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The Laplacian of a uniform hypergraph

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Abstract In this paper, we investigate the Laplacian, i.e., the normalized Laplacian tensor of a k-uniform hypergraph. We show that the real parts of all the eigenvalues of the Laplacian are in the interval [0, 2], and the real part is zero (respectively two) if and only if the eigenvalue is zero (respectively two). All the H$^+$-eigenvalues of the Laplacian and all the smallest H$^+$-eigenvalues of its sub-tensors are characterized through the spectral radii of some nonnegative tensors. All the H$^+$-eigenvalues of the Laplacian that are less than one are completely characterized by the spectral components of the hypergraph and vice verse. The smallest H-eigenvalue, which is also an H$^+$-eigenvalue, of the Laplacian is zero. When k is even, necessary and sufficient conditions for the largest H-eigenvalue of the Laplacian being two are given. If k is odd, then its largest H-eigenvalue is always strictly less than two. The largest H$^+$-eigenvalue of the Laplacian for a hypergraph having at least one edge is one; and its nonnegative eigenvectors are in one to one correspondence with the flower hearts of the hypergraph. The second smallest H$^+$-eigenvalue of the Laplacian is positive if and only if the hypergraph is connected. The number of connected components of a hypergraph is determined by the H$^+$-geometric multiplicity of the zero H$^+$-eigenvalue of the Lapalacian.

Keywords Tensor · Eigenvalue · Hypergraph · Laplacian

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1 Introduction

In This paper, we establish some basic facts on the spectrum of the normalized Laplacian tensor of a uniform hypergraph. It is an analogue of the spectrum of the normalized Laplacian matrix of a graph (Chung 1997). This work is derived by the recently rapid developments on both the spectral hypergraph theory (Lim 2007; Hu and Qi 2012a; Cooper andv Dutle 2012; Qi 2012; Pearson and Zhang 2012; Li et al. 2012; Rota Bulò and Pelillo 2009; Rota Bulò 2009; Xie and Chang 2012a, b, 2013c; Lim 2005) and the spectral theory of tensors (Chang et al. 2008, 2011; Friedland 2013; Hu et al. 2013, 2011; Lim 2005, 2007; Li et al. 2012; Hu and Qi 2012b; Ng et al. 2009; Qi 2005, 2006, 2007, 2012; Yang and Yang 2010, 2011; Zhang et al. 2012; Hu et al. 2012).

The study of the Laplacian tensor for a uniform hypergraph was initiated by Hu and Qi (2012a). The Laplacian tensor introduced there is based on the discretization of the higher order Laplace-Beltrami operator. Following this, Li, Qi and Yu proposed another definition of the Laplacian tensor (Li et al. 2012). Later, Xie and Chang introduced the signless Laplacian tensor for a uniform hypergraph (Xie and Chang 2012a, 2013c). All of these Laplacian tensors are in the spirit of the scheme of sums of powers. In formalism, they are not as simple as their matrix counterparts which can be written as $D - A$ or $D + A$ with A the adjacency matrix and D the diagonal matrix of degrees of a graph. Also, this approach only works for even-order hypergraphs.

Very recently, Qi (2012) proposed a simple definition $\mathcal{D}-\mathcal{A}$ for the Laplacian tensor and $\mathcal{D}+\mathcal{A}$ for the signless Laplacian tensor. Here $\mathcal{A} = (a_{i_1\ldots i_k})$ is the adjacency tensor of a k-uniform hypergraph and $\mathcal{D} = (d_{i_1\ldots i_k})$ the diagonal tensor with its diagonal elements being the degrees of the vertices. This is a natural generalization of the definition for $D - A$ and $D + A$ in spectral graph theory (Brouwer and Haemers 2011). The elements of the adjacency tensor, the Laplacian tensor and the signless Laplacian tensors are rational numbers. Some results were derived in Qi (2012). More results are expected along this simple and natural approach.

On the other hand, there is another approach in spectral graph theory for the Laplacian of a graph (Chung 1997). Suppose that G is a graph without isolated vertices. Let the degree of vertex i be d_i. The Laplacian, or the normalized Laplacian matrix, of G is defined as $L = I - \bar{A}$, where I is the identity matrix, $\bar{A} = (\bar{a}_{ij})$ is the normalized adjacency matrix, $\bar{a}_{ij} = \frac{1}{\sqrt{d_i d_j}}$, if vertices i and j are connected, and $\bar{a}_{ij} = 0$ otherwise. This approach involves irrational numbers in general. However, it is seen that λ is an eigenvalue of the Laplacian L if and only if $1 - \lambda$ is an eigenvalue of the normalized adjacency matrix $\bar{A}$. A comprehensive theory was developed based upon this by Chung (1997).

In this paper, we will investigate the normalized Laplacian tensor approach. A formal definition of the normalized Laplacian tensor and the normalized adjacency tensor will be given in Definition 2.7.

In the sequel, the normalized Laplacian tensor is simply called the Laplacian as in Chung (1997), and the normalized adjacency tensor simply as the adjacency tensor. In this paper, hypergraphs refer to k-uniform hypergraphs on n vertices. For a positive integer n, we use the convention $[n] := \{1, \ldots, n\}$. Let $G = (V, E)$ be a k-uniform hypergraph with vertex set $V = [n]$ and edge set E, and d_i be the degree of the vertex i. If $k = 2$, then G is a graph.

For a graph, let $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$ be the eigenvalues of L in increasing order. The following results are fundamental in spectral graph theory (Chung 1997, Sect. 1.3).

(i) $\lambda_0 = 0$ and $\sum_{i\in[n-1]} \lambda_i \leq n$ with equality holding if and only if G has no isolated vertices.
(ii) $0 \leq \lambda_i \leq 2$ for all $i \in [n-1]$, and $\lambda_{n-1} = 2$ if and only if a connected component of G is bipartite and nontrivial.
(iii) The spectrum of a graph is the union of the spectra of its connected components.
(iv) $\lambda_i = 0$ and $\lambda_{i+1} > 0$ if and only if G has exactly $i + 1$ connected components.

I. Our first major work is to show that the above results can be generalized to the Laplacian $\mathcal{L}$ of a uniform hypergraph. Let $c(n, k) := n(k-1)^{n-1}$. For a k-th order n-dimensional tensor, there are exactly $c(n, k)$ eigenvalues (with algebraic multiplicity) (Qi 2005; Hu et al. 2013). Let $\sigma(\mathcal{L})$ be the spectrum of $\mathcal{L}$ (the set of eigenvalues, which is also called the spectrum of G). Then, we have the followings.

(i) (Corollary 3.2) The smallest H-eigenvalue of $\mathcal{L}$ is zero.
(Proposition 3.1) $\sum_{\lambda\in\sigma(\mathcal{L})} m(\lambda)\lambda \leq c(n, k)$ with equality holding if and only if G has no isolated vertices. Here $m(\lambda)$ is the algebraic multiplicity of λ for all $\lambda \in \sigma(\mathcal{L})$.
(ii) (Theorem 3.1) For all $\lambda \in \sigma(\mathcal{L})$, $0 \leq \mathrm{Re}(\lambda)$ with equality holding if and only if $\lambda = 0$; and $\mathrm{Re}(\lambda) \leq 2$ with equality holding if and only if $\lambda = 2$.
(Corollary 6.2) When k is odd, we have that $\mathrm{Re}(\lambda) < 2$ for all $\lambda \in \sigma(\mathcal{L})$.
(Theorem 6.2/Corollary 6.5) When k is even, necessary and sufficient conditions are given for 2 being an eigenvalue/H-eigenvalue of $\mathcal{L}$.
(Corollary 6.6) When k is even and G is k-partite, 2 is an eigenvalue of $\mathcal{L}$.
(iii) (Theorem 3.1 together with Lemmas 2.1 and 3.3) Viewed as sets, the spectrum of G is the union of the spectra of its connected components.
Viewed as multisets, an eigenvalue of a connected component with algebraic multiplicity w contributes to G as an eigenvalue with algebraic multiplicity $w(k-1)^{n-s}$. Here s is the number of vertices of the connected component.
(iv) (Corollaries 3.2 and 4.1) Let all the H^+-eigenvalues of $\mathcal{L}$ be ordered in increasing order as $\mu_0 \leq \mu_1 \leq \cdots \leq \mu_{n(G)-1}$. Here $n(G)$ is the number of H^+-eigenvalues of $\mathcal{L}$ (with H^+-geometric multiplicity), see Definition 4.1.
Then $\mu_{n(G)-1} \leq 1$ with equality holding if and only if $|E| > 0$.
$\mu_0 = 0$; and $\mu_{i-2} = 0$ and $\mu_{i-1} > 0$ if and only if $\log_2 i$ is a positive integer and G has exactly $\log_2 i$ connected components. Thus, $\mu_1 > 0$ if and only if G is connected.

On top of these properties, we also show that the spectral radius of the adjacency tensor of a hypergraph with $|E| > 0$ is equal to one (Lemma 3.2). The linear subspace generated by the nonnegative H-eigenvectors of the smallest H-eigenvalue of the Laplacian has dimension exactly the number of the connected components of the hypergraph (Lemma 3.4). Equalities that the eigenvalues of the Laplacian should satisfy are given in Proposition 3.1. The only two H^+-eigenvalues of the Laplacian of a complete hypergraph are zero and one (Corollary 4.2). We give the H^+-geometric multiplicities of the H^+-eigenvalues zero and one of the Laplacian respectively in Lemma 4.4 and Proposition 4.2. We show that when k is odd and G is connected, the H-eigenvector of $\mathcal{L}$ corresponding to the H-eigenvalue zero is unique (Corollary 6.4). The spectrum of the adjacency tensor is invariant under multiplication by any s-th root of unity, here s is the primitive index of the adjacency tensor (Corollary 6.3). In particular, the spectrum of the adjacency tensor of a k-partite hypergraph is invariant under multiplication by any k-th root of unity (Corollary 6.6).

II. Our second major work is that we study the smallest H^+-eigenvalues of the sub-tensors of the Laplacian. We give variational characterizations for these H^+-eigenvalues (Lemma 5.1), and show that an H^+-eigenvalue of the Laplacian is the smallest H^+-eigenvalue of some sub-tensor of the Laplacian (Theorem 4.1 and 8). Bounds for these H^+-eigenvalues based on the degrees of the vertices and the second smallest H^+-eigenvalue of the Laplacian are given respectively in Propositions 5.1 and 5.2. We discuss the relations between these H^+-eigenvalues and the edge connectivity (Proposition 5.3) and the edge expansion (Proposition 5.5) of the hypergraph.

III. Our third major work is that we introduce the concept of spectral components of a hypergraph and investigate their intrinsic roles in the structure of the spectrum of the hypergraph. We simply interpret the idea of the spectral component first.

Let $G = (V, E)$ be a k-uniform hypergraph and $S \subset V$ be nonempty and proper. The set of edges $E(S, S^c) := \{e \in E \mid e \cap S \neq \emptyset,\ e \cap S^c \neq \emptyset\}$ is the *edge cut* with respect to S. Unlike the graph counterpart, the number of intersections $e \cap S^c$ may vary for different $e \in E(S, S^c)$. We say that $E(S, S^c)$ cuts S^c *with depth at least* $r \geq 1$ if $|e \cap S^c| \geq r$ for every $e \in E(S, S^c)$. A subset of V whose edge cut cuts its complement with depth at least two is closely related to an H^+-eigenvalue of the Laplacian. These sets are *spectral components* (Definition 2.5). With edge cuts of depth at least r, we define *r-th depth edge expansion* which generalizes the edge expansion for graphs (Definition 5.1). A *flower heart* of a hypergraph is also introduced (Definition 2.6), which is related to the largest H^+-eigenvalue of the Laplacian.

We show that the spectral components characterize completely the H^+-eigenvalues of the Laplacian that are less than one and vice verse, and the flower hearts are in one to one correspondence with the nonnegative eigenvectors of the H^+-eigenvalue one (Theorem 4.1). In general, the set of the H^+-eigenvalues of the Laplacian is strictly contained in the set of the smallest H^+-eigenvalues of its sub-tensors (Theorem 4.1 and Proposition 4.1). We introduce H^+-geometric multiplicity of an H^+-eigenvalue. The second smallest H^+-eigenvalue of the Laplacian is discussed, and a lower bound for it is given in Proposition 5.2. Bounds are given for the r-th depth edge expansion based on the second smallest H^+-eigenvalue of $\mathcal{L}$ for a connected hypergraph (Proposition 5.4 and Corollary 5.5). For a connected hypergraph, necessary and sufficient conditions for the second smallest H^+-eigenvalue of $\mathcal{L}$ being the largest H^+-eigenvalue (i.e., one) are given in Proposition 4.3.

The rest of this paper begins with some preliminaries in the next section. In Sect. 2.1, the eigenvalues of tensors and some related concepts are reviewed. Some basic facts about the spectral theory of symmetric nonnegative tensors are presented in Sect. 2.2. Some new observations are given. Some basic definitions on uniform hypergraphs are given in Sect. 2.3. The spectral components and the flower hearts of a hypergraph are introduced.

In Sect. 3.1, some facts about the spectrum of the adjacency tensor are discussed. Then some properties on the spectrum of the Laplacian are investigated in Sect. 3.2. We characterize all the H^+-eigenvalues of the Laplacian through the spectral components and the flower hearts of the hypergraph in Sect. 4.1. In Sect. 4.2, the H^+-geometric multiplicity is introduced, and the second smallest H^+-eigenvalue is explored.

The smallest H^+-eigenvalues of the sub-tensors of the Laplacian are discussed in Sect. 5. The variational characterizations of these eigenvalues are given in Sect. 5.1. Then their connections to the edge connectivity and the edge expansion are discussed in Sect. 5.2 and Sect. 5.3 respectively.

The eigenvectors of the eigenvalues on the spectral circle of the adjacency tensor are characterized in Sect. 6.1. It gives necessary and sufficient conditions under which the largest H-eigenvalue of the Laplacian is two. In Sect. 6.2, we reformulate the above conditions in the language of linear algebra over modules and give necessary and sufficient conditions under which the eigenvector of an eigenvalue on the spectral circle of the adjacency tensor is unique. Some final remarks are made in the last section.

2 Preliminaries

Some preliminaries on the eigenvalues and eigenvectors of tensors, the spectral theory of symmetric nonnegative tensors and basic concepts of uniform hypergraphs are presented in this section.

2.1 Eigenvalues of tensors

In this subsection, some basic facts about eigenvalues and eigenvectors of tensors are reviewed. For comprehensive references, see (Qi 2005, 2006, 2007; Hu et al. 2013) and references therein.

Let $\mathbb{C}$ ($\mathbb{R}$) be the field of complex (real) numbers and $\mathbb{C}^n$ ($\mathbb{R}^n$) the n-dimensional complex (real) space. The nonnegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$, the interior of $\mathbb{R}^n_+$ is denoted by $\mathbb{R}^n_{++}$. For integers $k \geq 3$ and $n \geq 2$, a real tensor $\mathcal{T} = (t_{i_1 \ldots i_k})$ of order k and dimension n refers to a multiway array (also called hypermatrix) with entries $t_{i_1 \ldots i_k}$ such that $t_{i_1 \ldots i_k} \in \mathbb{R}$ for all $i_j \in [n]$ and $j \in [k]$. Tensors are always referred to k-th order real tensors in this paper, and the dimensions will be clear from the content. Given a vector $\mathbf{x} \in \mathbb{C}^n$, define an n-dimensional vector $\mathcal{T}\mathbf{x}^{k-1}$ with its i-th element being $\sum_{i_2,\ldots,i_k \in [n]} t_{i i_2 \ldots i_k} x_{i_2} \cdots x_{i_k}$ for all $i \in [n]$. Let $\mathcal{I}$ be the identity tensor of appropriate dimension, e.g., $i_{i_1 \ldots i_k} = 1$ if and only if $i_1 = \cdots = i_k \in [n]$, and zero otherwise when the dimension is n. The following definitions are introduced by Qi (2005, 2012).

