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H^+-EIGENVALUES OF LAPLACIAN AND SIGNLESS LAPLACIAN TENSORS*

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Abstract. We propose a simple and natural definition for the Laplacian and the signless Laplacian tensors of a uniform hypergraph. We study their H^+-eigenvalues, i.e., H-eigenvalues with nonnegative H-eigenvectors, and H^{++}-eigenvalues, i.e., H-eigenvalues with positive H-eigenvectors. We show that each of the Laplacian tensor, the signless Laplacian tensor, and the adjacency tensor has at most one H^{++}-eigenvalue, but has several other H^+-eigenvalues. We identify their largest and smallest H^+-eigenvalues, and establish some maximum and minimum properties of these H^+-eigenvalues. We then define analytic connectivity of a uniform hypergraph and discuss its application in edge connectivity.

Key words. Laplacian tensor, signless Laplacian tensor, uniform hypergraph, H^+-eigenvalue.

AMS subject classifications. 05C65, 15A18.

1. Introduction

Recently, several papers appeared on spectral hypergraph theory via tensors [3, 6, 10, 15, 16, 19, 24, 25, 26]. These works are all on uniform hypergraphs [1]. In 2008, Lim [16] proposed to study spectral hypergraph theory via eigenvalues of tensors. In 2009, Bulò and Pelillo [3] gave new bounds on the clique number of a graph based on analysis of the largest eigenvalue of the adjacency tensor of a uniform hypergraph. In 2012, Hu and Qi [10] proposed a definition for the Laplacian tensor of an even uniform hypergraph, and analyzed its connection with edge and vertex connectivity. In the same year, Cooper and Dutle [6] analyzed the eigenvalues of the adjacency tensor (hypermatrix) of a uniform hypergraph, and proved a number of natural analogs of basic results in spectral graph theory. Li, Qi, and Yu [15] proposed another definition for the Laplacian tensor of an even uniform hypergraph, established a variational formula for its second smallest Z-eigenvalue, and used it to provide lower bounds for the bipartition width of the hypergraph. In [24, 26], Xie and Chang proposed a definition for the signless Laplacian tensor of an even uniform hypergraph, studied its largest and smallest H-eigenvalues and Z-eigenvalues, and its applications in the edge cut and the edge connectivity of the hypergraph. They also studied the largest and the smallest Z-eigenvalues of the adjacency tensor of a uniform hypergraph in [25]. In [19], Pearson and Zhang studied the H-eigenvalues and the Z-eigenvalues of the adjacency tensor of a uniform hypergraph.

Precisely speaking, the tensors mentioned above may be called hypermatrices. In physics and mechanics, tensors are physical quantities, while hypermatrices are multi-dimensional arrays. In geometry, a tensor to a hypermatrix is like a linear transformation to a matrix - the former objects are defined without choosing bases [21]. However, for most papers in tensor decomposition, spectral theory of tensors and spectral hypergraph theory, as most papers cited in this paper, the word "tensors" are used for those multi-dimensional arrays. Following this habit, we use the word

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"tensors" in this paper.

A uniform hypergraph is also called a k-graph [1, 2]. Let $G=(V,E)$ be a k-graph, where $V=\{1,2,\ldots,n\}$ is the vertex set, $E=\{e_1,e_2,\ldots,e_m\}$ is the edge set, $e_p\subset V$, and $|e_p|=k$ for $p=1,\ldots,m$, and $k\geq 2$. If $k=2$, then G is an ordinary graph. We assume that $e_p\neq e_q$ if $p\neq q$. Two vertices are called *adjacent* if they are in the same edge. Two vertices i and j are called *connected* if either i and j are adjacent, or there are vertices $i_1,\ldots,i_s$ such that i and i_1, i_k and j, i_r and i_{r+1} for $r=1,\ldots,s-1$, are adjacent respectively. A k-graph G is called connected if any pair of its vertices are connected. The **adjacency tensor** $\mathcal{A}=\mathcal{A}(G)$ of G, is a kth order n-dimensional symmetric tensor, with $\mathcal{A}=(a_{i_1i_2\cdots i_k})$, where $a_{i_1i_2\cdots i_k}=\frac{1}{(k-1)!}$ if $(i_1,i_2,\ldots,i_k)\in E$, and 0 otherwise. Thus, $a_{i_1i_2\cdots i_k}=0$ if two of its indices are the same. For $i\in V$, its degree $d(i)$ is defined as $d(i)=|\{e_p:i\in e_p\in E\}|$. We assume that every vertex has at least one edge. Thus, $d(i)>0$ for all i. The **degree tensor** $\mathcal{D}=\mathcal{D}(G)$ of G, is a kth order n-dimensional diagonal tensor, with its ith diagonal entry as $d(i)$. We denote the maximum degree, the minimum degree, and the average degree of G by Δ, δ, and $\bar{d}$ respectively. If $\bar{d}=\Delta=d$, then G is a regular graph, called a d-regular k-graph.

The definition of the adjacency tensor is natural. It was studied in [3, 6, 25]. On the other hand, the definitions of Laplacian and signless Laplacian tensors in [10, 15, 24, 26] are based upon some forms of sums of k-th powers. They are not simple and natural, and only work when k is even.

In this paper, we propose a simple and natural definition for the Laplacian and the signless Laplacian tensors of a k-graph G. Recall that when $k=2$, the Laplacian matrix and the signless Laplacian matrix of G are defined as $\mathcal{L}=\mathcal{D}-\mathcal{A}$ and $\mathcal{Q}=\mathcal{D}+\mathcal{A}$ [2]. Many results of spectral graph theory are based upon this definition. Thus, for $k\geq 3$, we propose to define the **Laplacian tensor** and the **signless Laplacian tensor** of G simply by $\mathcal{L}=\mathcal{D}-\mathcal{A}$ and $\mathcal{Q}=\mathcal{D}+\mathcal{A}$. This definition is simple and natural, and is closely related to the adjacency tensor $\mathcal{A}$. Furthermore, the signless Laplacian tensor $\mathcal{Q}$ is a symmetric nonnegative tensor, while the Laplacian tensor $\mathcal{L}$ is the limit of symmetric M-tensors in the sense of [29]. M-tensors are closely related with nonnegative tensors [29]. Thus, we may use the recently developed theory and algorithms on eigenvalues of nonnegative tensors [4, 5, 8, 9, 17, 18, 22, 27, 28] to study $\mathcal{L}$ and $\mathcal{Q}$.

We discover that $\mathcal{L}$ and $\mathcal{Q}$ have very nice spectral properties. They are not irreducible in the sense of [4]. But they are weakly irreducible in the sense of [8] if G is connected. When $k\geq 3$, each of them has at least $n+1$ H-eigenvalues with nonnegative H-eigenvectors. We call such H-eigenvalues **H$^+$-eigenvalues**. Furthermore, each of them has at most one H$^+$-eigenvalue with a positive eigenvector. We call such an H$^+$-eigenvalue an **H^{++}-eigenvalue**.

The remainder of this paper is distributed as follows. In the next section, we review the definition and properties of eigenvalues and H-eigenvalues of tensors, and introduce H$^+$-eigenvalues and H^{++}-eigenvalues. We study H$^+$-eigenvalues of $\mathcal{A}$, $\mathcal{L}$, and $\mathcal{Q}$ in Section 3. We show that each of $\mathcal{A}$, $\mathcal{L}$, and $\mathcal{Q}$ has at most one H^{++}-eigenvalue, but has several other H$^+$-eigenvalues. In Section 4, we study the smallest H-eigenvalue of $\mathcal{L}$, and its link with the connectedness of G. We identify the largest H$^+$-eigenvalue of $\mathcal{L}$, and establish a maximum property of this H$^+$-eigenvalue in Section 5. We establish some maximum properties of the largest H-eigenvalues of $\mathcal{Q}$ and $\mathcal{A}$, and discuss methods for computing them in Section 6. In Section 7, we identify the smallest H$^+$-eigenvalue of $\mathcal{Q}$, establish a minimum property of this H$^+$-eigenvalue, and discuss its applications in edge connectivity and maximum cut. In Section 8, we define **analytic connectivity** of G as a minimum quantity related to $\mathcal{L}$, and discuss its application in edge connectivity. Some final remarks are made in Section 9.

Denote by $\mathbf{1}$ the all 1 n-dimensional vector, $\mathbf{1}_j = 1$ for $j = 1, \ldots, n$. Denote by $\mathbf{e}^{(i)}$ the ith unit vector in $\Re^n$, i.e., $\mathbf{e}^{(i)}_j = 1$ if $i = j$ and $\mathbf{e}^{(i)}_j = 0$ if $i \neq j$, for $i, j = 1, \ldots, n$. For a vector x in $\Re^n$, we define its support as $\mathrm{supp}(x) = \{i \in V : x_i \neq 0\}$. Denote the set of all nonnegative vectors in $\Re^n$ by $\Re^n_+$ and the set of all positive vectors in $\Re^n$ by $\Re^n_{++}$. For a kth order n-dimensional tensor $\mathcal{C} = (c_{i_1 \cdots i_k})$, $|\mathcal{C}|$ is a kth order n-dimensional tensor $|\mathcal{C}| = (|c_{i_1 \cdots i_k}|)$. If both $\mathcal{C} = (c_{i_1 \cdots i_k})$ and $\mathcal{B} = (b_{i_1 \cdots i_k})$ are real kth order n-dimensional tensors, and $b_{i_1 \cdots i_k} \leq c_{i_1 \cdots i_k}$ for $i_1, \ldots, j_k = 1, \ldots, n$, then we write $\mathcal{B} \leq \mathcal{C}$. We use $\mathcal{J}$ to denote the kth order n-dimensional tensor with all of its entries being 1.

