CHAPTER I

EXTERIOR ALGEBRA

1.1. SCOPE OF THE CHAPTER

An operation that helps us to extend in some way the notion of vectorial product in the classical vector algebra to vector spaces with dimensions higher than three is called *the exterior product* and a vector space equipped with such an operation assigning a new vector to every pair of vectors in the vector space is called an *exterior algebra*. This operation was introduced in 1844 by German mathematician Hermann Günter Grassmann (1809-1877). Thus the exterior algebra is sometimes known as the ***Grassmann algebra***.

We first define in Sec. 1.2 linear vector spaces axiomatically over which the exterior algebra will be built. Some pertinent attributes of vector spaces to which we will have recourse frequently are briefly discussed there. These are concepts of linear independence and basis, linear operators, the algebraic dual space that is the linear vector space formed by linear functionals over this vector space and some significant properties of dual spaces of finite-dimensional vector spaces and finally exact sequences. Then, the multilinear functionals that are mappings from the finite Cartesian product of vector spaces into the field of scalars that are linear in each of their arguments are considered in Sec. 1.3. It is shown that by properly defining the operation of tensor product it becomes possible to endow the Cartesian products of vector spaces with a structure of a vector space and it is observed that multilinear (k-linear) functionals may be expressible in terms of elements of that space called tensors (contravariant on the vector spaces, covariant on their duals). Afterward we investigate briefly in Sec. 1.4 alternating k-linear functionals that are completely antisymmetric with respect to their arguments and the operation of alternation which help produce completely antisymmetric quantities. The generalised Kronecker deltas and Levi -Civita symbols that facilitate to a great extent the implementation of this operation are also discussed in detail. The exterior product of vectors are then defined by means of the operation of alternation on tensor products. It is then shown that a completely antisymmetric covariant tensor representing

Exterior Analysis, DOI: 10.1016/B978-0-12-415902-0.50001-3

an alternating k-linear functional is expressible by using exterior products. Such a tensor will be called an *exterior form*. Exterior products of exterior forms are defined in such a way that two exterior forms generate another form of different degree. Thus, this enables us to construct in Sec. 1.5 an exterior algebra over a vector space. This chapter ends in Sec. 1.6 with the discussion of the concept of rank of a form that makes it possible sometimes to reduce an exterior form to a simpler structure.

1.2. LINEAR VECTOR SPACES

In order to define a linear vector space abstractly we consider an *Abelian* (*commutative*) *group* $\{G, \#\}$ [after Norwegian mathematician Niels Henrik Abel (1802-1829)] and a *field* $\{\mathbb{F}, +, \times\}$. 1 is the identity element of the field with respect to multiplication. A binary operation $\mathbb{F} \times G \to G$ is denoted by $*$. Hence, this binary operation assigns a member $\alpha * x \in G$ of the group to an arbitrary scalar $\alpha \in \mathbb{F}$ and an arbitrary member $x \in G$ of the group. Furthermore, we shall assume that this binary operation $*$ will obey the following rules for all $\alpha, \beta \in \mathbb{F}$ and $x, y \in G$:

$$(i).\ (\alpha \times \beta) * x = \alpha * (\beta * x).$$
$$(ii).\ (\alpha + \beta) * x = (\alpha * x) \# (\beta * x), \quad \alpha * (x \# y) = (\alpha * x) \# (\alpha * y).$$
$$(iii).\ 1 * x = x.$$

The algebraic system $\mathcal{V} = \{G, \mathbb{F}, \#, +, \times, *\}$ satisfying these conditions is called a ***linear vector space*** over the field $\mathbb{F}$. Members of the group are named as ***vectors*** whereas members of the field as ***scalars***. The operation $\#$ is known as *vector addition* and the operation $*$ as *scalar multiplication*.

Sometimes it becomes advantageous to replace the field of scalars by a ring with identity in the system described above. Such an algebraic system is then called a ***module***. We will have opportunities to deal with modules in later parts of this work.

As far as we are concerned, the field of scalars $\mathbb{F}$ will either be the real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. Accordingly, we shall consider either *real* or *complex vector spaces*. However, in this work, we shall be mostly interested in real spaces. Moreover, in order to simplify the notation we prefer to use the same symbol $+$ to designate addition operations both in the group and in the field while identity elements with respect to these operations will be represented, respectively, by the symbols $\mathbf{0}$ and 0. Usually, we shall not use any symbol for scalar multiplication as well as for the product of two scalars of the field by adopting the familiar convention employed in the multiplication of real or complex numbers. Although one might think that representation of different operations by the same symbol would cause some complications, we should observe that the real nature of these symbols are unambiguously revealed within the context of expressions in which they are involved. Thus it is unlikely that misinterpretations may ever arise concerning these operations. Nevertheless, a much more detailed definition of a linear vector space can also be given as follows.

I. $+$ *is a binary operation on a set* V*, whose members are called vectors, having the following properties*:

(i). $u+v\in V$ *for all* $u,v\in V$ (*closed operation*).
(ii). $u+v=v+u$ *for all* $u,v\in V$ (*commutative operation*).
(iii). $(u+v)+w=u+(v+w)$ *for all* $u,v,w\in V$ (*associative operation*).
(iv). *There exists an* ***identity element*** $\mathbf{0}\in V$ *such that* $u+\mathbf{0}=u$.
(v). *There exists an* ***inverse element*** $-u\in V$ *for each* $u\in V$ *such that* $u+(-u)=\mathbf{0}$.

These properties are tantamount to say that the set V *is an Abelian group with respect to the operation* $+$. *The element* $\mathbf{0}$ *is called the* ***zero vector*** *and* $u+v$ *is called the* ***vector sum*** *of vectors* u *and* v. *We usually employ the abbreviated notation* $u-v$ *to denote* $u+(-v)$.

II. *Let* $\mathbb{F}$ *be a field of scalars. Scalar multiplication over the Abelian group* V *is so defined that it satisfies the following relations*:

For all $\alpha,\beta\in\mathbb{F}$ *and* $u,v\in V$ *we have*

(i). $\alpha u\in V$ (*closed operation*).
(ii). $(\alpha\beta)u=\alpha(\beta u)$ (*associative operation*).
(iii). $(\alpha+\beta)u=\alpha u+\beta u, \alpha(u+v)=\alpha u+\alpha v$ (*distributive operation*).
(iv). $1\cdot u=u$.

Here 1 *is the identity element of the field of scalars with respect to the multiplication. We call the set* V *satisfying all axioms in* **I** *and* **II** *a* ***linear vector space over the field*** $\mathbb{F}$. *The scalar multiplication is represented by the symbol* $\cdot$ *although we would often prefer to omit it.*

We can deduce some fundamental properties of linear vector spaces from the foregoing axioms:

(a). If we write $u=1\cdot u=(1+0)\cdot u=1\cdot u+0\cdot u=u+0\cdot u$, we immediately obtain

$$0\cdot u=\mathbf{0}$$

for all $u\in V$.

(b). From $\mathbf{0}=0\cdot u=(1-1)\cdot u=1\cdot u+(-1)\cdot u=u+(-1)\cdot u$, it follows that

$$(-1) \cdot u = -u$$

for all $u \in V$.

(c). Since $\alpha u = \alpha(u + \mathbf{0}) = \alpha u + \alpha \cdot \mathbf{0}$, we find that

$$\alpha \cdot \mathbf{0} = \mathbf{0}$$

for all $\alpha \in \mathbb{F}$.

Example 1.2.1. Let us consider the set $\mathbb{F}^n$ where n is a positive integer. $\mathbb{F}^n$ is the Cartesian product $\underbrace{\mathbb{F} \times \mathbb{F} \times \cdots \times \mathbb{F}}_{n}$. An element $u \in \mathbb{F}^n$ is an ordered n-tuple $u = (\alpha_1, \alpha_2, \dots, \alpha_n)$ where $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{F}$. For elements $v = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{F}^n$ and $\alpha \in \mathbb{F}$ let us define the vector addition and scalar multiplication by making use of the operations in the field $\mathbb{F}$ as follows

$$u + v = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_n + \beta_n),\ \alpha u = (\alpha\alpha_1, \alpha\alpha_2, \dots, \alpha\alpha_n).$$

It is then straightforward to see that the set $\mathbb{F}^n$ so equipped is a linear vector space. The zero vector $\mathbf{0} \in \mathbb{F}^n$ is the n-tuple $(0, 0, \dots, 0)$ and the inverse of the vector u is $-u = (-\alpha_1, \dots, -\alpha_n)$. With the same rules $\mathbb{R}^n$ becomes a real vector space while $\mathbb{C}^n$ is a complex vector space.

If we increase n indefinitely, the elements of the set $\mathbb{F}^\infty$ are *sequences* of scalars given by

$$u = (\alpha_1, \alpha_2, \dots, \alpha_n, \dots).$$

With the same rules $\mathbb{F}^\infty$ becomes also a linear vector space. ■

Example 1.2.2. Let us consider the set $\mathcal{F}(X, \mathbb{F})$ of all scalar-valued functions $f : X \to \mathbb{F}$ on an abstract set X. We define the sum of two functions in that set and the multiplication of a function with a scalar by the following rules

$$(f_1 + f_2)(x) = f_1(x) + f_2(x),\ \ (\alpha f)(x) = \alpha f(x).$$

We then see at once that this set acquires the structure of a vector space over the field $\mathbb{F}$. The zero vector $\mathbf{0}$ of this space corresponds naturally to zero function mapping all members of X to 0. ■

Let V be a vector space and $U \subseteq V$ be a subset. If the subset U is a linear vector space relative to operations in V, then the subset U is said to be a ***subspace*** of V. Subspaces are sometimes called *linear manifolds*. It may easily be verified that the necessary and sufficient conditions for a subset $U \subseteq V$ to be a subspace are (i) $u_1 + u_2 \in U$ for all $u_1, u_2 \in U$ and (ii) $\alpha u \in U$ for all $\alpha \in \mathbb{F}$ and $u \in U$. It is clear that we must have $\mathbf{0} \in U$. Every linear vector space has obviously two trivial subspaces: zero subspace $\{\mathbf{0}\}$ and the space itself.

As is well known, an ***equivalence relation*** R on an arbitrary set X is a subset $R \subseteq X^2$ of the Cartesian product $X^2 = X \times X$ which is *reflexive* $(x \in X \Rightarrow (x,x) \in R)$, *symmetric* $\big((x_1,x_2) \in R \Rightarrow (x_2,x_1) \in R\big)$ and *transitive* $\big((x_1,x_2) \in R,\ (x_2,x_3) \in R \Rightarrow (x_1,x_3) \in R\big)$. The set of all elements of X that are related to an element $x \in X$ by the equivalence relation is called an ***equivalence class*** $[x]$. It is readily seen that $\bigcup_{x\in X}[x] = X$ and *equivalence classes are all disjoint sets*. Therefore, equivalence classes constitute a ***partition*** on the set X. The set $X/R = \{[x] : x \in X\}$ is called the ***quotient set*** with respect to the equivalence relation R.

Let U be a subspace of the vector space V. We define a relation $\sim$ on V such that $u \sim v$ implies $u - v \in U$ for $u, v \in V$. Since $u - u = \mathbf{0} \in U$ we have $u \sim u$, i.e., the relation is reflexive. If $u \sim v$, namely if $u - v \in U$ we obtain $v - u = -(u - v) \in U$ and we see that $v \sim u$, i.e., the relation is symmetric. On the other hand, if $u \sim v$, $v \sim w$, namely, both $u - v \in U$ and $v - w \in U$, we then get $u - w = u - v + v - w \in U$. Hence we find that $u \sim w$, i.e., the relation is transitive. We then conclude that the relation so defined is an equivalence relation. Thus, this relation decomposes the vector space V into disjoint equivalence classes. Therefore an equivalence class, or a ***coset***, associated with a vector $v \in V$ is defined as the set

$$[v] = \{v + u : \forall u \in U\}. \tag{1.2.1}$$

Sometimes the notation $[v] = v + U$ is also used. We know that the set of all equivalence classes $V/U = \{[v] : v \in V\}$ is the quotient set. If we can devise appropriate rules for the addition of element of this set and for the scalar multiplication we are then able to endow the quotient set V/U with a vector space structure. To this end, we define vector addition and scalar multiplication on V/U by the following rules

$$[v_1] + [v_2] = [v_1 + v_2], \quad \alpha[v] = [\alpha v] \tag{1.2.2}$$

where the scalar α is an element of the field over which the vector space V is defined. The validity of this definition becomes evident if we note that

$$\begin{gathered}(v_1 + u_1) + (v_2 + u_2) = (v_1 + v_2) + u_1 + u_2 \in [v_1 + v_2]\\ \alpha(v + u) = \alpha v + \alpha u \in [\alpha v]\end{gathered}$$

for all $v_1, v_2 \in V$ and $u, u_1, u_2 \in U$. The set V/U equipped with such a structure is called the ***quotient space***, or more accurately, the ***quotient space of V modulo U***. The zero element of this vector space is the coset $U = [\mathbf{0}]$ and the inverse of an element $[v]$ is the coset $[-v]$. Since an equivalence class $[v] \in V/U$ is assigned to each vector $v \in V$, we can say that there exists a surjective mapping $\phi : V \to V/U$. ϕ is often called the ***canonical mapping*** of V onto V/U and we can write $[v] = \phi(v)$. Due to definitions (1.2.2) we immediately deduce that the mapping ϕ must satisfy the relations $\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2)$ and $\phi(\alpha v) = \alpha\phi(v)$. Thus *the canonical mapping is linear* [*see p.* 9]. Obviously, ϕ is not injective in general.

Let U_1 and U_2 be two subspaces of the vector space V. We define the set $U_1 + U_2$ by

$$U_1 + U_2 = \{u = u_1 + u_2 : \forall u_1 \in U_1, \forall u_2 \in U_2\} \subseteq V.$$

It is straightforward to see that this set is a *subspace* of V that is called the ***sum*** of subspaces U_1 and U_2. One must note that the sum of two subspaces is completely different from their *union* $U_1 \cup U_2$ as sets. It is easy to see that $U_1 \cup U_2$ is not in general a subspace. The ***intersection*** of two subspaces U_1 and U_2 is the set of all vectors belonging to both subspaces. It is then properly denoted by $U_1 \cap U_2$. In contrast to the union, one easily observes that the intersection of two subspaces, in fact, the intersection of a family of subspaces, is again a subspace. The intersection of subspaces cannot be empty since all subspaces must contain the zero vector. We say that two subspaces U_1 and U_2 of V are ***disjunct*** if $U_1 \cap U_2 = \{\mathbf{0}\}$.

Let U_1 and U_2 be two subspaces of the vector space V and let the subspace $U = U_1 + U_2 \subseteq V$ be the sum of these subspaces. If there corresponds to each vector $u \in U$ a *uniquely determined* pair of vectors $u_1 \in U_1$ and $u_2 \in U_2$ such that $u = u_1 + u_2$, we then say that the subspace U is the ***direct sum*** of subspaces U_1 and U_2 and we write $U = U_1 \oplus U_2$. It is quite easy to see that *the sum U of two subspaces U_1 and U_2 is a direct sum of these subspaces if and only if U_1 and U_2 are disjunct, that is, if and only if $U_1 \cap U_2 = \{\mathbf{0}\}$.*

Let V be a linear vector space and let V_1 be a subspace of V. If we can find another subspace V_2 of V such that

$$V = V_1 \oplus V_2$$

any such subspace V_2 is said to be ***complementary*** to V_1 in V. It can be shown by employing the celebrated ***Zorn lemma*** [German-American mathematician Max August Zorn (1906-1993)] that *there exists at least one subspace which is complementary to a given subspace of a linear vector space.* However, a complementary subspace is generally not uniquely determined. It is rather straightforward to observe that the restriction $\phi|_{V_2}$ of the canonical mapping $\phi : V \to V/V_1$ is injective, consequently, the function $\phi|_{V_2} : V_2 \to V/V_1$ is bijective. Therefore, $\phi|_{V_2}$ is an isomorphism between the spaces V_2 and V/V_1. We thus conclude that *any subspace of V which is complementary to a subspace V_1 is isomorphic to the quotient space V/V_1.*

This result reflects the fact that *all complementary subspaces of V_1 in V are isomorphic* to one another [*see p.* 10 for the definition of isomorphism].

Let $S_n = \{v_1, v_2, \dots, v_n\}$ be a non-empty set of a *finite*, say $n > 0$, number of elements of a vector space V. The vector l formed by the sum

$$l = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \in V$$

where $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{F}$ are arbitrary scalars is called a ***linear combination*** of the vectors in S_n. We call the set S_n as ***linearly independent*** if and only if the relation

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \mathbf{0} \tag{1.2.3}$$

is satisfied when all scalar coefficients vanish, namely, when $\alpha_i = 0$ for all $1 \le i \le n$. On the other hand, if the expression (1.2.3) is satisfied with scalar coefficients not all of which are zero, the set S_n is called as ***linearly dependent***. If *all non-empty finite subsets* of a possibly infinite set $\mathcal{A} \subseteq V$ are linearly independent, we say that the set $\mathcal{A}$ is ***linearly independent***. In such a set $\mathcal{A}$ no element of $\mathcal{A}$ can be expressed as a finite linear combination of some other elements of $\mathcal{A}$. *It is quite clear that a linearly independent set cannot be empty and cannot contain the zero vector.* Let us denote the *subspace* which is the collection of all finite linear combinations of vectors in $\mathcal{A}$ by $[\mathcal{A}]$. This subspace is called the ***linear hull*** of the set $\mathcal{A}$.

Theorem 1.2.1. *A subset $\mathcal{A}$ of a vector space V is linearly independent if and only if each vector in the subspace $[\mathcal{A}]$ can be uniquely represented as a finite linear combination of vectors in the set.*

Let the set $\mathcal{A}$ be linearly independent and let us assume that a vector $v \in [\mathcal{A}]$ is expressible as two different finite linear combinations of vectors in $\mathcal{A}$. But we can of course naturally combine vectors appearing in the first and the second representations into a single finite set such as $v_1, v_2, \dots, v_k$. We can then write

$$v = \sum_{i=1}^{k} \alpha_i v_i = \sum_{i=1}^{k} \beta_i v_i$$

where some of scalar coefficients $\{\alpha_i\}$ and $\{\beta_i\}$ may of course be zero. It then follows from the above expression that

$$\sum_{i=1}^{k} (\alpha_i - \beta_i) v_i = \mathbf{0}$$

which yields $\alpha_i = \beta_i$ for all $1 \le i \le k$ since all of the vectors involved are linearly independent. Hence the vector v has a unique representation. Conversely, let us assume that every vector in the subspace $[\mathcal{A}]$ has a *unique* representation in the form of finite linear combination of vectors in A. Since the set A is also contained in $[\mathcal{A}]$ this uniqueness should also be valid for all vectors in $\mathcal{A}$. This simply means that any element of $\mathcal{A}$ cannot be expressible as a linear combination of other vectors in $\mathcal{A}$. Hence $\mathcal{A}$ is a linearly independent set. □

If the linear hull of a linearly independent subset $\mathcal{B}$ of a vector space V is the entire space V, that is, if $[\mathcal{B}] = V$, then the set $\mathcal{B}$ is called a ***basis*** for the vector space V. In this case, every vector v in the vector space is expressible *in exactly one way* as a *finite linear combinations* of some vectors in B. Therefore, each vector $v \in V$ can be represented by the sum

$$v = \sum_{e_\lambda \in \mathcal{B}} \alpha_\lambda(v) e_\lambda \qquad (1.2.4)$$

where scalar coefficients $\alpha_\lambda(v) \in \mathbb{F}$ that are determined uniquely for any given vector v *do not vanish only for a finite number vectors* $e_\lambda \in \mathcal{B}$ and they are called ***components*** of the vector v with respect to the basis $\mathcal{B}$. The basis $\mathcal{B}$ might be an infinite, even uncountably infinite, set but the expression (1.2.4) must involve only a sum of finite number of vectors that may of course be different for each vector $v \in V$. Such a basis, if it exists, is called an ***algebraic basis*** or ***Hamel basis*** because it was first introduced, albeit in a limited framework, by German mathematician Georg Karl Wilhelm Hamel (1877-1954). We can also readily show that a linearly independent set $\mathcal{B}$ of V is a basis *if and only if* it is *maximal* with respect to linear independency. Here the term maximal is used to indicate that every subset of V containing the set $\mathcal{B}$ is *linearly dependent*. One can prove by resorting to the Zorn lemma that every *non-zero* vector space has an algebraic basis. However, like almost every proposition based on Zorn lemma, we have no algorithm at hand to determine such a basis although we definitely know that it exists. Furthermore, we cannot say that there exists a unique basis.