Definition 2.1 Let $\mathcal{T}$ be a k-th order n-dimensional real tensor. For some $\lambda \in \mathbb{C}$, if polynomial system $(\lambda \mathcal{I} - \mathcal{T})\mathbf{x}^{k-1} = 0$ has a solution $\mathbf{x} \in \mathbb{C}^n \setminus \{0\}$, then λ is called an eigenvalue of the tensor $\mathcal{T}$ and $\mathbf{x}$ an eigenvector of $\mathcal{T}$ associated with λ. If an eigenvalue λ has an eigenvector $\mathbf{x} \in \mathbb{R}^n$, then λ is called an H-eigenvalue and $\mathbf{x}$ an H-eigenvector. If $\mathbf{x} \in \mathbb{R}^n_+$ ($\mathbb{R}^n_{++}$), then λ is called an H^+-(H^{++}-)eigenvalue.

It is easy to see that an H-eigenvalue is real. We denote by $\sigma(\mathcal{T})$ the set of all eigenvalues of the tensor $\mathcal{T}$. It is called the spectrum of $\mathcal{T}$. We denoted by $\rho(\mathcal{T})$ the maximum module of the eigenvalues of $\mathcal{T}$. It is called the spectral radius of $\mathcal{T}$. In the sequel, unless stated otherwise, an eigenvector $\mathbf{x}$ would always refer to its normalization $\frac{\mathbf{x}}{\sqrt[k]{\sum_{i\in[n]}|x_i|^k}}$. This convention does not introduce any ambiguities, since the eigenvector defining equations are homogeneous. Hence, when $bfxin\mathbb{R}^n_+$, we always refer to $\mathbf{x}$ satisfying $\sum_{i=1}^n x_i^k = 1$. The *algebraic multiplicity* of an eigenvalue is defined as the multiplicity of this eigenvalue as a root of the characteristic polynomial $\chi_{\mathcal{T}}(\lambda)$. To give the definition of the characteristic polynomial, the determinant or the resultant theory is needed. For the determinant theory of a tensor, see (Hu et al. 2013). For the resultant theory of polynomial equations, see (Cox 2006, 1998).

Definition 2.2 Let $\mathcal{T}$ be a k-th order n-dimensional real tensor and λ be an indeterminate variable. The determinant $\mathrm{Det}(\lambda\mathcal{I} - \mathcal{T})$ of $\lambda\mathcal{I} - \mathcal{T}$, which is a polynomial in $\mathbb{C}[\lambda]$ and denoted by $\chi_{\mathcal{T}}(\lambda)$, is called the *characteristic polynomial* of the tensor $\mathcal{T}$.

It is shown that $\sigma(\mathcal{T})$ equals the set of roots of $\chi_{\mathcal{T}}(\lambda)$, see (Hu et al. 2013, Theorem 2.3). If λ is a root of $\chi_{\mathcal{T}}(\lambda)$ of multiplicity s, then we call s the algebraic multiplicity of the eigenvalue λ. Let $c(n,k) = n(k-1)^{n-1}$. By (Hu et al. 2013, Theorem 2.3) $\chi_{\mathcal{T}}(\lambda)$ is a monic polynomial of degree $c(n,k)$.

Definition 2.3 Let $\mathcal{T}$ be a k-th order n-dimensional real tensor and $s \in [n]$. The k-th order s-dimensional tensor $\mathcal{U}$ with entries $u_{i_1\dots i_k} = t_{j_{i_1}\dots j_{i_k}}$ for all $i_1,\dots,i_k \in [s]$ is called the *sub-tensor* of $\mathcal{T}$ associated to the subset $S := \{j_1,\dots,j_s\}$. We usually denoted $\mathcal{U}$ as $\mathcal{T}(S)$.

For a subset $S \subseteq [n]$, we denoted by $|S|$ its cardinality. For $bfxin\mathbb{C}^n$, $bfxS)$ is defined as an $|S|$-dimensional sub-vector of $\mathbf{x}$ with its entries being x_i for $i \in S$, and $\sup(\mathbf{x}) := \{i \in [n] \mid x_i \neq 0\}$ is its *support*. The following lemma follows from (Hu et al. 2013, Theorem 4.2).

Lemma 2.1 *Let $\mathcal{T}$ be a k-th order n-dimensional real tensor such that there exists an integer $s \in [n-1]$ satisfying $t_{i_1 i_2\dots i_k} \equiv 0$ for every $i_1 \in \{s+1,\dots,n\}$ and all indices $i_2,\dots,i_k$ such that $\{i_2,\dots,i_k\} \cap \{1,\dots,s\} \neq \emptyset$. Denote by $\mathcal{U}$ and $\mathcal{V}$ the sub-tensors of $\mathcal{T}$ associated to $[s]$ and $\{s+1,\dots,n\}$, respectively. Then it holds that*

$$\sigma(\mathcal{T}) = \sigma(\mathcal{U}) \cup \sigma(\mathcal{V}).$$

Moreover, if $\lambda \in \sigma(\mathcal{U})$ is an eigenvalue of the tensor $\mathcal{U}$ with algebraic multiplicity r, then it is an eigenvalue of the tensor $\mathcal{T}$ with algebraic multiplicity $r(k-1)^{n-s}$, and if $\lambda \in \sigma(\mathcal{V})$ is an eigenvalue of the tensor $\mathcal{V}$ with algebraic multiplicity p, then it is an eigenvalue of the tensor $\mathcal{T}$ with algebraic multiplicity $p(k-1)^s$.

2.2 Symmetric nonnegative tensors

The spectral theory of nonnegative tensors is a useful tool to investigate the spectrum of a uniform hypergraph (Cooper andv Dutle 2012; Pearson and Zhang 2012; Xie and Chang 2012a, b, 2013c; Qi 2012). A tensor is called nonnegative, if all of its entries are nonnegative. A tensor $\mathcal{T}$ is called symmetric, if $t_{\tau(i_1)\ldots\tau(i_k)} = t_{i_1\ldots i_k}$ for all permutations τ on $(i_1, \ldots, i_k)$ and all $i_1, \ldots, i_k \in [n]$. In this subsection, we present some basic facts about symmetric nonnegative tensors which will be used extensively in the sequel. For comprehensive references on this topic, see (Chang et al. 2008, 2011; Friedland 2013; Hu et al. 2011; Ng et al. 2009; Qi 2012; Yang and Yang 2010, 2011) and references therein.

By (Pearson and Zhang 2012, Lemma 3.1) hypergraphs are related to weakly irreducible nonnegative tensors. Essentially, weakly irreducible nonnegative tensors are introduced in Friedland (2013). In this paper, we adopt the following definition (Hu et al. 2011, Definition 2.2). For the definition of reducibility for a nonnegative matrix, see (Horn 1985, Chap. 8).

Definition 2.4 Suppose that $\mathcal{T}$ is a nonnegative tensor of order k and dimension n. We call an $n \times n$ nonnegative matrix $R(\mathcal{T})$ the *representation* of $\mathcal{T}$, if the (i, j)-th element of $R(\mathcal{T})$ is defined to be the summation of $t_{ii_2\ldots i_k}$ with indices $\{i_2, \ldots, i_k\} \ni j$.

We say that the tensor $\mathcal{T}$ is *weakly reducible* if its representation $R(\mathcal{T})$ is a reducible matrix. If $\mathcal{T}$ is not *weakly reducible*, then it is called *weakly irreducible*.

For convenience, a one dimensional tensor, i.e., a scalar, is regarded to be weakly irreducible.

We summarize the Perron–Frobenius theorem for nonnegative tensors which will be used in this paper in the next lemma. For comprehensive references on this theory, see (Chang et al. 2008; Yang and Yang 2010, 2011; Friedland 2013; Hu et al. 2011; Chang et al. 2011; Qi 2012) and references therein.

Lemma 2.2 *Let $\mathcal{T}$ be a nonnegative tensor. Then we have the followings.*

(i) $\rho(\mathcal{T})$ is an H^+-eigenvalue of $\mathcal{T}$.
(ii) If $\mathcal{T}$ is weakly irreducible, then $\rho(\mathcal{T})$ is the unique H^{++}-eigenvalue of $\mathcal{T}$.

Proof The conclusion (i) follows from (Yang and Yang 2010, Theorem 2.3). The conclusion (ii) follows from (Friedland 2013, Theorem 4.1). □

The next lemma is useful.

Lemma 2.3 *Let $\mathcal{B}$ and $\mathcal{C}$ be two nonnegative tensors, and $\mathcal{B} \geq \mathcal{C}$ in the sense of componentwise. If $\mathcal{B}$ is weakly irreducible and $\mathcal{B} \neq \mathcal{C}$, then $\rho(\mathcal{B}) > \rho(\mathcal{C})$. Thus, if $\mathbf{x} \in \mathbb{R}^n_+$ is an eigenvector of $\mathcal{B}$ corresponding to $\rho(\mathcal{B})$, then $\mathbf{x} \in \mathbb{R}^n_{++}$ is positive.*

Proof By (Yang and Yang 2011, Theorem 3.6) $\rho(\mathcal{B}) \geq \rho(\mathcal{C})$ and the equality holding implies that $|\mathcal{C}| = \mathcal{B}$. Since $\mathcal{C}$ is nonnegative and $\mathcal{B} \neq \mathcal{C}$, we must have the strict inequality.

The second conclusion follows immediately from the first one and the weak irreducibility of $\mathcal{B}$. For another proof, see (Yang and Yang 2011, Lemma 3.5). □

Note that the second conclusion of Lemma 2.3 is equivalent to that $\rho(\mathcal{S}) < \rho(\mathcal{B})$ for any sub-tensor $\mathcal{S}$ of $\mathcal{B}$ other than the trivial case $\mathcal{S} = \mathcal{B}$. By (Hu et al. 2011, Theorem 5.3) without the weakly irreducible hypothesis, it is easy to construct an example such that the strict inequality in Lemma 2.3 fails.

For general nonnegative tensors which are weakly reducible, there is a characterization on their spectral radii based on partitions, see (Hu et al. 2011, Theorems 5.2 and 5.3). As remarked before (Hu et al. 2011, Theorem 5.4) such partitions can result in diagonal block representations for symmetric nonnegative tensors. Recently, Qi proved that for a symmetric nonnegative tensor $\mathcal{T}$, it holds that (Qi 2012, Theorem 2)

$$\rho(\mathcal{T}) = \max\left\{\mathcal{T}\mathbf{x}^k := \mathbf{x}^T(\mathcal{T}\mathbf{x}^{k-1}) \mid \mathbf{x} \in \mathbb{R}^n_+,\ \sum_{i\in[n]} x_i^k = 1\right\}. \tag{1}$$

We summarize the above results in the next theorem with some new observations.

Theorem 2.1 *Let $\mathcal{T}$ be a symmetric nonnegative tensor of order k and dimension n. Then, there exists a pairwise disjoint partition $\{S_1, \ldots, S_r\}$ of the set $[n]$ such that every tensor $\mathcal{T}(S_j)$ is weakly irreducible. Moreover, we have the followings.*

(i) For any $\mathbf{x} \in \mathbb{C}^n$,

$$\mathcal{T}\mathbf{x}^k = \sum_{j\in[r]} \mathcal{T}(S_j)\mathbf{x}(S_j)^k, \ \textit{and}\ \rho(\mathcal{T}) = \max_{j\in[r]} \rho(\mathcal{T}(S_j)).$$

(ii) λ is an eigenvalue of $\mathcal{T}$ with eigenvector $\mathbf{x}$ if and only if λ is an eigenvalue of $\mathcal{T}(S_i)$ with eigenvector $\frac{\mathbf{x}(S_i)}{\sqrt[k]{\sum_{j\in S_i}|x_j|^k}}$ whenever $\mathbf{x}(S_i) \neq 0$.

(iii) $\rho(\mathcal{T}) = \max\{\mathcal{T}\mathbf{x}^k \mid \mathbf{x} \in \mathbb{R}^n_+,\ \sum_{i\in[n]} x_i^k = 1\}$. Furthermore, $\mathbf{x} \in \mathbb{R}^n_+$ is an eigenvector of $\mathcal{T}$ corresponding to $\rho(\mathcal{T})$ if and only if it is an optimal solution of the maximization problem 1.

Proof (i) By (Hu et al. 2011, Theorem 5.2) there exists a pairwise disjoint partition $\{S_1, \ldots, S_r\}$ of the set $[n]$ such that every tensor $\mathcal{T}(S_j)$ is weakly irreducible. Moreover, by the proof for (Hu et al. 2011, Theorem 5.2) and the fact that $\mathcal{T}$ is symmetric, $\{\mathcal{T}(S_j),\ j \in [r]\}$ encode all the possible nonzero entries of the tensor $\mathcal{T}$. After a reordering of the index set, if necessary, we get a diagonal block representation of the tensor $\mathcal{T}$. Thus, $\mathcal{T}\mathbf{x}^k = \sum_{j\in[r]} \mathcal{T}(S_j)\mathbf{x}(S_j)^k$ follows for every $\mathbf{x} \in \mathbb{C}^n$. The spectral radii characterization is (Hu et al. 2011, Theorem 5.3).

(ii) follows from the partition immediately.

(iii) Suppose that $\mathbf{x} \in \mathbb{R}^n_+$ is an eigenvector of $\mathcal{T}$ corresponding to $\rho(\mathcal{T})$, then $\rho(\mathcal{T}) = \mathbf{x}^T(\mathcal{T}\mathbf{x}^{k-1})$. Hence, $\mathbf{x}$ is an optimal solution of 1.

On the other side, suppose that $\mathbf{x}$ is an optimal solution of 1. Then, by (i), we have

$$\rho(\mathcal{T}) = \mathcal{T}\mathbf{x}^k = \mathcal{T}(S_1)\mathbf{x}(S_1)^k + \cdots + \mathcal{T}(S_r)\mathbf{x}(S_r)^k.$$

Whenever $\mathbf{x}(S_i) \neq 0$, we must have $\rho(\mathcal{T})(\sum_{j\in S_i}(x(S_i))_j^k) = \mathcal{T}(S_i)\mathbf{x}(S_i)^k$, since $\rho(\mathcal{T})(\sum_{j\in S_i}(y(S_i))_j^k) \geq \mathcal{T}(S_i)\mathbf{y}(S_i)^k$ for any $\mathbf{y} \in \mathbb{R}_+^n$ by 1. Hence, $\rho(\mathcal{T}(S_i)) = \rho(\mathcal{T})$. By Lemma 2.3, 1 and the weak irreducibility of $\mathcal{T}(S_i)$, we must have that $\mathbf{x}(S_i)$ is a positive vector whenever $\mathbf{x}(S_i) \neq 0$. Otherwise, $\rho([\mathcal{T}(S_i)](\sup(\mathbf{x}(S_i)))) = \rho(\mathcal{T}(S_i))$ with $\sup(\mathbf{x}(S_i))$ being the support of $\mathbf{x}(S_i)$, which is a contradiction. Thus,

$$\max\left\{\mathcal{T}(S_i)\mathbf{z}^k \mid \mathbf{z} \in \mathbb{R}_+^{|S_i|},\ \sum_{i\in S_i} z_i^k = 1\right\}$$

has an optimal solution $\mathbf{x}(S_i)$ in $\mathbb{R}_{++}^{|S_i|}$. By optimization theory (Bertsekas 1999), we must have that $(\mathcal{T}(S_i) - \rho(\mathcal{T})\mathcal{I})\mathbf{x}(S_i)^{k-1} = 0$. Then, by (ii), $\mathbf{x}$ is an eigenvector of $\mathcal{T}$. □

2.3 Uniform hypergraphs

In this subsection, we present some preliminary concepts of uniform hypergraphs which will be used in this paper. Please refer to Berge (1973), Chung (1997), Brouwer and Haemers (2011) for comprehensive references.

In this paper, unless stated otherwise, a hypergraph means an undirected simple k-uniform hypergraph G with vertex set V, which is labeled as $[n] = \{1, \ldots, n\}$, and edge set E. By k-uniformity, we mean that for every edge $e \in E$, the cardinality $|e|$ of e is equal to k. Throughout this paper, $k \geq 3$ and $n \geq k$.