2. H$^+$-eigenvalues and H^{++}-eigenvalues

In this section, we will review the definition and properties of eigenvalues and H-eigenvalues of tensors in [20], introduce H$^+$-eigenvalues and H^{++}-eigenvalues, and review the Perron-Frobenius Theorem for nonnegative tensors in [4, 8, 27]. We also discuss the reducibility and weak irreducibility of $\mathcal{L}$ and $\mathcal{Q}$ in this section.

Consider a real kth order n-dimensional tensor $\mathcal{T} = (t_{i_1 \cdots i_k})$. Let $x \in C^n$. Then

$$\mathcal{T}x^k = \sum_{i_1, \ldots, i_k = 1}^{n} t_{i_1 \cdots i_k} x_{i_1} \cdots x_{i_k},$$

and $\mathcal{T}x^{k-1}$ is a vector in C^n, with its ith component defined by

$$\left(\mathcal{T}x^{k-1}\right)_i = \sum_{i_2, \ldots, i_k = 1}^{n} t_{i i_1 \cdots i_k} x_{i_2} \cdots x_{i_k}.$$

Let r be a positive integer. Then $x^{[r]}$ is a vector in C^n, with its ith component defined by x_i^r. We say that $\mathcal{T}$ is symmetric if its entries $t_{i_1 \cdots i_k}$ are invariant under any permutation of its indices.

Suppose that $x \in C^n$, $x \neq 0$, $\lambda \in C$, and x and λ satisfy

$$\mathcal{T}x^{k-1} = \lambda x^{[k-1]}. \tag{2.1}$$

Then we call λ an **eigenvalue** of $\mathcal{T}$, and x its corresponding **eigenvector**. From (2.1), we may see that if λ is an eigenvalue of $\mathcal{T}$ and x is its corresponding eigenvector, then

$$\lambda = \frac{(\mathcal{T}x^{k-1})_j}{x_j^{k-1}}, \tag{2.2}$$

for some j with $x_j \neq 0$. In particular, if x is real, then λ is also real. In this case, we say that λ is an **H-eigenvalue** of $\mathcal{T}$ and x is its corresponding **H-eigenvector**. If $x \in \Re^n_+$, then we say that λ is an **H$^+$-eigenvalue** of $\mathcal{T}$. If $x \in \Re^n_{++}$, then we say that λ is an **H^{++}-eigenvalue** of $\mathcal{T}$. If λ is an **H$^+$-eigenvalue** but not an **H^{++}-eigenvalue** of $\mathcal{T}$, then we say that λ is a **strict H$^+$-eigenvalue** of $\mathcal{T}$.

We say that $\mathcal{T}$ is positive definite (semi-definite) if $\mathcal{T}x^k > 0$ ($\mathcal{T}x^k \geq 0$) for all $x \in \Re^n$, $x \neq 0$. Clearly, $\mathcal{T}$ is positive definite only if k is even, and when k is odd, $\mathcal{T}$ is positive semi-definite only if $\mathcal{T}$ is the zero tensor.

Note that (2.1) is a homogeneous system of x, with n variables and n equations. We may regard these variables as taking values in the complex field. According to algebraic geometry [7], the *resultant* of (2.1) is a polynomial in the coefficients of (2.1), hence a polynomial in λ, which vanishes if and only if (2.1) has a nonzero solution x. Denote this polynomial by $\phi_{\mathcal{T}}(\lambda)$, and call it the **characteristic polynomial** of $\mathcal{T}$.

The main properties of eigenvalues and H-eigenvalues of a real kth order n-dimensional symmetric tensor in [20] are summarized in the following theorem.

THEOREM 2.1. **(Eigenvalues of Real Symmetric Tensors). (Qi 2005)** *The following hold for the eigenvalues of a real kth order n-dimensional symmetric tensor $\mathcal{T}$:*

(a). A number $\lambda \in C$ is an eigenvalue of $\mathcal{T}$ if and only if it is a root of the characteristic polynomial $\phi_{\mathcal{T}}$. Hence, we regard the multiplicity of an eigenvalue λ of $\mathcal{T}$ as its multiplicity as a root of $\phi_{\mathcal{T}}$.

(b). The number of eigenvalues of $\mathcal{T}$, counting their multiplicities, is $n(k-1)^{n-1}$. Their product is equal to det($\mathcal{T}$), the resultant of $\mathcal{T}x^{k-1}=0$.

(c). The sum of all the eigenvalues of T is

$$(k-1)^{n-1}\mathrm{tr}(\mathcal{T}),$$

where tr($\mathcal{T}$) denotes the sum of the diagonal entries of $\mathcal{T}$.

(d). If k is even, then $\mathcal{T}$ always has H-eigenvalues. $\mathcal{T}$ is positive definite (positive semi-definite) if and only if all of its H-eigenvalues are positive (nonnegative).

(e). The eigenvalues of $\mathcal{T}$ lie in the following n disks:

$$|\lambda - t_{ii\cdots i}| \le \sum \{|t_{ii_2\cdots i_k}| : i_2,\ldots,i_k = 1,\ldots,n, (i_2,\ldots,i_k) \ne (i,\ldots,i)\},$$

for $i=1,\ldots,n$.

A substantial portion of this theorem is still true when $\mathcal{T}$ is not symmetric. As we are only concerned with real symmetric tensors, we do not go into this in detail.

We call $\sum\{t_{ii_2\cdots i_k} : i_2,\ldots,i_k = 1,\ldots,n, (i_2,\ldots,i_k) \ne (i,\ldots,i)\}$ the ith **off-diagonal sum** of $\mathcal{T}$.

The set of eigenvalues of $\mathcal{T}$ is called the **spectrum** of $\mathcal{T}$. The largest modulus of the eigenvalues of $\mathcal{T}$ is called the **spectral radius** of $\mathcal{T}$, denoted by $\rho(\mathcal{T})$.

Following [4], $\mathcal{T}$ is called **reducible** if there exists a proper nonempty subset I of $\{1,\ldots,n\}$ such that

$$t_{i_1\cdots i_k} = 0, \quad \forall i_1 \in I, \quad \forall i_2,\ldots,i_k \notin I.$$

If $\mathcal{T}$ is not reducible, then we say that $\mathcal{T}$ is **irreducible**. If we take $I=\{1,\ldots,n-1\}$, it is evident that $\mathcal{L}$ and $\mathcal{Q}$ are reducible.

Suppose that $\mathcal{T}=(t_{i_1\cdots i_k})$ is a kth order n-dimensional tensor. Construct a graph $\hat{G}(\mathcal{T})=(\hat{V},\hat{E})$, where $\hat{V}=\cup_{j=1}^n V_j, V_j$ is a copy of $\{1,\ldots,n\}$, for $j=1,\ldots,n$. Assume that $i_j \in V_j, i_l \in V_l, j \ne l$. The edge $(i_j,i_l) \in \hat{E}$ if and only if $t_{i_1\cdots i_k} \ne 0$ for some $k-2$ indices $\{i_1,\ldots,i_k\} \setminus \{i_j,i_l\}$. The tensor $\mathcal{T}$ is called **weakly irreducible** if $\hat{G}(\mathcal{T})$ is connected. The original definition in [8] for weakly irreducible tensors only applies to nonnegative tensors. Here we remove the nonnegativity restriction. As observed in [8], an irreducible tensor is always weakly irreducible. Very recently, Pearson and Zhang [19] proved that the adjacency tensor $\mathcal{A}$ is weakly irreducible if and only if the k-graph G is connected. Clearly, if the adjacency tensor $\mathcal{A}$ is weakly irreducible, then $\mathcal{L}$ and $\mathcal{Q}$ are weakly irreducible. This shows that if G is connected, then $\mathcal{A}$, $\mathcal{L}$, and Q are weakly irreducible.

If the entries $t_{i_1\cdots i_k}$ are nonnegative, $\mathcal{T}$ is called a **nonnegative tensor**. There is a rich theory on eigenvalues of a nonnegative tensor [4, 5, 8, 17, 18, 27, 28]. We now summarize the Perron-Frobenius theorem for nonnegative tensors, established in [4, 8, 27]. With the new definitions of H$^+$-eigenvalues and H^{++}-eigenvalues, this theorem can be stated concisely.

THEOREM 2.2. **(The Perron-Frobenius Theorem for Nonnegative Tensors).**

(1) **(Yang and Yang 2010).** *If $\mathcal{T}$ is a nonnegative tensor of order k and dimension n, then $\rho(\mathcal{T})$ is an H^+-eigenvalue of $\mathcal{T}$.*

(2) **(Friedland, Gaubert, and Han 2011).** *If furthermore $\mathcal{T}$ is weakly irreducible, then $\rho(\mathcal{T})$ is the unique H^{++}-eigenvalue of $\mathcal{T}$, with the unique eigenvector $x\in\Re^n_{++}$, up to a positive scaling coefficient.*

(3) **(Chang, Pearson, and Zhang 2008).** *If moreover $\mathcal{T}$ is irreducible, then $\rho(\mathcal{T})$ is the unique H^+-eigenvalue of $\mathcal{T}$.*

The tensors $\mathcal{L}$ and $\mathcal{Q}$ are reducible. This permits the possibility that they have some strict H$^+$ eigenvalues. In the next five sections, we will study their H$^+$ eigenvalues.