It is now quite clear that a non-zero vector space V might possess several, possibly infinitely many, bases. But it can be shown that all Hamel bases have the same cardinality. This cardinal number is called the ***dimension*** of the vector space V and is denoted by $\dim(V)$. If $V = \{\mathbf{0}\}$ we adopt the convention that its dimension is 0. If the dimension of a vector space is a finite integer, then this space is ***finite-dimensional***, otherwise it is ***infinite-dimensional***. In this work, we shall mostly be dealing with finite-dimensional vector spaces. When we would like to underline this fact we shall usually write, say, $V^{(n)}$.

In a vector space V, the ***line segment*** joining two vectors u and v is defined as the subset $\{\alpha u + (1-\alpha)v : 0 \leq \alpha \leq 1\} \subset V$. A non-empty subset A of a vector space V is called a ***convex*** subset if it contains every line segments joining any pair of vectors $u, v \in A$. In other words, a set $A \subseteq V$ is a convex set if $\{\alpha u + (1-\alpha)v : 0 \leq \alpha \leq 1\} \subset A$ for all

$u, v \in A$. *If A is a subspace of V, it is clear that it becomes automatically a convex set.*

Example 1.2.3. Let us consider the vector space $\mathbb{F}^n$ introduced in Example 1.2.1 and define the vectors $e_1, e_2, \dots, e_n \in \mathbb{F}^n$ as

$$e_1 = (1, 0, \dots, 0),\ \ e_2 = (0, 1, \dots, 0),\ \dots,\ e_n = (0, 0, \dots, 1).$$

It is obvious that an arbitrary vector $u = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{F}^n$ can now be expressed by the following linear combination

$$u = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n.$$

From the definitions of vectors $e_1, e_2, \dots, e_n$ we see at once that the relation

$$\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n = (\alpha_1, \alpha_2, \dots, \alpha_n) = \mathbf{0}$$

is satisfied if and only if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Hence the set $\mathcal{B} = \{e_1, e_2, \dots, e_n\} \subset \mathbb{F}^n$ is linearly independent and all linear combinations of vectors in $\mathcal{B}$ generate the vector space $\mathbb{F}^n$. Hence $\mathcal{B}$ is an algebraic basis for $\mathbb{F}^n$. Since the cardinal number of the set $\mathcal{B}$ is n, the dimension of the vector space $\mathbb{F}^n$ is n.

On the other hand, if we consider the vector space $\mathbb{F}^\infty$ we can easily verify that the countably infinite set $\{e_1, e_2, \dots, e_n, \dots\} \subset \mathbb{F}^\infty$ where

$$e_1 = (1, 0, \dots, 0, \dots),\ \dots, e_n = (0, 0, \dots, 0, 1, 0, \dots), \dots$$

are linearly independent and any vector $u = (\alpha_1, \alpha_2, \dots, \alpha_n, \dots)$ is uniquely represented by

$$u = \sum_{n=1}^{\infty} \alpha_n e_n.$$

However, it is quite evident that each vector $u \in \mathbb{F}^\infty$ cannot be expressed as a *finite* linear combinations of vectors $e_1,\ e_2, \dots, e_n, \dots$. Therefore, the countably infinite subset $\{e_1, e_2, \dots, e_n, \dots\} \subset \mathbb{F}^\infty$ cannot be a Hamel basis for the vector space $\mathbb{F}^\infty$. ■

If a function $A : U \to V$ between vector spaces U and V defined on the same scalar field $\mathbb{F}$ possesses the properties

$$A(u_1 + u_2) = A(u_1) + A(u_2) \in V,\ \ A(\alpha u) = \alpha A(u) \in V$$

for all $u, u_1, u_2 \in U$ and $\alpha \in \mathbb{F}$, then it is called a ***linear operator*** or a ***homomorphism*** since it preserves algebraic operations. It is evident that all linear operators of this kind constitute also a vector space $\mathcal{L}(U, V)$. If the inverse linear operator $A^{-1} : V \to U$ exists, then A is a ***regular linear operator***. The ***null space*** of a linear operator A is the subspace $\mathcal{N}(A) =$

$\{u \in U : Au = \mathbf{0}\} \subseteq U$ and its ***range*** is the subspace $\mathcal{R}(A) = \{v \in V : Au = v, \forall u \in U\} \subseteq V$. Sometimes $\mathcal{N}(A)$ is denoted by $\mathrm{Ker}\,(A)$, ***kernel*** of A, and $\mathcal{R}(A)$ by $\mathrm{Im}\,(A)$, ***image*** of U under A. We see that $\mathcal{N}(A) = \{\mathbf{0}\}$ if A is *injective* and $\mathcal{R}(A) = V$ if it is surjective. The necessary and sufficient condition for a linear operator to be regular is that it has to be bijective, i.e., $\mathcal{N}(A) = \{\mathbf{0}\}$ and $\mathcal{R}(A) = V$. A bijective linear mapping between two vector spaces preserving operations is called ***isomorphism*** and such spaces are said to be ***isomorphic***. It is straightforward to see that *compositions of isomorphisms is also an isomorphism*. It is a simple exercise to show that if $A : U \to V$ is an isomorphism and the set $\mathcal{B} \subseteq U$ is an algebraic basis for U, then the set $A(\mathcal{B})$ is an algebraic basis for V.

The ***rank*** $r(A)$ of a linear operator $A : U \to V$ is the dimension of its range and its ***nullity*** $n(A)$ is the dimension of its null space. Let N_A be a complementary subspace of the null space $\mathcal{N}(A)$ in U so that one writes $U = \mathcal{N}(A) \oplus N_A$. We consider the restriction $A^\dagger = A\big|_{N_A}$ of the linear transformation A to the subspace N_A. Each vector $u \in U$ is now expressed as a unique sum $u = u_1 + u_2$ where $u_1 \in \mathcal{N}(A)$ and $u_2 \in N_A$. We immediately notice that $\mathcal{R}(A^\dagger) = \mathcal{R}(A)$. Next, let us assume that $A^\dagger u_2 = \mathbf{0}$. We thus have $Au = Au_1 + Au_2 = Au_2 = A^\dagger u_2 = \mathbf{0}$. In consequence, we see that $u_2 \in \mathcal{N}(A)$. But, $\mathcal{N}(A) \cap N_A = \{\mathbf{0}\}$, therefore, it follows that $u_2 = \mathbf{0}$ which means that *the restriction of a linear transformation to the complementary subspace of its null space is injective*, *hence it is an isomorphism of* N_A *onto* $\mathcal{R}(A)$. *Consequently, if the set* $\mathcal{B}_1$ *is a basis for* N_A, *then* $A^\dagger(\mathcal{B}_1)$ *is a basis for* $\mathcal{R}(A)$. We thus conclude that if U is an n-dimensional vector space, then $\mathcal{R}(A)$ has to be finite-dimensional so that one gets the simple, but rather useful, relation

$$\dim\,(U) = n = n(A) + r(A).$$

If linearly independent vectors $e_1, \ldots, e_n \in U^{(n)}$ are chosen as a basis for a finite-dimensional vector space, then each vector $v \in U^{(n)}$ is *uniquely* expressible as

$$u = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n, \quad \alpha_i \in \mathbb{F}, \;\; 1 \le i \le n.$$

Let us denote $a = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{F}^n$. We then see that there exists a mapping $F : U^{(n)} \to \mathbb{F}^n$ determined by the relation $F(u) = a$. We deduce immediately from definition that F is a bijective linear operator. We thus conclude that the spaces $U^{(n)}$ and $\mathbb{F}^n$ are *isomorphic.*

Let U be a vector space defined over a field of scalars $\mathbb{F}$. A linear operator $f : U \to \mathbb{F}$ that assigns a scalar number $f(u)$ to each vector u in U is known as a ***linear functional***. The term functional was coined by French mathematician Jacques Salomon Hadamard (1865-1963) in 1903. The linear vector space $U^* = \mathcal{L}(U, \mathbb{F})$ formed by linear functionals is called the ***dual***, or more appropriately the ***algebraic dual***, of the vector space U.

Consider vector spaces $U^{(m)}$ and $V^{(n)}$ with bases $\{e_i\}$ and $\{f_i\}$ respectively. Let $A : U \to V$ be a linear operator. We can then write

$$v = Au = A\Big(\sum_{i=1}^{m} u^i e_i\Big) = \sum_{i=1}^{m} u^i A e_i = \sum_{i=1}^{m}\sum_{j=1}^{n} u^i a_i^j f_j = \sum_{j=1}^{n} v^j f_j$$

from which it follows that $v^j = \sum_{i=1}^{m} a_i^j u^i, j = 1, \ldots, n$. This relation is expressible in the matrix form $\mathbf{v} = \mathbf{Au}$ where $\mathbf{A}$ is the $m \times n$ matrix $[a_i^j]$ and $\mathbf{u}, \mathbf{v}$ are column matrices $[u^i], [v^j]$. The matrix $\mathbf{A}$ is a representation of the linear operator A with respect to some chosen bases in U and V.

Let us now consider a finite-dimensional vector space $U^{(n)}$. If a basis of this space is $\{e_1, e_2, \ldots, e_n\}$, then every vector $u \in U$ is written as $u = \sum_{i=1}^{n} u^i e_i$ where $u^i \in \mathbb{F}$. The value of a linear functional $f \in U^*$ on a vector u can now be evaluated as follows:

$$f(u) = \sum_{i=1}^{n} f(u^i e_i) = \sum_{i=1}^{n} u^i f(e_i) = \sum_{i=1}^{n} u^i \alpha_i \in \mathbb{F} \tag{1.2.5}$$

where the scalar numbers α_i are prescribed by

$$\alpha_i = f(e_i) \in \mathbb{F}, \quad i = 1, \ldots, n. \tag{1.2.6}$$

This means that the action of any linear functional on a vector u is completely determined by an ordered n-tuple $a = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{F}^n$. Thus, there is a mapping $T : U^* \to \mathbb{F}^n$ such that $T(f) = a$. If $T(f_1) = a_1,\ T(f_2) = a_2$, we then deduce from (1.2.5) that $T(f_1 + f_2) = a_1 + a_2,\ T(\alpha f) = \alpha a$. Hence, T turns out to be a linear operator. Each ordered n-tuple of scalars $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ determines a linear functional f. Therefore T is surjective. On other hand if $T(f_1) = T(f_2) = a$ we find $T(f_1 - f_2) = \mathbf{0} = (0, 0, \ldots, 0)$ and (1.2.5) leads to the conclusion that $(f_1 - f_2)(u) = 0$ for all $u \in U^{(n)}$. This simply implies that $f_1 - f_2 = 0$ or $f_1 = f_2$. Thus T is injective. Consequently, the linear operator T is bijective. This indicates that the vector space U^* is *isomorphic* to $\mathbb{F}^n$ just like the space $U^{(n)}$. Since isomorphic spaces must have the same dimension, the dimension of the space U^* is also n. Furthermore, U^* and $U^{(n)}$ must be isomorphic to one another because they are isomorphic to the same space $\mathbb{F}^n$. Let us now consider n linearly independent vectors $(0,\ldots,\underset{i}{1},\ldots,0), i=1,\ldots,n$ in the vector space $\mathbb{F}^n$ such that in the ith vector only its ith entry is 1 and all the others are zero. We can then obtain n linear functionals $f^i \in U^*, i = 1,\ldots,n$ corresponding to those vectors in $\mathbb{F}^n$ through the isomorphism $T^{-1}: \mathbb{F}^n \to U^*$.

The definition (1.2.6) leads now to relations

$$f^i(e_j) = \delta^i_j, \;\; i,j = 1,2,\ldots,n \tag{1.2.7}$$

where δ^i_j denotes the ***Kronecker delta*** [it is so named because it was first introduced by German mathematician Leopold Kronecker (1823-1891)]. It is equal to 1 if $i=j$ and to 0 if $i \neq j$. Hence, it essentially represents the $n \times n$ unit matrix. The set of linear functionals $\{f^i\}$ so obtained is linearly independent. To see this, we consider the zero functional given by

$$c_1 f^1 + c_2 f^2 + \cdots + c_i f^i + \cdots + c_n f^n = 0$$

where $c_1, c_2, \ldots, c_n \in \mathbb{F}$. Because the value of this functional on the basis vectors $e_j, j = 1,\ldots,n$ of the vector space U must be zero, we obtain

$$\sum_{i=1}^{n} c_i f^i(e_j) = \sum_{i=1}^{n} c_i \delta^i_j = c_j = 0, \;\; j = 1,2,\ldots,n.$$

This means that all linear functionals $f^1, f^2, \ldots, f^n$ are linearly independent and constitute a basis for the dual space U^* since its dimension is n. Hence, an arbitrary linear functional $f \in U^*$ can now be uniquely represented in the following form:

$$f = \sum_{i=1}^{n} \alpha_i f^i, \;\; \alpha_i \in \mathbb{F}, \;\; i = 1,\ldots,n.$$

Let $\{e_1,\ldots,e_n\}$ be the basis in U which we have employed to generate the basis $\{f^i\} \subset U^*$. Then the value of a functional $f \subset U^*$ on a vector $u \in U$ can be calculated as follows

$$f(u) = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i u^j f^i(e_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i u^j \delta^i_j = \sum_{i=1}^{n} u^i \alpha_i$$

which is the same as (1.2.5). We easily observe that the relations

$$f^i(u) = u^i, \quad f(e_i) = \alpha_i \tag{1.2.8}$$

are satisfied.

The two foregoing ordered sets of basis vectors $\{f^1, f^2, \dots, f^n\}$ of U^* and $\{e_1, e_2, \dots, e_n\}$ of U are called ***dual*** (or ***reciprocal***) bases. In view of the relations (1.2.7), we may also say that they form a set of ***biorthogonal bases***.

Sometimes it proves to be more convenient to use the notation $\langle f, u\rangle$ instead of $f(u)$. This symbolism is known as the ***duality pairing*** and it is clear that it describes a mapping $\langle \cdot\,, \cdot \rangle : U^* \times U \to \mathbb{F}$ which may be called a ***bilinear functional*** or a ***bilinear form*** due to the obvious reason that this functional has the following properties:

$$\begin{aligned} \langle f_1 + f_2, u\rangle &= \langle f_1, u\rangle + \langle f_2, u\rangle, \\ \langle f, u_1 + u_2\rangle &= \langle f, u_1\rangle + \langle f, u_2\rangle, \\ \langle \alpha f, u\rangle &= \langle f, \alpha u\rangle = \alpha\langle f, u\rangle. \end{aligned} \tag{1.2.9}$$

With this notation (1.2.7) can be rewritten as

$$\langle f^i, e_j\rangle = \delta^i_j. \tag{1.2.10}$$

Since one can write $f(u) = \sum_{i=1}^{n}\alpha_i u^i$, *it is obvious that if* $f(u) = 0$ *for all* $u \in U^{(n)}$, *we then get* $\alpha_1 = \cdots = \alpha_n = 0$, *namely,* $f = 0$; *conversely, if* $f(u) = 0$ *for all* $f \in U^{*(n)}$, *we then have to write* $u^1 = \cdots = u^n = 0$ *so that* $u = 0$.

We shall now discuss the change of basis in finite-dimensional vector spaces. We choose first a basis $\{e_i\}$ in a vector space U and consider another basis $\{e'_i\}$. Since both basis are to be linearly independent sets, this operation is obviously carried out by use of a *regular matrix* $\mathbf{A} = [a^i_j]$ *such that* $\det \mathbf{A} \neq 0$ as follows

$$e_j = a^i_j\, e'_i = \sum_{i=1}^{n} a^i_j\, e'_i, \tag{1.2.11}$$

where we have employed the celebrated *summation convention* proposed by the great German physicist Albert Einstein (1879-1955). *Repeated indices* (*usually superscripts and subscripts*), *that are sometimes called* ***dummy indices*** *because we can freely rename them without actually affecting the meaning of an expression, will imply a summation over the range of these indices. When we would like to suspend this rule we will underline the relevant indices*. Henceforth, we shall always resort to the Einstein summation convention to simplify the appearance of rather complicated expressions, at least notationally, by dispensing with the symbol Σ. It follows from the relations

$$u = u^j e_j = u^j a^i_j \, e'_i = u'^i e'_i$$

that the components of a vector u with respect to the new basis in terms of old components are given by

$$u'^i = a^i_j u^j. \tag{1.2.12}$$

Let the reciprocal basis in the dual space U^* to the basis $\{e'_i\}$ be $\{f'^i\}$. Hence the value of $f \in U^*$ on every vector $u \in U$ is found as

$$f(u) = u'^i \alpha'_i = u^j a^i_j \, \alpha'_i = u^j \alpha_j$$

from which it follows that

$$\alpha'_i = b^j_i \alpha_j, \qquad \{b^j_i\} = \mathbf{B} = \mathbf{A}^{-1}. \tag{1.2.13}$$

On the other hand, when we consider the relations

$$f'^i(u) = u'^i = a^i_j u^j = a^i_j \, f^j(u)$$

that must be satisfied for all vectors $u \in U$ we are led to the following transformation rules

$$f'^i = a^i_j \, f^j, \quad f^i = b^i_j \, f'^j. \tag{1.2.14}$$

Let us now consider a sequence of linear vector spaces $\{V_n\}$ and a sequence of linear operators $A_n : V_n \to V_{n+1}$, that is, homomorphisms, represented diagrammatically as

$$\cdots \longrightarrow V_{n-1} \stackrel{A_{n-1}}{\longrightarrow} V_n \stackrel{A_n}{\longrightarrow} V_{n+1} \longrightarrow \cdots. \tag{1.2.15}$$

The sequence $V^\bullet = \{V_n, A_n\}$ is called an ***exact sequence*** if $\mathcal{R}(A_{n-1}) = \mathcal{N}(A_n) \subset V_n$ for all n, This of course requires that $A_n \circ A_{n-1} = 0$. However, we observe easily that this condition alone is not sufficient for the above sequence to be exact. In fact, if $v_{n-1} \in V_{n-1}$, then $A_{n-1}(v_{n-1}) \in \mathcal{R}(A_{n-1}) \subseteq V_n$. Since by definition we assume $A_n\big(A_{n-1}(v_{n-1})\big) = 0$, we can only infer that $A_{n-1}(v_{n-1}) \in \mathcal{N}(A_n)$ implying merely that $\mathcal{R}(A_{n-1}) \subseteq \mathcal{N}(A_n)$. If, at each stage, the image of one homomorphism is contained in the kernel of the next homomorphism, this *increasing* sequence is called a ***cochain complex***. Clearly, an exact sequence is also a cochain complex, but the converse statement is generally not true. Let us consider two cochains $V^\bullet = \{V_n, A_n\}$ and $U^\bullet = \{U_n, B_n\}$. A ***cochain homomorphism*** $C^\bullet : V^\bullet \to U^\bullet$ is a set of homomorphisms $\{C_n : V_n \to U_n\}$ such that the following diagram commutes for all n:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & V_n & \xrightarrow{A_n} & V_{n+1} & \longrightarrow & \cdots \\
 & & \big\downarrow C_n & & \big\downarrow C_{n+1} & & \\
\cdots & \longrightarrow & U_n & \xrightarrow{B_n} & U_{n+1} & \longrightarrow & \cdots
\end{array}$$

We can thus write $C_{n+1} \circ A_n = B_n \circ C_n$ for all n.