For a subset $S \subset [n]$, we denoted by E_S the set of edges $\{e \in E \mid S \cap e \neq \emptyset\}$. For a vertex $i \in V$, we simplify $E_{\{i\}}$ as E_i. It is the set of edges containing the vertex i, i.e., $E_i := \{e \in E \mid i \in e\}$. The cardinality $|E_i|$ of the set E_i is defined as the *degree* of the vertex i, which is denoted by d_i. Then we have that $k|E| = \sum_{i\in[n]} d_i$. If $d_i = 0$, then we say that the vertex i is *isolated*. Two different vertices i and j are *connected* to each other (or the pair i and j is connected), if there is a sequence of edges $(e_1, \ldots, e_m)$ such that $i \in e_1$, $j \in e_m$ and $e_r \cap e_{r+1} \neq \emptyset$ for all $r \in [m-1]$. A hypergraph is called *connected*, if every pair of different vertices of G is connected. A set $S \subseteq V$ is a *connected component* of G, if every two vertices of S are connected and there is no vertices in $V \setminus S$ that are connected to any vertex in S. For the convenience, an isolated vertex is regarded as a connected component as well. Then, it is easy to see that for every hypergraph G, there is a partition of V as pairwise disjoint subsets $V = V_1 \cup \ldots \cup V_s$ such that every V_i is a connected component of G. Let $S \subseteq V$, the hypergraph with vertex set S and edge set $\{e \in E \mid e \subseteq S\}$ is called the *sub-hypergraph* of G induced by S. We will denoted it by G_S. In the sequel, unless stated otherwise, all the notations introduced above are reserved for the specific meanings.

Here are some convention. For a subset $S \subseteq [n]$, S^c denotes the complement of S in $[n]$. For a nonempty subset $S \subseteq [n]$ and $\mathbf{x} \in \mathbb{C}^n$, we denoted by $\mathbf{x}^S$ the monomial $\prod_{i\in S} x_i$.

Let $G = (V, E)$ be a k-uniform hypergraph. Let $S \subset V$ be a nonempty proper subset. Then, the edge set is partitioned into three pairwise disjoint parts: $E(S) := \{e \in E \mid e \subseteq S\}$, $E(S^c)$ and $E(S, S^c) := \{e \in E \mid e \cap S \neq \emptyset,\ e \cap S^c \neq \emptyset\}$. $E(S, S^c)$ is called the *edge cut* of G with respect to S.

When G is a usual graph (i.e., $k = 2$), for every edge in an edge cut $E(S, S^c)$ whenever it is nonempty, it contains exactly one vertex from S and the other one from S^c. When G is a k-uniform hypergraph with $k \geq 3$, the situation is much more complicated. We will say that an edge in $E(S, S^c)$ cuts S *with depth at least* r $(1 \leq r < k)$ if there are at least r vertices in this edge belonging to S. If every edge in the edge cut $E(S, S^c)$ cuts S with depth at least r, then we say that $E(S, S^c)$ cuts S *with depth at least* r.

Definition 2.5 Let $G = (V, E)$ be a k-uniform hypergraph. A nonempty subset $B \subseteq V$ is called a **spectral component** of the hypergraph G if either the edge cut $E(B, B^c)$ is empty or $E(B, B^c)$ cuts B^c with depth at least two.

It is easy to see that any nonempty subset $B \subset V$ satisfying $|B| \leq k - 2$ is a spectral component. Suppose that G has connected components $\{V_1, \ldots, V_r\}$, it is easy to see that $B \subset V$ is a spectral component of G if and only if $B \cap V_i$, whenever nonempty, is a spectral component of G_{V_i}. We will establish the correspondence between the H^+-eigenvalues that are less than one with the spectral components of the hypergraph, see Theorem 4.1.

Definition 2.6 Let $G = (V, E)$ be a k-uniform hypergraph. A nonempty proper subset $B \subseteq V$ is called a **flower heart** if B^c is a spectral component and $E(B^c) = \emptyset$.

If B is a flower heart of G, then G likes a flower with edges in $E(B, B^c)$ as leafs. It is easy to see that any proper subset $B \subset V$ satisfying $|B| \geq n - k + 2$ is a flower heart. There is a similar characterization between the flower hearts of G and these of its connected components. Theorem 4.1 will show that the flower hearts of a hypergraph correspond to its largest H^+-eigenvalue.

We here give the definition of the normalized Laplacian tensor of a uniform hypergraph.

Definition 2.7 Let G be a k-uniform hypergraph with vertex set $[n] = \{1, \ldots, n\}$ and edge set E. The **normalized adjacency tensor** $\mathcal{A}$, which is a k-th order n-dimension symmetric nonnegative tensor, is defined as

$$a_{i_1 i_2 \ldots i_k} := \begin{cases} \frac{1}{(k-1)!} \prod\limits_{j \in [k]} \frac{1}{\sqrt[k]{d_{i_j}}} & \textit{if } \{i_1, i_2 \ldots, i_k\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

The **normalized Laplacian tensor** $\mathcal{L}$, which is a k-th order n-dimensional symmetric tensor, is defined as

$$\mathcal{L} := \mathcal{J} - \mathcal{A},$$

where $\mathcal{J}$ is a k-th order n-dimensional diagonal tensor with the i-th diagonal element $j_{i \ldots i} = 1$ whenever $d_i > 0$, and zero otherwise.

When G has no isolated points, we have that $\mathcal{L} = \mathcal{I} - \mathcal{A}$. The spectrum of $\mathcal{L}$ is called the spectrum of the hypergraph G, and they are referred interchangeably.

The current definition is motivated by the formalism of the normalized Laplacian matrix of a graph investigated extensively by Chung (1997). We have a similar explanation for the normalized Laplacian tensor to the Laplacian tensor (i.e., $\mathcal{L} = P^k \cdot (\mathcal{D}–\mathcal{B})$ [1]) as that for the normalized Laplacian matrix to the Laplacian matrix (Chung 1997). Here P is a diagonal matrix with its i-th diagonal element being $\frac{1}{\sqrt[k]{d_i}}$ when $d_i > 0$ and zero otherwise.

We have already pointed out one of the advantages of this definition, namely, $\mathcal{L} = \mathcal{I} - \mathcal{A}$ whenever G has no isolated vertices. Such a special structure only happens for regular hypergraphs under the definition in Qi (2012). (A hypergraph is called regular if d_i is a constant for all $i \in [n]$.) By Definition 2.1, the eigenvalues of $\mathcal{L}$ are exactly a shift of the eigenvalues of $-\mathcal{A}$. Thus, we can establish many results on the spectra of uniform hypergraphs through the spectral theory of nonnegative tensors without the hypothesis of regularity. We note that, by Definition 2.1, $\mathcal{L}$ and $\mathcal{D} - \mathcal{B}$ do not share the same spectrum unless G is regular.

In the sequel, the normalized Laplacian tensor and the normalized adjacency tensor are simply called the Laplacian and the adjacency tensor respectively.

By Definition 2.4, the following lemma can be proved similarly to (Pearson and Zhang 2012, Lemma 3.1).

Lemma 2.4 *Let G be a k-uniform hypergraph with vertex set V and edge set E. G is connected if and only if $\mathcal{A}$ is weakly irreducible.*

For a hypergraph $G = (V, E)$, it can be partitioned into connected components $V = V_1 \cup \cdots \cup V_r$ for $r \geq 1$. Reorder the indices, if necessary, $\mathcal{L}$ can be represented by a block diagonal structure according to $V_1, \ldots, V_r$. By Definition 2.1, the spectrum of $\mathcal{L}$ does not change when reordering the indices. Thus, in the sequel, we assume that $\mathcal{L}$ is in the block diagonal structure with its i-th block tensor being the sub-tensor of $\mathcal{L}$ associated to V_i for $i \in [r]$. By Definition 2.7, it is easy to see that $\mathcal{L}(V_i)$ is the Laplacian of the sub-hypergraph G_{V_i} for all $i \in [r]$. Similar convention for the adjacency tensor $\mathcal{A}$ is assumed.

3 The spectrum of a uniform hypergraph

Basic properties of the eigenvalues of a uniform hypergraph are established in this section.

3.1 The adjacency tensor

In this subsection, some basic facts of the eigenvalues of the adjacency tensor are discussed.

[1] The matrix-tensor product is in the sense of (Qi 2005, p. 1321): $\mathcal{L} = (l_{i_1\ldots i_k}) := P^k \cdot (\mathcal{D} - \mathcal{A})$ is a k-th order n-dimensional tensor with its entries being $l_{i_1\ldots i_k} := \sum_{j_s \in [n],\, s \in [k]} p_{i_1 j_1} \cdots p_{i_k j_k} (d_{j_1\ldots j_k} - a_{j_1\ldots j_k})$.

By Definition 2.1, H$^+$-eigenvalues of $\mathcal{A}$ should be nonnegative, since $\mathcal{A}$ is nonnegative. For a connected hypergraph G, the following lemma says that the smallest H$^+$-eigenvalue of $\mathcal{A}$ is zero. Moreover, it establishes the relations between the nonnegative eigenvectors of the zero eigenvalue of $\mathcal{A}$ and the flower hearts of G.

Lemma 3.1 *Let G be a k-uniform connected hypergraph. Then zero is the smallest H^+-eigenvalue of $\mathcal{A}$. Moreover, a nonzero vector $\mathbf{x} \in \mathbb{R}^n_+$ is an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue zero if and only if $[sup(\mathbf{x})]^c$ is a flower heart of G.*

Proof Let $\mathbf{x}$ be the vector with its i-th element being one and the other entries being zero. Then, by Definition 2.7, it is easy to see

$$\mathcal{A}\mathbf{x}^{k-1} = 0.$$

Thus, zero is an H$^+$-eigenvalue of $\mathcal{A}$ by Definition 2.1. The minimality follows from the nonnegativity of H$^+$-eigenvalues.

For the necessity of the second half of this lemma, suppose that $\mathbf{x} \in \mathbb{R}^n_+$ is an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue zero. Since $\mathcal{A}\mathbf{x}^k = 0$ and G is connected with $n \geq k$, we must have that $\sup(\mathbf{x})$ is a proper subset of $[n]$. Thus, $[\sup(\mathbf{x})]^c$ is nonempty. Let $\tilde{\mathbf{d}} \in \mathbb{R}^n$ be the n-vector with its i-th element being $\sqrt[k]{d_i}$ for all $i \in [n]$. Then, by Definition 2.7, for all $i \in [\sup(\mathbf{x})]^c$,

$$\begin{aligned} 0 = (\mathcal{A}\mathbf{x}^{k-1})_i &= \sum_{e \in E([\sup(\mathbf{x})]^c),\ i \in e} \frac{\mathbf{x}^{e\setminus\{i\}}}{\tilde{\mathbf{d}}^e} + \sum_{e \in E(\sup(\mathbf{x}),[\sup(\mathbf{x})]^c),\ i \in e} \frac{\mathbf{x}^{e\setminus\{i\}}}{\tilde{\mathbf{d}}^e} \\ &= \sum_{e \in E(\sup(\mathbf{x}),[\sup(\mathbf{x})]^c),\ i \in e} \frac{\mathbf{x}^{e\setminus\{i\}}}{\tilde{\mathbf{d}}^e}. \end{aligned}$$

Thus, we have that $\mathbf{x}^{e\setminus\{i\}} = 0$ for all $e \in \{e \mid e \in E(\sup(\mathbf{x}), [\sup(\mathbf{x})]^c),\ i \in e\}$ whenever it is nonempty. Thus, the edge cut $E(\sup(\mathbf{x}), [\sup(\mathbf{x})]^c)$ must satisfy that either it is empty or it cuts $[\sup(\mathbf{x})]^c$ with depth at least two. Then, by Definition 2.5, $\sup(\mathbf{x})$ is a spectral component.

For the other $i \in \sup(\mathbf{x})$, we have

$$\begin{aligned} 0 = (\mathcal{A}\mathbf{x}^{k-1})_i &= \sum_{e \in E(\sup(\mathbf{x})),\ i \in e} \frac{\mathbf{x}^{e\setminus\{i\}}}{\tilde{\mathbf{d}}^e} + \sum_{e \in E(\sup(\mathbf{x}),[\sup(\mathbf{x})]^c),\ i \in e} \frac{\mathbf{x}^{e\setminus\{i\}}}{\tilde{\mathbf{d}}^e} \\ &= \sum_{e \in E(\sup(\mathbf{x})),\ i \in e} \frac{\mathbf{x}^{e\setminus\{i\}}}{\tilde{\mathbf{d}}^e}. \end{aligned}$$

Hence, $E(\sup(\mathbf{x}))$ must be empty. This, together with the previous result and Definition 2.6, implies that $[\sup(\mathbf{x})]^c$ is a flower heart.

For the sufficiency, suppose that there is a nonnegative nonzero vector $\mathbf{x}$ such that $[\sup(\mathbf{x})]^c$ is a flower heart of G. Reversing the above analysis, it is easy to see that $\mathcal{A}\mathbf{x}^{k-1} = 0$. Hence, $\mathbf{x} \in \mathbb{R}^n_+$ is an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue zero.

The proof is complete. □

By Lemma 2.2, $\rho(\mathcal{A})$ is the largest H$^+$-eigenvalue of $\mathcal{A}$. The next lemma says that $\rho(\mathcal{A}) = 1$ if and only if $|E| > 0$, and $\rho(\mathcal{A}) = 0$ if and only if $|E| = 0$.

Lemma 3.2 *Let G be a k-uniform hypergraph. Then $\mathcal{A}$ is a symmetric nonnegative tensor, and $\rho(\mathcal{A})$ is the largest H^+-eigenvalue of $\mathcal{A}$. Moreover, if $E = \emptyset$, then $\mathcal{A} = 0$ and hence $\rho(\mathcal{A}) = 0$; and if G has at least one edge, then $\rho(\mathcal{A}) = 1$.*

Proof The first half of the conclusion follows from Lemma 2.2 and Definition 2.7.

The trivial case $E = \emptyset$ is obvious. In the following, we assume that $E \neq \emptyset$ and prove that $\rho(\mathcal{A}) = 1$. Let $\mathbf{x}$ be a nonzero nonnegative vector. Then, we have that

$$\mathcal{A}\mathbf{x}^k = \sum_{e\in E} k \prod_{i\in e} \frac{x_i}{\sqrt[k]{d_i}} \le \sum_{e\in E} k \left(\frac{1}{k} \sum_{i\in e} \left(\frac{x_i}{\sqrt[k]{d_i}} \right)^k \right) = \sum_{e\in E}\sum_{i\in e} \frac{x_i^k}{d_i} = \sum_{i\in[n],\ d_i>0} \sum_{e\in E_i} \frac{x_i^k}{d_i}$$
$$= \sum_{i\in[n],\ d_i>0} x_i^k.$$

By Theorem 2.1 (iii), we then have that $\rho(\mathcal{A}) \le 1$.

Let $\tilde{\mathbf{d}} \in \mathbb{R}^n$ be the n-vector with its i-th element being $\sqrt[k]{d_i}$ for all $i \in [n]$. Then, by Definition 2.7, we have that

$$\mathcal{A}\tilde{\mathbf{d}}^k = \sum_{e\in E} k \tilde{\mathbf{d}}^e \prod_{i\in e} \frac{1}{\sqrt[k]{d_i}} = \sum_{e\in E} k \tilde{\mathbf{d}}^e \frac{1}{\tilde{\mathbf{d}}^e} = \sum_{e\in E} k = k|E| = \sum_{i\in[n]} d_i > 0.$$

Thus, $\mathcal{A}\left(\frac{\tilde{\mathbf{d}}}{\sqrt[k]{\sum_{i\in[n]} d_i}}\right)^k = 1$. This, together with $\rho(\mathcal{A}) \le 1$ and Theorem 2.1 (iii), implies that $\rho(\mathcal{A}) = 1$. □

The next lemma is a direct consequence of Lemmas 2.1, 3.1 and 3.2, and Definition 2.7.

Lemma 3.3 *Let G be a k-uniform hypergraph. Suppose that G has $s \ge 0$ isolated vertices $\{i_1, \ldots, i_s\}$ and $r \ge 0$ connected components $V_1, \ldots, V_r$ satisfying $|V_i| > 1$ for $i \in [r]$. Then we have the followings.*

(i) As sets,

$$\sigma(\mathcal{A}) = \sigma(\mathcal{A}_1) \cup \sigma(\mathcal{A}_2) \cup \cdots \cup \sigma(\mathcal{A}_r), \tag{2}$$

where $\mathcal{A}_i$ is the sub-tensor of $\mathcal{A}$ associated to V_i for $i \in [r]$, and the right hand side of 2 is understood as $\{0\}$ whenever $r = 0$.