3. H$^+$-eigenvalues of $\mathcal{A}$, $\mathcal{L}$, and Q

Theorem 2.1 establishes some basic properties of eigenvalues of the adjacency tensor $\mathcal{A}$, the Laplacian tensor $\mathcal{L}$, and the signless Laplacian tensors $\mathcal{Q}$. Note that they are all real kth order n-dimensional symmetric tensors. Both $\mathcal{A}$ and $\mathcal{Q}$ are nonnegative tensors. The diagonal entries of $\mathcal{A}$ are zero. The ith diagonal entry of $\mathcal{L}$ and $\mathcal{Q}$ is $d_i>0$. All the off-diagonal entries of $\mathcal{A}$ and $\mathcal{Q}$ are nonnegative. All the off-diagonal entries of $\mathcal{L}$ are non-positive. The ith off-diagonal sum of $\mathcal{A}$ and $\mathcal{Q}$ is d_i. The ith off-diagonal sum of $\mathcal{L}$ is $-d_i$.

THEOREM 3.1. **(Basic Properties of Eigenvalues of $\mathcal{A}$, $\mathcal{L}$, and $\mathcal{Q}$).** *Assume that $k\geq 3$. The following conclusions hold for eigenvalues of $\mathcal{A}$, $\mathcal{L}$, and $\mathcal{Q}$.*

(a). A number $\lambda\in C$ is an eigenvalue of $\mathcal{A}$ (respectively, $\mathcal{L}$ or $\mathcal{Q}$) if and only if it is a root of the characteristic polynomial $\phi_{\mathcal{A}}$ (respectively, $\phi_{\mathcal{L}}$ or $\phi_{\mathcal{Q}}$).

(b). The number of eigenvalues of $\mathcal{A}$ (respectively, $\mathcal{L}$ or $\mathcal{Q}$) is $n(k-1)^{n-1}$. Their product is equal to det($\mathcal{A}$) (respectively, det($\mathcal{L}$) or det($\mathcal{Q}$)).

(c). The sum of all the eigenvalues of $\mathcal{A}$ is zero. The sum of all the eigenvalues of $\mathcal{L}$ or $\mathcal{Q}$ is $(k-1)^{n-1}\sum_{i=1}^n d_i=k(k-1)^{n-1}m$.

(d). The eigenvalues of $\mathcal{A}$ lie in the disk $\{\lambda:|\lambda|\leq\Delta\}$. The eigenvalues of $\mathcal{L}$ and $\mathcal{Q}$ lie in the disk $\{\lambda:|\lambda-\Delta|\leq\Delta\}$.

(e). $\mathcal{L}$ and $\mathcal{Q}$ are positive semi-definite when k is even.

Proof. The conclusions (a), (b), (c), and (d) follow directly from Theorem 2.1 (a), (b), (c), and (e), and the basic structure of $\mathcal{A}$, $\mathcal{L}$, and $\mathcal{Q}$. By (d), the real parts of all the eigenvalues of $\mathcal{L}$ and $\mathcal{Q}$ are nonnegative. Then (e) follows from Theorem 2.1 (d). ∎

We now discuss H$^+$-eigenvalues of $\mathcal{L}$.

THEOREM 3.2. **(H$^+$-Eigenvalues of $\mathcal{L}$).** *Assume that $k\geq 3$. For $j=1,\cdots,n$, d_j is a strict H^+-eigenvalue of $\mathcal{L}$ with H-eigenvector $\mathbf{e}^{(j)}$. Zero is the unique H^{++}-eigenvalue of $\mathcal{L}$ with H-eigenvector $\mathbf{1}$, and is the smallest H-eigenvalue of $\mathcal{L}$.*

Proof. A real number μ is an H-eigenvalue of $\mathcal{L}$, with H-eigenvector x, if and only if $x\in\Re^n$, $x\neq 0$, and $\mathcal{L}x^{k-1}=\mu x^{[k-1]}$, i.e.,

$$d_i x_i^{k-1}-\sum\left\{\frac{1}{(k-1)!}x_{i_2}\cdots x_{i_k}:(i,i_2,\cdots,i_k)\in E\right\}=\mu x_i^{k-1}, \tag{3.1}$$

for $i=1\cdots,n$. We now may easily verify that for $j=1,\cdots,n$, d_j is an H$^+$-eigenvalue of $\mathcal{L}$ with H-eigenvector $\mathbf{e}^{(j)}$, and zero is an H^{++}-eigenvalue of $\mathcal{L}$ with H-eigenvector $\mathbf{1}$. By Theorem 3.1 (d), the real parts of all the eigenvalues of $\mathcal{L}$ are nonnegative. Thus, zero is the smallest H-eigenvalue of $\mathcal{L}$. Assume that x is a positive H-eigenvector of $\mathcal{L}$, associated with an H-eigenvalue μ. By Theorem 3.1 (d), $\mu\geq 0$. Let $x_j=\min_i\{x_i\}$. By (3.1), we have

$$\mu=d_j-\sum\left\{\frac{1}{(k-1)!}\frac{x_{i_2}}{x_j}\cdots\frac{x_{i_k}}{x_j}:(j,i_2,\cdots,i_k)\in E\right\}\leq d_j-d_j=0.$$

This shows that $\mu=0$. Thus, zero is the unique H^{++} eigenvalue of $\mathcal{L}$, and d_j is a strict H$^+$ eigenvalue of $\mathcal{L}$, for $j=1,\cdots,n$. ∎

As in spectral graph theory [2], we may call eigenvalues (respectively, H-eigenvalue or H$^+$-eigenvalue or H^{++}-eigenvalue or spectrum or spectral radius) of $\mathcal{A}$ as eigenvalues (respectively, H-eigenvalue or H$^+$-eigenvalue or H^{++}-eigenvalue or spectrum or spectral radius) of the k-graph G, or simply eigenvalues (respectively, H-eigenvalue or H$^+$-eigenvalue or H^{++}-eigenvalue or spectrum or spectral radius) if the context is clear. Similarly, we may call eigenvalues (respectively, H-eigenvalues or H$^+$-eigenvalue or H^{++}-eigenvalue or spectrum or spectral radius) of $\mathcal{L}$ and Q as Laplacian and signless Laplacian eigenvalues (respectively, H-eigenvalues or H$^+$-eigenvalue or H^{++}-eigenvalue or spectrum or spectral radius) of G, or simply Laplacian and signless Laplacian eigenvalues (respectively, H-eigenvalues or H$^+$-eigenvalue or H^{++}-eigenvalue or spectrum or spectral radius) if the context is clear.

Theorem 3.1 of [6] concerns the spectrum of the union of two disjoint hypergraphs. Checking its proof, we see that it also holds for Laplacian and signless Laplacian spectra. This will be useful for our further discussion. We state it here but omit its proof as the proof is the same as the proof of Theorem 3.1 of [6].

THEOREM 3.3. **(The Union of Two Disjoint Hypergraphs).** *Suppose $G=(V,E)$ is the union of two disjoint hypergraphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$, where $|V_1|=n_1$, $|V_2|=n_2$, $n_1+n_2=n=|V|$. Then the spectrum (respectively, the Laplacian spectrum or the signless Laplacian spectrum) of G is the union of the spectra (respectively, the Laplacian spectra or the signless Laplacian spectra) of G_1 and G_2, where, as multisets, an eigenvalue with multiplicity r in the spectrum (respectively, the Laplacian spectrum or the signless Laplacian spectrum) of G_1 occurs in the spectrum (respectively, the Laplacian spectrum or the signless Laplacian spectrum) of G with multiplicity $r(k-1)^{n_2}$.*

In general, G may be decomposed into components $G_r=(V_r,E_r)$ for $r=1,\ldots,s$. If $s=1$, then G is connected. Denote the adjacency tensor and the signless Laplacian tensor of G_r by $\mathcal{A}(G_r)$ and $\mathcal{Q}(G_r)$ respectively, for $r=1,\ldots,s$. Then by Theorem 3.3,

$$\rho(\mathcal{A})=\max_{r=1,\cdots,s}\{\rho(\mathcal{A}(G_r))\},\quad \rho(\mathcal{Q})=\max_{r=1,\ldots,s}\{\rho(\mathcal{Q}(G_r))\}.$$

With the above discussion, we are now ready to study H$^+$-eigenvalues of $\mathcal{Q}$ and $\mathcal{A}$.

