S,\varepsilon)}{-\ln \varepsilon} ;$$

the fact that the last two expressions are equal follows from (1). The notion of capacity, or rather ε-capacity, was originally used for $s(S,\varepsilon)$. This limit capacity is strongly related to the Hausdorff dimension, see [5 or 12], which is clear from the following equivalent definition. Let $\mathfrak{U}$ be a finite covering $\{U_i\}_{i\in I}$ of S. Then for $a > 0$ $D_{a,\mathfrak{U}} = \sum_{i\in I} (\text{diam } (U_i))^a$. Next we define $D_{a,\varepsilon}$ as the infinum of $D_{a,\mathfrak{U}}$ where $\mathfrak{U}$ runs over all finite covers of S each of whose elements has diameter ε. Notice that $D_{a,\varepsilon} \in [r\ (S,\varepsilon).\varepsilon^a, r(S,\frac{\varepsilon}{2}).\varepsilon^a]$. It is not hard to see that there is a unique number, which is in fact the limit capacity D(S), such that for $a > D(S)$, resp. $a < D(S)$, $\liminf_{\varepsilon\to 0} D_{a,\varepsilon}$ is zero, resp. infinite. This last definition of limit capacity goes over in the definition of Hausdorff dimension if we replace "each of whose elements has diameter ε" by "each of whose elements has diameter $\leq\varepsilon$.

For later reference we indicate a third definition of limit capacity. Let $\{b_i\}_{i=0}^{\infty}$ be some countable dense sequence in S. For $\varepsilon > 0$ we define the subset $J_\varepsilon \subset \mathbb{N}$ by :

$0 \in J_\varepsilon$; for $i > 0$:

$i \in J_\varepsilon$ if and only if for all j with $0 \leq j < i$ and $j \in J_\varepsilon$, we have $\rho(b_i, b_j) \geq \varepsilon$.

C_ε denotes the cardinality of J_ε. From these definitions it easily follows that whenever $0 < \varepsilon < \varepsilon'$,

$$r(S,\varepsilon') \leq C_\varepsilon \leq s(S,\varepsilon).$$

Hence we may also define D(S) by $D(S) = \liminf_{\varepsilon\to 0} \frac{\ln C_\varepsilon}{-\ln \varepsilon}$. From the literature, see [12], we know that the Hausdorff dimension is greater than or equal to the topological dimension and from the above considerations it is clear that the limit capacity is greater than or equal to the Hausdorff dimension. Both the Hausdorff dimension and the limit capacity depend on the metric (and not only on the topology). If however ρ and ρ' are metrics on S such that for some constant C and any $x, y \in S$, $C.\rho(x, y) \geq \rho'(x, y) \geq C^{-1}.\rho(x, y)$, then the limit capacity and the Hausdorff dimension are the same for the metrics ρ and ρ'. In this case the metrics ρ and ρ' are called metrically equivalent. Finally if S is a compact manifold with a metric ρ

which is metrically equivalent with a metric induced by a Riemannian structure, then the limit capacity equals the topological dimension.

Examples where the Hausdorff dimension is different from the limit capacity were given by Mañé [14]. It seems to be an open question whether a difference between Hausdorff dimension and limit capacity can occur for positive limit sets of smooth vector fields on compact manifolds; if the answer is no then for all our purposes the Hausdorff dimension and the limit capacity are the same.

Contrary to the topological dimension, the Hausdorff dimension and the limit capacity need not be integers. If we take for example for $\tilde{S}$ a Cantor set in $\mathbb{R}$, define as $\tilde{S} = \bigcap_{i=0}^{\infty} S_i$ where : $S_0 = [0,1]$; $S_{i+1} \subset S_i$; S_i has 2^i intervals of length α^i, $\alpha < \frac{1}{2}$; and S_{i+1} is obtained fron S_i by removing in the middle of each segment of S_i a segment of length $\alpha^i.(1-2\alpha)$. We takeas countable dense subset S the union of the left endpoints of the intervals of S_i for all i. If we compute C_ε for $\varepsilon = \alpha^i$, we find $C_{\alpha^i} = 2^i$. From this it is not hard to deduce that

$$D(S) = -\frac{\ln 2}{\ln \alpha} .$$

In determining the limit capacity of a closed subset of a compact manifold it is important to note that there is only one metric equivalence class on the manifold which contains a metric induced by a Riemannian structure. Limit capacity is always assumed to be defined with respect to a metric in this class.