(ii) $\mathcal{A}_i$ defined above is the adjacency tensor of the sub-hypergraph G_i of G induced by V_i for all $i \in [r]$. Thus, $\rho(\mathcal{A}_i) = 1$.

(iii) Let $m_i(\lambda)$ be the algebraic multiplicity of λ as an eigenvalue of $\mathcal{A}_i$. As multisets, we have that zero is an eigenvalue of $\mathcal{A}$ with algebraic multiplicity

$$s(k-1)^{n-1} + \sum_{i\in[r]} m_i(0)(k-1)^{n-|V_i|},$$

and $\lambda \in \sigma(\mathcal{A}_i) \setminus \{0\}$ *is an eigenvalue of* $\mathcal{A}$ *with algebraic multiplicity*

$$\sum_{i\in[r]} m_i(\lambda)(k-1)^{n-|V_i|}.$$

The next corollary follows from Lemmas 2.2, 2.4 and 3.3, and Theorem 2.1 (ii).

Corollary 3.1 *Let* G *be a* k*-uniform hypergraph. Then,* 1 *is the unique* H^{++}*-eigenvalue of* $\mathcal{A}$ *if and only if* G *has no isolated vertices.*

3.2 The Laplacian

In this subsection, we discuss some facts on the eigenvalues of the Laplacian of a uniform hypergraph. We start with the following theorem.

Theorem 3.1 *Let* G *be a* k*-uniform hypergraph. Then, we have the followings.*

(i) If G *has at least one edge, then* $\lambda \in \sigma(\mathcal{L})$ *if and only if* $1-\lambda \in \sigma(\mathcal{A})$*. Otherwise,* $\sigma(\mathcal{L}) = \sigma(\mathcal{A}) = \{0\}$.
(ii) If $\lambda \in \sigma(\mathcal{L})$*, then* $Re(\lambda) \geq 0$ *with equality holding if and only if* $\lambda = 0$*, and* $2 \geq Re(\lambda)$ *with equality holding if and only if* $\lambda = 2$.

Proof Suppose that G has $s \geq 0$ isolated vertices $\{i_1, \ldots, i_s\}$ and $r \geq 0$ connected components $V_1, \ldots, V_r$ satisfying $|V_i| > 1$ for $i \in [r]$. Let $\mathcal{A}_i$ be the adjacency tensor and $\mathcal{L}_i$ the Laplacian of the sub-hypergraph G_{V_i} of G induced by V_i for all $i \in [r]$.

For the conclusion (i), if $s = n$, then $\mathcal{L} = \mathcal{A} = 0$. Thus, $\sigma(\mathcal{L}) = \sigma(\mathcal{A}) = \{0\}$. If $s = 0$, then $\mathcal{L} = \mathcal{I} - \mathcal{A}$ by Definition 2.7. We get the conclusion (i) by Definition 2.1. In the following, suppose that G has at least one edge and $s \geq 1$. We then have that $r \geq 1$. By Lemma 2.1, Theorem 2.1 and Definition 2.7, $\mathcal{L}$ has a block diagonal structure with diagonal sub-tensors $\{0, \mathcal{L}_1, \ldots, \mathcal{L}_r\}$, and moreover

$$\sigma(\mathcal{L}) = \{0\} \cup \sigma(\mathcal{L}_1) \cup \cdots \cup \sigma(\mathcal{L}_r).$$

Since every G_{V_i} is connected, by the established results, we have that $\lambda \in \sigma(\mathcal{L}_i)$ if and only if $1-\lambda \in \sigma(\mathcal{A}_i)$ for all $i \in [r]$. By Lemmas 3.2 and 3.3, we have that $\rho(\mathcal{A}_i) = 1$ for all $i \in [r]$. Hence, $\{0\} \subset \sigma(\mathcal{L}_i)$ for all $i \in [r]$. Combining these results, (i) follows.

For the conclusion (ii), if G has no edges, then the results are trivial. In the sequel, suppose that $s < n$. If $\lambda \in \sigma(\mathcal{L})$, then $1-\lambda \in \sigma(\mathcal{A}_i)$ for some $\mathcal{A}_i$ by (i) and Lemma 3.3. Then, by the definition for the spectral radius, it follows that $|1-\lambda| \leq \rho(\mathcal{A}_i) = 1$. Thus, we must have that $0 \leq \mathrm{Re}(\lambda) \leq 2$. By the same reason, we have the necessary and sufficient characterizations, since we must have $\mathrm{Im}(\lambda) = 0$ whenever the equalities are fulfilled. □

In Sect. 6, we will show that $\mathrm{Re}(\lambda) < 2$ if k is odd, i.e., it is impossible that $\lambda = 2$ is an eigenvalue of $\mathcal{L}$ when k is odd. Necessary and sufficient conditions are given for $\lambda = 2$ being an eigenvalue of $\mathcal{L}$ when k is even.

The next corollary says that the H^+-eigenvalues of $\mathcal{L}$ have a much more modest behavior than the eigenvalues.

Corollary 3.2 *Let G be a k-uniform hypergraph. Then, we have the followings.*

(i) Zero is the unique H^{++}-eigenvalue of $\mathcal{L}$. The smallest H-eigenvalue of $\mathcal{L}$ is zero.
(ii) All the H^{+}-eigenvalues of $\mathcal{L}$ are in the interval $[0, 1]$. The largest H^{+}-eigenvalue of $\mathcal{L}$ is one if and only if $|E| > 0$, and it is zero if and only if $|E| = 0$.
(iii) All the H-eigenvalues of $\mathcal{L}$ are nonnegative. If k is even, then $\mathcal{L}$ is positive semi-definite (i.e., $\mathcal{L}\mathbf{x}^k \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$), and $\mathcal{L}\mathbf{x}^k$ can be written as a sum of squares (i.e., $\mathcal{L}\mathbf{x}^k = \sum_{i\in[r]} p_i(\mathbf{x})^2$ for some integer r and polynomials p_i).

Proof For (i), zero is an H^{++}-eigenvalue follows from Definition 2.1, Lemma 2.2 and 3.2 and Theorem 3.1 (i) immediately. The uniqueness follows from Lemma 2.2, Corollary 3.1 and Theorem 3.1 (i), since 1 is the unique H^{++}-eigenvalue of the connected components that have more than one vertices, and the spectra of the isolated vertices are the same set $\{0\}$. Finally, the minimality follows from Theorem 3.1 (ii), since all the H-eigenvalues are real.

For the conclusion (ii), first we have that all the H^{+}-eigenvalues of $\mathcal{L}$ are in the interval $[0, 2]$ by Theorem 3.1 (ii). Suppose that $\lambda > 1$ is an H^{+}-eigenvalue of $\mathcal{L}$. Then, by Definition 2.1, Theorems 2.1 and 3.1 and Lemma 3.3, $1 - \lambda < 0$ is an H^{+}-eigenvalue of some connected component of G. This is a contradiction to Lemma 3.1. Thus, $\lambda \in [0, 1]$. The remaining conclusions follow from Theorem 3.1 (i), and Lemmas 3.1 and 3.3 immediately.

By Theorem 3.1 (ii), all the H-eigenvalues of $\mathcal{L}$ are nonnegative. When k is even, it is further equivalent to that $\mathcal{L}$ is positive semidefinite by (Qi 2005, Theorem 5). Thus, the first two statements of the conclusion (iii) follow. This, together with (Fidalgo and Kovacec 2011, Corollary 2.8) implies the last statement of the conclusion. □

We remark that, unlike the graph counterpart (Chung 1997), there would be little hope to write $\mathcal{L}\mathbf{x}^k$ as a sum of powers of linear forms.

The next lemma is on the H-eigenvectors of the smallest H-eigenvalue of $\mathcal{L}$.

Lemma 3.4 *Let G be a k-uniform hypergraph. Suppose that G has connected components $V_1, \ldots, V_r$. We have the followings.*

(i) Let $L \subseteq \mathbb{R}^n$ be the subspace generated by the H-eigenvectors of $\mathcal{L}$ corresponding to the H-eigenvalue zero. Let $\mathcal{L}_i$ be the Laplacian of G_{V_i}. Let $\tilde{L}_i$ be the subspace generated by the H-eigenvectors of $\mathcal{L}_i$ corresponding to the H-eigenvalue zero, and L_i be the canonical embedding of $\tilde{L}_i$ into $\mathbb{R}^n$ with respect to V_i. Then L has a direct sum decomposition:

$$L = L_1 \oplus \cdots \oplus L_r. \tag{3}$$

(ii) Let $M \subseteq \mathbb{R}^n$ be the subspace generated by the nonnegative H-eigenvectors of $\mathcal{L}$ corresponding to the H-eigenvalue zero. Let $\tilde{M}_i$ and M_i be defined similarly. Then $M = M_1 \oplus \cdots \oplus M_r$, $\dim(M_i) = 1$ for all $i \in [r]$, and hence $\dim(M) = r$.

Proof Let $\mathcal{A}_i$ be the adjacency tensor of the sub-hypergraph G_{V_i} of G induced by V_i for all $i \in [r]$. When V_i is a singleton, then $\mathcal{A}_i$ is the scalar zero. When $|V_i| > 1$, by Lemma 2.4, $\mathcal{A}_i$ is a weakly irreducible nonzero tensor.

(i) Suppose $\mathbf{x} \in \mathbb{R}^n$ is an H-eigenvector of $\mathcal{L}$ corresponding to the eigenvalue zero. By Definition 2.1, and Theorems 2.1 (ii) and 3.1 (i), whenever $\mathbf{x}(V_i) \neq 0$, $\mathbf{x}(V_i)$ an H-eigenvector of $\mathcal{L}_i$ corresponding to the eigenvalue zero. Thus, $\mathbf{x}(V_i) \in \tilde{L}_i$ for all $i \in [r]$. The reverse of the statement is true as well: if $0 \neq \mathbf{z} \in \mathbb{R}^{|V_i|}$ is an H-eigenvector of $\mathcal{L}_i$ corresponding to the eigenvalue zero, then its embedding into $\mathbb{R}^n$ is an H-eigenvector of $\mathcal{L}$.

Suppose that $\mathbf{y} \in L$ is nonzero and for some positive integer s, $\mathbf{y} = \sum_{i\in[s]} \mathbf{x}_i$ with $\mathbf{x}_i$ being H-eigenvectors of $\mathcal{L}$ corresponding to the eigenvalue zero. Then, $\mathbf{x}_i(V_j) \in \tilde{L}_j$ for all $j \in [r]$ and $i \in [s]$ by the preceding discussion. Thus,

$$\mathbf{y} = \sum_{i\in[s]} \mathbf{x}_i(V_1) \oplus \cdots \oplus \sum_{i\in[s]} \mathbf{x}_i(V_r) \in L_1 \oplus \cdots \oplus L_r.$$

Here we use the same notation $\mathbf{x}_i(V_j)$ for both $\mathbf{x}_i(V_j) \in \mathbb{R}^{|V_j|}$ and its embedding in $\mathbb{R}^n$.

On the contrary, suppose that $\mathbf{y}_i \in L_i$ is nonzero for $i \in [r]$. Then, we have that $\mathbf{y}_i = \sum_{j\in[s_i]} \mathbf{x}_{i,j}$ for some positive integer s_i and H-eigenvectors $\mathbf{x}_{i,j}(V_i)$ of $\mathcal{L}_i$. Moreover, $\mathbf{x}_{i,j}(V_l) = 0$ whenever $l \neq i$ by the definition of L_i. Thus, $\mathbf{x}_{i,j} \in L$ by the preceding discussion. Hence, $\mathbf{y} = \mathbf{y}_1 \oplus \cdots \oplus \mathbf{y}_r = \sum_{j\in[1_j]} \mathbf{x}_{1,j} \oplus \cdots \oplus \sum_{j\in[r_j]} \mathbf{x}_{r,j} = \sum_{i\in[r]} \sum_{j\in[s_i]} \mathbf{x}_{i,j} \in L$. Combining these results, the direct sum decomposition 3 follows.

(ii) Note that $\tilde{M}_i$ is the subspace generated by the nonnegative eigenvectors of $\mathcal{L}_i$ corresponding to the eigenvalue zero. If $|V_i| = 1$, then $\tilde{M}_i = \mathbb{R}$. When $|V_i| > 1$, by Lemmas 2.2, 2.3 and 2.4, the nonnegative eigenvectors of $\mathcal{A}_i$ corresponding to $\rho(\mathcal{A}_i) = 1$ is unique and positive. Thus, by Theorem 3.1 (i), the nonnegative eigenvector of $\mathcal{L}_i$ corresponding to the eigenvalue zero is unique and positive. Hence, $\dim(\tilde{M}_i) = \dim(M_i) = 1$. A similar proof as that for (i) shows that $M = M_1 \oplus \cdots \oplus M_r$ and hence $\dim(M) = \sum\limits_{i\in[r]} \dim(M_i) = r$. □

Lemma 3.4 says that the dimension of the linear subspace generated by the nonnegative eigenvectors of the eigenvalue zero of the Laplacian is exactly the number of the connected components of the hypergraph. By Corollary 6.4, we will see that if k is odd, then $\dim(L_i) = 1$ for all $i \in [r]$ and hence $\dim(L) = r$.

The next proposition gives equations that the eigenvalues of the Laplacian should satisfy.

Proposition 3.1 *Let G be a k-uniform hypergraph. We have the followings.*

(i) Let $m(\lambda)$ be the algebraic multiplicity of $\lambda \in \sigma(\mathcal{L})$ and $c(n,k) = n(k-1)^{n-1}$. Then

$$\sum_{\lambda\in\sigma(\mathcal{L})} m(\lambda)\lambda \leq c(n,k)$$

with equality holding if and only if G has no isolated vertices.

(ii) Suppose that G has no isolated vertices. Let $\{\lambda_0, \lambda_1, \ldots, \lambda_h\}$ be the H-eigenvalues of $\mathcal{L}$ in increasing order with algebraic multiplicity; and $\{\alpha_i \pm \sqrt{-1}\beta_i,\ i \in [w]\}$ be the remaining eigenvalues[2] of $\mathcal{L}$ with algebraic multiplicity. Then,

$$\sum_{j\in[h]} \lambda_j + 2\sum_{j\in[w]} \alpha_j = c(n,k), \tag{4}$$

and

$$\sum_{j\in[h]} \lambda_j^2 + 2\sum_{j\in[w]} \alpha_j^2 - 2\sum_{j\in[w]} \beta_j^2 = c(n,k). \tag{5}$$

If furthermore $k \geq 4$, then we also have

$$\sum_{j\in[h]} \lambda_j^3 + 2\sum_{j\in[w]} \alpha_j^3 - 6\sum_{j\in[w]} \alpha_j\beta_j^2 = c(n,k). \tag{6}$$

Proof (i) follows from Definition 2.7 and (Hu et al. 2013, Corollary 6.5 (i)) which says that the summation of the eigenvalues is equal to $(k-1)^{n-1}$ times the summation of the diagonal elements of $\mathcal{L}$.

(ii) First note that $\lambda_0 = 0$ by Corollary 3.2 (i). Second, by (Hu et al. 2013, Theorem 2.3) the degree of $\chi_{\mathcal{T}}(\lambda)$ is $c(n,k)$. Hence, $\sum_{\lambda\in\sigma(\mathcal{L})} m(\lambda) = c(n,k)$. Third, by Definition 2.7 and the proof of (Cooper andv Dutle 2012, Corollary 3.14) we see that the h-th order traces of the tensor $\mathcal{A}$ is zero for all $h \in [k-1]$. (Hu et al. 2013, Theorem 6.10) says that the summation of the h-th powers of all the eigenvalues of $\mathcal{A}$ is equal to the h-th trace of $\mathcal{A}$ for all $h \leq c(n,k)$. Thus, by Theorem 3.1 (i) and (Hu et al. 2013, Theorem 6.10) we have that

$$\sum_{\lambda\in\sigma(\mathcal{L})} m(\lambda)(1-\lambda)^h = 0, \quad \forall h \in [k-1].$$

Then, 4, 5 and 6 are just the expansions of the corresponding equalities for $h = 1, 2$ and 3 respectively. □

We can derive more equalities for the other $h \in [k-1]$ similarly.