THEOREM 3.4. **(H^+-Eigenvalues of $\mathcal{Q}$).** *Assume that $k\geq 3$. Suppose that G has s components $G_r=(V_r,E_r)$ for $r=1,\dots,s$. For $j=1,\dots,n$, d_j is a strict H^+-eigenvalue of $\mathcal{Q}$ with an H-eigenvector $\mathbf{e}^{(j)}$. Let $\nu_1=\rho(\mathcal{Q})$. If $\nu_1\equiv\rho(\mathcal{Q}(G_r))$ for $r=1,\dots,s$, then ν_1 is the unique H^{++}-eigenvalue of $\mathcal{Q}$. Otherwise, $\mathcal{Q}$ has no H^{++}-eigenvalue, and for $r=1,\dots,s$, $\rho(\mathcal{Q}(G_r))$ is a strict H^+-eigenvalue of $\mathcal{Q}$.*

Proof. A real number ν is an H-eigenvalue of $\mathcal{Q}$, with an H-eigenvector x, if and only if $x\in\Re^n$, $x\neq 0$, and $\mathcal{Q}x^{k-1}=\nu x^{[k-1]}$, i.e.,

$$d_ix_i^{k-1}+\sum\left\{\frac{1}{(k-1)!}x_{i_2}\cdots x_{i_k}:(i,i_2,\dots,i_k)\in E\right\}=\nu x_i^{k-1}, \tag{3.2}$$

for $i=1,\dots,n$. Then, we may easily verify that for $j=1,\dots,n$, d_j is an H^+-eigenvalue of Q with an H-eigenvector $\mathbf{e}^{(j)}$.

For $r=1,\dots,s$, as G_r is connected, $\mathcal{Q}(G_r)$ is weakly irreducible by [19]. By Theorem 2.2, $\rho(\mathcal{Q}(G_r))$ is the unique H^{++}-eigenvalue of $\mathcal{Q}(G_r)$, with a positive H-eigenvector $x^{(r)}\in\Re^{|V_r|}$. In (3.2), let $\nu=\rho(\mathcal{Q}(G_r))$, $x_i=x_i^{(r)}$ if $i\in V_r$, and $x_i=0$ if $i\notin V_r$. Then we see that (3.2) is satisfied for $i=1,\dots,n$. This shows that for $r=1,\dots,s$, $\rho(\mathcal{Q}(G_r))$ is an H^+-eigenvalue of $\mathcal{Q}$.

Assume that ν is an H^{++}-eigenvalue of $\mathcal{Q}$ with a positive H-eigenvector x. For $r=1,\dots,s$, define $x^{(r)}\in\Re^{|V_r|}$ by $x_i^{(r)}=x_i$ for $i\in V_r$. Then $x^{(r)}$ is a positive H-eigenvector in $\Re^{|V_r|}$. By (3.2), ν is an H^{++}-eigenvalue of $\mathcal{Q}(G_r)$. Because $\mathcal{Q}(G_r)$ is weakly irreducible, by Theorem 2.2, $\nu=\rho(\mathcal{Q}(G_r))$. Thus, if $\mathcal{Q}$ has an H^{++}-eigenvalue, then it must be $\nu_1=\rho(\mathcal{Q})\equiv\rho(\mathcal{Q}(G_r))$ for $r=1,\dots,s$. This completes our proof. ∎

THEOREM 3.5. **(H^+-Eigenvalues of $\mathcal{A}$).** *Assume that $k\geq 3$. Then zero is a strict H^+-eigenvalue of $\mathcal{A}$. Suppose that G has s components $G_r=(V_r,E_r)$ for $r=1,\dots,s$. Let $\lambda_1=\rho(\mathcal{A})$. If $\lambda_1\equiv\rho(\mathcal{A}(G_r))$ for $r=1,\dots,s$, then λ_1 is the unique H^{++}-eigenvalue of $\mathcal{A}$. Otherwise, $\mathcal{A}$ has no H^{++}-eigenvalue, and for $r=1,\dots,s$, $\rho(\mathcal{A}(G_r))$ is a strict H^+-eigenvalue of $\mathcal{A}$.*

Proof. Zero is an H-eigenvalue of $\mathcal{A}$, with an H-eigenvector x, if and only if $x\in\Re^n$, $x\neq 0$, and $\mathcal{A}x^{k-1}=0$, i.e.,

$$\sum\left\{\frac{1}{(k-1)!}x_{i_2}\cdots x_{i_k}:(i,i_2,\dots,i_k)\in E\right\}=0,$$

for $i=1\cdots,n$. Let x be a vector in $\Re^n_+$ with $1\leq \mathrm{supp}(x)\leq k-2$. Then we see that x is a nonnegative H-eigenvector of $\mathcal{A}$, corresponding to the zero H-eigenvalue. Thus, zero is an H^+-eigenvalue of $\mathcal{A}$. The proof of the remaining conclusions of this theorem is similar to the last part of the proof of the last theorem, so we omit it. ∎

For some k-graph G, $\mathcal{L}$, $\mathcal{Q}$, and $\mathcal{A}$ may have more strict H^+-eigenvalues. For example, let $k=3$, $n=8$, $m=8$, and $E=\{(1,2,3),(1,4,5),(2,4,5),(3,4,5),(4,5,6),(4,5,7),(4,5,8),(6,7,8)\}$. Then $d_1=d_2=d_3=d_6=d_7=d_8=2$ and $d_4=d_5=6$ are strict H^+-eigenvalues of $\mathcal{L}$ and $\mathcal{Q}$, 0 is a strict H^+-eigenvalue of $\mathcal{A}$. It is easy to verify that $\mu=1$, $\nu=3$, and $\lambda=1$ are also strict H^+-eigenvalues of $\mathcal{L}$, $\mathcal{Q}$, and $\mathcal{A}$, with an H-eigenvector $(1,1,1,0,0,0,0,0)$.

We will not identify all strict H^+-eigenvalues of $\mathcal{L}$, $\mathcal{Q}$, and $\mathcal{A}$, but we will identify the largest and the smallest H^+-eigenvalues of $\mathcal{L}$ and $\mathcal{Q}$, and establish their maximum or minimum properties in the next few sections. They are the most important H^+-eigenvalues of $\mathcal{L}$ and $\mathcal{Q}$.

There are also H-eigenvalues of $\mathcal{L}$ and $\mathcal{Q}$ which are not H$^+$-eigenvalues. We will give such an example in sections 5 and 7.

Theorems 3.2, 3.4, and 3.5 say that each of $\mathcal{L}$, $\mathcal{Q}$, and $\mathcal{A}$ has at most one H^{++}-eigenvalue. Actually, a real symmetric matrix has at most one H^{++}-eigenvalue. By Theorem 2.2, a weakly irreducible nonnegative tensor has at most one H^{++}-eigenvalue. By extending the proof of Theorem 3.4, probably this is also true for a general nonnegative tensor. We may also show that this is true for a real diagonal tensor. However, by numerical experiments, we found that this is not true for some real symmetric tensors. Thus, we ask the following question.

Question 1. Is there a reasonable class of real symmetric tensors, which includes the above cases, such that any tensor in this class has at most one H^{++}-eigenvalue?

4. The smallest Laplacian H-eigenvalue

The smallest Laplacian H-eigenvalue of G is $\mu_1=0$. By Theorem 3.2, $\mathbf{1}$ is an H-eigenvector of $\mathcal{L}$, associated with the H^{++}-eigenvalue $\mu_1=0$. We say that $x\in\Re^n$ is a **binary vector** if x_i is either 0 or 1 for $i=1,\dots,n$. Thus, $\mathbf{1}$ is a binary H-eigenvector of $\mathcal{L}$, associated with the H-eigenvalue $\mu_1=0$. We say that a binary H-eigenvector x of $\mathcal{L}$, associated with an H-eigenvalue μ, is a **minimal binary H-eigenvector** of $\mathcal{L}$, associated with μ, if there does not exist another binary H-eigenvector y of $\mathcal{L}$, associated with μ, such that supp(y) is a proper subset of supp(x).

Let $G=(V,E)$ be a k-graph. For $e_p=(i_1,\dots,i_k)\in E$, define a kth order n-dimensional symmetric tensor $\mathcal{L}(e_p)$ by

$$\mathcal{L}(e_p)x^k=\sum_{j=1}^k x_{i_j}^k-kx_{i_1}\cdots x_{i_k}$$

for any $x\in C^n$. Then, for any $x\in C^n$, we have

$$\mathcal{L}x^k=\sum_{e_p\in E}\mathcal{L}(e_p)x^k.$$

THEOREM 4.1. **(The Smallest Laplacian H-Eigenvalue).** *For a k-graph G, we have the following conclusions.*

(a). For any $x\in\Re^+$, $\mathcal{L}x^k\ge 0$. We have

$$0=\min\{\mathcal{L}x^k:x\in\Re^n_+,\sum_{i=1}^n x_i^k=1\}.$$

(b). A binary vector $x\in\Re^n$ is a minimal binary H-eigenvector of $\mathcal{L}$ associated with the H-eigenvalue $\mu_1=0$ if and only if supp(x) is the vertex set of a component of G.

(c). A vector $x\in\Re^n$ is an H-eigenvector of $\mathcal{L}$ associated with the H-eigenvalue $\mu_1=0$ if it is a nonzero linear combination of minimal binary H-eigenvectors of $\mathcal{L}$ associated with the H-eigenvalue $\mu_1=0$.

Proof.