An exact sequence of the form

$$0 \longrightarrow U \xrightarrow{A} V \xrightarrow{B} W \longrightarrow 0 \tag{1.2.16}$$

is called a ***short exact sequence***. Obviously A is injective because $\mathcal{N}(A) = \{0\}$ whereas B is surjective since $\mathcal{R}(B) = W$. Hence, A has left inverses and B right inverses so that there are homomorphisms $L : V \to U$ and $R : W \to V$ such that $L \circ A = i_U, B \circ R = i_W$ where i_U and i_W are identity mappings A simple example to a short exact sequence is provided by the quotient space V/U produced by a subspace $U \subseteq V$:

$$0 \longrightarrow U \xrightarrow{\mathcal{I}} V \xrightarrow{\phi} V/U \longrightarrow 0$$

where $\mathcal{I} : U \to V$ is the inclusion mapping, i.e., $u = \mathcal{I}(u) \in V$ for all $u \in U$ and $\phi : V \to V/U$ is the canonical mapping [*see* p. 6]. We know that $\mathcal{N}(V/U) = U$ so that we may write $\phi \circ \mathcal{I} = 0$.

A salient property of exact sequences is revealed in the following theorem known as ***the five lemma***.

Theorem 1.2.2. *Let $V^\bullet = \{V_n, A_n\}$ and $U^\bullet = \{U_n, B_n\}$ be two exact sequences and $C^\bullet = \{C_n : V_n \to U_n\}$ be a cochain homomorphism. Let us consider the five consecutive elements of these sequences corresponding to $n-2, n-1, n, n+1, n+2$. If $C_{n-2}, C_{n-1}, C_{n+1}, C_{n+2}$ are isomorphisms, then C_n must also be an isomorphism.*

The commutativity of the diagram below with rows of exact sequences

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & V_{n-2} & \xrightarrow{A_{n-2}} & V_{n-1} & \xrightarrow{A_{n-1}} & V_n & \xrightarrow{A_n} & V_{n+1} & \xrightarrow{A_{n+1}} & V_{n+2} & \longrightarrow & \cdots \\
 & & \big\downarrow C_{n-2} & & \big\downarrow C_{n-1} & & \big\downarrow C_n & & \big\downarrow C_{n+1} & & \big\downarrow C_{n+2} & & \\
\cdots & \longrightarrow & U_{n-2} & \xrightarrow{B_{n-2}} & U_{n-1} & \xrightarrow{B_{n-1}} & U_n & \xrightarrow{B_n} & U_{n+1} & \xrightarrow{B_{n+1}} & U_{n+2} & \longrightarrow & \cdots
\end{array}$$

requires that $C_{n+1} \circ A_n = B_n \circ C_n : V_n \to U_{n+1}$ for each n.

Let us first show that the homomorphism C_n is injective. Let $v_n \in V_n$ and assume that $C_n(v_n) = 0 \in U_n$. Then $C_{n+1}\big(A_n(v_n)\big) = B_n\big(C_n(v_n)\big) = 0$. Since C_{n+1} is an isomorphism, we obtain $A_n(v_n) = 0$. Therefore,

$v_n \in \mathcal{N}(A_n) = \mathcal{R}(A_{n-1})$ so that there exists $v_{n-1} \in V_{n-1}$ such that $v_n = A_{n-1}(v_{n-1})$. Then $B_{n-1}\big(C_{n-1}(v_{n-1})\big) = C_n\big(A_{n-1}(v_{n-1})\big) = C_n(v_n) = 0$ implying that $C_{n-1}(v_{n-1}) \in \mathcal{N}(B_{n-1}) = \mathcal{R}(B_{n-2})$ so that we may choose $u_{n-2} \in U_{n-2}$ such that $B_{n-2}(u_{n-2}) = C_{n-1}(v_{n-1})$. Since C_{n-2} is an isomorphism, there exists $v_{n-2} \in V_{n-2}$ such that $C_{n-2}(v_{n-2}) = u_{n-2}$. Then, we obtain $C_{n-1}\big(A_{n-2}(v_{n-2})\big) = B_{n-2}\big(C_{n-2}(v_{n-2})\big) = B_{n-2}(u_{n-2}) = C_{n-1}(v_{n-1})$. Because C_{n-1} is an isomorphism, we get $v_{n-1} = A_{n-2}(v_{n-2})$. Since $A_{n-1} \circ A_{n-2} = 0$ because $\{V_n, A_n\}$ is an exact sequence, we thus find $0 = A_{n-1}\big(A_{n-2}(v_{n-2})\big) = A_{n-1}(v_{n-1}) = v_n$. Hence, $v_n = 0$ which amounts to say that C_n is injective.

We shall now show that C_n is surjective. Let $u_n \in U_n$ be an arbitrary vector. We then have $u_{n+1} = B_n(u_n) \in U_{n+1}$. Since C_{n+1} is an isomorphism, there exists a vector $v_{n+1} \in V_{n+1}$ so that $B_n(u_n) = C_{n+1}(v_{n+1})$. We thus have $C_{n+2}\big(A_{n+1}(v_{n+1})\big) = B_{n+1}\big(C_{n+1}(v_{n+1})\big) = B_{n+1}\big(B_n(u_n)\big) = 0$ because $B_{n+1} \circ B_n = 0$ since $\{U_n, B_n\}$ is an exact sequence. We thus find $A_{n+1}(v_{n+1}) = 0$ because C_{n+2} is an isomorphism. Since v_{n+1} belongs to the null space of A_{n+1}, then there exists a vector $v_n \in V_n$ such that $v_{n+1} = A_n(v_n)$ because $\mathcal{N}(A_{n+1}) = \mathcal{R}(A_n)$. Let us now consider the vector $u_n - C_n(v_n) \in U_n$. Recalling that $B_n(u_n) = C_{n+1}(v_{n+1})$, We readily observe that

$$\begin{aligned} B_n\big(u_n - C_n(v_n)\big) &= B_n(u_n) - B_n\big(C_n(v_n)\big) \\ &= B_n(u_n) - C_{n+1}\big(A_n(v_n)\big) \\ &= B_n(u_n) - C_{n+1}(v_{n+1}) = 0. \end{aligned}$$

Since $u_n - C_n(v_n) \in \mathcal{N}(B_n)$, there exists a vector $u_{n-1} \in U_{n-1}$ satisfying the relation $u_n - C_n(v_n) = B_{n-1}(u_{n-1})$ and we have $u_{n-1} = C_{n-1}(v_{n-1})$ for some $v_{n-1} \in V_{n-1}$ because C_{n-1} is an isomorphism. Let now consider the vector $v_n + A_{n-1}(v_{n-1}) \in V_n$. We can then write

$$\begin{aligned} C_n\big(v_n + A_{n-1}(v_{n-1})\big) &= C_n(v_n) + C_n\big(A_{n-1}(v_{n-1})\big) \\ &= C_n(v_n) + B_{n-1}\big(C_{n-1}(v_{n-1})\big) \\ &= C_n(v_n) + B_{n-1}(u_{n-1}) = u_n \end{aligned}$$

implying that C_n is surjective. Since this linear operator is both injective and surjective, then C_n is an isomorphism. □

Let $V^\bullet$ be a cochain given by (1.2.15) such that $\mathcal{R}(A_{n-1}) \subseteq \mathcal{N}(A_n)$. The quotient space of $\mathcal{N}(A_n)$ with respect to its subspace $\mathcal{R}(A_{n-1})$ is the vector space

$$H^n(V^\bullet) = \mathcal{N}(A_n)/\mathcal{R}(A_{n-1}) = \mathrm{Ker}\,(A_n)/\mathrm{Im}\,(A_{n-1}). \quad (1.2.17)$$

$H^n(V^\bullet)$ is called the ***nth cohomology group*** due to the fact that a vector space is an Abelian group. An element of the vector space $H^n(V^\bullet)$, called a ***cohomology class***, is an equivalence class $[v_n] = \{v_n + A_{n-1}v_{n-1}\}$ involving all vectors $v_{n-1} \in V_{n-1}$ where $A_n v_n = 0$. We shall now demonstrate the following theorem commonly known as ***the zigzag lemma***.

Theorem 1.2.3. *Let us consider the following short exact sequence*

$$0 \longrightarrow U^\bullet \xrightarrow{A^\bullet} V^\bullet \xrightarrow{B^\bullet} W^\bullet \longrightarrow 0 \tag{1.2.18}$$

where $U^\bullet = (U_n, d), V^\bullet = (V_n, d), W^\bullet = (W_n, d)$ *are cochains so that* $d^2 = 0$ *and* $A^\bullet = \{A_n\}, B^\bullet = \{B_n\}$ *are cochain homomorphisms. Then there exists a homomorphism* $\Gamma : H^n(W^\bullet) \to H^{n+1}(U^\bullet)$ *such that the sequence*

$$\cdots \xrightarrow{\Gamma} H^n(U^\bullet) \xrightarrow{A_n} H^n(V^\bullet) \xrightarrow{B_n} H^n(W^\bullet) \xrightarrow{\Gamma} H^{n+1}(U^\bullet) \xrightarrow{A_{n+1}} \cdots \tag{1.2.19}$$

is exact.

We consider the following commutative diagram whose rows are short exact sequences and columns are cochains:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \longrightarrow & U_n & \xrightarrow{A_n} & V_n & \xrightarrow{B_n} & W_n & \longrightarrow & 0 \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \longrightarrow & U_{n+1} & \xrightarrow{A_{n+1}} & V_{n+1} & \xrightarrow{B_{n+1}} & W_{n+1} & \longrightarrow & 0 \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \longrightarrow & U_{n+2} & \xrightarrow{A_{n+2}} & V_{n+2} & \xrightarrow{B_{n+2}} & W_{n+2} & \longrightarrow & 0 \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

We thus infer that for all n, the homomorphism A_n is injective and B_n is surjective and $\mathcal{R}(A_n) = \mathcal{N}(B_n)$. Similarly, we have $\mathcal{R}_n(d) \subseteq \mathcal{N}_{n+1}(d)$ and this gives rise to cohomology groups $H^n(U^\bullet)$, $H^n(V^\bullet)$, $H^n(W^\bullet)$ for all n along columns of cochains. The linear operator $A_n : U_n \to \mathcal{N}(B_n) = \mathcal{R}(A_n) \subseteq V_n$ is evidently bijective so that it is an isomorphism, hence its inverse $A_n^{-1} : \mathcal{N}(B_n) \to U_n$ exists. Equivalence classes in the quotient space $V_n/\mathcal{N}(B_n)$ are given by $[v_n] = \{v_n + A_n u_n : u_n \in U_n\}$. Then the operator B_n interpreted as $B_n : V_n/\mathcal{N}(B_n) \to W_n$ becomes an isomorphism so that one has the inverse $B_n^{-1}w_n = [v_n]$. Therefore, we may define a linear operator Γ by

$$\Gamma = A_{n+1}^{-1} \circ d \circ B_n^{-1} : W_n \to U_{n+1} \tag{1.2.20}$$

which is unique within the precepts of the cohomology. Due to the commutativity of the diagram, we infer from (1.2.20) that

$$d \circ \Gamma = d \circ A_{n+1}^{-1} \circ d \circ B_n^{-1} = A_{n+2}^{-1} \circ d^2 \circ B_n^{-1} = 0.$$

It straightforward to see that we also get the relation $0 = A_{n+1}^{-1} \circ d^2 \circ B_{n+1}^{-1} = \Gamma \circ d$. Let us now consider a representative w_n of the equivalence class $[w_n] \in H^n(W^\bullet)$ so that $dw_n = 0$. We then obtain $d(\Gamma w_n) = 0$. Hence, $\Gamma w_n \in H^{n+1}(U^\bullet)$. Thus, Γ is a homomorphism as follows

$$\Gamma : H^n(W^\bullet) \to H^{n+1}(U^\bullet).$$

Let us take a vector $w_n \in W_n$. Since B_n is surjective, there exists a representative vector $v_n \in V_n$ of an equivalence class $[v_n]$ such that $B_n v_n = w_n$. Because we have to consider the cochain $W^\bullet$, let us assume that $w_n \in \mathcal{N}_n(d) \subseteq W_n$ so that $dw_n = 0$. Due to the commutativity of the above diagram we find that $dB_n v_n = B_{n+1} dv_n = 0$. Thus, $dv_n \in \mathcal{N}(B_{n+1}) = \mathcal{R}(A_{n+1})$. Since A_{n+1} is injective, there is a unique vector u_{n+1} such that $A_{n+1}u_{n+1} = dv_n$. It follows from the commutativity of the above diagram that $A_{n+2}\, du_{n+1} = dA_{n+1}u_{n+1} = d^2 v_n = 0$ so that $du_{n+1} \in \mathcal{N}(A_{n+2})$. Since A_{n+2} is injective, we get $du_{n+1} = 0$. Hence, u_{n+1} belongs to a cohomology class. Obviously, it is expressed as $u_{n+1} = \Gamma w_n$. However, we have to show that this result is independent of the choice of representative of the equivalence class. Let us consider another vector $v_n' \in [v_n]$. We then must write $v_n - v_n' \in \mathcal{N}(B_n)$. Exactness requires that there exists a $u_n \in U_n$ such that $A_n u_n = v_n - v_n'$. Now the commutativity of the diagram implies that

$$A_{n+1} du_n = dA_n u_n = d(v_n - v_n').$$

It then follows from cochain and exact sequence properties that there are $u_{n+1}, u_{n+1}' \in U_{n+1}$ such that $A_{n+1}u_{n+1} = dv_n$ and $A_{n+1}u_{n+1}' = dv_n'$. Since A_{n+1} is injective, the relation $A_{n+1}(u_{n+1} - u_{n+1}' - du_n) = 0$ yields $du_n = u_{n+1} - u_{n+1}'$, hence we get $du_{n+1} = du_{n+1}'$. Consequently, u_{n+1} and u_{n+1}' belong to the same cohomology class.

We now consider an element $w_n = dw_{n-1} \in W_n$ where $w_{n-1} \in W_{n-1}$. Since $dw_n = 0$, we get $w_n \in H^n(W^\bullet)$. In view of the surjectivity of B_{n-1} we can write $B_{n-1}v_{n-1} = w_{n-1}$ for a vector $v_{n-1} \in V_{n-1}$. Let $v_n = dv_{n-1}$ so that $dv_n = d^2 v_{n-1} = 0 \in V_{n+1}$. We have seen above that there exists a unique vector $u_{n+1} \in U_{n+1}$ such that $A_{n+1}u_{n+1} = dv_n = 0$. Since A_{n+1} is injective, we have $u_{n+1} = 0$. This of course implies that all elements in the equivalence class $[w_n] \in H^n(W^\bullet)$ are mapped under the operator Γ onto the same equivalence class $[u_{n+1}] \in H^{n+1}(U^\bullet)$. Hence, Γ is a well defined operator.

Finally, we have to show that the sequence

$$\cdots \xrightarrow{\Gamma} H^n(U^\bullet) \xrightarrow{A_n} H^n(V^\bullet) \xrightarrow{B_n} H^n(W^\bullet) \xrightarrow{\Gamma} H^{n+1}(U^\bullet) \xrightarrow{A_{n+1}} \cdots$$

is exact. To this end, it suffices to prove exactness at $H^n(U^\bullet)$. Because, the sequence is exact at $H^n(V^\bullet)$ since $\mathcal{R}(A_n) = \mathcal{N}(B_n)$ and proof at $H^n(W^\bullet)$may be accomplish in the same fashion. Let $[w_{n-1}] \in H^{n-1}(W^\bullet)$ and take the element $\Gamma[w_{n-1}] \in H^n(U^\bullet) = \mathcal{R}_{n-1}(\Gamma)$ into account. It then immediately follows from (1.2.20) that $A_n\Gamma[w_{n-1}] = [dv_{n-1}] = [0]$ where $[v_{n-1}] = B_{n-1}^{-1}[w_{n-1}]$. Consequently, we obtain $\mathcal{R}_{n-1}(\Gamma) \subseteq \mathcal{N}(A_n)$. Conversely, let us now consider an equivalence class $[u_n] \in \mathcal{N}(A_n)$ of the cohomology group $H^n(U^\bullet)$. Since $A_n[u_n] = [0] \in H^n(V^\bullet)$, we find that $A_n[u_n] = [dv_{n-1}]$. We then define $w_{n-1} = B_{n-1}v_{n-1} \in W_{n-1}$ and the cohomology class $[w_{n-1}] \in H^{n-1}(W^\bullet)$. Since $\Gamma[w_{n-1}] = A_n^{-1}dB_{n-1}^{-1}[w_{n-1}]$ $= A_n^{-1}dB_{n-1}^{-1}B_{n-1}[v_{n-1}] = A_n^{-1}[dv_{n-1}] = A_n^{-1}A_n[u_n] = [u_n] \in \mathcal{N}(A_n)$, we see that $[u_n]$ is the image of an equivalence class $[w_{n-1}]$ under Γ. Thus, we get $\mathcal{N}(A_n) \subseteq \mathcal{R}_{n-1}(\Gamma)$ and we finally find $\mathcal{R}_{n-1}(\Gamma) = \mathcal{N}(A_n)$ Hence, the sequence is exact at $H^n(U^\bullet)$. We shall not repeat the analysis to prove exactness at $H^n(W^\bullet)$. □

Finally, for later applications, we have to emphasise the fact that what we have said so far are equally valid for *modules*.

1.3. MULTILINEAR FUNCTIONALS

Let $(U_1, U_2, \dots, U_k)$ be ordered k-tuple of linear vector spaces defined over the same field of scalars $\mathbb{F}$. Let us consider a scalar-valued function $\mathcal{T} : U_1 \times U_2 \times \cdots \times U_k \to \mathbb{F}$ on the Cartesian product of these vector spaces. If the function $\mathcal{T}(u_{(1)}, u_{(2)}, \dots, u_{(k)}) \in \mathbb{F}$, where $u_{(\alpha)} \in U_\alpha$, $\alpha = 1, 2, \dots, k$, is *linear in each one of its arguments*, that is, if the following relations

$$\begin{aligned} \mathcal{T}(\dots, u_{(i)} + v_{(i)}, \dots) &= \mathcal{T}(\dots, u_{(i)}, \dots) + \mathcal{T}(\dots, v_{(i)}, \dots) \quad (1.3.1) \\ \mathcal{T}(\dots, \alpha u_{(i)}, \dots) &= \alpha \mathcal{T}(\dots, u_{(i)}, \dots),\ \alpha \in \mathbb{F} \end{aligned}$$

are satisfied for all $1 \le i \le k$, then the function $\mathcal{T}$ is called a ***multilinear functional*** (or a ***k-linear functional***). In finite-dimensional vector spaces whose dimensions and bases are $n_1, \dots, n_k$ and $\{e_i^{(\alpha)}\} \in U_\alpha$, $i = 1, \dots, n_\alpha$,

$\alpha = 1, \dots, k$, we can then write $u_{(\alpha)} = \sum_{i=1}^{n_\alpha} u^i_{(\alpha)} e^{(\alpha)}_i$, without having recourse to the summation convention. Multilinearity then leads to the following value of the functional at vectors $u_{(1)} \in U_1, u_{(2)} \in U_2, \dots, u_{(k)} \in U_k$

$$\mathcal{T}(u_{(1)}, u_{(2)}, \dots, u_{(k)}) = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_k=1}^{n_k} t_{i_1 i_2 \cdots i_k} u^{i_1}_{(1)} u^{i_2}_{(2)} \cdots u^{i_k}_{(k)} \tag{1.3.2}$$

where $n_1 \times n_2 \times \cdots \times n_k$ number of scalar $t_{i_1 i_2 \cdots i_k}$ are defined by

$$t_{i_1 i_2 \cdots i_k} = \mathcal{T}(e^{(1)}_{i_1}, e^{(2)}_{i_2}, \dots, e^{(k)}_{i_k}) \in \mathbb{F}. \tag{1.3.3}$$

We thus conclude that the set of scalars $\{t_{i_1 i_2 \cdots i_k}\}$ completely determines the action of a k-linear functional on any set of k number of vectors $u_{(1)} \in U_1$, $u_{(2)} \in U_2, \ \dots, u_{(k)} \in U_k$. We can thus say that they unambiguously characterise a multilinear functional.