4 H$^+$-eigenvalues of the Laplacian

In this section, we discuss the H$^+$-eigenvalues of the Laplacian. We denote by $\sigma^+(\mathcal{L})$ the set of all the H$^+$-eigenvalues of $\mathcal{L}$. By Corollary 3.2, it is nonempty. We characterize all the H$^+$-eigenvalues and the corresponding nonnegative eigenvectors through

[2] By the discussion on (Qi 2005, p. 1315) they must appear in conjugate complex pairs. They are called N-eigenvalues in that paper.

the spectral components and the flower hearts of G in Sect. 4.1. Then, in the other subsection, we introduce the H^+-geometric multiplicity of an H^+-eigenvalue and discuss the second smallest H^+-eigenvalue of $\mathcal{L}$.

4.1 Characterizations

The next lemma characterizes all the H^+-eigenvalues of $\mathcal{L}$.

Lemma 4.1 *Let G be a k-uniform hypergraph. Suppose that G has connected components $V_1, \ldots, V_r$ for some positive integer r. Let $\mathcal{L}_i$ be the Laplacian of the sub-hypergraph G_i of G induced by V_i. Then, we have the followings.*

(i) *$\lambda = 0$ is an H^+-eigenvalue of $\mathcal{L}$ with nonnegative eigenvector $\mathbf{x}$ if and only if $\mathbf{x}(V_i)$ is the unique positive eigenvector of $\mathcal{L}_i$ whenever $\mathbf{x}(V_i) \neq 0$.*
(ii) *$1 > \lambda > 0$ is an H^+-eigenvalue of $\mathcal{L}$ with nonnegative eigenvector $\mathbf{x}$ if and only if $\mathbf{x}(V_i) = 0$ whenever $|V_i| = 1$, and $1 - \lambda$ is an H^+-eigenvalue of $\mathcal{A}_i$ with eigenvector $\mathbf{x}(V_i)$ whenever $|V_i| > 1$ and $\mathbf{x}(V_i) \neq 0$.*
(iii) *$\lambda = 1$ is an H^+-eigenvalue of $\mathcal{L}$ with nonnegative eigenvector $\mathbf{x}$ if and only if $\mathbf{x}(V_i) = 0$ whenever $|V_i| = 1$, and $[sup(\mathbf{x}(V_i))]^c$ is a flower heart of G_i whenever $|V_i| > 1$ and $\mathbf{x}(V_i) \neq 0$.*

Proof (i) By Definition 2.1 and Theorem 2.1, it is easy to see that $\lambda = 0$ is an H^+-Eigenvalue of $\mathcal{L}$ with nonnegative eigenvector $\mathbf{x}$ if and only if $\mathbf{x}(V_i)$ is a nonnegative eigenvector of $\mathcal{L}_i$ whenever $\mathbf{x}(V_i) \neq 0$. In this situation, when $|V_i| = 1$, $\mathbf{x}(V_i) > 0$ is a scalar; and when $|V_i| > 1$, $\mathbf{x}(V_i)$ is a nonnegative eigenvector of the adjacency tensor of the connected sub-hypergraph G_i corresponding to its spectral radius 1. By Lemmas 2.3 and 2.4, and Theorem 3.1, it follows that $\mathbf{x}(V_i)$ is the unique positive eigenvector of $\mathcal{L}_i$. The converse is also true.
(ii) follows from Definitions 2.1 and 2.7, and Theorem 3.1 immediately.
(iii) follows from Definitions 2.1 and 2.7, Lemma 3.1 and Theorem 3.1.

□

By Lemma 4.1 (i) and (iii), the H^+-eigenvalues zero and one of $\mathcal{L}$ and their corresponding nonnegative eigenvectors are clear. In the following, without loss of generality by Lemma 4.1 (ii), we consider a connected hypergraph G.

The next lemma, together with Theorem 2.1, says that the spectral radius λ of the sub-tensor of $\mathcal{A}$ associated to a spectral component of G contributes to an H^+-eigenvalue $1 - \lambda$ of $\mathcal{L}$.

Lemma 4.2 *Let G be a k-uniform connected hypergraph. Let $S \subseteq [n]$ be a nonempty subset. Suppose that S is a spectral component of G. Let*

$$\lambda = \max \left\{ \mathcal{A}\mathbf{y}^k \mid \mathbf{y} \in \mathbb{R}^n_+,\ \sum_{i \in [n]} y_i^k = 1,\ y_i = 0, \forall i \in S^c \right\}. \tag{7}$$

Then, $1 \geq \lambda \geq 0$ and $1 - \lambda$ is an H^+-eigenvalue of $\mathcal{L}$. Moreover, the optimal solutions of 7 and the nonnegative eigenvectors of $\mathcal{L}$ with support being contained in S corresponding to the H^+-eigenvalue $1 - \lambda$ are in one to one correspondence.

Proof If $S = V$, then $\lambda = \rho(\mathcal{A}) = 1$ by Theorem 2.1 and Lemma 3.2. By Corollary 3.2 (i), $1 - \lambda = 0$ is an H^+-eigenvalue of $\mathcal{L}$. The eigenvector correspondence follows from Theorem 2.1 (iii).

In the sequel, suppose that $S \neq V$ is proper. Let $\mathcal{B}$ be the k-th order $|S|$-dimensional sub-tensor of $\mathcal{A}$ corresponding to the set S. Then, we have that

$$\begin{aligned}\lambda &= \max\left\{\mathcal{A}\mathbf{y}^k \mid \mathbf{y} \in \mathbb{R}^n_+,\ \sum_{i\in[n]} y_i^k = 1,\ y_i = 0, \forall i \in S^c\right\}\\ &= \max\left\{\mathcal{B}\mathbf{z}^k \mid \mathbf{z} \in \mathbb{R}^{|S|}_+,\ \sum_{i\in[|S|]} z_i^k = 1\right\}.\end{aligned}$$

By Theorem 2.1, $\lambda = \rho(\mathcal{B})$. Suppose that $\mathbf{y}$ is an optimal solution to 7 with the optimal value λ. Then, the sub-vector $\mathbf{z}$ of $\mathbf{y}$ corresponding to S is an eigenvector of $\mathcal{B}$ corresponding to λ by Theorem 2.1 (iii). Hence, we have

$$\mathcal{B}\mathbf{z}^{k-1} = \lambda\mathbf{z}^{[k-1]},$$

where $\mathbf{z}^{[k-1]}$ is a vector with its i-th entry being z_i^{k-1}. For $i \in S^c$, we have that

$$(\mathcal{A}\mathbf{y}^{k-1})_i = \sum_{\{i,i_2,\ldots,i_k\}\in E_i} \frac{1}{\sqrt[k]{d_i}} \prod_{j=2}^{k} \frac{y_{i_j}}{\sqrt[k]{d_{i_j}}} = \sum_{\{i,i_2,\ldots,i_k\}\in E_i\cap E(S,S^c)} \frac{1}{\sqrt[k]{d_i}} \prod_{j=2}^{k} \frac{y_{i_j}}{\sqrt[k]{d_{i_j}}} = 0,$$

since S is spectral component which implies that $\{i_2, \ldots, i_k\} \cap S^c \neq \emptyset$ for every $\{i, i_2, \ldots, i_k\} \in E(S, S^c)$. For $i \in S$, we have

$$\begin{aligned}(\mathcal{A}\mathbf{y}^{k-1})_i &= \sum_{\{i,i_2,\ldots,i_k\}\in E_i} \frac{1}{\sqrt[k]{d_i}} \prod_{j=2}^{k} \frac{y_{i_j}}{\sqrt[k]{d_{i_j}}} = \sum_{\{i,i_2,\ldots,i_k\}\in E_i\cap E(S)} \frac{1}{\sqrt[k]{d_i}} \prod_{j=2}^{k} \frac{y_{i_j}}{\sqrt[k]{d_{i_j}}}\\ &= (\mathcal{B}\mathbf{z}^{k-1})_i = \lambda y_i^{k-1}.\end{aligned}$$

Thus, λ is an H^+-eigenvalue of $\mathcal{A}$ with eigenvector $\mathbf{y}$. Then, $1 - \lambda$ being an H^+-eigenvalue of $\mathcal{L}$ with the eigenvector $\mathbf{y}$ with support being contained in S follows immediately .

The conclusion that a nonnegative eigenvector with support being contained in S of $\mathcal{L}$ corresponding to the eigenvalue $1 - \lambda$ is an optimal solution of 7 follows immediately. □

The next lemma says that the converse of Lemma 4.2 is also true.

Lemma 4.3 *Let G be a k-uniform connected hypergraph. If $\mathbf{x} \in \mathbb{R}^n_+$ is an eigenvector of $\mathcal{L}$ corresponding to an H^+-eigenvalue λ, then sup$(\mathbf{x})$ is a spectral component of G and $1 - \lambda$ is the spectral radius of the sub-tensor of $\mathcal{A}$ corresponding to sup$(\mathbf{x})$.*

Proof Let $S := \sup(\mathbf{x})$ and S^c be its complement. If $S = V$, then by Lemma 4.1 (i), $\lambda = 0$ and $1 = 1-\lambda$ is the spectral radius of $\mathcal{A}$. Obviously, V is a spectral component of G by Definition 2.5.

If S is a proper subset of V, then $\{e \in E \mid e \cap S^c \neq \emptyset\}$ is nonempty, since G is connected. By Corollary 3.2, we have that $1 \geq \lambda \geq 0$. By the hypothesis that $(\mathcal{L}\mathbf{x}^{k-1})_i = 0$ for all $i \in S^c$, we have

$$\sum_{\{i,i_2,\ldots,i_k\}\in E_i} \frac{1}{\sqrt[k]{d_i}} \prod_{j=2}^{k} \frac{x_{i_j}}{\sqrt[k]{d_{i_j}}} = \sum_{\{i,i_2,\ldots,i_k\}\in E_i\cap E(S,S^c)} \frac{1}{\sqrt[k]{d_i}} \prod_{j=2}^{k} \frac{x_{i_j}}{\sqrt[k]{d_{i_j}}} = 0.$$

Thus, we must have that $|e \cap S^c| \geq 2$ for every $e \in E(S, S^c)$. Hence, S is a spectral component of G. Let $\mathcal{B}$ be the sub-tensor of $\mathcal{A}$ corresponding to S, and $\mathbf{y}$ be the sub-vector of $\mathbf{x}$ corresponding to S. Then, for all $i \in S$, we have

$$\begin{aligned}(1-\lambda)y_i^{k-1} = (1-\lambda)x_i^{k-1} = (\mathcal{A}\mathbf{x}^{k-1})_i &= \sum_{\{i,i_2,\ldots,i_k\}\in E_i} \frac{1}{\sqrt[k]{d_i}} \prod_{j=2}^{k} \frac{x_{i_j}}{\sqrt[k]{d_{i_j}}} \\ &= \sum_{\{i,i_2,\ldots,i_k\}\in E_i\cap E(S)} \frac{1}{\sqrt[k]{d_i}} \prod_{j=2}^{k} \frac{x_{i_j}}{\sqrt[k]{d_{i_j}}} = (\mathcal{B}\mathbf{y}^{k-1})_i.\end{aligned}$$

Hence, $\mathbf{y}$ is a positive eigenvector of $\mathcal{B}$. Then, $1-\lambda$ is an H^{++}-eigenvalue of $\mathcal{B}$. By Lemma 2.2 (ii), and Theorem 2.1 (i) and (ii) (see also Qi 2012, Theorem 4) a symmetric nonnegative tensor has at most one H^{++}-eigenvalue. If it has one, then it should be the spectral radius of this tensor. Hence, we have that $1-\lambda = \rho(\mathcal{B})$. □

By Lemmas 4.1, 4.2 and 4.3, we have the following theorem which characterizes all the nonnegative eigenvectors of $\mathcal{L}$.

Theorem 4.1 *Let G be a k-uniform hypergraph. Suppose that G has $r \geq 1$ connected components $V_1, \ldots, V_r$. Let $\mathcal{L}_i$ and $\mathcal{A}_i$ be respectively the Laplacian and the adjacency tensor of the sub-hypergraph G_i of G induced by V_i. Then $\mathbf{x} \in \mathbb{R}^n_+$ is an eigenvector of $\mathcal{L}$ corresponding to an H^+-eigenvalue λ if and only if*

- *(i) when $\lambda = 0$, then $\mathbf{x}(V_i)$ is the unique positive eigenvector of $\mathcal{L}_i$ whenever $\mathbf{x}(V_i) \neq 0$;*
- *(ii) when $1 > \lambda > 0$, then $\mathbf{x}(V_i) = 0$ whenever $|V_i| = 1$, and $sup(\mathbf{x}(V_i))$ is a spectral component of G_i and $1-\lambda$ is the spectral radius of the sub-tensor of $\mathcal{A}_i$ corresponding to $sup(\mathbf{x}(V_i))$ whenever $\mathbf{x}(V_i) \neq 0$ and $|V_i| > 1$;*
- *(iii) when $\lambda = 1$, then $\mathbf{x}(V_i) = 0$ whenever $|V_i| = 1$, and $[sup(\mathbf{x}(V_i))]^c$ is a flower heart of G_i whenever $\mathbf{x}(V_i) \neq 0$ and $|V_i| > 1$.*

By Theorem 4.1, all the H^+-eigenvalues can be computed out, since they correspond to the spectral radii of certain nonnegative tensors. The algorithm proposed in Hu et al. (2011) can be applied. It is globally R-linearly convergent.

By Theorem 4.1, when G has no isolated vertices, if μ is the spectral radius of the sub-tensor of $\mathcal{A}$ corresponding to a spectral component S, then

$$1 - \max\left\{\mathcal{A}\mathbf{y}^k \mid \mathbf{y} \in \mathbb{R}^n_+,\ \sum_{i\in[n]} y_i^k = 1,\ y_i = 0, \forall i \in S^c\right\} = 1 - \mu \in \sigma^+(\mathcal{L}).$$

Equivalently, we have

$$\begin{aligned} 1 - \mu &= 1 + \min\left\{-\mathcal{A}\mathbf{y}^k \mid \mathbf{y} \in \mathbb{R}^n_+,\ \sum_{i\in[n]} y_i^k = 1,\ y_i = 0, \forall i \in S^c\right\} \\ &= \min\left\{\mathcal{L}\mathbf{y}^k \mid \mathbf{y} \in \mathbb{R}^n_+,\ \sum_{i\in[n]} y_i^k = 1,\ y_i = 0, \forall i \in S^c\right\}. \end{aligned} \tag{8}$$

Define

$$\sigma_s(\mathcal{L}) := \left\{\lambda | \lambda = \min\left\{\mathcal{L}\mathbf{y}^k \middle| \sum_{i=1}^n y_i^k = 1,\ \mathbf{y} \in \mathbb{R}^n_+,\ y_i = 0,\ \forall i \in A^c\right\}, A \in 2^V \setminus \{\emptyset\}\right\}. \tag{9}$$

Then, Theorem 4.1, together with Theorem 3.1, says that $\sigma^+(\mathcal{L}) \subseteq \sigma_s(\mathcal{L})$. Here a natural question arises. Are the two sets equal to each other? The next proposition gives a negative answer, it says that the hypothesis in Lemma 4.2 is necessary, i.e., if S is not a spectral component, then the optimal value of 8 may not be an H$^+$-eigenvalue of $\mathcal{L}$. More properties on the set $\sigma_s(\mathcal{L})$ are discussed in Sect. 5.

A hypergraph $G = (V, E)$ is complete if E contains all the possible edges.

Proposition 4.1 *Let G be a k-uniform complete hypergraph with $n > k$. Then,*

$$\sigma^+(\mathcal{L}) \neq \sigma_s(\mathcal{L}). \tag{10}$$

Proof Since G is complete, it is easy to see that the sub-tensors of $\mathcal{A}$ corresponding to the sets with the same cardinality are the same. Thus, there are at most n values in $\sigma_s(\mathcal{L})$. By Lemmas 2.2 and 2.3 and the fact that G is complete, the values corresponding to sets with different cardinalities larger than $k-1$ are strictly smaller than one and different. The values corresponding to sets with cardinalities not larger than $k-1$ are one, since the sub-tensors are all the identity tensors with appropriate dimensions. Hence, there are exactly $n-k+2$ values in $\sigma_s(\mathcal{L})$.