(a). For any $e_p=(i_1,\dots,i_k)\in E$ and $x\in\Re^n_+$, we know that the arithmetic mean of $x_{i_1}^k,\cdots,x_{i_k}^k$ is greater than or equal to their geometric mean, i.e.,

$$\frac{1}{k}\sum_{j=1}^k x_{i_j}^k\ge x_{i_1}\cdots x_{i_k}.$$

This implies that $\mathcal{L}(e_p)x^k \geq 0$. Thus, $\mathcal{L}x^k \geq 0$ for any $x \in \Re^n_+$. As $\mathcal{L}y^k = 0$, where $y = \frac{l}{n^{\frac{1}{k}}}$, we have

$$0 = \min\{\mathcal{L}x^k : x \in \Re^n_+, \sum_{i=1}^n x_i^k = 1\}.$$

(b). A nonzero vector $x \in \Re^n$ is an H-eigenvector of $\mathcal{L}$, associated the the H-eigenvalue $\mu_1 = 0$, if and only $\mathcal{L}x^{k-1} = 0$, i.e.,

$$d_i x_i^{k-1} = \sum \left\{ \frac{1}{(k-1)!} x_{i_2} \cdots x_{i_k} : (i, i_2, \ldots, i_k) \in E \right\}, \tag{4.1}$$

for $i = 1, \ldots, n$.

Suppose that x is a binary vector and $\mathrm{supp}(x)$ is the vertex set of a component of G. Then the equation (4.1) reduces to $d_i = d_i$ if $i \in \mathrm{supp}(x)$, and $0 = 0$ if $i \notin \mathrm{supp}(x)$. Thus, x is a binary H-eigenvector of $\mathcal{L}$ associated with H-eigenvalue $\mu_1 = 0$. Suppose that y is a binary vector and $\mathrm{supp}(y)$ is a proper subset of $\mathrm{supp}(x)$. Then there are $i \in \mathrm{supp}(y)$ and an edge $(i, i_2, \ldots, i_k) \in E$ such that one of the indices $\{i_2, \ldots, i_k\}$ is not in $\mathrm{supp}(y)$. Then, for this i, by replacing x by y in (4.1), the left hand side of (4.1) becomes d_i, while the right hand side of (4.1) is strictly less than d_i, i.e., (4.1) does not hold under this replacement. This shows that y cannot be a binary H-eigenvector of $\mathcal{L}$ associated with H-eigenvalue $\mu_1 = 0$, i.e., x is a minimal binary H-eigenvector of $\mathcal{L}$ associated with the H-eigenvalue $\mu_1 = 0$.

On the other hand, suppose that x is a binary H-eigenvector of $\mathcal{L}$ associated with the H-eigenvalue $\mu_1 = 0$. Let $i \in \mathrm{supp}(x)$. Then, in order that the equation (4.1) holds for i, for any $(i, i_2, \ldots, i_k) \in E$, we must have $i_2, \ldots, i_k \in \mathrm{supp}(x)$. This shows that $\mathrm{supp}(x)$ is either the vertex set of a component of G, or the union of the vertex sets of several components of G. This proves (b).

(c). Let $\left\{y^{(1)}, \ldots, y^{(s)}\right\}$ be the set of binary H-eigenvectors of $\mathcal{L}$ associated with H-eigenvalue $\mu_1 = 0$.

Suppose that x is a nonzero linear combination of $y^{(1)}, \ldots, y^{(s)}$, $x = \sum_{r=1}^s \alpha_r y(r)$, where α_r are real numbers. If $i \in \mathrm{supp}(y^{(r)})$ for some r, then the equation (4.1) is $\alpha_r^{k-1} d_i = \alpha_r^{k-1} d_i$. Otherwise, the equation (4.1) is $0 = 0$. Thus, x is an H-eigenvector of $\mathcal{L}$ associated with the H-eigenvalue $\mu_1 = 0$. This proves (c). ∎

COROLLARY 4.2. *The following two statements are equivalent.*

(a). The k-graph G is connected.

(b). The vector $\mathbf{1}$ *is the unique minimal binary H-eigenvector of* $\mathcal{L}$ *associated with the H-eigenvalue* $\mu_1 = 0$.

5. The largest Laplacian H$^+$-eigenvalue

In Section 3, we showed that zero is the unique Laplacian H^{++}-eigenvalue of G, and d_j is a strict H$^+$-eigenvalue of G, for $j = 1, \ldots, n$. We now identify the largest Laplacian H$^+$-eigenvalue of G, and establish a maximum property of this Laplacian H$^+$-eigenvalue.

THEOREM 5.1. **(The largest Laplacian H$^+$-eigenvalue).** *Assume that* $k \geq 3$. *The largest Laplacian H$^+$-eigenvalue of G is* $\Delta = \max_i\{d_i\}$. *We have*

$$\Delta = \max\{\mathcal{L}x^k : x \in \Re^n_+, \sum_{i=1}^n x_i^k = 1\}. \tag{5.1}$$

Proof. Suppose that μ is a Laplacian H$^+$-eigenvalue of G associated with nonnegative H-eigenvector x. Assume that $x_j > 0$. By (3.1), we have

$$\mu x_j^{k-1} = d_j x_j^{k-1} - \sum\left\{\frac{1}{(k-1)!} x_{i_2}\cdots x_{i_k} : (i,i_2,\ldots,i_k)\in E\right\} \le d_j x_j^{k-1}.$$

This implies that

$$\mu \le d_j \le \Delta.$$

By Theorem 3.2, Δ is an H$^+$-eigenvalue of $\mathcal{L}$. Thus, Δ is the largest H$^+$-eigenvalue of $\mathcal{L}$.

Suppose that $\Delta = d_j$. Let $x = \mathbf{e}^{(j)}$. Then x is a feasible point of the maximization problem in (5.1). We have

$$\mathcal{L}x^k = \sum_{i=1}^n \left[d_i x_i^k - \sum\left\{\frac{1}{(k-1)!} x_i x_{i_2}\cdots x_{i_k} : (i,i_2,\ldots,i_k)\in E\right\}\right] = \Delta.$$

This shows that

$$\Delta \le \max\{\mathcal{L}x^k : x\in\Re^n_+, \sum_{i=1}^n x_i^k = 1\}.$$

On the other hand, suppose x^* is a maximizer of the maximization problem in (5.1). As the feasible set is compact, and the objective function is continuous, such a maximizer exists. By optimization theory, for $i=1,\cdots,n$, either $x_i^* = 0$ and

$$d_i(x_i^*)^{k-1} - \sum\left\{\frac{1}{(k-1)!} x^*_{i_2}\cdots x^*_{i_k} : (i,i_2,\ldots,i_k)\in E\right\} \ge \mu (x_i^*)^{k-1}, \tag{5.2}$$

or $x_i^* > 0$ and

$$d_i(x_i^*)^{k-1} - \sum\left\{\frac{1}{(k-1)!} x^*_{i_2}\cdots x^*_{i_k} : (i,i_2,\ldots,i_k)\in E\right\} = \mu (x_i^*)^{k-1}, \tag{5.3}$$

where μ is a Lagrange multiplier. As x^* is feasible for the maximization problem, (5.3) holds for at least one i, say i_0. We have

$$d_{i_0}(x^*_{i_0})^{k-1} \ge \mu (x^*_{i_0})^{k-1}.$$

As $x^*_{i_0} > 0$, we have $\mu \le d_{i_0} \le \Delta$. Multiplying (5.2) and (5.3) by x_i^* and summing over $i=1,\ldots,n$, we have

$$\mathcal{L}(x^*)^k = \mu \sum_{i=1}^n (x_i^*)^k = \mu.$$

Thus,

$$\mu = \max\{\mathcal{L}x^k : x\in\Re^n_+, \sum_{i=1}^n x_i^k = 1\}.$$

This shows that

$$\Delta \geq \max\{\mathcal{L}x^k : x \in \Re^n_+, \sum_{i=1}^n x_i^k = 1\}.$$

Hence, (5.1) holds. ∎

In general, Δ may not be the largest H-eigenvalue of $\mathcal{L}$. For example, let $n=k=6$, $m=1$, and $E=\{(1,2,3,4,5,6)\}$. Then $\Delta=1$, while $\mu=2$ is an H-eigenvalue of $\mathcal{L}$ with an H-eigenvector $(1,1,1,-1,-1,-1)$.

6. The largest H-eigenvalue and the largest signless Laplacian H-eigenvalue

The largest H-eigenvalue is $\lambda_1 = \rho(\mathcal{A})$. The largest signless Laplacian H-eigenvalue is $\nu_1 = \rho(\mathcal{Q})$. As both $\mathcal{A}$ and $\mathcal{Q}$ are nonnegative tensors, their properties are similar. We thus discuss them together.

When k is even, by [20], we know that

$$\lambda_1 = \max\{\mathcal{A}x^k : x \in \Re^n, \sum_{i=1}^n x_i^k = 1\},$$

and

$$\nu_1 = \max\{\mathcal{Q}x^k : x \in \Re^n, \sum_{i=1}^n x_i^k = 1\}.$$

The feasible sets of the above two maximization problems are the same. It is a compact set when k is even. When k is odd, it is not compact. We intend to establish some maximum properties of λ_1 and ν_1, which hold whenever k is even or odd.

Corollary 3.4 of [6] indicates that when G is connected,

$$\lambda_1 = \max\{\mathcal{A}x^k : x \in \Re^n_+, \sum_{i=1}^n x_i^k = 1\}. \tag{6.1}$$

Using a similar argument, we may show that when G is connected,

$$\nu_1 = \max\{\mathcal{Q}x^k : x \in \Re^n_+, \sum_{i=1}^n x_i^k = 1\}. \tag{6.2}$$

We wish to show that (6.1) and (6.2) hold even if G is not connected.