4. Determination of dimension and entropy.

We consider the following situation : M is a compact manifold with a smooth vector field X, a smooth function $y:M \to \mathbb{R}$ and a point $p \in M$. We assume that p is part of its own positive limit set $L^+(p)$; also we assume that for some fixed $\alpha > 0$, the sequence $\{\varphi_{i\alpha}(p)\}_{i=0}^{\infty}$ is dense in $L^+(p)$ and that (φ_α, y) is generic in the sense of theorem 1; φ_t denotes the flow of X. Note that the only non-generic assumption we made on (M, X, y, p, α) is $p \in L^+(p)$. This assumption can in some sense be justified : if the orbit $\varphi_t(q)$ goes to an "attractor for $t \to \infty$", then if we replace q by $\varphi_T(q) = \tilde{q}$, $T \gg 1$, it is almost true that $\tilde{q} \in L^+(\tilde{q})$. So the assumption $p \in L^+(p)$ can be seen as a way to include in the description (see the introduction) the fact that we can only start measuring after the experiment is already going for quite a time (with fixed Ω).

In this situation we have the sequence $\{a_i = y(\varphi_{i.\alpha}(p))\}_{i=0}^{\infty}$ which represents the experimental output (so for the moment we assume the experiment has been carried out for an infinite amount of time). From this sequence we obtain subsets $J_{n,\varepsilon} \subset \mathbb{N}$ by the following inductive definition (see also the end of section 3) :

$$0 \in J_{n,\varepsilon}; \text{ for } i > 0 :$$

$$i \in J_{n,\varepsilon} \text{ if and only if for all } 0 \leq j < i, \text{ with } j \in J_{n,\varepsilon},$$
$$\max(|a_i - a_j|, |a_{i+1} - a_{j+1}|, \ldots, |a_{i+n} - a_{j+n}|) \geq \varepsilon .$$

$C_{n,\varepsilon}$ denotes the cardinality of $J_{n,\varepsilon}$.

Main theorem. The limit capacity of $L^+(p)$ equals

$$D(L^+(p)) = \lim_{n\to\infty} (\liminf_{\varepsilon\to 0} \frac{\ln C_{n,\varepsilon}}{-\ln \varepsilon}),$$

where $\lim_{n\to\infty}$ reaches the limit value for every $n \geq 2(\dim(M))$.

The topological entropy of $\varphi_\alpha | L^+(p)$ equals

$$H(L^+(p)) = \lim_{\varepsilon\to 0} (\limsup_{n\to 0} (\frac{\ln C_{n,\varepsilon}}{n})) ,$$

where $\lim_{\varepsilon\to 0}$ often (e.g. if $L^+(p)$ is an expansive basic set [3]) reaches the limit value for every $0 < \varepsilon < \varepsilon_0$ for some ε_0.

Proof. We take some $N \geq 2.\dim(M)$. The map $\Phi : M \to \mathbb{R}^{N+1}$, defined by

$$q \mapsto (y(q), y(\varphi_\alpha(q)), \ldots, y(\varphi_{N.\alpha}(q))$$

is an embedding. On $\Phi(M)$ we use the metric

$$\rho((x_0, \ldots, x_N), (x'_0, \ldots, x'_N)) = \max_i |x_i - x'_i| .$$

This metric is equivalent in the metric sense to any metric on $\Phi(M)$ derived from a Riemannian metric. Se we may use ρ to compute $D(L^+(p)) = D(\Phi(L^+(p)))$.

The first statement in the main theorem now follows by applying section 3 to to the sequence $\{\varphi_{i.\alpha}(p)\}_{i=1}^{\infty}$ in $L^{+}(p)$.

Next we come to the determination of the topological entropy of $\varphi_{\alpha}|L^{+}(p)$. For this we have to find the cardinality of a minimal ε-spanning set of orbits of length n, see Bowen [2]. A minimal ε-spanning set of orbits of length n of φ_{α} is a finite set $\{q_i\}_{i\in I}$ in $L^{+}(p)$ such that :

(i) for every $q \in L^{+}(p)$ there is some $i_0 \in I$ such that $\rho(\varphi_{i.\alpha}(q), \varphi_{i.\alpha}(q_{i_0})) < \varepsilon$ for all $0 \leq i \leq n$;

(ii) among all subsets of $L^{+}(p)$ satisfying (i), $\{q_i\}_{i\in I}$ has minimal cardinality.