Since G is complete, every set A satisfying $|A| \geq k-1$ cannot be a spectral component. Hence, the value corresponding to $\{i\}^c$ for every i cannot be in $\sigma^+(\mathcal{L})$ by Theorem 4.1. Since otherwise, this value can be expressed by some spectral component. It should be one by the preceding discussion. This would contradict the fact that

$\rho(\mathcal{A}(\{i\}^c)) > 0$ (which implies $1-\rho(\mathcal{A}(\{i\}^c)) < 1$) by Lemma 2.3 and Theorem 2.1, since $\mathcal{A}(\{i\}^c)$ is nonzero.

Hence, the result 10 follows. The proof is complete. □

Actually, by Proposition 4.3 and Corollary 4.2 in Sect. 4.2, $\sigma^+(\mathcal{L}) = \{0, 1\}$ for a complete hypergraph. While, by the proof of Lemma 3.2, it can be calculated that $\sigma_s(\mathcal{L}) = \{1 - \frac{d(s)}{d(n)},\quad s \in \{k-1, \ldots, n\}\}$ with $d(s) := \binom{s-1}{k-1}$.

4.2 H$^+$-geometric multiplicity

In this subsection, we discuss the second smallest H$^+$-eigenvalue of the Laplacian. To this end, we need to order the H$^+$-eigenvalues first. Since the eigenvectors of an eigenvalue of a tensor do not form a linear subspace of $\mathbb{C}^n$ like its matrix counterpart in general, it is subtle to define geometric multiplicity of an eigenvalue of a tensor. However, by Theorem 4.1 and the fact that the number of the spectral components of a hypergraph is always finite, we can define the H$^+$-*geometric multiplicity* of an H$^+$-eigenvalue of $\mathcal{L}$ in the following way.

Definition 4.1 Let G be a k-uniform hypergraph. Let μ be an H$^+$-eigenvalue of $\mathcal{L}$. The H$^+$**-geometric multiplicity** of the H$^+$-eigenvalue μ is defined to be the number of nonnegative eigenvectors (up to scalar multiplication) corresponding to μ.

For a hypergraph G, we denote by $n(G)$ the number of the H$^+$-eigenvalues of $\mathcal{L}$ (with H$^+$-geometric multiplicity). By Corollary 3.2 (i), $\mathcal{L}$ always has the H$^+$-eigenvalue zero and hence $n(G) \geq 1$. When $|E| > 0$, by Lemma 3.1 and Theorem 3.1, 1 is also an H$^+$-eigenvalue of $\mathcal{L}$. Then, in this case $n(G) \geq 2$. By Definition 4.1 and Corollary 3.2 (ii), we can order all the H$^+$-eigenvalues (with H$^+$-geometric multiplicity) of $\mathcal{L}$ in increasing order as:

$$0 = \mu_0 \leq \mu_1 \leq \ldots \leq \mu_{n(G)-1} \leq 1. \tag{11}$$

The next lemma establishes the relation between the number of the connected components of a hypergraph G and the H$^+$-geometric multiplicity of the eigenvalue zero.

Lemma 4.4 *Let G be a k-uniform hypergraph. Suppose that G has r connected components. Then, the H$^+$-geometric multiplicity $c(G, 0)$ of the H$^+$-eigenvalue zero of $\mathcal{L}$ is $c(G, 0) = 2^r - 1$.*

Proof Suppose that $\{V_1, \ldots, V_r\}$ are the connected components of G. For all $i \in [r]$, let $\mathcal{L}_i$ be the Laplacian of the sub-hypergraph G_i of G induced by V_i. For any choice of s $(1 \leq s \leq r)$ connected components $\{V_{i_1}, \ldots, V_{i_s}\}$ of G, let $\mathbf{x}(V_{i_j})$ be the unique positive eigenvector of $\mathcal{L}_{i_j}$ by Theorem 4.1 (i). Let $\mathbf{x}(V_i) = 0$ for the other V_i. By Theorem 4.1 (i), the vector $\mathbf{x}$ formed by the components $\mathbf{x}(V_i)$ is a nonnegative eigenvector of $\mathcal{L}$ corresponding to the eigenvalue zero. By Theorem 4.1 (i) again, the correspondence between the choices of the connected components of G and the nonnegative eigenvectors of $\mathcal{L}$ corresponding to the eigenvalue zero in the above sense is one to one. Thus, by Definition 4.1, the H$^+$-geometric multiplicity $c(G, 0)$ of the H$^+$-eigenvalue zero of $\mathcal{L}$ is $\sum_{s\in[r]}\binom{r}{s} = 2^r - 1$. □

The next corollary is a direct consequence of Lemma 4.4.

Corollary 4.1 *Let G be a k-uniform hypergraph. Then, $\mu_{i-2} = 0$ and $\mu_{i-1} > 0$ if and only if $\log_2 i$ is a positive integer and G has exactly $\log_2 i$ connected components. In particular, $\mu_1 > 0$ if and only if G is connected.*

The next proposition gives the H$^+$-geometric multiplicity of the H$^+$-eigenvalue one of $\mathcal{L}$.

Proposition 4.2 *Let G be a k-uniform hypergraph and $|E| > 0$. Suppose that G has $r \geq 0$ connected components $\{V_1, \ldots, V_r\}$ with $|V_i| > 1$. Let G_i be the sub-hypergraph of G induced by V_i. Suppose that G_i has $t_i \geq 0$ flower hearts for all $i \in [r]$. Then the H$^+$-geometric multiplicity $c(G, 1)$ of the H$^+$-eigenvalue one of $\mathcal{L}$ is*

$$c(G, 1) = \sum_{i\in[r]} s_i(t_1, \ldots, t_r),$$

where $s_i(t_1, \ldots, t_r)$ is the elementary symmetric polynomial on the variables $\{t_1, \ldots, t_r\}$ of degree i, and the vacuous summation is understood as zero.

Proof Note that $s_i(t_1, \ldots, t_r) = \sum_{1\leq j_1<\ldots<j_i\leq r} t_{j_1}\cdots t_{j_i}$. By Theorem 4.1 (iii), the result follows from a similar proof to that for Lemma 4.4. □

Proposition 4.2 says that $c(G, 1)$ is independent of the number of isolated vertices of the hypergraph G. For the other H$^+$-eigenvalues, by Theorem 4.1, their H$^+$-geometric multiplicities are determined by the number of the connected components, and the spectral components of every connected component. Similarly, these H$^+$-geometric multiplicities are independent of the number of isolated vertices of the hypergraph.

The next proposition gives necessary and sufficient conditions for $\mu_1 = \mu_{n(G)-1} = 1$. By Corollary 4.1, the underlying hypergraph should be connected.

Proposition 4.3 *Let G be a k-uniform connected hypergraph. Then, $\mu_1 = 1$ if and only if the complements of all the proper spectral components are the flower hearts. In this situation, we have $\sigma^+(\mathcal{L}) = \{0, 1\}$.*

Proof The first half follows from Theorem 4.1 immediately. The result $\sigma^+(\mathcal{L}) = \{0, 1\}$ in this situation follows from the fact that $\mu_{n(G)-1} = 1$ when $|E| > 0$. □

The next corollary completes Proposition 4.1.

Corollary 4.2 *Let G be a k-uniform complete hypergraph. Then, $\sigma^+(\mathcal{L}) = \{0, 1\}$.*

Proof Suppose that $A \neq V$ is a nonempty subset of $\{1, \ldots, n\}$. Since G is complete, A is a spectral component if and only if $|A| \leq k - 2$. On the other side, we also have that $E(A) = \emptyset$ whenever $|A| \leq k - 2$. Hence, A^c is a flower heart by Definition 2.6. Then, the result follows from Proposition 4.3. □

We will give lower bounds for μ_1 in Sect. 5.1.

5 The smallest H$^+$-eigenvalues of the sub-tensors of the Laplacian

Suppose that G is a k-uniform hypergraph without isolated vertices. By Theorem 4.1, if λ is an H$^+$-eigenvalue of the Laplacian $\mathcal{L}$, there exists a spectral component of G such that λ has the characterization 8. However, Proposition 4.1 says that $\sigma^+(\mathcal{L}) \neq \sigma_s(\mathcal{L})$ in general. In this section, we show that every $\lambda \in \sigma_s(\mathcal{L})$ is the smallest H$^+$-eigenvalue of some sub-tensor of $\mathcal{L}$. This is the mean of the subscript "s" of $\sigma_s(\mathcal{L})$. Then, we discuss the relations between these H$^+$-eigenvalues and μ_1, the edge connectivity and the edge expansion.

5.1 Characterization

We establish the equivalence between the smallest H$^+$-eigenvalues of the sub-tensors of $\mathcal{L}$ and $\sigma_s(\mathcal{L})$. Let $S \subseteq [n]$ be nonempty and $\kappa(S)$ the smallest H$^+$-eigenvalue of $\mathcal{L}(S)$.

The next lemma says that $\{\kappa(S) \mid S \in 2^V \setminus \{\emptyset\}\} = \sigma_s(\mathcal{L})$.

Lemma 5.1 *Let G be a k-uniform hypergraph without isolated vertices and $S \subseteq [n]$ be nonempty. We have that $\kappa(S) = 1 - \rho(\mathcal{A}(S)) \in [0, 1]$, and*

$$\kappa(S) = \min \left\{ \mathcal{L}\mathbf{y}^k \mid \mathbf{y} \in \mathbb{R}^n_+,\ \sum_{i \in [n]} y_i^k = 1,\ y_i = 0, \forall i \in S^c \right\}. \tag{12}$$

Proof Note that $\mathcal{L}(S) = \mathcal{I} - \mathcal{A}(S)$. Hence, λ is an H$^+$-eigenvalue of $\mathcal{L}(S)$ if and only if $1 - \lambda$ is an H$^+$-eigenvalue of $\mathcal{A}(S)$. Thus, by Lemmas 2.2, 2.3 and 3.2, we have that $\kappa(S) = 1 - \rho(\mathcal{A}(S)) \in [0, 1]$. This, together with Theorem 2.1, further implies that

$$\begin{aligned}
\kappa(S) &= 1 - \max \left\{ \mathcal{A}(S)\mathbf{y}^k \mid \mathbf{y} \in \mathbb{R}^{|S|}_+,\ \sum_{i \in [|S|]} y_i^k = 1 \right\} \\
&= 1 - \max \left\{ \mathcal{A}\mathbf{y}^k \mid \mathbf{y} \in \mathbb{R}^n_+,\ \sum_{i \in [n]} y_i^k = 1,\ y_i = 0, \forall i \in S^c \right\} \\
&= 1 + \min \left\{ -\mathcal{A}\mathbf{y}^k \mid \mathbf{y} \in \mathbb{R}^n_+,\ \sum_{i \in [n]} y_i^k = 1,\ y_i = 0, \forall i \in S^c \right\} \\
&= \min \left\{ 1 - \mathcal{A}\mathbf{y}^k \mid \mathbf{y} \in \mathbb{R}^n_+,\ \sum_{i \in [n]} y_i^k = 1,\ y_i = 0, \forall i \in S^c \right\} \\
&= \min \left\{ \mathcal{L}\mathbf{y}^k \mid \mathbf{y} \in \mathbb{R}^n_+,\ \sum_{i \in [n]} y_i^k = 1,\ y_i = 0, \forall i \in S^c \right\}.
\end{aligned}$$

Thus, 12 follows. The proof is complete. □

The next corollary is a direct consequence of Lemma 5.1.

Corollary 5.1 *Let G be a k-uniform hypergraph without isolated vertices and $S, T \subseteq [n]$ be nonempty such that $S \subset T$. Then, $\kappa(T) \leq \kappa(S)$.*

Corollary 5.2 *Let G be a k-uniform hypergraph without isolated vertices. Then, $\mu_1 = \min\{\kappa(S) \mid S\ is\ a\ proper\ spectral\ component\}$.*

Proof We first prove that there is a proper spectral component S of G such that $\mu_1 = \kappa(S)$. Then, the minimality follows immediately from Theorem 4.1 and 11.

By Theorem 4.1 and Lemma 5.1, there is a spectral component of G such that $\kappa(S) = \mu_1$. If G is connected, then $\mu_1 > 0$ by Corollary 4.1. While, $\kappa(V) = 0$ by Lemmas 3.2 and 5.1. Thus, $S \neq V$. If G has at least two connected components V_1 and V_2, then V_1 is a proper spectral component and $\kappa(V_1) = 0$ by Lemmas 5.1 and 3.2. Since $\mu_1 = 0$ by Corollary 4.1, V_1 can be chosen as S. □

Recall that d_i is the degree of the vertex i. In the following, we define $d_{\min} := \min_{i\in[n]} d_i$ and $d_{\max} := \max_{i\in[n]} d_i$. For a nonempty subset $S \subset V$, define $\mathrm{vol}(S) := \sum_{i\in S} d_i$ as the *volume* of S. The volume $\mathrm{vol}([n])$ of the hypergraph is simply denoted as d_{vol}.

Proposition 5.1 *Let G be a k-uniform hypergraph without isolated vertices. For any nonempty subset $S \subseteq V$, we have*

$$\kappa(S) \leq \frac{(k-1)vol(S^c)}{vol(S)} \tag{13}$$

with the convention that $vol(\emptyset) = 0$. In particular, for any $i \in [n]$, we have that

$$\kappa(\{i\}^c) \leq \frac{(k-1)d_{\max}}{d_{vol} - d_{\max}}. \tag{14}$$

Proof When $S = V$, then $\kappa(S) = \kappa(V) = 0$ by 12 and Lemma 3.2. Thus, the result follows. In the following, we assume that $S \neq V$. Let $\tilde{\mathbf{d}} \in \mathbb{R}^n$ be the n-vector with its i-th element being $\sqrt[k]{d_i}$ for all $i \in [n]$. Let $\mathbf{y}$ be the vector with its j-th element being $\frac{\tilde{d}_j}{\sqrt[k]{\mathrm{vol}(S)}}$ for $j \in S$ and $y_j = 0$ for $j \in S^c$. Then, by Lemma 5.1, we have that

$$\begin{aligned} \kappa(S) \leq \mathcal{L}\mathbf{y}^k &= 1 - k\sum_{e\in E\setminus E_{S^c}} \frac{\mathbf{y}^e}{\tilde{\mathbf{d}}^e} = 1 - k\sum_{e\in E\setminus E_{S^c}} \frac{1}{\mathrm{vol}(S)} = 1 - \frac{k|E| - k|E_{S^c}|}{\mathrm{vol}(S)} \\ &= \frac{\mathrm{vol}(S) + k|E_{S^c}| - d_{vol}}{\mathrm{vol}(S)} = \frac{k|E_{S^c}| - \mathrm{vol}(S^c)}{\mathrm{vol}(S)} \leq \frac{(k-1)\mathrm{vol}(S^c)}{\mathrm{vol}(S)}. \end{aligned}$$

Here, the fourth equality follows from the fact that $k|E| = d_{vol}$, and the last inequality from the fact that: for every $e \in E_{S^c}$, e contributes to $\mathrm{vol}(S^c)$ at least one. Thus, the number of edges in E_{S^c} is at most $\mathrm{vol}(S^c)$. Thus, $k|E_{S^c}| \leq k\mathrm{vol}(S^c)$.

14 follows from the fact that $d_i \leq d_{\max}$ for any $i \in [n]$. □

Note that 13 is nontrivial only if $\mathrm{vol}(S) > (k-1)\mathrm{vol}(S^c)$; and the bound is tight.

A hypergraph is d-regular, if $d_i = d \geq 0$ for all $i \in [n]$. The following corollary is a direct consequence of Proposition 5.1.

Corollary 5.3 *Let G be a k-uniform hypergraph and d-regular for some $d > 0$. For any $i \in [n]$, we have that*

$$\kappa(\{i\}^c) \leq \frac{k-1}{n-1}.$$

By the proof of Proposition 5.1, if $d_i = d_{\min}$, then $\kappa(\{i\}^c) \leq \frac{k-1}{n-1}$, since $d_{vol} - d_{min} \geq (n-1)d_{min}$. Hence, the next corollary follows.