Theorem 6.1. **(The largest H-eigenvalue and the largest signless Laplacian H-eigenvalue).** *Assume that $k \geq 3$. Then (6.1) and (6.2) always hold.*

Proof. We now prove (6.1). Suppose that G is decomposed to some components $G_r = (V_r, E_r)$ for $r = 1, \ldots, s$. Then $\lambda_1 = \max\{\rho(\mathcal{A}(G_r)) : r = 1, \ldots, s\}$, and for $r = 1, \ldots, s$,

$$\rho(\mathcal{A}(G_r)) = \max\{\mathcal{A}(G_r)(x^{(r)})^k : x^{(r)} \in \Re_+^{|V_r|}, \sum_{i \in V_r} \left(x_i^{(r)}\right)^k = 1\}.$$

Suppose that $\lambda_1 = \rho(\mathcal{A}(G_j))$ for some j. Define $x \in \Re^n_+$ by $x_i = x_i^{(r)}$ if $i \in V_r$ and $x_i = 0$ otherwise. Then $\sum_{i=1}^n x_i^k = 1$, and $\mathcal{A}x^k = \mathcal{A}(G_j)(x^{(j)})^k$. We see that $\lambda_1 = \mathcal{A}x^k$ and x

is a feasible point of the maximization problem in (6.1). This shows that

$$\lambda_1 \le \max\{\mathcal{A}x^k : x \in \Re^n_+, \sum_{i=1}^n x_i^k = 1\}.$$

On the other hand, suppose that x_* is a maximizer of the maximization problem in (6.1). Then,

$$\max\{\mathcal{A}x^k : x \in \Re^n_+, \sum_{i=1}^n x_i^k = 1\} = \mathcal{A}x_*^k = \sum_{r=1}^s \mathcal{A}(G_r)(\bar{x}^{(r)})^k,$$

where $\bar{x}^{(r)} \in \Re_+^{|V_r|}$ and $\bar{x}_i^{(r)} = (x_*)_i$ for $i \in V_r$, for $r = 1, \ldots, s$. For $r = 1, \ldots, s$, assume that $\alpha_r = \sum_{i \in V_r} (x_*)_i^k$. Then $\alpha_r \ge 0$ for $r = 1, \ldots, s$, and $\sum_{r=1}^s \alpha_r = 1$. If $\alpha_r > 0$, then define $x^{(r)} \in \Re_+^{|V_r|}$ by $x^{(r)} = \frac{1}{(\alpha_r)^{\frac{1}{k}}} \bar{x}^{(r)}$. Then $\sum_{i \in V_r} \left(x_i^{(r)}\right)^k = 1$. We now have

$$\begin{aligned} \mathcal{A}x_*^k = \sum \{\mathcal{A}(G_r)(\bar{x}^{(r)})^k : \alpha_r > 0\} &= \sum \{\alpha_r \mathcal{A}(G_r)(x^{(r)})^k : \alpha_r > 0\} \\ &\le \sum \{\alpha_r \rho(\mathcal{A}(G_r)) : \alpha_r > 0\} \\ &\le \sum \{\alpha_r \lambda_1 : \alpha_r > 0\} = \lambda_1. \end{aligned}$$

Thus, we have

$$\lambda_1 \ge \max\{\mathcal{A}x^k : x \in \Re^n_+, \sum_{i=1}^n x_i^k = 1\}.$$

Hence, (6.1) holds.

Similarly, we may show that (6.2) holds. □

COROLLARY 6.2. **(Bounds for ν_1).** *We always have*

$$\max\{\Delta, 2\bar{d}\} \le \nu_1 \le 2\Delta. \tag{6.3}$$

Proof. By Theorem 3.1 (d), we have that

$$0 \le \nu_1 \le 2\Delta.$$

In (6.2), letting $x = \frac{l}{n^{\frac{1}{k}}}$, we see that $\nu_1 \ge 2\bar{d}$. Assume that $d_j = \Delta$. In (6.2), letting $x = \mathbf{e}^{(j)}$, we see that $\nu_1 \ge \Delta$. Thus, we always have

$$\nu_1 \ge \max\{\Delta, 2\bar{d}\}.$$

These prove (6.3). □

It was established in [6] that $\bar{d} \le \lambda_1 \le \Delta$.

Question 2. Are there any formulas related to λ_1 and ν_1?

We may compare ν_1, λ_1, and $\rho(\mathcal{L})$. We prove a lemma first.

LEMMA 6.3. *If $\mathcal{C}$ is a nonnegative tensor of order k and dimension n, and $\mathcal{B}$ is a tensor of order k and dimension n, satisfying $|\mathcal{B}| \le \mathcal{C}$, then $\rho(\mathcal{B}) \le \rho(\mathcal{C})$.*

Proof. Let $\mathcal{C}_\epsilon = \mathcal{C} + \epsilon J$, with $\epsilon > 0$. Then $\mathcal{C}_\epsilon$ is a positive tensor, thus irreducible, and $|\mathcal{B}| \le \mathcal{C}_\epsilon$. By Lemma 3.2 of [27], we have $\rho(\mathcal{B}) \le \rho(\mathcal{C}_\epsilon)$. Let $\epsilon \to 0$. As the eigenvalues of a tensor are roots of the characteristic polynomial, whose coefficients are polynomials in the entries of that tensor [20], the spectral radius of that tensor is continuous in its entries. Then we have $\rho(\mathcal{B}) \le \rho(\mathcal{C})$. ∎

With this lemma, we immediately have the following proposition.

PROPOSITION 6.4. *For a k-graph G, we have*

$$\nu_1 = \rho(\mathcal{Q}) \ge \rho(\mathcal{L}), \text{ and } \nu_1 = \rho(\mathcal{Q}) \ge \lambda_1 = \rho(\mathcal{A}).$$

Note that it is possible that $\nu_1 = \rho(\mathcal{Q}) = \rho(\mathcal{L})$. For example, let $n = k = 6$, $m = 1$ and $E = \{(1,2,3,4,5,6)\}$. Then G is connected. Thus, $\mathcal{A}$, $\mathcal{L}$, and $\mathcal{Q}$ are weakly irreducible. We have $\mathcal{L}x^6 = \sum_{i=1}^6 x_i^6 - 6x_1\cdots x_6$ and $\mathcal{Q}x^6 = \sum_{i=1}^6 x_i^6 - 6x_1\cdots x_6$. We see that $\nu = 2$ is an H^{++} eigenvalue of $\mathcal{Q}$ with an H-eigenvector $l = (1,1,1,1,1,1)$. By Theorem 2.2 (b), we have $\rho(\mathcal{Q}) = 2$. On the other hand, we see that $\mu = 2$ is an H-eigenvalue of $\mathcal{L}$ with an H-eigenvector $l = (1,1,1,-1,-1,-1)$. By Proposition 6.4, we have $\rho(\mathcal{L}) = \rho(\mathcal{Q}) = 2$. Thus, it is a research topic to identify the conditions under which $\rho(\mathcal{L}) = \rho(\mathcal{Q})$.

We now discuss algorithms for computing ν_1. As $\mathcal{Q}$ is a nonnegative tensor, we may use algorithms for finding the largest eigenvalue of a nonnegative tensor to compute it. However, the convergence of the NQZ algorithm [18] needs the condition that $\mathcal{Q}$ is primitive [5], and the convergence of the LZI algorithm needs the condition that $\mathcal{Q}$ is irreducible [17]. These conditions are somewhat strong. The linear convergence of the LZI algorithm needs the condition that $\mathcal{Q}$ is weakly positive [28]. A nonnegative tensor $\mathcal{T} = (t_{i_1\cdots i_k})$ is weakly positive if $t_{ij\cdots j} > 0$ for all $i \ne j, i, j = 1, \ldots, n$. We see that $\mathcal{Q}$ cannot be weakly positive. Thus, it may not be a good choice to use these two algorithms for computing ν_1. Instead, one may use the HHQ algorithm proposed in [9] to compute ν_1. The HHQ algorithm is globally R-linearly convergent if Q is weakly irreducible in the sense of [8]. As discussed above, if G is connected, then Q is weakly irreducible. Thus, the HHQ algorithm is practical for computing ν_1 when G is connected. If G is not connected, the HHQ algorithm may be used for components (and then the maximum value chosen), by the observation at the beginning of the proof of Theorem 11. This argument is also valid for computing λ_1.

Thus, we may use the HHQ algorithm to compute λ_1 and ν_1, and we have global R-linear convergence.

7. The smallest signless Laplacian H^+-eigenvalue

We now identify the smallest signless Laplacian H^+-eigenvalue of G, and establish a minimum property of this signless Laplacian H^+-eigenvalue.