Let us now suppose that $U_1 = U_2 = \cdots = U_k = U^{(n)}$. The value of a multilinear functional $\mathcal{T} : U^k \to \mathbb{F}$ on vectors $u_{(1)}, u_{(2)}, \dots, u_{(k)} \in U$ can now be found from (1.3.2) and (1.3.3) as follows

$$\begin{aligned} \mathcal{T}(u_{(1)}, u_{(2)}, \dots, u_{(k)}) &= t_{i_1 i_2 \cdots i_k} u^{i_1}_{(1)} u^{i_2}_{(2)} \cdots u^{i_k}_{(k)}, \\ t_{i_1 i_2 \cdots i_k} &= \mathcal{T}(e_{i_1}, e_{i_2}, \dots, e_{i_k}), \ \ 1 \le i_1, i_2, \cdots, i_k \le n \end{aligned} \tag{1.3.4}$$

where we experience no difficulty in resorting to the summation convention because the range of all indices is the same now, from 1 to n. In this case, we can introduce a more advantageous representation of a multilinear functional as an operator. To this end, we shall first introduce the tensor product of two vector spaces.

Let U and V be two linear vector spaces defined on the same field of scalars $\mathbb{F}$. As is well known, the Cartesian product $U \times V$ of these spaces is formed by ordered pairs (u, v) where $u \in U$ and $v \in V$. There is initially no algebraic structure on this product set. However, by making use of known operations on vector spaces U and V, we may define appropriate operations on the set $U \times V$ so that it may be equipped with a structure of a linear vector space. The resulting vector space will be called the ***tensor product*** of spaces U and V and will be denoted by $W = U \otimes V$. Let us choose operations of vector addition and scalar multiplication on W in such a way that tensor product of vectors $u \otimes v \in U \otimes V$ has to satisfy the following bilinearity conditions:

$$(i).\ u \otimes (v_1 + v_2) = u \otimes v_1 + u \otimes v_2,$$

$$(ii).\ (u_1 + u_2) \otimes v = u_1 \otimes v + u_2 \otimes v,$$
$$(iii).\ (\alpha u) \otimes v = u \otimes (\alpha v) = \alpha(u \otimes v), \quad \alpha \in \mathbb{F}.$$

Let us note that the same symbol $+$ in the foregoing expressions represents, in fact, different addition operations in three different vector spaces U, V and W. We can thus write

$$(u_1 + u_2) \otimes (v_1 + v_2) = u_1 \otimes v_1 + u_1 \otimes v_2 + u_2 \otimes v_1 + u_2 \otimes v_2.$$

The space W is then defined as the collection of all *finite sums* $\sum\limits_i u_i \otimes v_i$ where $u_i \in U$ and $v_i \in V$. If we consider finite-dimensional vector spaces $U^{(m)}$ and $V^{(n)}$ with respective bases $\{e_i\}$ and $\{f_j\}$, a vector $w \in W$ is evidently expressible as $w = w^{ij} e_i \otimes f_j$.Hence, W is an mn-dimensional vector space with a basis $\{e_i \otimes f_j\}$. The tensor product can evidently be extended on Cartesian products of arbitrary number of vector spaces.

Let us now consider the n-dimensional dual space U^* of an n-dimensional vector space U. It is quite clear that an element, or a vector, of the tensor product $\otimes\, ^k U^*$ can now be represented by

$$\mathcal{T} = t_{i_1 i_2 \cdots i_k} f^{i_1} \otimes f^{i_2} \otimes \cdots \otimes f^{i_k} \tag{1.3.5}$$

where $\{f^i\}$ is the reciprocal basis in U^* corresponding to the basis $\{e_i\}$ in U. We define the value of the element $\mathcal{T}$ on an ordered k-tuple of vectors $(u_{(1)}, u_{(2)}, \ldots, u_{(k)}) \in U^k$ as

$$\mathcal{T}(u_{(1)}, \ldots, u_{(k)}) = t_{i_1 \cdots i_k} u^{j_1}_{(1)} \cdots u^{j_k}_{(k)} f^{i_1}(e_{j_1}) \cdots f^{i_k}(e_{j_k})$$

In view of (1.2.7), we then find that

$$\mathcal{T}(u_{(1)}, u_{(2)}, \ldots, u_{(k)}) = t_{i_1 i_2 \cdots i_k} u^{i_1}_{(1)} u^{i_2}_{(2)} \cdots u^{i_k}_{(k)}.$$

We immediately see that the above relation leads to $(1.3.4)_2$ for vectors e_{i_1}, $e_{i_2}, \ldots, e_{i_k}$. Hence (1.3.5) does in fact play the part of a k-linear functional on U^k and the tensor product $\otimes\, ^k U^*$ is the vector space in which such k-linear functionals inhabit. We say that the elements of this vector space are ***k-covariant tensors*** and the number k is known as the ***order*** of the tensor. The scalar coefficients $t_{i_1 i_2 \cdots i_k}$ are then called the ***components*** *of such a tensor* with respect to bases $f^{i_1} \otimes \cdots \otimes f^{i_k}$. It is easily observed that the tensor product $f^{i_1} \otimes \cdots \otimes f^{i_k}$ of basis vectors constitutes a basis for the space $\otimes\, ^k U^*$. Indeed the value of the zero element in $\otimes\, ^k U^*$

$$t_{i_1 i_2 \cdots i_k} f^{i_1} \otimes f^{i_2} \otimes \cdots \otimes f^{i_k} = 0$$

on vectors $e_{j_1}, e_{j_2}, \dots, e_{j_k} \in U$ vanishes naturally so that one obtains

$$t_{i_1 i_2 \cdots i_k} f^{i_1}(e_{j_1}) f^{i_2}(e_{j_2}) \cdots f^{i_k}(e_{j_k}) = t_{j_1 j_2 \cdots j_k} = 0$$

for all coefficients. Hence, the dimension of this vector space is n^k. Obviously, the sum of two tensors of the same kind and multiplication of a tensor by a scalar are again the following tensors of the same kind:

$$\begin{aligned} \mathcal{T}_1 + \mathcal{T}_2 &= (t^{(1)}_{i_1 i_2 \cdots i_k} + t^{(2)}_{i_1 i_2 \cdots i_k}) f^{i_1} \otimes f^{i_2} \otimes \cdots \otimes f^{i_k} \\ \alpha \mathcal{T} &= (\alpha t_{i_1 i_2 \cdots i_k}) f^{i_1} \otimes f^{i_2} \otimes \cdots \otimes f^{i_k}. \end{aligned}$$

This is of course a direct consequence of $\otimes^k U^*$ being a linear vector space. We can now naturally define the tensorial product of a k-covariant tensor and an l-covariant tensor by

$$\mathcal{T}_1 \otimes \mathcal{T}_2 = t^{(1)}_{i_1 \cdots i_k} t^{(2)}_{j_1 \cdots j_l} f^{i_1} \otimes \cdots \otimes f^{i_k} \otimes f^{j_1} \otimes \cdots \otimes f^{j_l}.$$

The result is obviously a $(k + l)$-covariant tensor.

Let us now change the basis $\{e_i\}$ in the vector space U to another basis $\{e'_i\}$ as in (1.2.11). We know that the reciprocal basis $\{f^i\}$ in the dual space U^* changes to a reciprocal basis $\{f'^i\}$ through the relations (1.2.14). Consequently, the same tensor $\mathcal{T}$ is represented with respect to two different bases as follows

$$\begin{aligned} \mathcal{T} &= t_{j_1 j_2 \cdots j_k} f^{j_1} \otimes f^{j_2} \otimes \cdots \otimes f^{j_k} = t'_{i_1 i_2 \cdots i_k} f'^{i_1} \otimes f'^{i_2} \otimes \cdots \otimes f'^{i_k} \\ &= t_{j_1 j_2 \cdots j_k} b^{j_1}_{i_1} b^{j_2}_{i_2} \cdots b^{j_k}_{i_k} f'^{i_1} \otimes f'^{i_2} \otimes \cdots \otimes f'^{i_k} \end{aligned}$$

from which we immediately deduce that the following rule of transformation between components of a k-covariant tensor must be valid:

$$t'_{i_1 i_2 \cdots i_k} = b^{j_1}_{i_1} b^{j_2}_{i_2} \cdots b^{j_k}_{i_k} t_{j_1 j_2 \cdots j_k}. \tag{1.3.6}$$

In a similar fashion we may define a multilinear (k-linear) functional on the dual space U^* of a vector space. Such a functional $\mathcal{T} : (U^*)^k \to \mathbb{F}$ assigns a scalar number $\mathcal{T}(f^{(1)}, f^{(2)}, \dots, f^{(k)}) \in \mathbb{F}$ to an ordered k-tuple of linear functionals $(f^{(1)}, f^{(2)}, \dots, f^{(k)}) \in (U^*)^k$ and obeys the rules

$$\begin{aligned} \mathcal{T}(\dots, f^{(i)} + g^{(i)}, \dots) &= \mathcal{T}(\dots, f^{(i)}, \dots) + \mathcal{T}(\dots, g^{(i)}, \dots) \\ \mathcal{T}(\dots, \alpha f^{(i)}, \dots) &= \alpha \mathcal{T}(\dots, f^{(i)}, \dots), \alpha \in \mathbb{F}. \end{aligned}$$

By resorting to the reciprocal basis $\{f^i\} \in U^*$ corresponding to the basis $\{e_i\} \in U$, we can of course write $f^{(m)} = \alpha^{(m)}_i f^i, \alpha^{(m)}_i \in \mathbb{F}, 1 \le m \le k$ and we obtain

$$\mathcal{T}(f^{(1)}, f^{(2)}, \dots, f^{(k)}) = t^{i_1 i_2 \cdots i_k} \alpha^{(1)}_{i_1} \alpha^{(2)}_{i_2} \cdots \alpha^{(k)}_{i_k}, \qquad (1.3.7)$$
$$t^{i_1 i_2 \cdots i_k} = \mathcal{T}(f^{i_1}, f^{i_2}, \dots, f^{i_k}).$$

The ensemble of scalar numbers $t^{i_1 i_2 \cdots i_k}$, $1 \le i_1, i_2, \cdots, i_k \le n$ entirely determines the action of a multilinear functional $\mathcal{T}$ on $(U^*)^k$. Let us now define an element in the tensor product $\otimes\, ^k U$ by

$$\mathcal{T} = t^{i_1 i_2 \cdots i_k} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}.$$

$\mathcal{T}$ is called a ***k-contravariant tensor***. It is evident that the linearly independents elements $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ constitute a basis for the vector space $\otimes\, ^k U$. n^k number of scalars $t^{i_1 i_2 \cdots i_k}$ are said to be *components* of this tensor with respect to bases $e_{i_1} \otimes \cdots \otimes e_{i_k}$. Let us define the value of the tensor $\mathcal{T}$ on k linear functionals $f^{(1)},\ f^{(2)}, \dots, f^{(k)}$ by the relation

$$\mathcal{T}(f^{(1)}, f^{(2)}, \dots, f^{(k)}) = t^{i_1 i_2 \cdots i_k} f^{(1)}(e_{i_1}) f^{(2)}(e_{i_2}) \cdots f^{(k)}(e_{i_k}).$$

In view of (1.2.6) we find that

$$\mathcal{T}(f^{(1)}, f^{(2)}, \dots, f^{(k)}) = t^{i_1 i_2 \cdots i_k} \alpha^{(1)}_{i_1} \alpha^{(2)}_{i_2} \cdots \alpha^{(k)}_{i_k}.$$

It is clear that the product of a k-contravariant tensor and an l-contravariant tensor is a $(k+l)$-contravariant tensor. We now consider a change of basis in the vector space U. We then obtain

$$\begin{aligned}\mathcal{T} &= t^{j_1 j_2 \cdots j_k} e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k} = t'^{i_1 i_2 \cdots i_k} e'_{i_1} \otimes e'_{i_2} \otimes \cdots \otimes e'_{i_k} \\ &= t^{j_1 j_2 \cdots j_k} a^{i_1}_{j_1} a^{i_2}_{j_2} \cdots a^{i_k}_{j_k} e'_{i_1} \otimes e'_{i_2} \otimes \cdots \otimes e'_{i_k}\end{aligned}$$

from which we deduce the following rule of transformation for components of a contravariant tensor

$$t'^{i_1 i_2 \cdots i_k} = a^{i_1}_{j_1} a^{i_2}_{j_2} \cdots a^{i_k}_{j_k} t^{j_1 j_2 \cdots j_k}. \qquad (1.3.8)$$

We can also easily define tensors of mixed type. A k-contravariant and l-covariant ***mixed tensor*** is an element of the vector space $\otimes\, ^k U \otimes\, ^l U^*$ and can be written in the form

$$\begin{aligned}\mathcal{T} &= t^{i_1 i_2 \cdots i_k}_{j_1 j_2 \cdots j_l} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \otimes f^{j_1} \otimes f^{j_2} \otimes \cdots \otimes f^{j_l}, \\ t^{i_1 i_2 \cdots i_k}_{j_1 j_2 \cdots j_l} &= \mathcal{T}(f^{i_1}, f^{i_2}, \dots, f^{i_k}, e_{j_1}, e_{j_2}, \dots, e_{j_l}), \\ &1 \le i_1, i_2, \cdots, i_k \le n,\ 1 \le j_1, j_2, \cdots, j_l \le n.\end{aligned}$$

The value of this tensor on linear functionals $f^{(1)}, f^{(2)}, \dots, f^{(k)} \in U^*$ and vectors $u_{(1)},\ u_{(2)}, \dots, u_{(l)} \in U$ is given by

$$\mathcal{T}(f^{(1)},\dots,f^{(k)},u_{(1)},\dots,u_{(l)})=t^{i_1i_2\cdots i_k}_{j_1j_2\cdots j_l}\alpha^{(1)}_{i_1}\alpha^{(2)}_{i_2}\cdots\alpha^{(k)}_{i_k}u^{j_1}_{(1)}u^{j_2}_{(2)}\cdots u^{j_l}_{(l)}.$$

It is quite obvious that we do not have to select the ordering in the tensor products in the foregoing way. We may, of course, consider a different ordering such as $U\otimes U^*\otimes U^*\otimes U\otimes U^*\otimes\cdots$. The indices of components of this type of a tensor occupy accordingly proper upper and lower positions. It is evident that different ordering of spaces in the tensor product will give rise to different types of tensors of the same order.

If, in a mixed tensor of order $k+l$, we remove the tensor product between the functional f^{j_m} and the vector e_{i_n}, then the relation $f^{j_m}(e_{i_n})=\delta^{j_m}_{i_n}$ between reciprocal basis vectors reduces the order of the tensor. We thus obtain a $(k-1)$-contravariant and $(l-1)$-covariant tensor, in other words, a tensor of order $k+l-2$ defined by the relation

$$\mathcal{T}_c=t^{i_1\cdots i_n\cdots i_k}_{j_1\cdots i_n\cdots j_l}e_{i_1}\otimes\cdots\otimes e_{i_{n-1}}\otimes e_{i_{n+1}}\otimes\cdots\otimes e_{i_k}\\ \otimes f^{j_1}\otimes\cdots\otimes f^{j_{m-1}}\otimes f^{j_{m+1}}\otimes\cdots\otimes f^{j_l}.$$

This operation is called a ***contraction***. The components of the contracted tensor are given as follows:

$${}_ct^{i_1\cdots i_{n-1}i_{n+1}\cdots i_k}_{j_1\cdots j_{m-1}j_{m+1}\cdots j_l}=t^{i_1\cdots i_{n-1}ii_{n+1}\cdots i_k}_{j_1\cdots j_{m-1}ij_{m+1}\cdots j_l}.$$

1.4. ALTERNATING *k*-LINEAR FUNCTIONALS

Let us consider a multilinear functional $\omega: U^k\to\mathbb{F}$ where U is a finite-dimensional vector space so that for vectors $u_i\in U,\ i=1,\dots,k$ we have $\omega(u_1,u_2,\dots,u_k)\in\mathbb{F}$. We know from Sec 1.3 that the multilinear functional ω may be represented by a k-covariant tensor. We say that the multilinear functional ω is an ***alternating k-linear functional*** or a ***k-vector*** or a ***multivector*** if it becomes zero whenever any two of its arguments are equal. It can be shown that such an alternating multilinear functional enjoys the following properties:

1. *An alternating k-linear functional is completely antisymmetric in the sense that its value changes only its sign whenever any two of its arguments are interchanged.*

To understand the effect of interchanging the argument vectors u_i and u_j let us take into account the expansion

$$\omega(u_1,\dots,u_i+u_j,\dots,u_i+u_j,\dots,u_k)=\\ \omega(u_1,\dots,u_i,\dots,u_i,\dots,u_k)$$

$$+\,\omega(u_1,\ldots,u_i,\ldots,u_j,\ldots,u_k)$$
$$+\,\omega(u_1,\ldots,u_j,\ldots,u_i,\ldots,u_k)$$
$$+\,\omega(u_1,\ldots,u_j,\ldots,u_j,\ldots,u_k) = 0.$$

If we note that the first and the fourth terms in the above expression is zero by definition, we obtain from the middle lines the following property of *complete antisymmetry* for every pair of arguments:

$$\omega(u_1,\ldots,u_i,\ldots,u_j,\ldots,u_k) = \,-\,\omega(u_1,\ldots,u_j,\ldots,u_i,\ldots,u_k)$$

Thus if $U = U^{(n)}$, then the value of an alternating k-linear functional on vectors $u_1, u_2, \ldots, u_k \in U$ are given by

$$\omega(u_1, u_2, \ldots, u_k) = \omega_{i_1 i_2\cdots i_k} u_1^{i_1} u_2^{i_2}\cdots u_k^{i_k} \tag{1.4.1}$$

where the scalars $\omega_{i_1 i_2\cdots i_k} = \omega(e_{i_1}, e_{i_2}, \ldots, e_{i_k}) \in \mathbb{F}$ are *completely antisymmetric* with respect to k indices $i_1, i_2, \cdots, i_k$ taking the values from 1 to n. Hence, for every pair of indices the relation

$$\omega_{i_1\cdots i_p\cdots i_q\cdots i_k} = \,-\,\omega_{i_1\cdots i_q\cdots i_p\cdots i_k} \tag{1.4.2}$$

is satisfied. It is then straightforward to see that the number of independent components of such coefficients are given by $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$.

2. *The value of an alternating k-linear functional on linearly dependent vectors is zero.*

Let us assume that at least one of the k vectors is a linear combination of the remaining $k-1$ vectors. When we expand the functional by employing multilinearity, we see that it is expressible as a sum of terms in each of which at least two arguments in the functional are equal. Hence the value of the functional becomes zero. Consequently *if $k > n$ all k-linear functionals on a vector space of dimension n are identically zero*.

3. *Any alternating n-linear functional on a linear vector space $U^{(n)}$ that vanishes on an ordered basis $\{e_1, e_2, \ldots, e_n\}$ of $U^{(n)}$ is identically zero.*

If we insert ordered vectors $u_i = u_i^j\,e_j, i = 1, \ldots, n$ into the functional, expand the resulting expression by making use of multilinearity, equate to zero the terms involving repeated arguments and exploit the property of antisymmetry, we see that the value of the functional is a linear combination of terms in the form $\pm\,\omega(e_1, e_2, \ldots, e_n)$. In case $\omega(e_1, e_2, \ldots, e_n) = 0$, the value of the functional becomes eventually zero on every ordered n-tuple of vectors.

We can generate a completely antisymmetric quantity from a quantity with k indices, say $a_{i_1 i_2 \cdots i_k}$, through the ***alternation mapping***. Let us denote a permutation of indices $i_1, \ldots, i_k$ by $\sigma_m(i_1, i_2, \ldots, i_k)$. As is well known the total number of all such permutations is $k!$. We now introduce the following quantity through the alternation mapping

$$a_{[i_1 i_2 \cdots i_k]} = \frac{1}{k!} \sum_{m=1}^{k!} (-1)^{\kappa(\sigma_m)} a_{\sigma_m(i_1, i_2, \ldots, i_k)} \tag{1.4.3}$$

where $\kappa(\sigma_m) = 0$ if $\sigma_m(i_1, i_2, \ldots, i_k)$ is an even permutation whereas $\kappa(\sigma_m) = 1$ if it is odd. We know that a permutation is realised by means of a number of transpositions performed by interchanging successive indices. A specified permutation is called an even permutation if the number of transpositions performed is even and odd if that number is odd. We can immediately verify that the quantity $a_{[i_1 i_2 \cdots i_k]}$ is completely antisymmetric. *Henceforth, the indices inside a square bracket will always represent the completely antisymmetric part.* As an example, let us consider a quantity a_{ijk} with three indices. We then find that

$$a_{[ijk]} = \frac{1}{3!}(a_{ijk} + a_{jki} + a_{kij} - a_{ikj} - a_{kji} - a_{jik}).$$

If $a_{i_1 i_2 \cdots i_k}$ is already completely antisymmetric, then it is clearly understood that $a_{i_1 i_2 \cdots i_k} = a_{[i_1 i_2 \cdots i_k]}$.