Corollary 5.4 *Let G be a k-uniform hypergraph without isolated vertices. We have that*

$$\min_{i\in[n]} \kappa(\{i\}^c) \leq \frac{k-1}{n-1}.$$

The next proposition gives lower bounds on μ_1 in terms of $\kappa(\{i\}^c)$.

Proposition 5.2 *Let G be a k-uniform hypergraph without isolated vertices. Then, for any proper spectral component S of G such that $\mu_1 = \kappa(S)$,*

$$\mu_1 \geq \max_{i\in S^c} \kappa(\{i\}^c) \geq \min_{i\in[n]} \kappa(\{i\}^c). \tag{15}$$

Proof The result follows from Theorem 4.1, Lemma 5.1 and Corollaries 5.1 and 5.2. □

In the next subsection, $\min_{i\in[n]} \kappa(\{i\}^c)$ is related to the edge connectivity of the hypergraph. This value is similar to the algebraic connectivity defined in (Qi 2012, Sect. 8).

5.2 Edge connectivity

In this short subsection, we discuss the relation between the smallest H$^+$-eigenvalues of the sub-tensors of $\mathcal{L}$ and the edge connectivity. Recall that the minimum of the cardinalities of the edge cuts corresponding to nonempty proper subsets is called the **edge connectivity** of G. We denote it by $e(G)$. Note that G is disconnected if and only if $e(G) = 0$. It is also easy to see that $e(G) \leq d_{\min}$.

Proposition 5.3 *Let G be a k-uniform hypergraph without isolated vertices. We have that*

$$\min_{i\in[n]} \kappa(\{i\}^c) \leq \frac{k}{d_{vol}} e(G). \tag{16}$$

Proof Let $\tilde{\mathbf{d}} \in \mathbb{R}^n$ be the n-vector with its i-th element being $\sqrt[k]{d_i}$ for all $i \in [n]$. Let S be a nonempty proper subset of $[n]$. Let $\mathbf{y}$ be the vector with its j-th element being $\frac{\tilde{d}_j}{\sqrt[k]{\sum_{i\in S} d_i}}$ for $j \in S$ and $y_j = 0$ for $j \in S^c$. Then, by Lemma 5.1,

$$\kappa(S) \leq \mathcal{L}\mathbf{y}^k = 1 - k \sum_{e\in E(S)} \frac{\mathbf{y}^e}{\tilde{\mathbf{d}}^e} = 1 - k \sum_{e\in E(S)} \frac{1}{\mathrm{vol}(S)} = 1 - \frac{k|E(S)|}{\mathrm{vol}(S)}.$$

Similarly, we have that $\kappa(S^c) \leq 1 - \frac{k|E(S^c)|}{\mathrm{vol}(S^c)}$. Thus,

$$\begin{aligned}\mathrm{vol}(S)\kappa(S) + \mathrm{vol}(S^c)\kappa(S^c) &\leq \mathrm{vol}(S) + \mathrm{vol}(S^c) - k(|E(S)| + |E(S^c)|)\\ &= d_{vol} - k(|E| - |E(S, S^c)|)\\ &= k|E(S, S^c)|.\end{aligned}$$

Since both S and S^c are nonempty, we have that $S \subseteq \{r\}^c$ and $S^c \subseteq \{s\}^c$ for some r and s respectively. By Corollary 5.1, we have that

$$\begin{aligned}d_{vol} \min_{i\in[n]} \kappa(\{i\}^c) \leq d_{vol} \min\{\kappa(\{r\}^c), \kappa(\{s\}^c)\} &\leq \sum_{i\in S} d_i\kappa(S) + \sum_{i\in S^c} d_i\kappa(S^c)\\ &\leq k|E(S, S^c)|.\end{aligned}$$

Thus, 16 follows. □

5.3 Edge expansion

In this subsection, we define and discuss the edge expansion of a hypergraph.

The next definition is a generalization of the edge expansion of a graph.

Definition 5.1 Let G be a k-uniform hypergraph without isolated vertices and $r \in [k-1]$. The r**-th depth edge expansion**, denoted by $h_r(G)$, of G is defined as

$$h_r(G) = \min_{S\subset V,\ \mathrm{vol}(S)\leq\lceil\frac{d_{vol}}{2}\rceil} \frac{|E(S, S^c)|}{\mathrm{vol}(S)}, \tag{17}$$

where the minimum takes additionally over all nonempty subsets S such that either $E(S, S^c)$ is empty or it cuts S^c with depth at least r.

When $r = 1$ and G reduces to a usual graph, this definition is the same as that in (Chung 1997, Sect. 2.2). Moreover, in this situation, it is easy to see that

$$h_1(G) = \min_{S\subset V} \frac{|E(S, S^c)|}{\min\{\mathrm{vol}(S), \mathrm{vol}(S^c)\}},$$

since in this case, $E(S, S^c)$, whenever nonempty, cuts both S and S^c with depth exactly one. For a hypergraph, the situation is more complicated. Thus, we need the generalized definition 17.

Definition 5.1 is well defined for all $r \in [k-1]$, since $E(\{i\}, \{i\}^c)$, whenever nonempty, always cuts $\{i\}^c$ with depth $k-1 \geq r$ for all $i \in [n]$.

The next proposition gives bound on $h_2(G)$ in terms of μ_1.

Proposition 5.4 *Let G be a k-uniform hypergraph without isolated vertices and $r \in [k-1]$. We have that $\kappa(S) \leq \frac{(k-2)|E(S,S^c)|}{vol(S)}$ for any spectral component S. Thus,*

$$\mu_1 \leq (k-2)h_2(G).$$

Proof Let $\tilde{\mathbf{d}} \in \mathbb{R}^n$ be the n-vector with its i-th element being $\sqrt[k]{d_i}$ for all $i \in [n]$. Let S be a spectral component, then either $E(S, S^c)$ is empty or it cuts S^c with depth at least two. The empty case is trivial, in the following, we assume that $E(S, S^c)$ is nonempty. Let $\mathbf{y}$ be the vector with its j-th entry being $\frac{\tilde{d}_j}{\sqrt[k]{\mathrm{vol}(S)}}$ for $j \in S$ and $y_j = 0$ for $j \in S^c$. By Lemma 5.1, we have

$$\begin{aligned}
\kappa(S) &= \min\left\{\mathcal{L}\mathbf{z}^k \mid \mathbf{z} \in \mathbb{R}^n_+,\ \sum_{i\in[n]} z_i^k = 1,\ z_i = 0, \forall i \in S^c\right\} \\
&\leq \mathcal{L}\mathbf{y}^k = 1 - k\sum_{e\in E(S)} \frac{\mathbf{y}^e}{\tilde{\mathbf{d}}^e} = 1 - k\sum_{e\in E(S)} \frac{1}{\mathrm{vol}(S)} \\
&= 1 - \frac{k|E(S)|}{\mathrm{vol}(S)} \leq \frac{(k-2)|E(S, S^c)|}{\mathrm{vol}(S)}.
\end{aligned}$$

The last inequality follows from the fact that $k|E(S)| \geq \mathrm{vol}(S) - (k-2)|E(S, S^c)|$, since $E(S, S^c)$ cuts S^c with depth at least two. Then, the first conclusion follows. This, together with Definition 5.1 and Corollary 5.2, implies that

$$\mu_1 \leq (k-2)h_2(G),$$

since $d_{vol} > \lceil \frac{d_{vol}}{2} \rceil$ which implies that the minimum 17 involves only proper subsets. □

By Proposition 5.2 and a similar proof of Proposition 5.4 and the fact that $\min_{i\in[n]} \kappa(\{i\}^c) \leq \kappa(S)$ for any nonempty proper subset S, we have the following proposition.

Proposition 5.5 *Let G be a k-uniform hypergraph without isolated vertices. We have that for all $r \in [k-1]$, $\kappa(S) \leq \frac{(k-r)|E(S,S^c)|}{vol(S)}$ for any nonempty subset S such that either $E(S, S^c)$ is empty or it cuts S^c with depth at least r. In particular,*

$$\min_{i\in[n]} \kappa(\{i\}^c) \leq (k-r)h_r(G). \tag{18}$$

Note that any nonempty subset S such that either $E(S, S^c)$ is empty or it cuts S^c with depth at least r with $r \geq 2$ is a spectral component by Definition 2.5. This, together with Corollary 5.2 and Proposition 5.5, implies the following corollary immediately.

Corollary 5.5 *Let G be a k-uniform hypergraph without isolated vertices. For all r such that $2 \le r \le k-1$, we have that $\mu_1 \le (k-r)h_r(G)$.*

6 The largest H-eigenvalue of the Laplacian

By Theorem 3.1, if λ is an H-eigenvalue of $\mathcal{L}$, then $\lambda \le 2$. Does $\mathcal{L}$ has an eigenvalue 2? If it does, is it an H-eigenvalue of $\mathcal{L}$? In this section, we discuss these questions. By Lemma 3.3, it is sufficient to consider connected hypergraphs. Note that when $\lambda = 2$, we have that -1 is an eigenvalue of the adjacency tensor $\mathcal{A}$.

6.1 Eigenvectors of eigenvalues on the spectral circle of the adjacency tensor

As $\rho(\mathcal{A}) = 1$, the set of complex numbers with module one is called the *spectral circle* of the adjacency tensor $\mathcal{A}$. By (Yang and Yang 2011, Theorem 3.9) if there are $r \ge 1$ eigenvalues of $\mathcal{A}$ with module one, then they are uniformly distributed on the spectral circle, i.e., they appear in the form $\exp(\frac{2s\pi\sqrt{-1}}{r})$ for $s \in [r]$. In this subsection, we establish necessary and sufficient conditions for a nonzero vector being an eigenvector of an eigenvalue on the spectral circle.

The next technical lemma is useful.

Lemma 6.1 *Let G be a k-uniform connected hypergraph. If $\mathbf{x} \in \mathbb{C}^n$ is an eigenvector of $\mathcal{A}$ with eigenvalue $\exp(\sqrt{-1}\theta)$, then there exist $\theta_i \in \mathbb{R}$ such that*

$$x_i = \exp(\sqrt{-1}\theta_i)\frac{\sqrt[k]{d_i}}{\sqrt[k]{d_{vol}}}, \quad \forall i \in [n], \tag{19}$$

and for all $i \in [n]$, there exists $\gamma_i \in \mathbb{C}$ such that

$$\exp(\sqrt{-1}\theta_{i_2})\cdots\exp(\sqrt{-1}\theta_{i_k}) = \gamma_i, \quad \forall e = \{i, i_2, \ldots, i_k\} \in E_i. \tag{20}$$

Proof Let $\tilde{\mathbf{d}} \in \mathbb{R}^n$ be the n-vector with its i-th element being $\sqrt[k]{d_i}$ for all $i \in [n]$. Then we have the following

$$\begin{aligned}\sum_{i\in[n]}|x_i||(\mathcal{A}\mathbf{x}^{k-1})_i| &= \sum_{i\in[n]}|x_i|\left|\sum_{e\in E_i}\frac{1}{\sqrt[k]{d_i}}\prod_{j\in e\setminus\{i\}}\frac{x_j}{\sqrt[k]{d_j}}\right| \\ &\le \sum_{e\in E}k\prod_{i\in e}\frac{|x_i|}{\sqrt[k]{d_i}} \le \sum_{e\in E}k\left(\frac{1}{k}\sum_{i\in e}(\frac{|x_i|}{\sqrt[k]{d_i}})^k\right) \\ &= \sum_{e\in E}\sum_{i\in e}\frac{|x_i|^k}{d_i} = \sum_{i\in[n]}\sum_{e\in E_i}\frac{|x_i|^k}{d_i} = \sum_{i\in[n]}|x_i|^k.\end{aligned}$$

This, together with the hypothesis $\sum_{i\in[n]}|x_i||(\mathcal{A}\mathbf{x}^{k-1})_i| = \sum_{i\in[n]}|x_i|^k$, implies that all of the inequalities should be equalities.

By the fact that the second (the arithmetic-geometric mean) inequality is an equality, we have that $|x_i| = \alpha|\sqrt[k]{d}_i|$ for some $\alpha > 0$ for all $i \in [n]$, since G is connected. When normalizing the vectors $\mathbf{x}$ and $\tilde{\mathbf{d}}$, we get 19.

By the fact that first inequality is an equality, we have 20. □

Let $\mathbb{Z}$ be the ring of integers. For a positive integer k, let $\langle k\rangle$ be the ideal in $\mathbb{Z}$ generated by k. Let $\urcorner := \{\bar{0}, \bar{1}, \ldots, \overline{k-1}\}$ be the quotient ring $\mathbb{Z}/\langle k\rangle$. The image of $\alpha \in \mathbb{Z}$ under the natural homomorphism $\mathbb{Z} \to \urcorner$ is denoted by $\bar{\alpha}$. For basic definitions, see (Lang 2002; Cox 1998, 2006).

The next theorem gives necessary and sufficient conditions for $\exp(\sqrt{-1}\theta)$ being an eigenvalue of $\mathcal{A}$.

Theorem 6.1 *Let G be a k-uniform connected hypergraph. Then,* $\exp(\sqrt{-1}\theta)$ *is an eigenvalue of $\mathcal{A}$ if and only if $\theta = \frac{2\alpha\pi}{k}$ for some integer α, and there exist integers α_i for $i \in [n]$ such that*

$$\sum_{j\in e} \bar{\alpha}_j = \bar{\alpha}, \quad \forall e \in E. \tag{21}$$

Proof Suppose that $\exp(\sqrt{-1}\theta)$ is an eigenvalue of $\mathcal{A}$ with an eigenvector $\mathbf{x}$. By Lemma 6.1, for all $i \in [n]$, there exist $\theta_i \in \mathbb{R}$ such that $x_i = \exp(\sqrt{-1}\theta_i)\frac{\sqrt[k]{d_i}}{\sqrt[k]{d_{vol}}}$, and

$$\exp(\sqrt{-1}\theta_{i_2})\cdots\exp(\sqrt{-1}\theta_{i_k}) = \gamma_i, \quad \forall e = \{i, i_2, \ldots, i_k\} \in E_i$$

for some $\gamma_i \in \mathbb{C}$. Let $\tilde{\mathbf{d}} \in \mathbb{R}^n$ be the n-vector with its i-th element being $\sqrt[k]{d_i}$ for all $i \in [n]$. Since

$$\exp(\sqrt{-1}\theta)x_i^{k-1} = (\mathcal{A}\mathbf{x}^{k-1})_i = \gamma_i \frac{(\mathcal{A}\tilde{\mathbf{d}}^{k-1})_i}{(\sqrt[k]{d_{vol}})^{k-1}} = \gamma_i \left(\frac{\sqrt[k]{d_i}}{\sqrt[k]{d_{vol}}}\right)^{k-1},$$

we have that for all $i \in [n]$,

$$\exp(\sqrt{-1}\theta_{i_2})\cdots\exp(\sqrt{-1}\theta_{i_k}) = \exp(\sqrt{-1}\theta)[\exp(\sqrt{-1}\theta_i)]^{k-1},$$
$$\forall e = \{i, i_2, \ldots, i_k\} \in E_i.$$

Thus, likewise, we must have

$$\sum_{j\in e} \theta_j = \theta + k\theta_i + 2\alpha_{i,e}\pi, \quad \forall i \in e, \ \forall e \in E, \tag{22}$$

for some integer $\alpha_{i,e}$. Since the eigenvalue equations are homogeneous, we can scale $\mathbf{x}$ such that $\theta_1 = 0$ without loss of generality. Consequently,

$$\theta + k\theta_i + 2\alpha_{i,e}\pi = \sum_{j\in e} \theta_j = \theta + 2\alpha_{1,e}\pi, \quad \forall i \in e, \forall e \in E_1.$$

Hence, $\theta_i = \alpha_i \frac{2\pi}{k}$ for some integer α_i for all $i \in V(1)$ (the set of vertices that share an edge with the vertex 1). Since G is connected, by a similar proof, we can show that $\theta_i = \alpha_i \frac{2\pi}{k}$ for some integer α_i for all $i \in [n]$. Then, we have

$$\theta + 2\alpha_{1,e}\pi = \sum_{j\in e} \theta_j = \left(\sum_{j\in e} \alpha_j\right) \frac{2\pi}{k}, \quad \forall e \in E_1.$$

Hence, $\theta = \alpha \frac{2\pi}{k}$ for some integer α. With these, we have that 22 becomes

$$\left(\sum_{j\in e} \alpha_j\right) \frac{2\pi}{k} = \alpha \frac{2\pi}{k} + 2(\alpha_i + \alpha_{i,e})\pi, \quad \forall i \in e, \quad \forall e \in E.$$

Equivalently,

$$\sum_{j\in e} \bar{\alpha}_j = \bar{\alpha}, \quad \forall e \in E.$$

Reversing the above proof, we see that the converse statement is true as well. □

As we remarked at the beginning of this subsection, $\theta = \frac{2\alpha\pi}{k}$ with some integer α is not by accident. The eigenvalues of $\mathcal{A}$ with module $\rho(\mathcal{A}) = 1$ distribute uniformly on the spectral circle $\{\lambda \mid |\lambda| = 1\}$ (Yang and Yang 2011). The number of the eigenvalues on this circle is called the **primitive index** of the tensor $\mathcal{A}$.