THEOREM 7.1. **(The smallest signless Laplacian H^+-eigenvalue).** *The smallest signless Laplacian H^+-eigenvalue of G is δ. We always have*

$$\delta = \min\{\mathcal{Q}x^k : x \in \Re^n_+, \sum_{i=1}^n x_i^k = 1\}. \tag{7.1}$$

Proof. Suppose that ν is an H^+-eigenvalue of $\mathcal{Q}$, with a nonnegative H-eigenvector x. Suppose that $x_j > 0$. By (3.2), we have

$$d_j x_j^{k-1} + \sum\left\{\frac{1}{(k-1)!} x_{i_2}\cdots x_{i_k} : (j, i_2, \ldots, i_k) \in E\right\} = \nu x_j^{k-1}.$$

This implies that $d_j x_j^{k-1} \le \nu x_j^{k-1}$, i.e., $\nu \ge d_j \ge \delta$. As δ is an H$^+$-eigenvalue of $\mathcal{Q}$ by Theorem 3.4, this shows that δ is the smallest H$^+$-eigenvalue of $\mathcal{Q}$.

We now prove (7.1). Suppose that $d_j = \delta$. Let $x = \mathbf{e}^{(j)}$ in (7.1). Then we have

$$\delta \ge \min\{\mathcal{Q}x^k : x \in \Re^n_+, \sum_{i=1}^n x_i^k = 1\}. \tag{7.2}$$

Suppose that x^* is an optimal solution of the minimization problem in (7.1). By the optimization theory, there are Lagrange multipliers $u \in \Re^n$ and $\nu \in \Re$ such that for $i = 1, \cdots, n$,

$$\left(\mathcal{Q}(x^*)^{k-1}\right)_i = \nu (x_i^*)^{k-1} + u_i, \tag{7.3}$$

$$x_i^* \ge 0,\ u_i \ge 0,\ x_i^* u_i = 0,$$

and

$$\sum_{i=1}^n (x_i^*)^k = 1. \tag{7.4}$$

Let $I = \mathrm{supp}(x^*)$. By (7.4), $I \ne \emptyset$. Then for $i \in I, u_i = 0$ and for $i \notin I, x_i^* = 0$. Multiplying (7.3) by x_i^* and summing from $i = 1$ to n, we have

$$\nu = \mathcal{Q}(x^*)^k.$$

Now assume that $x_j^* = \max\{x_i^* : i \in I\}$. Then $x_j^* > 0$ and $u_j = 0$. By (7.3), we have

$$\left(\mathcal{Q}(x^*)^{k-1}\right)_j = \nu \left(x_j^*\right)^{k-1},$$

which implies that

$$d_j \left(x_j^*\right)^{k-1} \le \nu \left(x_j^*\right)^{k-1}.$$

Thus,

$$\nu = \mathcal{Q}(x^*)^k \ge d_j \ge \delta.$$

Hence,

$$\delta \le \min\{\mathcal{Q}x^k : x \in \Re^n_+, \sum_{i=1}^n x_i^k = 1\}.$$

Combining this with (7.2), we have (7.1). ∎

In general, δ may not be the smallest H-eigenvalue of $\mathcal{Q}$. For example, let $n = k = 6$, $m = 1$, and $E = \{(1,2,3,4,5,6)\}$. Then $\delta = 1$, while $\nu = 0$ is an H-eigenvalue of $\mathcal{Q}$ with an H-eigenvector $(1,1,1,-1,-1,-1)$. In general, we may show that $\mathcal{Q}$ has a zero H-eigenvalue if and only if $k = 4j+2$ for some integer j, and there is a vector $x \in \Re^k$ such that for any edge $e_p = (i_1, \ldots, i_k) \in E$, half of $x_{i_1}, \ldots, x_{i_k}$ are j, and the other half are -1. Hence, if $k = 4j$ or if $k = 4j+2$ but such an x does not exist, then $\mathcal{Q}$ is positive definite.

We now give an application of Theorem 7.1. Suppose that S is a proper nonempty subset of V. Denote $\bar{S}=V\setminus S$. Then $\bar{S}$ is also a proper nonempty subset of V. The edge set E is now partitioned into three parts $E(S),E(\bar{S})$ and $E(S,\bar{S})$. The edge set $E(S)$ consists of edges whose vertices are all in S. The edge set $E(\bar{S})$ consists of edges whose vertices are all in $\bar{S}$. The edge set $E(S,\bar{S})$ consists of edges whose vertices are in both S and $\bar{S}$. We call $E(S,\bar{S})$ an **edge cut** of G. If we delete $E(S,\bar{S})$ from G, then G is separated into two k-graphs $G[S]=(S,E(S))$ and $G[\bar{S}]=(\bar{S},E(\bar{S}))$. For a vertex $i\in S$, we denote its degree at $G[S]$ by $d_i(S)$. Similarly, for a vertex $i\in\bar{S}$, we denote its degree at $G[\bar{S}]$ by $d_i(\bar{S})$. We denote the maximum degrees, the minimum degrees, the average degrees of $G[S]$ and $G[\bar{S}]$ by $\Delta(S)$, $\Delta(\bar{S})$, $\delta(S)$, $\delta(\bar{S})$, $\bar{d}(S)$, and $\bar{d}(\bar{S})$ respectively. For an edge $e_p\in E(S,\bar{S})$, $t(e_p)$ of its vertices are in S, where $1\le t(e_p)\le k-1$. For all edges $e_p\in E(S,\bar{S})$, the average value of such $t(e_p)$ is denoted $t(S)$. Then $1\le t(S)\le k-1$. Similarly, we may define $t(\bar{S})$. Then $t(S)+t(\bar{S})=k$. We call the minimum or maximum cardinality of such an edge cut the **edge connectivity** or **maximum cut** of G, and denote it by $e(G)$ or $c(G)$ respectively.

For $e_p=(i_1,\ldots,i_k)\in E$, define a kth order n-dimensional symmetric tensor $\mathcal{Q}(e_p)$ by

$$\mathcal{Q}(e_p)x^k=\sum_{j=1}^k x_{i_j}^k+kx_{i_1}\cdots x_{i_k}$$

for any $x\in C^n$. Then, for any $x\in C^n$, we have

$$\mathcal{Q}x^k=\sum_{e_p\in E}\mathcal{Q}(e_p)x^k.$$

Proposition 7.2. *For a k-graph G, we have the following conclusions.*

(a). The edge connectivity satisfies $e(G)\le\delta$.

(b). We have

$$c(G)\le\frac{n}{k}(2\bar{d}-\delta).$$

(c). If $n\le 2k-1$, then $e(G)=\delta$.

Proof.

(a). Assume that $d_j=\delta$. Let $S=\{j\}$. Then $|E(S,\bar{S})|=d_j=d_{min}$. This proves (a).

(b). Let S be a nonempty proper subset of V. Let $x=\frac{1}{|S|^{\frac{1}{k}}}\sum_{i\in S}\mathbf{e}^{(i)}$. For $e_p\in E(S)$, we have

$$\mathcal{Q}(e_p)x^k=\frac{2k}{|S|}.$$

For $e_p\in E(\bar{S})$, we have

$$\mathcal{Q}(e_p)x^k=0.$$

For $e_p\in E(S,\bar{S})$, we have

$$\mathcal{Q}(e_p)x^k=\frac{t(e_p)}{|S|}.$$

As

$$\mathcal{Q}x^k = \left(\sum_{e_p\in E(S)} + \sum_{e_p\in E(\bar{S})} + \sum_{e_p\in E(S,\bar{S})}\right)\mathcal{Q}(e_p)x^k,$$

we have

$$\mathcal{Q}x^k = \frac{2k}{|S|}|E(S)| + \frac{t(S)}{|S|}|E(S,\bar{S})|. \tag{7.5}$$

Similarly, letting $y = \frac{1}{|\bar{S}|^{\frac{1}{k}}}\sum_{i\in\bar{S}}\mathbf{e}^{(i)}$, we have

$$\mathcal{Q}y^k = \frac{2k}{|\bar{S}|}|E(\bar{S})| + \frac{t(\bar{S})}{|\bar{S}|}|E(S,\bar{S})|. \tag{7.6}$$

By (7.1) and (7.5), we have

$$|S|\delta \le 2k|E(S)| + t(S)|E(S,\bar{S})|. \tag{7.7}$$

By (7.1) and (7.6), we have

$$|\bar{S}|\delta \le 2k|E(\bar{S})| + t(\bar{S})|E(S,\bar{S})|. \tag{7.8}$$

Summing (7.7) and (7.8), we have

$$n\delta \le 2k\left(|E(S)| + |E(\bar{S})|\right) + k|E(S,\bar{S})|,$$

i.e.,

$$n\delta \le 2k\left(m - |E(S,\bar{S})|\right) + k|E(S,\bar{S})|,$$

which implies that

$$\delta \le \frac{2km}{n} - \frac{k}{n}|E(S,\bar{S})|.$$

Noticing that $\bar{d} = \frac{km}{n}$, we have

$$|E(S,\bar{S})| \le \frac{n}{k}(2\bar{d} - d_{min}).$$

This proves (b).

(c). When $n \le 2k-1$, either $|S| < k$ or $|\bar{S}| < k$. Without loss of generality, assume that $|S| < k$. Then $E(S) = \emptyset$ and $|E(S)| = 0$. From (7.7), we have

$$|S|\delta \le t(S)|E(S,\bar{S})|.$$

We always have $t(S) \le |S|$. Thus, we have

$$\delta \le |E(S,\bar{S})|.$$

Combining this with conclusion (a), we have conclusion (c). ∎

8. Analytic connectivity

We define the **analytic connectivity** $\alpha(G)$ of the k-graph G by

$$\alpha(G)=\min_{j=1,\cdots,n}\min\{\mathcal{L}x^k : x\in\Re^n_+, \sum_{i=1}^n x_i^k=1, x_j=0\}.$$

By Theorem 4.1, $\mathcal{L}x^k\ge 0$ for any $x\in\Re^n_+$. Thus, $\alpha(G)\ge 0$. We first prove the following proposition.