Since the coefficients $\omega_{i_1 i_2 \cdots i_k}$ are completely antisymmetric, only the completely antisymmetric parts of terms $u_1^{i_1} u_2^{i_2} \cdots u_k^{i_k}$ in a k-fold sum as in (1.4.1) can contribute to the sum so that we can write

$$\begin{aligned} \omega(u_1, u_2, \ldots, u_k) &= \omega_{i_1 i_2 \cdots i_k} u_1^{i_1} u_2^{i_2} \cdots u_k^{i_k} \\ &= \omega_{i_1 i_2 \cdots i_k} u_1^{[i_1} u_2^{i_2} \cdots u_k^{i_k]}. \end{aligned} \tag{1.4.4}$$

The components of a completely antisymmetric quantity $\omega_{i_1 i_2 \cdots i_k}$ whose indices satisfy inequalities $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ will be called its *essential components*. Because all other components are either zero or determined by essential components, sometimes, only with a change of sign. The expression (1.4.4) can then be written in the following form by using essential components

$$\omega(u_1, u_2, \ldots, u_k) = k! \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \omega_{i_1 i_2 \cdots i_k} u_1^{[i_1} u_2^{i_2} \cdots u_k^{i_k]}. \tag{1.4.5}$$

As an example, we consider a 2-linear alternating functional $\omega(u_1, u_2) = \omega_{ij} u_1^i u_2^j$ and $n = 3$. Since $\omega_{ij} = -\omega_{ji}$ we obtain at once with $k = 2$

$$\begin{aligned}
\omega(u_1,u_2) &= \omega_{12}u_1^1u_2^2+\omega_{21}u_1^2u_2^1+\omega_{13}u_1^1u_2^3\\
&\qquad+\omega_{31}u_1^3u_2^1+\omega_{23}u_1^2u_2^3+\omega_{32}u_1^3u_2^2\\
&=\omega_{12}(u_1^1u_2^2-u_1^2u_2^1)+\omega_{13}(u_1^1u_2^3-u_1^3u_2^1)+\omega_{23}(u_1^2u_2^3-u_1^3u_2^2)\\
&=2(\omega_{12}u_1^{[1}u_2^{2]}+\omega_{13}u_1^{[1}u_2^{3]}+\omega_{23}u_1^{[2}u_2^{3]})\\
&=\omega_{12}u_1^{[1}u_2^{2]}+\omega_{21}u_1^{[2}u_2^{1]}+\omega_{13}u_1^{[1}u_2^{3]}\\
&\qquad\qquad+\omega_{31}u_1^{[3}u_2^{1]}+\omega_{23}u_1^{[2}u_2^{3]}+\omega_{32}u_1^{[3}u_2^{2]}\\
&=\omega_{ij}u_1^{[i}u_2^{j]}.
\end{aligned}$$

The operation of alternation can be performed much more systematically by introducing the ***generalised Kronecker delta***. We shall define in an n-dimensional space the generalised Kronecker delta of order $k \le n$ by means of the following *symbolic determinant*

$$\delta^{i_1i_2\cdots i_k}_{j_1j_2\cdots j_k}=\begin{vmatrix}\delta^{i_1}_{j_1} & \delta^{i_1}_{j_2} & \cdots & \delta^{i_1}_{j_k}\\ \delta^{i_2}_{j_1} & \delta^{i_2}_{j_2} & \cdots & \delta^{i_2}_{j_k}\\ \vdots & \vdots & & \vdots\\ \delta^{i_k}_{j_1} & \delta^{i_k}_{j_2} & \cdots & \delta^{i_k}_{j_k}\end{vmatrix} \tag{1.4.6}$$

where the range of all indices $i_1,i_2,\cdots,i_k$ and $j_1,j_2,\cdots,j_k$ is, of course, from 1 to n. Since a determinant changes only its sign when we interchange either its two columns or its two rows we immediately notice that n^{2k} number of quantities $\delta^{i_1i_2\cdots i_k}_{j_1j_2\cdots j_k}$ are completely antisymmetric with respect to its superscripts or its subscripts so that only the sign of the relevant quantity changes when we interchange any two of its upper indices or lower indices and it becomes zero when any two indices in upper or lower positions are equal. If the indices $\{i_1,i_2,\cdots,i_k\}$ and $\{j_1,j_2,\cdots,j_k\}$ are *not* chosen from a same subset of the set $\{1,\dots,n\}$ involving k distinct numbers, then at least one row of the determinant (1.4.6) is zero owing to the definition of the Kronecker delta. Hence, the corresponding generalised Kronecker delta vanishes. On the other hand, if the upper and lower indices are *both even or odd permutations* of the same distinct k numbers the generalised Kronecker delta becomes $+1$ whereas it becomes -1 if *one is an even and the other is the odd permutations* of these k numbers. To see this, it suffices to note that when we choose upper and lower indices from the same set of distinct indices we can obviously set $i_1=j_1,\ i_2=j_2,\dots,i_k=j_k$ by properly interchanging row and columns in the determinant, in other words, by properly permuting upper and lower indices. In this case the determinant reduces simply to

$$\mp\begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} = \mp 1.$$

It is clear that if it is necessary to make either even or odd permutations in both upper and lower indices then the value of the generalised Kronecker delta would be $+1$. However, if it is required to make even permutation in one set of indices and odd permutation in the other set the value would, of course, be -1. It is clear that if $k > n$, the generalised Kronecker delta becomes identically zero.

Since the generalised Kronecker delta is completely antisymmetric with respect to both upper and lower indices, it follows from the definition (1.4.6) that

$$\delta^{i_1 i_2 \cdots i_k}_{j_1 j_2 \cdots j_k} = k!\, \delta^{i_1}_{[j_1} \delta^{i_2}_{j_2} \cdots \delta^{i_k}_{j_k]} = k!\, \delta^{[i_1}_{j_1} \delta^{i_2}_{j_2} \cdots \delta^{i_k]}_{j_k}. \tag{1.4.7}$$

Indeed, we can readily observe this property in two simple examples below for $k = 2$ and $k = 3$

$$\delta^{ij}_{kl} = \begin{vmatrix} \delta^i_k & \delta^i_l \\ \delta^j_k & \delta^j_l \end{vmatrix} = \delta^i_k \delta^j_l - \delta^i_l \delta^j_k = 2\, \delta^i_{[k} \delta^j_{l]} = 2\, \delta^{[i}_k \delta^{j]}_l$$

$$\begin{aligned} \delta^{ijk}_{lmn} &= \begin{vmatrix} \delta^i_l & \delta^i_m & \delta^i_n \\ \delta^j_l & \delta^j_m & \delta^j_n \\ \delta^k_l & \delta^k_m & \delta^k_n \end{vmatrix} = \delta^i_l \delta^j_m \delta^k_n - \delta^i_l \delta^j_n \delta^k_m + \delta^i_m \delta^j_n \delta^k_l - \delta^i_m \delta^j_l \delta^k_n \\ &\quad + \delta^i_n \delta^j_l \delta^k_m - \delta^i_n \delta^j_m \delta^k_l \\ &= 3!\, \delta^i_{[l} \delta^j_m \delta^k_{n]} = 3!\, \delta^{[i}_l \delta^j_m \delta^{k]}_n. \end{aligned}$$

Consider a quantity $A^{i_1 i_2 \cdots i_k}$ with k indices. It is rather straightforward to see that (1.4.7) leads to the relation

$$\delta^{i_1 i_2 \cdots i_k}_{j_1 j_2 \cdots j_k} A^{j_1 j_2 \cdots j_k} = k!\, A^{[i_1 i_2 \cdots i_k]} \tag{1.4.8}$$

Let us now rewrite the expression (1.4.4) defining an alternating k-linear functional in the form

$$\omega(u_1, u_2, \ldots, u_k) = \omega_{i_1 i_2 \cdots i_k} f^{[i_1}(u_1) f^{i_2}(u_2) \cdots f^{i_k]}(u_k)$$

where, as usual, the vectors, or linear functionals $\{f^i\} \subset U^*$ constitutes the reciprocal basis in the dual space with respect to the basis $\{e_i\} \subset U$. Thus we can represent this alternating k-linear functional acting on an element $(u_1, u_2, \ldots, u_k)$ of the Cartesian product U^k [*see* (1.3.5)] by the following expression

$$\omega = \omega_{i_1 i_2 \cdots i_k} f^{[i_1} \otimes f^{i_2} \otimes \cdots \otimes f^{i_k]} \tag{1.4.9}$$

by employing the tensor product. Resorting to the relation (1.4.8) we can transform the expression (1.4.9) into

$$\omega = \frac{1}{k!}\, \omega_{i_1 i_2 \cdots i_k} \delta^{i_1 i_2 \cdots i_k}_{j_1 j_2 \cdots j_k} f^{j_1} \otimes f^{j_2} \otimes \cdots \otimes f^{j_k}.$$

We now define the ***exterior product***, or ***wedge product***, of k basis vectors in the dual space U^* by the relation

$$\begin{aligned} f^{i_1} \wedge f^{i_2} \wedge \cdots \wedge f^{i_k} &= \delta^{i_1 i_2 \cdots i_k}_{j_1 j_2 \cdots j_k} f^{j_1} \otimes f^{j_2} \otimes \cdots \otimes f^{j_k} \\ &= k!\, f^{[i_1} \otimes f^{i_2} \otimes \cdots \otimes f^{i_k]}. \end{aligned} \tag{1.4.10}$$

We can then represent (1.4.9) in the form

$$\omega = \frac{1}{k!}\, \omega_{i_1 i_2 \cdots i_k} f^{i_1} \wedge f^{i_2} \wedge \cdots \wedge f^{i_k}. \tag{1.4.11}$$

For instance, we find that

$$\begin{aligned} f^i \wedge f^j &= f^i \otimes f^j - f^j \otimes f^i, \\ f^i \wedge f^j \wedge f^k &= f^i \otimes f^j \otimes f^k + f^j \otimes f^k \otimes f^i + f^k \otimes f^i \otimes f^j \\ &\quad - f^i \otimes f^k \otimes f^j - f^k \otimes f^j \otimes f^i - f^j \otimes f^i \otimes f^k. \end{aligned}$$

It is clear that the exterior product introduced by (1.4.10) is completely antisymmetric. In view of the representation (1.4.11), we call an alternating k-linear functional as an ***exterior form of degree k*** or simply a ***k-form***. *Such a form is obviously a completely antisymmetric k-covariant tensor*. The value of a k-form on linearly independent k vectors $u_1, u_2, \ldots, u_k \in U$ is given by (1.4.4). However, if we recall the definition of a determinant we can immediately recognise that a quantity $u_1^{[i_1} u_2^{i_2} \cdots u_k^{i_k]}$ is expressible by a determinant as follows:

$$k!\, u_1^{[i_1} u_2^{i_2} \cdots u_k^{i_k]} = \begin{vmatrix} u_1^{i_1} & u_2^{i_1} & \cdots & u_k^{i_1} \\ u_1^{i_2} & u_2^{i_2} & \cdots & u_k^{i_2} \\ \vdots & \vdots & & \vdots \\ u_1^{i_k} & u_2^{i_k} & \cdots & u_k^{i_k} \end{vmatrix}.$$

We can thus write

$$\omega(u_1,u_2,\dots,u_k) = \omega_{i_1i_2\cdots i_k}u_1^{[i_1}u_2^{i_2}\cdots u_k^{i_k]} = \frac{1}{k!}\,\omega_{i_1i_2\cdots i_k}\begin{vmatrix} u_1^{i_1} & u_2^{i_1} & \cdots & u_k^{i_1} \\ u_1^{i_2} & u_2^{i_2} & \cdots & u_k^{i_2} \\ \vdots & \vdots & & \vdots \\ u_1^{i_k} & u_2^{i_k} & \cdots & u_k^{i_k} \end{vmatrix}$$

$$= \frac{1}{k!}\,\omega_{i_1i_2\cdots i_k}\begin{vmatrix} f^{i_1}(u_1) & f^{i_1}(u_2) & \cdots & f^{i_1}(u_k) \\ f^{i_2}(u_1) & f^{i_2}(u_2) & \cdots & f^{i_2}(u_k) \\ \vdots & \vdots & & \vdots \\ f^{i_k}(u_1) & f^{i_k}(u_2) & \cdots & f^{i_k}(u_k) \end{vmatrix}$$

By employing essential components, we can also transform this expression into the form

$$\omega(u_1,u_2,\dots,u_k) = \sum_{1\le i_1<i_2<\cdots<i_k\le n} \omega_{i_1i_2\cdots i_k}V^{i_1i_2\cdots i_k}(u_1,u_2,\dots,u_k) \qquad (1.4.12)$$

Here

$$V^{i_1i_2\cdots i_k}(u_1,u_2,\dots,u_k) = \begin{vmatrix} u_1^{i_1} & u_2^{i_1} & \cdots & u_k^{i_1} \\ u_1^{i_2} & u_2^{i_2} & \cdots & u_k^{i_2} \\ \vdots & \vdots & & \vdots \\ u_1^{i_k} & u_2^{i_k} & \cdots & u_k^{i_k} \end{vmatrix} \qquad (1.4.13)$$

may be interpreted as the *k-dimensional volume*[1] of the projection of k-dimensional parallelepiped formed by vectors $u_1,u_2,\dots,u_k$ in n-dimensional vector space on a subspace generated by *axes* $i_1<i_2<\cdots<i_k$. As an example, let us consider $n=3,\ k=2$ and a 2-form

$$\omega = \frac{1}{2}\,\omega_{ij}\,f^i\wedge f^j$$

whose value on vectors u_1 and u_2 is given by

$$\omega(u_1,u_2) = \omega_{12}V^{12}(u_1,u_2) + \omega_{13}V^{13}(u_1,u_2) + \omega_{23}V^{23}(u_1,u_2)$$

where one identifies the numbers $V^{12}(u_1,u_2) = u_1^1u_2^2 - u_1^2u_2^1$, $V^{13}(u_1,u_2) = u_1^1u_2^3 - u_1^3u_2^1$ and $V^{23}(u_1,u_2) = u_1^2u_2^3 - u_1^3u_2^2$ as *areas* of parallelograms that are projections of the parallelogram formed by vectors u_1 and u_2 in the 3-dimensional space, respectively, on the planes generated by 12-, 13- and

[1] One must notice the fact that this number does not correspond to the real invariant geometric volume. As is easily observed, this number is dependent on the selected basis of the vector space U. But it is non-zero for linearly independent vectors.

23-axes. We can now say that a k-form defined on an n-dimensional vector space U makes it possible for us to evaluate certain linear combinations, with coefficients of that form, of *k-dimensional volumes* projected onto k-dimensional subspaces from a k-dimensional parallelepiped formed by k linearly independent vector in U.

Let us now consider an n-form as follows

$$\omega = \frac{1}{n!}\,\omega_{i_1 i_2 \cdots i_n} f^{i_1} \wedge f^{i_2} \wedge \cdots \wedge f^{i_n}. \tag{1.4.14}$$

Since the indices have to be permutations of the numbers $1, 2, \ldots, n$, the only essential component is $\omega_{12\cdots n}$. In order to express this situation more systematically we now introduce the ***Levi-Civita symbol*** [after Italian mathematician Tullio Levi-Civita (1873-1941)] with covariant indices as

$$e_{i_1 i_2 \cdots i_n} = \begin{cases} \;\;\, 0, \text{if any two indices are equal,} \\ +\,1, \text{if indices } (i_1, \cdots, i_n) \text{ is an even permutation of } (1, \ldots, n), \\ -\,1, \text{if indices } (i_1, \cdots, i_n) \text{ is an odd permutation of } (1, \ldots, n),. \end{cases}$$

The symbol $e^{i_1 i_2 \cdots i_n}$ with contravariant indices is defined in exactly the same fashion. On the other hand, it is easy to see that we have the relation

$$e_{i_1 i_2 \cdots i_n} e^{i_1 i_2 \cdots i_n} = n! \tag{1.4.15}$$

since each term in the above sum will take the value $+\,1$ for every permutation. We can thus write for an n-form

$$\begin{aligned} \omega &= \frac{1}{n!}\, e_{i_1 i_2 \cdots i_n} \omega_{12\cdots n} e^{i_1 i_2 \cdots i_n} f^1 \wedge f^2 \wedge \cdots \wedge f^n \\ &= \omega_{12\cdots n} f^1 \wedge f^2 \wedge \cdots \wedge f^n \end{aligned}$$

Since $\binom{n}{n} = 1$ there exists indeed only one linearly independent form, for instance, $f^1 \wedge f^2 \wedge \cdots \wedge f^n$. *All other n-forms are scalar multiples of that form*. The value of this form ω on linearly independent n vectors $u_1, u_2, \ldots, u_n \in U$ are given by

$$\begin{aligned} \omega(u_1, u_2, \ldots, u_n) &= \omega_{12\cdots n} \begin{vmatrix} u_1^1 & u_2^1 & \cdots & u_n^1 \\ u_1^2 & u_2^2 & \cdots & u_n^2 \\ \vdots & \vdots & & \vdots \\ u_1^n & u_2^n & \cdots & u_n^n \end{vmatrix} \\ &= \omega_{12\cdots n} V_n(u_1, u_2, \ldots, u_n). \end{aligned}$$

We may interpret the determinant V_n as the volume of an n-dimensional parallelepiped formed n vectors in the space U. If these vectors are linearly independent we know that the above determinant cannot vanish so that we have $V_n(u_1, u_2, \ldots, u_n) \neq 0$. If we rename the basis vectors $e_1, \ldots, e_n$ in U properly we can set $V_n(e_1, \ldots, e_n) = +1$ and we find

$$\omega_{12\cdots n} = \omega(e_1, e_2, \ldots, e_n)$$

as it should be. If $k = n$, the generalised Kronecker deltas are obviously expressible in terms of Levi-Civita symbols in the following way

$$\delta^{i_1 i_2 \cdots i_n}_{j_1 j_2 \cdots j_n} = e^{i_1 i_2 \cdots i_n} e_{j_1 j_2 \cdots j_n}. \tag{1.4.16}$$

The determinant V_n can now be written as

$$V_n = n!\, u_1^{[1} \cdots u_n^{n]} = e_{i_1 i_2 \cdots i_n} u_1^{i_1} u_2^{i_2} \cdots u_n^{i_n}.$$

But this expression is completely antisymmetric with respect to indices $1, \ldots, n$. Therefore, we can also write

$$V_n = \frac{1}{n!} e_{i_1 \cdots i_n} e^{j_1 \cdots j_n} u_{j_1}^{i_1} \cdots u_{j_n}^{i_n} = \frac{1}{n!}\, \delta^{j_1 \cdots j_n}_{i_1 \cdots i_n} u_{j_1}^{i_1} \cdots u_{j_n}^{i_n}. \tag{1.4.17}$$

It then readily follows from (1.4.17) that the relation

$$\begin{aligned} e_{k_1 \cdots k_n} V_n &= \frac{1}{n!}\, \delta^{j_1 \cdots j_n}_{k_1 \cdots k_n} e_{i_1 \cdots i_n} u_{j_1}^{i_1} \cdots u_{j_n}^{i_n} \\ &= e_{i_1 \cdots i_n} u_{[k_1}^{i_1} \cdots u_{k_n]}^{i_n} = e_{i_1 \cdots i_n} u_{k_1}^{i_1} \cdots u_{k_n}^{i_n} \end{aligned} \tag{1.4.18}$$

is valid for determinants.