If $\exp(\frac{2\alpha\pi}{k}\sqrt{-1})$ is an eigenvalue of $\mathcal{A}$, then there exist integers α_i for $i \in [n]$ such that for all $e \in E$, $\sum_{j\in e} \bar{\alpha}_j = \bar{\alpha}$. It is then easy to see that for all $s \in [k]$, $\sum_{j\in e} \overline{s\alpha_j} = \overline{s\alpha}$ for all $e \in E$. Thus, $\exp(\frac{2s\alpha\pi}{k}\sqrt{-1})$ is an eigenvalue of $\mathcal{A}$ for all $s \in [k]$ by Theorem 6.1. Thus, the primitive index must be a factor of k. We include it in the next corollary. Recall that (r, s) denotes the greatest common divisor of the integers r and s.

Corollary 6.1 *Let G be a k-uniform connected hypergraph. If $\alpha \in [k]$ is the smallest positive integer such that $\exp(\frac{2\alpha\pi}{k}\sqrt{-1})$ is an eigenvalue of $\mathcal{A}$, then the primitive index of $\mathcal{A}$ is $\frac{k}{(k,\alpha)}$. Thus, the primitive index of $\mathcal{A}$ is a factor of k; when k is a prime number, either 1 is the unique eigenvalue of $\mathcal{A}$ on the spectral circle or $\exp(j\frac{2\pi}{k}\sqrt{-1})$ is an eigenvalue of $\mathcal{A}$ for all $j \in [k]$.*

The next corollary, together with Theorem 3.1, says that when k is odd, $\mathcal{L}$ does not have an eigenvalue being 2.

Corollary 6.2 *Let G be a k-uniform connected hypergraph and k be odd. Then for any $\lambda \in \sigma(\mathcal{L})$, we have $Re(\lambda) < 2$.*

Proof Note that by Theorem 3.1 (ii), $\mathrm{Re}(\lambda) \le 2$ with equality holding if and only if $\lambda = 2$. In this case -1 is an eigenvalue of $\mathcal{A}$. While, Theorem 6.1 says that -1 cannot be an eigenvalue of $\mathcal{A}$, since k is odd. Thus, the result follows. □

The next corollary says that the spectrum of the adjacency tensor is invariant under the multiplication by $\exp(\frac{2j\pi}{s}\sqrt{-1})$ for all $j \in [s]$ with $s \geq 1$ being the primitive index of $\mathcal{A}$.

Corollary 6.3 *Let G be a k-uniform connected hypergraph and the primitive index of $\mathcal{A}$ be $s \geq 1$. Let $(\alpha_1, \ldots, \alpha_n)$ be a set of integers satisfying the equations 21 for $\alpha = \frac{k}{s}$. If λ is an eigenvalue of $\mathcal{A}$ with an eigenvector $\mathbf{x}$, then $\exp(\frac{2\pi}{s}\sqrt{-1})\lambda$ is also an eigenvalue of $\mathcal{A}$ with an eigenvector $\mathbf{z}$ with $z_i := \exp(\frac{2\alpha_i\pi}{k}\sqrt{-1})x_i$ for all $i \in [n]$.*

Proof Suppose that λ is an eigenvalue of $\mathcal{A}$ with an eigenvector $\mathbf{x} \in \mathbb{C}^n$. Then, we have that $\mathcal{A}\mathbf{x}^{k-1} = \lambda\mathbf{x}^{[k-1]}$. By the hypothesis, we have that

$$\sum_{j\in e} \bar{\alpha}_j = \overline{\left(\frac{k}{s}\right)}, \quad \forall e \in E. \tag{23}$$

Let $\mathbf{z} \in \mathbb{C}^n$ be an n-vector with $z_i := \exp(\frac{2\alpha_i\pi}{k}\sqrt{-1})x_i$ for all $i \in [n]$. Then, for all $i \in [n]$,

$$\begin{aligned}(\mathcal{A}\mathbf{z}^{k-1})_i &= \sum_{e\in E_i} \frac{1}{\sqrt[k]{d_i}} \prod_{j\in e\setminus\{i\}} \frac{z_j}{\sqrt[k]{d_j}} = \sum_{e\in E_i} \frac{\exp(\frac{2\pi}{s}\sqrt{-1})}{\exp(\frac{2\alpha_i\pi}{k}\sqrt{-1})} \frac{1}{\sqrt[k]{d_i}} \prod_{j\in e\setminus\{i\}} \frac{x_j}{\sqrt[k]{d_j}} \\ &= \frac{\exp(\frac{2\pi}{s}\sqrt{-1})}{\exp(\frac{2\alpha_i\pi}{k}\sqrt{-1})}\lambda x_i^{k-1} = \frac{\exp(\frac{2\pi}{s}\sqrt{-1})}{\exp\left(\left[\frac{2\alpha_i\pi}{k} + (k-1)\frac{2\alpha_i\pi}{k}\right]\sqrt{-1}\right)}\lambda z_i^{k-1} \\ &= \exp\left(\frac{2\pi}{s}\sqrt{-1}\right)\lambda z_i^{k-1},\end{aligned}$$

where the second equality follows from 23. Thus, $\exp(\frac{2\pi}{s}\sqrt{-1})\lambda$ is an eigenvalue of $\mathcal{A}$ with the eigenvector $\mathbf{z}$. The proof is complete. □

The invariant of the eigenvalues under the multiplication by $\exp(\frac{2j\pi}{s}\sqrt{-1})$ in Corollary 6.3 follows from (Yang and Yang 2011, Theorem 3.17) as well. While, our proof is more constructive, and it reveals the relations between the eigenvectors of the eigenvalues on the same orbit.

By the proof of Theorem 6.1, we actually get the following theorem, which characterizes all the eigenvectors of $\mathcal{A}$ corresponding to the eigenvalues on the spectral circle.

Theorem 6.2 *Let G be a k-uniform connected hypergraph and $\alpha \in [k]$. Then, a nonzero vector $\mathbf{x}$ is an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue $\exp(\frac{2\alpha\pi}{k}\sqrt{-1})$ if and only if there exist θ and integers α_i such that $x_i = \exp(\sqrt{-1}\theta)\exp(\frac{2\alpha_i\pi}{k}\sqrt{-1})\frac{\sqrt[k]{d_i}}{\sqrt[k]{d_{vol}}}$ for $i \in [n]$, and*

$$\sum_{j\in e} \bar{\alpha}_j = \bar{\alpha}, \quad \forall e \in E.$$

The next corollary says that when k is odd, the H-eigenvector of $\mathcal{A}$ corresponding to the spectral radius is unique up to scalar multiplication.

Corollary 6.4 *Let G be a k-uniform connected hypergraph and k be odd. If $\mathbf{x} \in \mathbb{R}^n$ is an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue one, then $\mathbf{x}$ or $-\mathbf{x}$ is the unique positive eigenvector.*

Proof By Theorem 6.2, the real vector $\mathbf{x}$ that is an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue one should satisfy

$$\exp(\sqrt{-1}\theta)\exp\left(\frac{2\alpha_i\pi}{k}\sqrt{-1}\right) = \pm 1, \quad \forall i \in [n].$$

These constraints say that $\theta + \frac{2\alpha_i\pi}{k} = \beta_i\pi$ for some integers β_i for all $i \in [n]$. Hence, $\frac{2(\alpha_i-\alpha_j)\pi}{k} = (\beta_i - \beta_j)\pi$ for all $i, j \in [n]$. Since k is odd, we must have that $\beta_i - \beta_j \in \langle 2\rangle \subset \mathbb{Z}$. Thus, $\exp(\sqrt{-1}\theta)\exp(\frac{2\alpha_i\pi}{k}\sqrt{-1}) = 1$ for all $i \in [n]$ or $\exp(\sqrt{-1}\theta)\exp(\frac{2\alpha_i\pi}{k}\sqrt{-1}) = -1$ for all $i \in [n]$, since $\exp(\beta_i\pi) = \exp(\beta_j\pi)$ for all $i, j \in [n]$. The result now follows. □

The next corollary gives necessary and sufficient conditions for 2 being an H-eigenvalue of $\mathcal{L}$.

Corollary 6.5 *Let G be a k-uniform connected hypergraph and k be even. Then, 2 is an H-eigenvalue of $\mathcal{L}$ if and only if there exists a pairwise disjoint partition of the vertex set $V = V_1 \cup V_2$ with $V_1 \neq \emptyset$ such that for every edge $e \in E$, $|e \cap V_1|$ is an odd number.*

Proof The sufficiency is obvious: let $\theta = 0$ and $\alpha_i := \frac{k}{2}$ whenever $i \in V_1$ and $\alpha_i = 0$ whenever $i \in V_2$. Since then $\sum_{j\in e}\bar{\alpha}_j = \frac{\bar{k}}{2}$ for all $e \in E$, by Theorem 6.2, we see that -1 is an H-eigenvalue of $\mathcal{A}$. Hence, 2 is an H-eigenvalue of $\mathcal{L}$.

For the necessity, suppose that -1 is an H-eigenvalue of $\mathcal{A}$ with an H-eigenvector $\mathbf{x}$. By Theorem 6.2, we have that there exist θ and integers α_i satisfying

$$\exp(\sqrt{-1}\theta)\exp\left(\frac{2\alpha_i\pi}{k}\sqrt{-1}\right) = \pm 1, \quad \forall i \in [n],$$

and

$$\sum_{j\in e}\bar{\alpha}_j = \frac{\bar{k}}{2}, \quad \forall e \in E.$$

The former constraints say that $\theta + \frac{2\alpha_i\pi}{k} = \beta_i\pi = \frac{\beta_i k\pi}{k} = \frac{2(\frac{k}{2}\beta_i)\pi}{k}$ for some integers β_i for all $i \in [n]$. Thus, $\theta = \frac{2\beta\pi}{k}$ for some integer β. Since $\overline{k\beta} = \bar{0}$, we can absorb θ into α_i for all $i \in [n]$. Without loss of generality, we denote the absorbed integers still by α_i. Then, we have

$$\exp\left(\frac{2\alpha_i\pi}{k}\sqrt{-1}\right) = \pm 1, \quad \forall i \in [n] \Longleftrightarrow \bar{\alpha}_i = \bar{0}, \text{ or } \bar{\alpha}_i = \frac{\bar{k}}{2}, \quad \forall i \in [n],$$

and still

$$\sum_{j\in e}\bar{\alpha}_j=\frac{\bar{k}}{2},\quad \forall e\in E.$$

Since $\overline{2\alpha_i}=\bar{0}$ for all $i\in[n]$, the latter constraints imply that there exists a pairwise disjoint partition of the vertex set $V=V_1\cup V_2$ with $V_1\neq\emptyset$ such that for every edge $e\in E$, $|e\cap V_1|$ is an odd number. Actually, V_1 can be chosen as $\{i\in[n]\mid \bar{\alpha}_i=\frac{\bar{k}}{2}\}$. □

A hypergraph is called k-partite, if there is a pairwise disjoint partition of $V=V_1\cup\cdots\cup V_k$ such that every edge $e\in E$ intersects V_i nontrivially (i.e., $e\cap V_i\neq\emptyset$) for all $i\in[k]$.

Corollary 6.6 *Let G be a k-uniform connected hypergraph. If G is k-partite, then the primitive index of $\mathcal{A}$ is k.*

Proof Since G is k-partite, let $V_1,\ldots,V_k$ be one of its k-partition. For any $j\in[k]$, let $\theta=0$, $\bar{\alpha}_i=\bar{j}$ for $i\in V_1$, and $\bar{\alpha}_i=\bar{0}$ for all $i\notin V_1$. Thus, we fulfill $\sum_{i\in e}\bar{\alpha}_i=\bar{j}$ for all $e\in E$. Hence, for all $j\in[k]$, $\exp(j\frac{2\pi}{k}\sqrt{-1})$ is an eigenvalue of $\mathcal{A}$ by Theorem 6.2. □

Corollaries 6.6 and 6.3 imply that the spectrum of the adjacency tensor of a k-partite hypergraph is invariant under multiplication by any k-th root of unity. Thus, we recover (Cooper andv Dutle 2012, Theorem 4.2) for the spectrum of the normalized adjacency tensor of a k-partite hypergraph.

6.2 Algebraic reformulation

In this short subsection, we reformulate Theorems 6.1 and 6.2 into the language of linear algebra over modules.

For a positive integer k, let the ring $\mathbb{K}$ be defined as above. Let $\mathbb{A}:=\mathbb{K}^n$ be the free $\mathbb{K}$-module of rank n. For every hypergraph $G=(V,E)$, we can associated it a submodule $\mathbb{G}\subseteq\mathbb{A}$ with $\mathbb{G}$ being generated by $\tilde{G}:=\{\mathbf{z}(e)\in\mathbb{A}\mid z(e)_i=\bar{1},\ i\in e, z(e)_i=\bar{0},\ i\notin e,\ \forall e\in E\}$. Let $\hat{G}$ be the $|E|\times n$ representation matrix of $\mathbb{K}^{|E|}\rightarrow\mathbb{G}$ with its rows being given by $\mathbf{z}^T$ with $\mathbf{z}\in\tilde{G}$. Denote by $\mathbf{1}\in\mathbb{A}$ the vector consisting of all one. Then we have the following result.

Theorem 6.3 *Let G be a k-uniform connected hypergraph. Let $\theta=\frac{2\alpha\pi}{k}$ with some nonnegative integer α. Then, $\exp(\sqrt{-1}\theta)$ is an eigenvalue of $\mathcal{A}$ if and only if there is a vector $\mathbf{y}\in\mathbb{A}$ such that $\mathbf{z}^T\mathbf{y}=\bar{\alpha}$ for all $\mathbf{z}\in\tilde{G}$. Moreover, when $\exp(\sqrt{-1}\theta)$ is an eigenvalue of $\mathcal{A}$, it has a unique eigenvector (up to scalar multiplication) if and only if the kernel of $\hat{G}$ is $\langle\mathbf{1}\rangle$.*

The merit of Theorem 6.3 is that it states the nonlinear eigenvalue problem of tensors as a linear algebra problem. In the classic linear algebra over fields, for a matrix $A\in\mathbb{R}^{m\times n}$, we see that $\dim(\ker(A))+\dim(\mathrm{im}(A^T))=n$ (Horn 1985). Here

$\ker(A)$ and $\mathrm{im}(A)$ mean respectively the kernel and the image of the linear map $A : \mathbb{R}^n \to \mathbb{R}^m$. However, in above case for $\hat{G}$, the situation is more complicated, since the set of the first syzygies of $\mathbb{G}$ could be nontrivial (i.e., other than $\{0\}$) (Cox 2006, 1998). The next further work would be necessary.

7 Final remarks

This paper, the Laplacian of a uniform hypergraph is introduced and investigated. Various basic facts about the spectrum of the Laplacian are explored. These basic facts are related to the structures of the hypergraph. Among them, the sets of the H^+-eigenvalues and the nonnegative eigenvectors of the Laplacian are characterized through the spectral components and the flower hearts of the hypergraph. It is shown that all the H^+-eigenvalues of the Laplacian can be computed out efficiently. Thus, they are applicable. We also characterized the eigenvectors of the eigenvalues on the spectral circle of the adjacency tensor. It is formulated in the language of linear algebra over modules. That would be our next topic to investigate.

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