PROPOSITION 8.1. *The k-graph G is connected if and only if the analytic connectivity $\alpha(G)>0$.*

Proof. Suppose that G is not connected. Let $G_1=(V_1,E_1)$ be a component of G. Then there is a $j\in V\setminus V_1$. Let $x=\frac{1}{|V_1|}\sum_{i\in V_1}\mathbf{e}^{(i)}$. Then x is a feasible point of $\min\{\mathcal{L}x^k : x\in\Re^n_+, \sum_{i=1}^n x_i^k=1, x_j=0\}$, and we see that $\min\{\mathcal{L}x^k : x\in\Re^n_+, \sum_{i=1}^n x_i^k=1, x_j=0\}=0$. This implies that $\alpha(G)=0$.

Suppose that $\alpha(G)=0$. There is a j such that $\min\{\mathcal{L}x^k : x\in\Re^n_+, \sum_{i=1}^n x_i^k=1, x_j=0\}=0$. Suppose that x^* is a minimizer of this minimization problem. Then $x_j^*=0$, $\mathcal{L}(x^*)^k=0$ and by optimization theory, there is a Lagrange multiplier μ such that for $i=1,\ldots n$, $i\ne j$, either $x_i^*=0$ and

$$d_i(x_i^*)^{k-1}-\sum\left\{\frac{1}{(k-1)!}x_{i_2}^*\cdots x_{i_k}^* : (i,i_2,\ldots,i_k)\in E\right\}\ge \mu(x_i^*)^{k-1}, \tag{8.1}$$

or $x_i^*>0$ and

$$d_i(x_i^*)^{k-1}-\sum\left\{\frac{1}{(k-1)!}x_{i_2}^*\cdots x_{i_k}^* : (i,i_2,\ldots,i_k)\in E\right\}= \mu(x_i^*)^{k-1}. \tag{8.2}$$

In (8.1) and (8.2), we always have $x^*\in\Re^n_+$, $\sum_{i=1}^n(x_i^*)^k=1$ and $x_j^*=0$. Multiplying (8.1) and (8.2) with x_i^* and summing them together, we have $\mu\sum_{i=1}^n(x_i^*)^k=\mathcal{L}(x^*)^k=0$, i.e., $\mu=0$. Then for $i=1,\ldots n$, $i\ne j$, either $x_i^*=0$ or

$$d_i(x_i^*)^{k-1}-\sum\left\{\frac{1}{(k-1)!}x_{i_2}^*\cdots x_{i_k}^* : (i,i_2,\ldots,i_k)\in E\right\}=0. \tag{8.3}$$

Let $x_r^*=\max\{x_i^* : i=1,\ldots,n\}$. Then by (8.3), we have

$$0=d_r-\sum\left\{\frac{1}{(k-1)!}\frac{x_{i_2}^*}{x_r^*}\cdots\frac{x_{i_k}^*}{x_r^*} : (r,i_2,\ldots,i_k)\in E\right\}.$$

Note that

$$d_r=\sum\left\{\frac{1}{(k-1)!} : (r,i_2,\ldots,i_k)\in E\right\}.$$

Thus, we have $x_i=x_r$ as long as i and r are in the same edge. From this, we see that $x_i=x_r$ as long as i and r are in the same component of G. Because $x_j^*=0$, we see that j and r are in the different components of G, i.e., G is not connected. This proves the proposition. ∎

We now further explore an application of $\alpha(G)$.

PROPOSITION 8.2. *For a k-graph G, we have*

$$e(G) \geq \frac{n}{k}\alpha(G).$$

Proof. Let S be a nonempty proper subset of V. Then there is a $j \notin S$ such that

$$\min\{\mathcal{L}x^k : x \in \Re^n_+, \sum_{i=1}^n x_i^k = 1, x_j = 0\} \geq \alpha(G). \tag{8.4}$$

Let $x = \frac{1}{|S|^{\frac{1}{k}}} \sum_{i \in S} \mathbf{e}^{(i)}$. Then x is a feasible point of the minimization problem in (8.4). For $e_p \in E(S)$ and $e_p \in E(\bar{S})$, we have

$$\mathcal{L}(e_p)x^k = 0,$$

where $\mathcal{L}(e_p)$ is defined in Section 4. For $e_p \in E(S,\bar{S})$, we have

$$\mathcal{L}(e_p)x^k = \frac{t(e_p)}{|S|}.$$

As

$$\mathcal{L}x^k = \left(\sum_{e_p \in E(S)} + \sum_{e_p \in E(\bar{S})} + \sum_{e_p \in E(S,\bar{S})} \right) \mathcal{L}(e_p)x^k,$$

we have

$$\mathcal{L}x^k = \frac{t(S)}{|S|}|E(S,\bar{S})|. \tag{8.5}$$

Similarly, letting $y = \frac{1}{|\bar{S}|^{\frac{1}{k}}} \sum_{i \in \bar{S}} \mathbf{e}^{(i)}$, we have

$$\mathcal{L}y^k = \frac{t(\bar{S})}{|\bar{S}|}|E(S,\bar{S})|. \tag{8.6}$$

By (8.4) and (8.5), we have

$$|S|\alpha(G) \leq t(S)|E(S,\bar{S})|. \tag{8.7}$$

By (8.4) and (8.6), we have

$$|\bar{S}|\alpha(G) \leq t(\bar{S})|E(S,\bar{S})|. \tag{8.8}$$

Summing up (8.7) and (8.8), we have

$$n\alpha(G) \leq k|E(S,\bar{S})|,$$

i.e.,

$$\frac{n}{k}\alpha(G) \leq |E(S,\bar{S})|.$$

This implies that

$$e(G) \geq \frac{n}{k}\alpha(G).$$

□

We now give an upper bound for $\alpha(G)$.

PROPOSITION 8.3. *For a k-graph G, we have*

$$0 \leq \alpha(G) \leq \delta.$$

Proof. We know $\alpha(G) \geq 0$. It suffices to prove that $\alpha(G) \leq \delta$. Suppose that $d_r = \delta$ and $j \neq r$. Then $l^{(r)}$ is a feasible point of

$$\min\{\mathcal{L}x^k : x \in \Re^n_+, \sum_{i=1}^n x_i^k = 1, x_j = 0\},$$

and $\mathcal{L}(l^{(r)})^k = \delta$. This implies that

$$\alpha(G) \leq \min\{\mathcal{L}x^k : x \in \Re^n_+, \sum_{i=1}^n x_i^k = 1, x_j = 0\} \leq \delta.$$

□

By Proposition 8.1, when G is not connected, $\alpha(G) = 0$. Let $n = k$, $m = 1$, and $E = \{(1, 2, \ldots, k)\}$. Then $\mathcal{L}x^k = \sum_{i=1}^k x_i^k - kx_1 \cdots x_k$, and we see that $\alpha(G) = 1 = \delta$. Thus, both the lower bound 0 and the upper bound δ in Proposition 8.3 are attainable. However, it is possible that $0 < \alpha(G) < \delta$. Let $k = 3$, $n = 4$, $m = 2$, and $E = \{(1,2,3),(2,3,4)\}$. Then G is connected and $\alpha(G) > 0$. We have $\mathcal{L}x^3 = x_1^3 + 2x_2^3 + 2x_3^3 + x_4^3 - 3x_1x_2x_3 - 3x_2x_3x_4$. Consider

$$\begin{aligned} &\min\{\mathcal{L}x^3 : x \in \Re^4_+, \sum_{i=1}^4 x_i^3 = 1, x_4 = 0\} \\ &= \min\{x_1^3 + 2x_2^3 + 2x_3^3 - 3x_1x_2x_3 : x_1^3 + x_2^3 + x_3^3 = 1, x_1, x_2, x_3 \geq 0\}. \end{aligned}$$

Let $y = \left(\frac{1}{3}\right)^{\frac{1}{3}}(1,1,1)$. Then we see that

$$\alpha(G) \leq \min\{\mathcal{L}x^3 : x \in \Re^4_+, \sum_{i=1}^4 x_i^3 = 1, x_4 = 0\} \leq y_1^3 + 2y_2^3 + 2y_3^3 - 3y_1y_2y_3 = \frac{2}{3} < 1 = \delta.$$

Actually, the exact value of $\alpha(G)$ for this example is $\alpha(G) = 1 - \beta^2$, where β satisfies $\beta + \beta^3 = 1$ and $0.5 < \beta < 1$.

Question 3. In general, how can we calculate $\alpha(G)$?

9. Final remarks

In this paper, we propose a simple and natural definition for the Laplacian and the signless Laplacian tensors of a uniform hypergraph. We show that they have very nice spectral properties. This sets the base for further exploring their applications in spectral hypergraph theory. Several further questions are raised. We expect that the research on these two Laplacian tensors will also motivate further development of the spectral theory of tensors. Some very recent papers [11, 12, 13, 14, 22, 23] demonstrated the impacts on these two aspects.

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