It is straightforward to realise that the addition of k-forms on a vector space U and their multiplication with scalars are again k-forms. To see this let us consider two k-forms α and β:

$$\alpha = \frac{1}{k!}\, \alpha_{i_1 \cdots i_k} f^{i_1} \wedge \cdots \wedge f^{i_k}, \;\; \beta = \frac{1}{k!}\, \beta_{i_1 \cdots i_k} f^{i_1} \wedge \cdots \wedge f^{i_k}.$$

The sum $\gamma = \alpha + \beta$ of these forms will naturally be

$$\gamma = \frac{1}{k!}\, \gamma_{i_1 \cdots i_k} f^{i_1} \wedge \cdots \wedge f^{i_k}, \quad \gamma_{i_1 \cdots i_k} = \alpha_{i_1 \cdots i_k} + \beta_{i_1 \cdots i_k}.$$

Similarly, for an arbitrary scalar λ the form $\eta = \lambda\alpha$ is given by

$$\eta = \frac{1}{k!}\, \eta_{i_1 \cdots i_k} f^{i_1} \wedge \cdots \wedge f^{i_k}, \quad \eta_{i_1 \cdots i_k} = \lambda \alpha_{i_1 \cdots i_k}.$$

Hence k-forms constitute a linear vector space which will be denoted by $\Lambda^k(U)$. This vector space is well defined for $1 < k \leq n$. Obviously, there are $\binom{n}{k}$ linearly independent k-forms in this space. All forms whose degrees satisfying $k > n$ are identically zero. If we define exterior forms for $k = 1$ by the expression

$$\omega = \omega_i f^i, \ \ \omega_i \in \mathbb{F} \tag{1.4.19}$$

the spaces $\Lambda^k(U)$ will be completely determined for $1 \leq k \leq n$. There are evidently n linearly independent 1-form since $\binom{n}{1} = n$.

1.5. EXTERIOR ALGEBRA

We shall now try to define the product of two exterior forms in such a way that the result will again be an exterior form. Thus, we will be able to construct an exterior algebra. Let us consider the forms $\alpha \in \Lambda^p(U)$ and $\beta \in \Lambda^q(U)$ given below such that $p \leq n$, $q \leq n$ and $p + q \leq n$:

$$\begin{aligned} \alpha &= \frac{1}{p!}\,\alpha_{i_1 i_2 \cdots i_p} f^{i_1} \wedge f^{i_2} \wedge \cdots \wedge f^{i_p}, \\ \beta &= \frac{1}{q!}\beta_{j_1 j_2 \cdots j_q} f^{j_1} \wedge f^{j_2} \wedge \cdots \wedge f^{j_q}. \end{aligned}$$

The ***exterior product*** $\alpha \wedge \beta$ of forms α and β will now be defined in the following fashion

$$\alpha \wedge \beta = \frac{1}{p!\,q!}\,\alpha_{i_1 i_2 \cdots i_p} \beta_{j_1 j_2 \cdots j_q} f^{i_1} \wedge f^{i_2} \wedge \cdots \wedge f^{i_p} \wedge f^{j_1} \wedge f^{j_2} \wedge \cdots \wedge f^{j_q}$$

where the exterior product of basis vectors is, of course, determined by

$$\begin{aligned} & f^{i_1} \wedge f^{i_2} \wedge \cdots \wedge f^{i_p} \wedge f^{j_1} \wedge f^{j_2} \wedge \cdots \wedge f^{j_q} = \\ & \qquad \delta^{i_1 i_2 \cdots i_p j_1 j_2 \cdots j_q}_{k_1 k_2 \cdots k_p l_1 l_2 \cdots l_q} f^{k_1} \otimes f^{k_2} \otimes \cdots \otimes f^{k_p} \otimes f^{l_1} \otimes f^{l_2} \otimes \cdots \otimes f^{l_q}. \end{aligned}$$

With this definition we are obviously led to the result $\alpha \wedge \beta \in \Lambda^{p+q}(U)$. The coefficients of the form $\alpha \wedge \beta$ should be completely antisymmetric with respect to $p + q$ indices. But they are already completely antisymmetric with respect to the first p and the last q indices. Therefore, the number of independent components will be $(p + q)!/p!\,q!$ and if we define

$$\gamma_{i_1 i_2 \cdots i_p j_1 j_2 \cdots j_q} = \frac{(p+q)!}{p!q!}\alpha_{[i_1 i_2 \cdots i_p}\beta_{j_1 j_2 \cdots j_q]} \tag{1.5.1}$$

we obtain

$$\begin{aligned}\gamma &= \alpha \wedge \beta \\ &= \frac{1}{(p+q)!}\,\gamma_{i_1\cdots i_p j_1\cdots j_q} f^{i_1}\wedge\cdots\wedge f^{i_p}\wedge f^{j_1}\wedge\cdots\wedge f^{j_q}.\end{aligned} \tag{1.5.2}$$

If $p+q>n$ we clearly find $\alpha\wedge\beta=0$. As an example, consider

$$\alpha=\alpha_i f^i\in\Lambda^1(U),\quad \beta=\frac{1}{2!}\,\beta_{jk}f^j\wedge f^k\in\Lambda^2(U)$$

where β_{jk} are antisymmetric. For $n>3$ we obtain

$$\gamma=\alpha\wedge\beta=\frac{1}{2!}\alpha_i\beta_{jk}\,f^i\wedge f^j\wedge f^k=\frac{1}{2!}\alpha_{[i}\beta_{jk]}\,f^i\wedge f^j\wedge f^k.$$

On the other hand, we find that

$$\begin{aligned}\alpha_{[i}\beta_{jk]} &= \frac{1}{3!}(\alpha_i\beta_{jk}+\alpha_j\beta_{ki}+\alpha_k\beta_{ij}-\alpha_i\beta_{kj}-\alpha_k\beta_{ji}-\alpha_j\beta_{ik}) \\ &= \frac{1}{3}(\alpha_i\beta_{jk}+\alpha_j\beta_{ki}+\alpha_k\beta_{ij})=\frac{1}{3}\,\gamma_{ijk}.\end{aligned}$$

Hence the exterior product $\alpha\wedge\beta$ has the standard structure

$$\gamma=\frac{1}{3!}\,\gamma_{ijk}\,f^i\wedge f^j\wedge f^k\in\Lambda^3(U).$$

Just from the definition of the exterior product of forms, we conclude that the exterior product is *distributive*, namely

$$\alpha\wedge(\beta+\gamma)=\alpha\wedge\beta+\alpha\wedge\gamma,\;(\alpha+\beta)\wedge\gamma=\alpha\wedge\gamma+\beta\wedge\gamma. \tag{1.5.3}$$

Here we have, naturally, considered the addition of forms of the same degree. It is evident that the exterior product so defined is *associative*:

$$\alpha\wedge(\beta\wedge\gamma)=(\alpha\wedge\beta)\wedge\gamma=\alpha\wedge\beta\wedge\gamma. \tag{1.5.4}$$

However, the exterior product is not generally *commutative*. Let us consider the forms $\alpha\in\Lambda^p(U)$ and $\beta\in\Lambda^q(U)$. We can show that the relation

$$\beta\wedge\alpha=(-1)^{pq}\alpha\wedge\beta \tag{1.5.5}$$

is valid. Indeed, in order to transform the form $\alpha\wedge\beta$ into the form $\beta\wedge\alpha$, we are compelled to interchange the exterior products $f^{i_1}\wedge\cdots\wedge f^{i_p}$ and $f^{j_1}\wedge\cdots\wedge f^{j_q}$ as blocks. To this end, we first put the vector f^{i_p} at the end of the second sequence by successively interchanging it with vectors $f^{j_1}, f^{j_2}, \ldots, f^{j_q}$. Every transposition gives rise to the multiplication by -1. Thus the form is eventually multiplied by $(-1)^q$. Since this operation should be repeated p times for vectors $f^{i_p},\ f^{i_{p-1}}, \dots, f^{i_1}$ we obtain the relation (1.5.5). It follows now from (1.5.5) that if $\alpha \in \Lambda^1(U)$, we then of course find $\alpha \wedge \alpha = 0$.

The vector space of k-forms $\Lambda^k(U)$ on an n-dimensional vector space U is not an algebra since it is not closed with respect to the exterior product. If we use the notation $\mathbb{R} = \Lambda^0(U)$ to denote the field of real numbers, the sequence of spaces $\Lambda^k(U)$ starts then with $\Lambda^0(U)$ and ends with $\Lambda^n(U)$. Let us now define a vector space $\Lambda(U)$ by the following direct sum:

$$\Lambda(U) = \Lambda^0(U) \oplus \Lambda^1(U) \oplus \Lambda^2(U) \oplus \cdots \oplus \Lambda^n(U). \tag{1.5.6}$$

It is obvious that the vector space $\Lambda(U)$ now becomes an algebra under the exterior product. In other words, for all forms $\alpha, \beta \in \Lambda(U)$ we find $\alpha \wedge \beta \in \Lambda(U)$. We call the algebra $\Lambda(U)$ as the ***exterior algebra***. However, this vector space is constructed as a direct sum of some linear vector spaces. Therefore, it is called a ***graded algebra***.

We are now going to show that the k-forms $f^{i_1} \wedge f^{i_2} \wedge \cdots \wedge f^{i_k}$, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ constitute a basis for the vector space $\Lambda^k(U)$. To this end, it suffices to prove that those forms are linearly independent. With arbitrary scalars $\alpha_{i_1 i_2 \cdots i_k}$, let us write

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \alpha_{i_1 i_2 \cdots i_k} f^{i_1} \wedge f^{i_2} \wedge \cdots \wedge f^{i_k} = 0$$

Let us choose an arbitrary index set of k distinct numbers $\{i'_1, i'_2, \cdots, i'_k\}$ out of the set $\{1, 2, \dots, n\}$. Let the index set of $n-k$ natural numbers that is the complement of this subset with respect to the set $\{1, 2, \dots, n\}$ be the subset $\{j'_{k+1}, \cdots, j'_n\}$. The exterior product of the foregoing expression by the $(n-k)$-form $f^{j'_{k+1}} \wedge \cdots \wedge f^{j'_n}$ will be

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \alpha_{i_1 i_2 \cdots i_k} f^{i_1} \wedge f^{i_2} \wedge \cdots \wedge f^{i_k} \wedge f^{j'_{k+1}} \wedge \cdots \wedge f^{j'_n} = 0.$$

However, the set $\{j'_{k+1}, \cdots, j'_n\}$ is the complement of the set $\{i'_1, \cdots, i'_k\}$ with respect to the set $\{1, 2, \dots, n\}$. Consequently, all terms in the above sum except the one corresponding to those indices vanish because at least two basis vectors (actually 1-forms) would be equal. We thus see that only the term

$$\alpha_{i'_1 \cdots i'_k} f^{i'_1} \wedge \cdots \wedge f^{i'_k} \wedge f^{j'_{k+1}} \wedge \cdots \wedge f^{j'_n} = \pm\, \alpha_{i'_1 \cdots i'_k} f^1 \wedge \cdots \wedge f^n = 0$$

survives in that zero form. The value of that form on n linearly independent vectors $u_1, u_2, \dots, u_n \in U$ is given by

$$V^n(u_1, u_2, \ldots, u_n)\, \alpha_{i'_1 \cdots i'_k} = 0.$$

Since $V^n(u_1, u_2, \ldots, u_n) \neq 0$ we find that $\alpha_{i'_1 \cdots i'_k} = 0$. Since the choice of indices is entirely arbitrary, we conclude that all scalar coefficients must vanish. Hence, the set of forms $\{f^{i_1} \wedge \cdots \wedge f^{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$ constitutes a basis for the vector space $\Lambda^k(U)$. The cardinality of this set is $\binom{n}{k}$ implying that the dimension of the vector space $\Lambda^k(U)$ is $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$. The basis of the vector space $\Lambda(U)$, which is defined by the direct sum (1.5.6), is clearly determined by the union of bases of component vector spaces. Since the basis of the vector space $\Lambda^0(U)$ is 1, the basis of $\Lambda(U)$ is prescribed by

$$\{1\} \cup \{f^i\} \cup \cdots \cup \{f^{i_1} \wedge \cdots \wedge f^{i_k} : i_1 < \cdots < i_k\} \cup \cdots \cup \{f^1 \wedge \cdots \wedge f^n\}.$$

Therefore the *dimension of the exterior algebra* $\Lambda(U)$ on a vector space $U^{(n)}$ is given by the integer

$$N = \sum_{k=0}^{n} \binom{n}{k} = 2^n. \tag{1.5.7}$$

We say that a k-form is a ***simple form*** if it is expressible as an exterior product of k linearly independent 1-forms, that is, if a k-form is simple it can be written as follows

$$\omega = \omega^{(1)} \wedge \omega^{(2)} \wedge \cdots \wedge \omega^{(k)}, \;\; \omega^{(i)} \in \Lambda^1(U), \;\; \omega \in \Lambda^k(U) \tag{1.5.8}$$

where $\omega^{(m)} = \omega_i^{(m)} f^i$, $m = 1, 2, \ldots, k$. We thus obtain

$$\omega = \omega^{(1)}_{[i_1} \cdots \omega^{(k)}_{i_k]} f^{i_1} \wedge f^{i_2} \wedge \cdots \wedge f^{i_k} = \frac{1}{k!}\, \omega_{i_1 i_2 \cdots i_k} f^{i_1} \wedge f^{i_2} \wedge \cdots \wedge f^{i_k}.$$

Here the scalar numbers $\omega_{i_1 i_2 \cdots i_k} = k!\, \omega^{(1)}_{[i_1} \cdots \omega^{(k)}_{i_k]}$ are components of the form ω. The value of a simple k-form on k linearly independent vectors u_1, $u_2, \ldots, u_k \in U$ can now be evaluated as follows

$$\omega(u_1, u_2, \ldots, u_k) = \omega^{(1)}_{i_1} \cdots \omega^{(k)}_{i_k} \begin{vmatrix} u_1^{i_1} & u_2^{i_1} & \cdots & u_k^{i_1} \\ u_1^{i_2} & u_2^{i_2} & \cdots & u_k^{i_2} \\ \vdots & \vdots & & \vdots \\ u_1^{i_k} & u_2^{i_k} & \cdots & u_k^{i_k} \end{vmatrix}$$

$$= \begin{vmatrix} \omega^{(1)}_{i_1} u^{i_1}_1 & \omega^{(1)}_{i_1} u^{i_1}_2 & \cdots & \omega^{(1)}_{i_1} u^{i_1}_k \\ \omega^{(2)}_{i_2} u^{i_2}_1 & \omega^{(2)}_{i_2} u^{i_2}_2 & \cdots & \omega^{(2)}_{i_2} u^{i_2}_k \\ \vdots & \vdots & & \vdots \\ \omega^{(k)}_{i_k} u^{i_k}_1 & \omega^{(k)}_{i_k} u^{i_k}_2 & \cdots & \omega^{(k)}_{i_k} u^{i_k}_k \end{vmatrix} = \begin{vmatrix} \omega^{(1)}(u_1) & \omega^{(1)}(u_2) & \cdots & \omega^{(1)}(u_k) \\ \omega^{(2)}(u_1) & \omega^{(2)}(u_2) & \cdots & \omega^{(2)}(u_k) \\ \vdots & \vdots & & \vdots \\ \omega^{(k)}(u_1) & \omega^{(k)}(u_2) & \cdots & \omega^{(k)}(u_k) \end{vmatrix}$$

1.6. RANK OF AN EXTERIOR FORM

Let us consider a form $\omega \in \Lambda^k(U)$ on an n-dimensional vector space U (unless stated otherwise we shall always consider a finite dimensional vector space):

$$\omega = \frac{1}{k!}\,\omega_{i_1 i_2 \cdots i_k} f^{i_1} \wedge f^{i_2} \wedge \cdots \wedge f^{i_k}. \tag{1.6.1}$$

We now choose a certain linear combinations of reciprocal basis vectors in the dual space U^* as follows

$$g^\alpha = c^\alpha_i f^i, \quad i = 1, 2, \ldots, n; \quad \alpha = 1, 2, \ldots, m. \tag{1.6.2}$$

c^α_i are some scalar coefficients. We shall assume that the vectors g^α are linearly independent. In other words, the rank of the rectangular matrix $[c^\alpha_i]$ should be m. Therefore, the transformations (1.6.2) will be meaningful if only $m \le n$. Let us suppose that these transformations reduce the form (1.6.1) into the following k-form

$$\omega = \frac{1}{k!}\,\Omega_{\alpha_1 \alpha_2 \cdots \alpha_k} g^{\alpha_1} \wedge g^{\alpha_2} \wedge \cdots \wedge g^{\alpha_k}.$$

The least integer m found in this fashion, that is, $r = \min m$, is called the ***rank*** of the form ω. In order to determine the rank of a form, we have to look for the nontrivial, linearly independent solutions of the following homogeneous equations

$$\omega_{i_1 i_2 \cdots i_k} h^{i_1} = 0, \quad h^{i_1} \in U^*. \tag{1.6.3}$$

If we find linearly independent $n - r$ solutions $h^a, a = r+1, r+2, \ \ldots, n$ we can then write $h^\alpha = \gamma^\alpha_a h^a, \alpha = 1, 2, \ldots, r$. Hence, the rank of the rectangular matrix $[\gamma^\alpha_a]$ must be r for vectors h^α to be *linearly independent among themselves*. We will see that this number denotes also the rank of the form ω. If $r = n$, then we clearly get $h^i = 0, \ i = 1, 2, \ldots, n$. In this case, we cannot reduce the number of basis vectors or forms appearing in (1.6.1).