Let $r(n,\varepsilon)$ be this cardinality. There is also a maximal cardinality of ε-separated orbits of length n, denoted by $s(n,\varepsilon)$ (see [1]). The entropy can now be defined as

$$H(L^{+}(p)) = \lim_{\varepsilon\to 0} (\lim_{n\to\infty} \sup(\frac{\ln\ r(n,\varepsilon)}{n})) = \lim_{\varepsilon\to 0} (\lim_{n\to\infty} \sup(\frac{\ln(s(n,\varepsilon))}{n})).$$

If we use the metric ρ, defined above, we can replace $s(n,\varepsilon)$ or $r(n,\varepsilon)$ by $C_{n+N,\varepsilon}$; see section 3. From this we obtain :

$$H(L^{+}(p)) = \lim_{\varepsilon\to 0} (\lim_{n\to\infty} \sup (\frac{\ln(C_{n+N,\varepsilon})}{n})) = \lim_{\varepsilon\to 0} (\lim_{n\to\infty} \sup(\frac{\ln(C_{n,\varepsilon})}{n})) .$$

Observation. Application of this main theorem to the output of the Taylor-Couette experiment, described in the introduction, gives some complications due to the fact that $\{a_i\}_{i=1}^{\bar{N}}$ is finite in this case. For such a finite sequence one should proceed as follows: for n,ε,m with $n + m \leq \bar{N}$ we define subsets $J_{n,\varepsilon,m} \subset \mathbb{N}$ as follows :

(i) $0 \in J_{n,\varepsilon,m}$; for $i > 0$:

(ii) $i \in J_{n,\varepsilon,m}$ if and only if both :

  (a) $i \leq m$;

  (b) for all $j < i$, $j \in J_{n,\varepsilon,m}$, $\max_{0\leq k\leq n} |a_{i+k} - a_{j+k}| \geq \varepsilon$.

$C_{n,\varepsilon,m}$ denotes the cardinality of $J_{n,\varepsilon,m}$. For $\bar{N} = \infty$, one would have $\lim_{m\to\infty} C_{n,\varepsilon,m} = C_{n,\varepsilon}$. $C_{n,\varepsilon,m}$ is non-decreasing in m. Hence it seems reasonable to take $C_{n,\varepsilon,\bar{N}-n}$ as an approximation of $C_{n,\varepsilon}$ provided the difference between $C_{n,\varepsilon,\bar{N}-n}$ and say, $C_{n,\varepsilon,[\frac{1}{2}(\bar{N}-n)]}$

is sufficiently small, say of the order of 1 or 2%. In this way we have the possibility of calculating $C_{n,\varepsilon}$ in a certain region of the (n,ε)- plane; also one should consider these values for $C_{n,\varepsilon}$ only reliable if ε is well above the expected errors in the measurement. From these numerical values for $C_{n,\varepsilon}$ one should decide, on the basis of the main theorem what the values of $D(L^+(p))$ and $H(L^+(p))$ are or whether the limits defining these values "do not exist numerically".

If, in the calculation of $D(L^+(p))$, the $\lim_{n\to\infty}$ would have the tendency of going to infinity, this would imply that representing the evolution on a finite dimensional manifold is a mistake. If on the other hand this limit would go to a non-integer, this would be evidence in favour of a strange attractor. Namely, as we have seen in section 3, for a Cantor set C we may have D(C) a non integer, and strange attractors have in general a Cantor set like structure, e.g. see [3].

If the experimental data do not clearly indicate the limits in the calculation of $D(L^+(p))$ and $H(L^+(P))$ to exist and to be finite, then both the Landau-Lifschitz and the Ruelle-Takens picture are to be rejected as explanation of the experimental data.

Final remarks.

1. It does not seem to be known whether, for differentiable dynamical systems the "inf" and "sup" in the definition of limit capacity and entropy can be omitted. If they can omitted, one has a better test on the validity of the assumptions "finite dimensional and deterministic" : also the first limit has "to exist numerically".

2. Yorke pointed out to the author that he and others had made calculations of limit capacities in relation with a conjecture on Lyapunov numbers and dimension for attractors, see [7]. His calculating scheme is different from ours and probably faster. The calculations indicate that the computing time rapidly increases with dimension, which probably also holds for our computing scheme.

3. It should be noticed that the defining formulas for dimension and entropy become more alike when we write them in the following form .

$$D(L^+(p)) = \lim_{n\to\infty} (\lim_{\varepsilon\to 0}\inf (\frac{\ln C_{n,\varepsilon}}{n - \ln \varepsilon})$$

$$H(L^+(p)) = \lim_{\varepsilon\to 0} (\lim_{n\to\infty}\sup (\frac{\ln C_{n,\varepsilon}}{n - \ln \varepsilon})) .$$

If we denote $\frac{\ln C_{n,\varepsilon}}{n - \ln \varepsilon}$ by $Z(n, -\ln \varepsilon)$ and regard both n and $-\ln \varepsilon$ as continuous variables one can see from a few examples (Anosov automorphisms on the torus and horseshoes) that often $\lim_{\substack{\alpha,\beta\to\infty \\ \alpha/\beta\to\gamma}} Z(\alpha,\beta)$ exists for all positive γ, forming a one-parameter family of "topologically invariants" connecting entropy with limit capacity. It would be interesting to investigate the existence of these limits for more general attractors. This might be connected with the above mentioned conjecture of Yorke.

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