Let us now assume that the rank of the form is satisfying the condition $r < n$. It follows from equations (1.6.3) that

$$\omega_{\alpha i_2 \cdots i_k} h^\alpha + \omega_{a i_2 \cdots i_k} h^a = (\omega_{\alpha i_2 \cdots i_k} \gamma^\alpha_a + \omega_{a i_2 \cdots i_k}) h^a = 0.$$

Since we supposed that the vectors h^a are linearly independent, we then see that the relations

$$\omega_{\alpha i_2 \cdots i_k} \gamma^\alpha_a + \omega_{a i_2 \cdots i_k} = 0, \tag{1.6.4}$$

where $a = r+1, \dots, n; i_m = 1, \dots, n; m \geq 2,$ should be satisfied. Let us now define the linearly independent vectors

$$g^\alpha = f^\alpha - \gamma^\alpha_a f^a, \;\; \alpha = 1, 2, \dots, r \tag{1.6.5}$$

and insert the vectors $f^\alpha = g^\alpha + \gamma^\alpha_a f^a$ into the first factor in the exterior product in (1.6.1). In the first step we obtain

$$\begin{aligned}
\omega &= \frac{1}{k!}\, \omega_{i_1 i_2 \cdots i_k} f^{i_1} \wedge f^{i_2} \wedge \cdots \wedge f^{i_k} \\
&= \frac{1}{k!} (\, \omega_{\alpha_1 i_2 \cdots i_k} f^{\alpha_1} + \omega_{a_1 i_2 \cdots i_k} f^{a_1}) \wedge f^{i_2} \wedge \cdots \wedge f^{i_k} \\
&= \frac{1}{k!} \big[\, \omega_{\alpha_1 i_2 \cdots i_k} (g^{\alpha_1} + \gamma^{\alpha_1}_{a_1} f^{a_1}) + \omega_{a_1 i_2 \cdots i_k} f^{a_1} \big] \wedge f^{i_2} \wedge \cdots \wedge f^{i_k} \\
&= \frac{1}{k!} \big[\, \omega_{\alpha_1 i_2 \cdots i_k} g^{\alpha_1} + (\omega_{\alpha_1 i_2 \cdots i_k} \gamma^{\alpha_1}_{a_1} + \omega_{a_1 i_2 \cdots i_k}) f^{a_1} \big] \wedge f^{i_2} \wedge \cdots \wedge f^{i_k} \\
&= \frac{1}{k!}\, \omega_{\alpha_1 i_2 \cdots i_k} g^{\alpha_1} \wedge f^{i_2} \wedge \cdots \wedge f^{i_k}
\end{aligned}$$

where we made use of the relation (1.6.4) in the fourth line. In the second step, we are led to

$$\begin{aligned}
\omega = &-\frac{1}{k!}\, \omega_{i_2 \alpha_1 \cdots i_k} g^{\alpha_1} \wedge f^{i_2} \wedge \cdots \wedge f^{i_k} = \\
-&\frac{1}{k!} g^{\alpha_1} \wedge \big[\, \omega_{\alpha_2 \alpha_1 i_3 \cdots i_k} g^{\alpha_2} + (\omega_{\alpha_2 \alpha_1 i_3 \cdots i_k} \gamma^{\alpha_2}_{a_2} + \omega_{a_2 \alpha_1 i_3 \cdots i_k}) f^{a_2} \big] \wedge f^{i_3} \wedge \cdots \wedge f^{i_k} \\
\mathrm{k} = &\frac{1}{k!} \omega_{\alpha_1 \alpha_2 i_3 \cdots i_k} g^{\alpha_1} \wedge g^{\alpha_2} \wedge f^{i_3} \wedge \cdots \wedge f^{i_k}.
\end{aligned}$$

Continuing this way, we arrive at the following result in the kth step

$$\omega = \frac{1}{k!}\, \omega_{\alpha_1 \alpha_2 \cdots \alpha_k} g^{\alpha_1} \wedge g^{\alpha_2} \wedge \cdots \wedge g^{\alpha_k}. \tag{1.6.6}$$

This clearly means that the k-form ω is now generated by basis forms $\{g^{\alpha_1} \wedge g^{\alpha_2} \wedge \cdots \wedge g^{\alpha_k}\}$. The cardinality $\binom{r}{k}$ of this set is of course less than the cardinality $\binom{n}{k}$ of the original basis set. If $r = k$, i.e., if the rank of the form is equal to its degree, we get $\begin{pmatrix} k \\ k \end{pmatrix} = 1$ and the form ω can be represented by

$$\omega = \omega_{1\cdots k}\, g^1 \wedge g^2 \wedge \cdots \wedge g^k. \tag{1.6.7}$$

In order to see this, it suffices to note that one has

$$\omega_{\alpha_1\cdots\alpha_k} = e_{\alpha_1\cdots\alpha_k}\omega_{1\cdots k}, \quad g^{\alpha_1} \wedge \cdots \wedge g^{\alpha_k} = e^{\alpha_1\cdots\alpha_k} g^1 \wedge \cdots \wedge g^k$$

and $e_{\alpha_1\cdots\alpha_k}e^{\alpha_1\cdots\alpha_k} = k!$. If we write $\widetilde{g}^{\,1} = \omega_{1\cdots k}\, g^1$, (1.6.7) now becomes

$$\omega = \widetilde{g}^{\,1} \wedge g^2 \wedge \cdots \wedge g^k.$$

We thus conclude that *every k-form whose rank is equal to its degree can be reduced to a simple form*. Conversely, if a k-form is simple it can be written in the form (1.5.8) as follows

$$\omega = \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^k$$

where 1-forms $\omega^\alpha = \omega_i^\alpha f^i, \alpha = 1, \ldots, k, i = 1, \ldots, n$ are of course linearly independent. Therefore, the rank of the rectangular matrix $[\omega_i^\alpha]$ must be k. Thus we can state the following theorem:

Theorem 1.6.1. *A form $\omega \in \Lambda^k(U), k \leq n$ is a simple form if and only if its rank is equal to its degree.* □

We now apply the general approach which we have developed above to a 2-form owing to its rather simple structure. We know that an arbitrary form $\omega \in \Lambda^2(U^{(n)})$ is expressible as

$$\omega = \frac{1}{2}\,\omega_{ij}\, f^i \wedge f^j \tag{1.6.8}$$

where ω_{ij} constitutes an antisymmetric $n \times n$ matrix of real numbers. In order to find the rank of the form ω we have to determine nontrivial, linearly independent solutions $h^i \in U^*$ of the homogeneous equations

$$\omega_{ij}\, h^j = 0. \tag{1.6.9}$$

Since ω_{ij} is an antisymmetric matrix, its rank is always an even number, say, $r = 2m$ where m is a positive integer. Therefore, the dimension of the null space of the linear operator represented by the matrix ω_{ij}, or the number of linearly independent vectors spanning this subspace would be $n - 2m$. In other words, $2m$ vectors out of n vectors satisfying the equations (1.6.9) are expressible as linear combinations of the remaining

$n - 2m$ vectors. Thus the rank of the form ω becomes $2m$. If n is an even number, it may happen that the rank of the form may be equal to n. In this case, it will not be possible to reduce the form. However if n is an odd number, $2m$ will, of course, always be smaller than n. Consequently, in this case a 2-form is always reducible.

Example 1.6.1. Let us first begin with a relatively simple case of $n = 3$. By using the essential components, we can express a 2-form by the following expression

$$\omega = \omega_{12}\, f^1 \wedge f^2 + \omega_{13}\, f^1 \wedge f^3 + \omega_{23}\, f^2 \wedge f^3.$$

Obviously the rank of this form is 2. Indeed, the equations $\omega_{ij}\, h^j = 0$ are now written in the form

$$\begin{aligned} \omega_{12}\, h^2 + \omega_{13}\, h^3 &= 0 \\ -\, \omega_{12}\, h^1 + \omega_{23}\, h^3 &= 0 \\ -\, \omega_{13}\, h^1 - \omega_{23}\, h^2 &= 0 \end{aligned}$$

whence we deduce by the assumption $\omega_{12} \neq 0$ that

$$h^1 = \frac{\omega_{23}}{\omega_{12}} h^3, \qquad h^2 = -\frac{\omega_{13}}{\omega_{12}} h^3.$$

Let us now define 1-forms

$$g^1 = \omega_{12}\Big(f^1 - \frac{\omega_{23}}{\omega_{12}} f^3\Big), \;\; g^2 = f^2 + \frac{\omega_{13}}{\omega_{12}} f^3.$$

We immediately see that the form ω reduces to

$$\omega = g^1 \wedge g^2 \qquad \blacksquare$$

Example 1.6.2. In order to explore a little bit more complicated case, let us now choose $n = 4$. By using the essential antisymmetric components we can express a 2-form as follows

$$\begin{aligned} \omega = \omega_{12}\, f^1 \wedge f^2 + \omega_{13}\, f^1 \wedge f^3 + \omega_{14}\, f^1 \wedge f^4 + \omega_{23}\, f^2 \wedge f^3 \\ + \omega_{24}\, f^2 \wedge f^4 + \omega_{34}\, f^3 \wedge f^4. \end{aligned}$$

The rank of the form ω can now be either 4 or 2. If the rank is 4, ω is evidently not reducible. Let us consider the equations

$$\begin{aligned} \omega_{12}\, h^2 + \omega_{13} h^3 + \omega_{14}\, h^4 &= 0 \\ -\, \omega_{12}\, h^1 + \omega_{23}\, h^3 + \omega_{24}\, h^4 &= 0 \end{aligned}$$

$$
\begin{aligned}
-\,\omega_{13}\,h^1-\omega_{23}\,h^2+\omega_{34}\,h^4&=0\\
-\,\omega_{14}\,h^1-\omega_{24}\,h^2-\omega_{34}\,h^3&=0.
\end{aligned}
$$

$\Delta=(\omega_{12}\omega_{34}-\omega_{13}\omega_{24}+\omega_{14}\omega_{23})^2$ is the determinant of the coefficient of these linear equations. If $\Delta\neq 0$, then the rank of the form is 4. If only $\Delta=0$, then the rank reduces to 2. When $\Delta=0$ the solution of the above homogeneous equations is given by

$$
h^1=\frac{\omega_{23}}{\omega_{12}}h^3+\frac{\omega_{24}}{\omega_{12}}h^4,\quad h^2=-\frac{\omega_{13}}{\omega_{12}}h^3-\frac{\omega_{14}}{\omega_{12}}h^4
$$

with the assumption $\omega_{12}\neq 0$. Hence, the transformations

$$
g^1=\omega_{12}\Big[f^1-\frac{1}{\omega_{12}}\big(\omega_{23}f^3+\omega_{24}f^4\big)\Big],\ g^2=f^2+\frac{1}{\omega_{12}}\big(\omega_{13}f^3+\omega_{14}f^4\big)
$$

lead to the expression

$$
\omega=g^1\wedge g^2. \qquad \blacksquare
$$

It is possible to introduce a *canonical structure* for 2-forms imposed by their ranks.

Theorem 1.6.2. *Let ω be a 2-form whose rank is $2m$. There exist linearly independent 1-forms $g^1,g^2,\dots,g^{2m}$ such that ω is expressible in the following canonical form*

$$
\begin{aligned}
\omega&=g^1\wedge g^{m+1}+g^2\wedge g^{m+2}+\cdots+g^m\wedge g^{2m}\\
&=\sum_{i=1}^m g^i\wedge g^{m+i}
\end{aligned}
\tag{1.6.10}
$$

We can easily prove this theorem by resorting to mathematical induction. By employing the essential components we can write the form ω in the following manner

$$
\begin{aligned}
\omega=\omega_{12}\,f^1\wedge f^2+\omega_{13}\,f^1\wedge f^3+\cdots+\omega_{1n}\,f^1\wedge f^n\\
+\,\omega_{23}\,f^2\wedge f^3+\omega_{24}\,f^2\wedge f^4+\cdots+\omega_{2n}\,f^2\wedge f^n+\overline{\Phi}_1
\end{aligned}
$$

where $\overline{\Phi}_1$ is a quadratic form depending only to basis forms $f^3,f^4,\dots,f^n$. Let us then rewrite it as follows

$$
\begin{aligned}
\omega=f^1\wedge(\omega_{12}\,f^2+\omega_{13}\,f^3+\cdots+\omega_{1n}\,f^n)\\
+\,f^2\wedge(\omega_{23}\,f^3+\omega_{24}\,f^4+\cdots+\omega_{2n}\,f^n)+\overline{\Phi}_1.
\end{aligned}
\tag{1.6.11}
$$

If we assume that $\omega_{12}\neq 0$, we can define 1-forms g^1 and g^{m+1} by

$$
\begin{aligned}
g^1 &= f^1 - \frac{1}{\omega_{12}}(\omega_{23}\,f^3 + \omega_{24}\,f^4 + \cdots + \omega_{2n}\,f^n) \\
g^{m+1} &= \omega_{12}\,f^2 + \omega_{13}\,f^3 + \cdots + \omega_{1n}\,f^n.
\end{aligned} \tag{1.6.12}
$$

When we insert the forms (1.6.12) into the expression (1.6.11) we conclude after some manipulations that

$$
\begin{aligned}
\omega = \Big[g^1 &+ \frac{1}{\omega_{12}}(\omega_{23}\,f^3 + \omega_{24}\,f^4 \cdots + \omega_{2n}\,f^n)\Big] \wedge g^{m+1} + \\
&\frac{1}{\omega_{12}}(g^{m+1} - \omega_{13}\,f^3 - \cdots - \omega_{1n}\,f^n) \wedge (\omega_{23}\,f^3 + \omega_{24}\,f^4 \cdots + \omega_{2n}\,f^n) \\
&+ \overline{\Phi}_1 = g^1 \wedge g^{m+1} - \frac{1}{\omega_{12}}\,g^{m+1} \wedge (\omega_{23}\,f^3 + \cdots + \omega_{2n}\,f^n) \\
&\qquad + \frac{1}{\omega_{12}}\,g^{m+1} \wedge (\omega_{23}\,f^3 + \cdots + \omega_{2n}\,f^n) \\
&\qquad - \frac{1}{\omega_{12}}(\omega_{13}\,f^3 + \cdots + \omega_{1n}\,f^n) \wedge (\omega_{23}\,f^3 + \cdots + \omega_{2n}\,f^n) + \overline{\Phi}_1
\end{aligned}
$$

or

$$
\omega = g^1 \wedge g^{m+1} + \Phi_1.
$$

The new quadratic form Φ_1 will evidently involve only $n-2$ number of 1-forms $f^3, f^4, \ldots, f^n$. Thus its rank will be at most $2m-2$. If this number is not zero, namely, if $\Phi_1 \neq 0$, we then repeat this operation this time for the form Φ_1. After repeating this operation k number of times, we reach to the conclusion

$$
\omega = \sum_{i=1}^{k} g^i \wedge g^{m+i} + \Phi_k.
$$

The rank of the quadratic form Φ_k depending on $n-2k$ number of 1-forms will now at most $2m-2k$. Therefore, when we repeat this operation a sufficient number of times the form Φ_k will eventually vanish and we shall arrive at the relation (1.6.10). □

Example 1.6.3. We consider a 2-form on a 4-dimensional vector space given by its essential components:

$$
\begin{aligned}
\omega = \omega_{12}\,f^1 \wedge f^2 &+ \omega_{13}\,f^1 \wedge f^3 + \omega_{14}\,f^1 \wedge f^4 + \omega_{23}\,f^2 \wedge f^3 \\
&+ \omega_{24}\,f^2 \wedge f^4 + \omega_{34}\,f^3 \wedge f^4.
\end{aligned}
$$

The number m can now be at most 2. We define as above

$$g^1 = f^1 - \frac{1}{\omega_{12}}(\omega_{23}\, f^3 + \omega_{24}\, f^4),$$
$$g^3 = \omega_{12}\, f^2 + \omega_{13}\, f^3 + \omega_{14}\, f^4.$$

When $\omega_{12} \neq 0$, we then easily find that

$$\omega = g^1 \wedge g^3 + \frac{\omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23}}{\omega_{12}}\, f^3 \wedge f^4.$$

Let us now write

$$g^2 = \frac{\omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23}}{\omega_{12}}\, f^3,$$
$$g^4 = f^4$$

we obtain

$$\omega = g^1 \wedge g^3 + g^2 \wedge g^4.$$

On the other hand, if the relation $\omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23} = 0$ is satisfied, then the rank of the form ω reduces to 2 and the canonical form becomes

$$\omega = g^1 \wedge g^3.$$ ■

In view of Theorem 1.6.2, it is now understood that any 2-form on a vector space $U^{(n)}$ whose rank is an even number is always expressible in the following canonical form

$$\omega = \sum_{\alpha=1}^{m} g^\alpha \wedge g^{m+\alpha}.$$

$r = 2m$ is the rank of the form and $g^1, g^2, \ldots, g^{2m}$ are linearly independent 1-forms. We now define 2-forms

$$\omega_\alpha = g^\alpha \wedge g^{m+\alpha} \in \Lambda^2(U), \quad \alpha = 1, 2, \ldots, m. \qquad (1.6.13)$$

Due to properties of the exterior product we immediately observe that the relations

$$\omega_{\underline{\alpha}} \wedge \omega_{\underline{\alpha}} = 0, \qquad \omega_\alpha \wedge \omega_\beta = \omega_\beta \wedge \omega_\alpha \qquad (1.6.14)$$

are satisfied. We can now write

$$\omega = \sum_{\alpha=1}^{m} \omega_\alpha.$$

Let us next consider the form $\omega^k = \underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_{k} \in \Lambda^{2k}(U)$. Owing to the commutation rule $(1.6.14)_2$ we readily realise that the well known multinomial expansion

$$\begin{aligned} \omega^k &= \Big(\sum_{\alpha=1}^{m} \omega_\alpha\Big)^k \\ &= \sum_{k_1+k_2+\cdots+k_m=k} \frac{k!}{k_1!k_2!\cdots k_m!}\, \omega_1^{k_1} \wedge \omega_2^{k_2} \wedge \cdots \wedge \omega_m^{k_m} \end{aligned} \tag{1.6.15}$$

would be valid just like in the classical algebra. But, if $k_\alpha > 1$, then we have $\omega_\alpha^{k_\alpha} = 0$, $\alpha = 1, 2, \ldots, m$ due to $(1.6.14)_1$. Hence only the terms corresponding to $k_1 = k_2 = \cdots = k_m = 1$ and involving only the exponents k_α meeting the restriction $k_1 + k_2 + \cdots + k_m = k$ will survive. When we take $k = m$, this expansion will of course yield

$$\begin{aligned} \omega^m &= m!\, \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_m \\ &= m!\, g^1 \wedge g^{m+1} \wedge g^2 \wedge g^{m+2} \wedge \cdots \wedge g^m \wedge g^{2m}. \end{aligned}$$

Hence ω^m is a simple form. This result should be anticipated because the rank of the form ω^m is equal to its degree. The relation (1.6.15) implies clearly that $\omega^k = 0$ if $k > m$. This scheme suggests a rather simple method to determine the rank of a quadratic form: If $\omega^m \neq 0$, but $\omega^{m+1} = 0$, then the rank of the quadratic form ω is $r = 2m$. If $k < m$, it then follows from (1.6.15) that

$$\omega^k = k! \sum \textit{k-fold exterior products of forms } (\omega_1, \omega_2, \cdots, \omega_m)$$

whence we deduce with a little care that ω^k is represented by

$$\begin{aligned} \omega^k &= k! \sum_{\alpha_1=1}^{m-k+1} \sum_{\alpha_2=\alpha_1+1}^{m-k+2} \cdots \sum_{\alpha_k=\alpha_{k-1}+1}^{m} \omega_{\alpha_1} \wedge \omega_{\alpha_2} \wedge \cdots \wedge \omega_{\alpha_k} \\ &= k! \sum_{\alpha_1=1}^{m-k+1} \sum_{\alpha_2=\alpha_1+1}^{m-k+2} \cdots \sum_{\alpha_k=\alpha_{k-1}+1}^{m} g^{\alpha_1} \wedge g^{m+\alpha_1} \wedge g^{\alpha_2} \wedge g^{m+\alpha_2} \wedge \cdots \wedge g^{\alpha_k} \wedge g^{m+\alpha_k}. \end{aligned}$$

As an application of what we have obtained so far let us try to answer this question: under what conditions a quadratic form

$$\omega = \frac{1}{2}\, \omega_{ij}\, f^i \wedge f^j$$

is expressible as $\omega = g^1 \wedge g^2$, i.e., as an exterior product of two 1-forms? In order to realise this situation, the rank of the form must be 2, namely, we must have $m = 1$, and consequently $\omega^2 = \omega \wedge \omega = 0$. This relation then gives rise to

$$\omega^2 = \frac{1}{4}\,\omega_{ij}\,\omega_{kl}\, f^i \wedge f^j \wedge\ f^k \wedge f^l = \frac{1}{4}\,\omega_{[ij}\,\omega_{kl]}\, f^i \wedge f^j \wedge\ f^k \wedge f^l = 0$$

or $\omega_{[ij}\,\omega_{kl]} = 0$. By making use of the relation (1.4.8), we should note that one can write

$$\omega_{[ij}\,\omega_{kl]} = \frac{1}{4!}\,\delta^{pqrs}_{ijkl}\,\omega_{pq}\,\omega_{rs}.$$

Moreover, it follows from the definition of the generalised Kronecker delta that we arrive at the expansion

$$\begin{aligned}
\delta^{pqrs}_{ijkl} = \begin{vmatrix} \delta^p_i & \delta^p_j & \delta^p_k & \delta^p_l \\ \delta^q_i & \delta^q_j & \delta^q_k & \delta^q_l \\ \delta^r_i & \delta^r_j & \delta^r_k & \delta^r_l \\ \delta^s_i & \delta^s_j & \delta^s_k & \delta^s_l \end{vmatrix} &= \delta^p_l\delta^q_k\delta^r_j\delta^s_i - \delta^p_k\delta^q_l\delta^r_j\delta^s_i - \delta^p_l\delta^q_j\delta^r_k\delta^s_i \\
&+ \delta^p_j\delta^q_l\delta^r_k\delta^s_i + \delta^p_k\delta^q_j\delta^r_l\delta^s_i - \delta^p_j\delta^q_k\delta^r_l\delta^s_i - \delta^p_l\delta^q_k\delta^r_i\delta^s_j + \delta^p_k\delta^q_l\delta^r_i\delta^s_j \\
&+ \delta^p_l\delta^q_i\delta^r_k\delta^s_j - \delta^p_i\delta^q_l\delta^r_k\delta^s_j - \delta^p_k\delta^q_i\delta^r_l\delta^s_j + \delta^p_i\delta^q_k\delta^r_l\delta^s_j + \delta^p_l\delta^q_j\delta^r_i\delta^s_k \\
&- \delta^p_j\delta^q_l\delta^r_i\delta^s_k - \delta^p_l\delta^q_i\delta^r_j\delta^s_k + \delta^p_i\delta^q_l\delta^r_j\delta^s_k + \delta^p_j\delta^q_i\delta^r_l\delta^s_k - \delta^p_i\delta^q_j\delta^r_l\delta^s_k \\
&- \delta^p_k\delta^q_j\delta^r_i\delta^s_l + \delta^p_j\delta^q_k\delta^r_i\delta^s_l + \delta^p_k\delta^q_i\delta^r_j\delta^s_l - \delta^p_i\delta^q_k\delta^r_j\delta^s_l \\
&- \delta^p_j\delta^q_i\delta^r_k\delta^s_l + \delta^p_i\delta^q_j\delta^r_k\delta^s_l.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\omega_{[ij}\,\omega_{kl]} &= \frac{1}{24}(\omega_{lk}\,\omega_{ji} - \omega_{kl}\,\omega_{ji} - \omega_{lj}\,\omega_{ki} + \omega_{jl}\,\omega_{ki} + \omega_{kj}\,\omega_{li} - \omega_{jk}\,\omega_{li} \\
&\quad - \omega_{lk}\,\omega_{ij} + \omega_{kl}\,\omega_{ij} + \omega_{li}\,\omega_{kj} - \omega_{il}\,\omega_{kj} - \omega_{ki}\,\omega_{lj} + \omega_{ik}\,\omega_{lj} \\
&\quad + \omega_{lj}\,\omega_{ik} - \omega_{jl}\,\omega_{ik} - \omega_{li}\,\omega_{jk} + \omega_{il}\,\omega_{jk} + \omega_{ji}\,\omega_{lk} - \omega_{ij}\,\omega_{lk} \\
&\quad - \omega_{kj}\,\omega_{il} + \omega_{jk}\,\omega_{il} + \omega_{ki}\,\omega_{jl} - \omega_{ik}\,\omega_{jl} - \omega_{ji}\,\omega_{kl} + \omega_{ij}\,\omega_{kl}) \\
&= \frac{1}{3}(\omega_{ij}\,\omega_{kl} - \omega_{ik}\,\omega_{jl} + \omega_{il}\,\omega_{jk}).
\end{aligned}$$

Hence, the conditions which we are looking for turn out to be

$$\omega_{ij}\,\omega_{kl} - \omega_{ik}\,\omega_{jl} + \omega_{il}\,\omega_{jk} = 0.$$

A non-degenerate quadratic form $\omega \in \Lambda^2(U)$ with maximal rank on a linear vector space $U^{(n)}$ is called a ***symplectic form***. The maximality of the rank implies that $r = n$ if the dimension n of the vector space is an even number, and $r = n - 1$ if it is an odd number. Non-degeneracy means that the relation $\omega(u) = 0$ for a vector $u \in U$, or in explicit form, the set of equations

$$\omega_{ij} u^j = 0 \tag{1.6.16}$$

has only the trivial solution $u = 0$. If n is an even number, then the maximal rank will imply the existence of non-degeneracy. However, if n is an odd number, then the maximal rank should be less than n so that equations (1.6.16) will be satisfied by a vector $u \neq 0$. Consequently, the form ω will be degenerate. We thus conclude that *a symplectic form can only be defined on vector spaces with even dimensions.*

Exterior forms have several other algebraic properties. However, we prefer to postpone to treat them on differentiable manifolds later in Chapter V within a much more general context.

I. EXERCISES

1.1. Let U be a linear vector space. U_1 and U_2 are its finite-dimensional subspaces. (a) Show that their sum $U_1 + U_2$ is also a subspace whose dimension is given by

$$\dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2).$$

(b) Find the basis set of the subspace $U_1 \cap U_2$. (c) Show that the subset $U_1 \cup U_2$ is generally not a subspace. (d) Show that $[U_1 \cup U_2] = U_1 + U_2$. (e) Show that $U_1 \cup U_2 = U_1 + U_2$ if and only if one of the relations $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$ are satisfied.

1.2. U is a vector space and $u_0 \in U$ is a given *fixed* vector. If $\mathbb{F}$ is the field of scalars over which this vector space is defined, then we introduce two new operations # and $*$ that can be interpreted as the vector addition and scalar multiplication as follows

$$u_1 \# u_2 = u_1 + u_2 + u_0, \quad \alpha * u = \alpha u + (\alpha - 1) u_0$$

for all $u_1, u_2, u \in U$ and $\alpha \in \mathbb{F}$. Show that the triple $(U, \#, *)$ is also a linear vector space.

1.3. If n is a positive integer, show that the set $\mathbb{Q}^n$ is a linear vector space over the field of rational numbers $\mathbb{Q}$.

1.4. Construct explicitly three subspaces U_1, U_2 and U_3 of the vector space $\mathbb{R}^3$ such that

$$U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = \{\mathbf{0}\}$$

but $U_1 \cap (U_2 + U_3) \neq \{\mathbf{0}\}$.

1.5. If the subspaces U_1, U_2 and U_3 of a vector space V satisfy the relations $U_1 \cap U_2 = U_1 \cap U_3, U_1 + U_2 = U_1 + U_3$ and $U_2 \subset U_3$, show that we have necessarily $U_2 = U_3$.

1.6. Let us consider the linear vector space of functions differentiable up to the nth order on an open interval $\mathcal{I}$ of $\mathbb{R}$. Show that the necessary and sufficient condition for the set of such functions $\{f_1(x), f_2(x), \dots, f_n(x)\}$ to be linearly independent at the point x is that the following determinant does not vanish at that point

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix} \neq 0.$$

The above determinant $W(x)$ is known as the *Wronskian* of the set of functions $\{f_1(x), f_2(x), \dots, f_n(x)\}$ [after Polish-French mathematician Josef-Maria Hoëné Wronski (1778-1853)].

1.7. Are the functions $\{1, \sin x, \cos x, \sin 2x, \cos 2x\}$ linearly independent ?

1.8. The complex numbers $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ are satisfying the conditions $\alpha_i \neq \alpha_j$ when $i \neq j$. Show that the set of functions $\{e^{\alpha_i x} : i = 1, 2, \dots, n\}$ is linearly independent.

1.9. Show that the set P_n of polynomials with real coefficients whose degrees are less than or equal to n constitute a linear vector space. Is the subset

$$S = \Big\{ p(x) \in P_n : \int_0^1 p(x)\, dx = 0 \Big\} \subseteq P_n$$

a subspace?

1.10. Show that $m \times n$ real matrices constitute a vector space $M_{mn}(\mathbb{R})$ with dimension mn and determine a basis for this space.

1.11. A square matrix satisfying the relation $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$ is ***symmetric*** whereas if it satisfies the relation $\mathbf{A} = -\mathbf{A}^{\mathrm{T}}$ it is ***antisymmetric***. $\mathbf{A}^{\mathrm{T}}$ is the transpose matrix. Show that a symmetric matrix and an antisymmetric matrix of the same order are linearly independent.

1.12. Let U be a finite-dimensional vector space and let $A : U \to U$ be a linear transformation. Show that the following statements are equivalent:

$$\begin{gathered} (i).\ \mathcal{N}(A) \cap \mathcal{R}(A) = \{\mathbf{0}\}. \\ (ii).\ \mathcal{N}(A^2) \subset \mathcal{N}(A). \\ (iii).\ \mathcal{N}(A) \oplus \mathcal{R}(A) = U. \end{gathered}$$

1.13. A is a linear transformation which maps the vector space $V^{(n)}$ into itself. For a given basis $\{e_1, \dots, e_n\}$, let us suppose that the transformation A satisfies the relations

$$Ae_i = e_1 + e_2 + \cdots + e_n,\ \ i = 1, \dots, n$$

What is the value of A at a vector $v = v_1 e_1 + \cdots + v_n e_n$? Find the null space and the range of A.

1.14. U and V are vector spaces and $A : U \to V$ is a linear transformation. Let a finite-dimensional subspace of U be U_1. Show that

$$\dim [A(U_1)] = \dim U_1 - \dim [\mathcal{N}(A) \cap U_1]$$

1.15. Let K be a convex subset of a vector space U. For a finite number of vectors $u_1, u_2, \ldots, u_n$ arbitrarily chosen from the set K and for scalars $\alpha_i \geq 0$, $i = 1, \ldots, n$ obeying the condition $\sum_{i=1}^{n} \alpha_i = 1$, show that their linear combination belongs to K, namely, $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n \in K$. If A is a linear transformation from U into a vector space V, prove that A maps convex sets in U onto convex sets in V.

1.16. Let U, V, W be linear vector spaces and let $A : U \to V$ and $B : V \to W$ be linear transformations. Show that

$$r(BA) \leq \min \{r(A), r(B)\}, \quad n(BA) \leq n(A) + n(B).$$

1.17. Show that V is a zero vector space if and only if the sequence $0 \to V \to 0$ is exact.

1.18. Show that the linear operator $A : U \to V$ is an isomorphism if and only if the sequence $0 \to U \xrightarrow{A} V \to 0$ is exact.

1.19. Let $0 \to U \to V \to W \to 0$ be an exact sequence where U, V, W are finite-dimensional vector spaces. Show that $\dim (V) = \dim (U) + \dim (W)$.

1.20. Let $0 \to V_1 \to V_2 \to \cdots \to V_n \to 0$ be an exact sequence where each V_i is finite-dimensional. Show that $\sum_{i=1}^{n} (-1)^i \dim (V_i) = 0$.

1.21. $\mathcal{T} : U \times V \to \mathbb{F}$ is a bilinear functional. One can define two kernels or null spaces for $\mathcal{T}$: the subspace $\mathcal{N}_U(\mathcal{T}) = \{u \in U : \mathcal{T}(u, v) = 0, \forall v \in V\} \subseteq U$ and the subspace $\mathcal{N}_V(\mathcal{T}) = \{v \in V : \mathcal{T}(u, v) = 0, \ \forall u \in U\} \subseteq V$. $\mathcal{T}$ is called a *non-degenerate* transformation if $\mathcal{N}_U(\mathcal{T}) = \{\mathbf{0}\}$ and $\mathcal{N}_V(\mathcal{T}) = \{\mathbf{0}\}$. We denote quotient spaces of U and V with respect to these subspaces by $U/\mathcal{N}_U(\mathcal{T})$ and $V/\mathcal{N}_V(\mathcal{T})$, respectively. We define a bilinear functional on the Cartesian product of these spaces, i.e., $\mathcal{S} = U/\mathcal{N}_U(\mathcal{T}) \times V/\mathcal{N}_V(\mathcal{T}) \to \mathbb{F}$ by the relation $\mathcal{S}([u], [v]) = \mathcal{T}(u, v)$. Show that the functional $\mathcal{S}$ is non-degenerate.

1.22. $\mathcal{T} : U \times U \to \mathbb{F}$ is a symmetric bilinear functional, i.e., for each $u_1, u_2 \in U$ one has $\mathcal{T}(u_1, u_2) = \mathcal{T}(u_2, u_1)$. Show that $\mathcal{T}$ satisfies the *polarisation identity*

$$4\mathcal{T}(u_1, u_2) = \mathcal{T}(u_1 + u_2, u_1 + u_2) - \mathcal{T}(u_1 - u_2, u_1 - u_2).$$

A real functional $Q : U \to \mathbb{R}$ is called a *quadratic functional* if it satisfies the relation $Q(\alpha u) = \alpha^2 Q(u)$ for all $\alpha \in \mathbb{R}$ and $u \in U$. A quadratic functional is derivable from a symmetric bilinear functional in the following manner

$$Q(u) = \mathcal{T}(u, u), \quad u \in U.$$

Conversely, show that a symmetric bilinear functional can be generated from such a quadratic functional through the relation

$$\mathcal{T}(u_1, u_2) = Q\Big(\frac{u_1 + u_2}{2}\Big) - Q\Big(\frac{u_1 - u_2}{2}\Big) = \frac{1}{4}\big[Q(u_1 + u_2) - Q(u_1 - u_2)\big].$$

1.23. Let U and V be linear vector spaces. Show that the tensor products $U \otimes V$ and $V \otimes U$ are isomorphic vector spaces.

1.24. If $u_i \in U_i, i = 1, 2, \ldots, n$, show that the equality $u_1 \otimes u_2 \otimes \cdots \otimes u_n = 0$ is satisfied if and only if anyone vector is zero, i.e., if $u_i = 0$ for at least one $1 \le i \le n$.

1.25. If $u_i, u_i' \in U_i, i = 1, 2, \ldots, n$, then verify that the equality $u_1 \otimes u_2 \otimes \cdots \otimes u_n = u_1' \otimes u_2' \otimes \cdots \otimes u_n' \neq 0$ is satisfied if and only if $u_i' = \alpha_{\underline{i}} u_{\underline{i}}$, $\alpha_i \in \mathbb{F}$, $\alpha_i \neq 0$, $i = 1, 2, \ldots, n$ such that $\alpha_1 \alpha_2 \cdots \alpha_n = 1$.

1.26. U and V are vector spaces, and $U_1 \subset U$ and $V_1 \subset V$ are subspaces. Verify that the relation $(U_1 \otimes V) \cap (U \otimes V_1) = U_1 \otimes V_1$ is valid.

1.27. U and V are vector spaces, and $U_1, U_2 \subset U$ and $V_1, V_2 \subset V$ are subspaces. Verify that the following relation is valid

$$(U_1 \otimes V_1) \cap (U_2 \otimes V_2) = (U_1 \cap U_2) \otimes (V_1 \cap V_2).$$

1.28. A 2-covariant tensor $\mathcal{T}$ on a vector space $U^{(n)}$ is called a *symmetric tensor* if $\mathcal{T}(u_1, u_2) = \mathcal{T}(u_2, u_1)$ for all $u_1, u_2 \in U$, and an *antisymmetric tensor* if $\mathcal{T}(u_1, u_2) = -\mathcal{T}(u_2, u_1)$. If the set $\{f^i\}$ is a basis for the dual space U^*, we write $\mathcal{T} = t_{ij} f^i \otimes f^j$. Show that the components of the tensor $\mathcal{T}$ must satisfy the conditions $t_{ij} = t_{ji}$ if it is symmetric, and the conditions $t_{ij} = -t_{ji}$ if it is antisymmetric. Show further that these conditions do not depend on the choice of bases in U^*.

1.29. Show that any 2-covariant tensor is expressible *uniquely* as the sum of one symmetric and one antisymmetric tensor.

1.30. Let $\mathcal{T} = t_{ij} f^i \otimes f^j$ and $\mathcal{S} = s_{ij} f^i \otimes f^j$ be symmetric non-zero tensors on a vector space $U^{(n)}$. Show that if components of these tensors satisfy the equality

$$t_{ij} s_{kl} - t_{il} s_{jk} + t_{jk} s_{il} - t_{kl} s_{ij} = 0,$$

then the relation $t_{ij} = (t_{kk}/s_{ll})\, s_{ij}$ is valid. This result is known as *Schouten's theorem* [Dutch mathematician Jan Arnoldus Schouten (1883-1971)].

1.31. The components of a 2-covariant tensor are satisfying the relations

$$\alpha t_{ij} + \beta t_{ji} = 0$$

where $\alpha, \beta \in \mathbb{F}, \alpha, \beta \neq 0$. Show that this tensor must be either symmetric or antisymmetric.

1.32. A 3-tensor $\mathcal{T} = t^i_{kl} e_i \otimes f^k \otimes f^l$ on a vector space $U^{(2)}$ is explicitly given by

$$\mathcal{T} = -e_2 \otimes f^1 \otimes f^2 + 6\,e_1 \otimes f^2 \otimes f^2 - 3\,e_2 \otimes f^2 \otimes f^1.$$

Find all contracted tensors.

1.33. A mixed 3-tensor on a vector space $U^{(2)}$ is given by

$$\mathcal{T} = 3\,e_1 \otimes e_2 \otimes f^1 - e_2 \otimes e_2 \otimes f^2 + e_2 \otimes e_1 \otimes f^2.$$

A new basis for $U^{(2)}$ is determined by transformations

$$e_1' = e_1 - 2e_2,\; e_2' = e_1 + e_2.$$

Find the components of this tensor with respect to the new basis.

1.34. Evaluate the quantities $e_{ijk}e^{lmn}, e_{ijk}e^{imn}, e_{ijk}e^{ijn}, e_{ijk}e^{ijk}$ where the indices take the values $1, 2, 3$.

1.35. Using the definition of generalised Kronecker delta, show that one can write

$$(a).\ \delta^{ij}_{kl}\,\omega_{ij} = \omega_{kl} - \omega_{lk}$$
$$(b).\ \delta^{ijk}_{lmn}\,\omega_{ijk} = \omega_{lmn} - \omega_{lnm} + \omega_{mnl} - \omega_{nml} + \omega_{nlm} - \omega_{mln}.$$

1.36. Find the values of $\delta^{k\,k+1\cdots n\,1\,2\cdots k-1}_{1\,2\cdots n-1\,n}$ and $\delta^{1\;2\;\cdots\;2n-1\;2n}_{1\;n+1\,2\;n+2\;3\;n+3\;\cdots\;n\;2n}$.

1.37. Let the basis and its reciprocal for a vector space $U^{(n)}$ and its dual $U^{*(n)}$ be $\{e_i, f^i, i = 1, \ldots, n\}$, respectively. Then verify that for $1 \le k \le n$, one finds $f^{i_1} \wedge f^{i_2} \wedge \cdots \wedge f^{i_k}(e_{j_1}, e_{j_2}, \ldots, e_{j_k}) = \delta^{i_1 i_2 \cdots i_k}_{j_1 j_2 \cdots j_k}$.

1.38. We consider the following members of the exterior algebra $\Lambda(U^{(4)})$: $\alpha = \alpha_{13}f^1 \wedge f^3 + \alpha_{24}f^2 \wedge f^4$, $\beta = \beta_1 f^1 + \beta_4 f^4$, $\gamma = \gamma_{14}f^1 \wedge f^4 + \gamma_{23}f^2 \wedge f^3$, $\theta = \theta_{123}f^1 \wedge f^2 \wedge f^3 + \theta_{234}f^2 \wedge f^3 \wedge f^4$ where all coefficients are scalars. Evaluate the forms (a) $\alpha \wedge \beta - \beta \wedge \gamma + \theta$, (b) $\alpha \wedge \alpha + 3\gamma \wedge \gamma - 2\theta \wedge \beta$, (c) $\beta \wedge \theta + \alpha \wedge \gamma$.

1.39. Let us consider the forms $\alpha \in \Lambda^2(U), \beta \in \Lambda^1(U)$. Show that one can write

$$(\alpha \wedge \beta)(u_1, u_2, u_3) = \alpha(u_1, u_2)\beta(u_3) - \alpha(u_1, u_3)\beta(u_2) - \alpha(u_2, u_3)\beta(u_1)$$

for all $u_1, u_2, u_3 \in U$.

1.40. If we choose to omit the factor $1/k!$ in the definition (1.4.11) of an exterior form $\omega \in \Lambda^k(U)$, show that the exterior product of such types of forms turns out to be no longer associative.

1.41. Let us consider an exterior form $\omega \in \Lambda^{n-1}(U^{(n)})$, $\omega \neq 0$ on a vector space $U^{(n)}$. Show that the forms α satisfying the equality $\alpha \wedge \omega = 0$ constitute an $(n-1)$-dimensional subspace of $\Lambda(U^{(n)})$ and there exist 1-forms $\alpha^1, \alpha^2, \ldots,$ α^{n-1} such that ω is expressible as $\omega = \alpha^1 \wedge \alpha^2 \wedge \cdots \wedge \alpha^{n-1}$.

1.42. If U is finite-dimensional, then show that the vector spaces $\big(\Lambda^k(U)\big)^*$ and $\Lambda^k(U^*)$ are isomorphic.

1.43. The exterior form $\omega \in \Lambda^3(U^{(4)})$ is given by

$$\omega = a_1 f^1 \wedge f^2 \wedge f^3 + a_2 f^1 \wedge f^2 \wedge f^4 + a_3 f^1 \wedge f^3 \wedge f^4 + a_4 f^2 \wedge f^3 \wedge f^4.$$

Find its rank and its reduced form .