CHAPTER II

DIFFERENTIABLE MANIFOLDS

2.1. SCOPE OF THE CHAPTER

The concept of manifold is essentially propounded to extend the definition of surfaces in classical differential geometry to higher dimensional spaces. This relatively new concept was first introduced into mathematics by German mathematician Friedrich Bernhard Riemann (1826-1866) who was the first one to do extensive work generalising the idea of a surface in a three-dimensional space to higher dimensions. The term manifold is derived from Riemann's original German term, *Mannigfaltigkeit*. This term is translated into English as *manifoldness* by English mathematician William Kingdon Clifford (1845-1879). Riemann's intuitive notion of a Mannigfaltigkeit evolved into what is formalised today as the concept of manifold. German mathematician Herman Klaus Hugo Weyl (1885-1955) gave an intrinsic definition for differentiable manifolds in his lecture course on Riemann surfaces in 1911–1912 at Göttingen University uniting analysis, geometry and topology. However, it was American mathematician Hassler Whitney (1907-1989) who clarified the foundational aspects of differentiable manifolds during the 1930s. Especially, the Whitney embedding theorems provided a firm connection between manifolds and Euclidean spaces.

In Sec. 2.2 we first briefly review topological spaces to which differentiable manifolds also belong. We define fundamental notions and focus on various relevant properties of topological spaces. We then introduce a metric space as a special topological space and finally the Euclidean space that proves to be very important for our investigation. A manifold, also a differentiable manifold, is defined as a topological space that is *locally* equivalent to the Euclidean space. This amounts to say that each point of the manifold belongs to an open set which is homeomorphic to an open set of the Euclidean space. These open sets covering the manifolds are called charts and an atlas is a collection of charts. Certain operations such as differentiation are not allowed on manifolds as topological spaces. However, the local equivalence with the Euclidean space enables us to perform these

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operations on manifolds by means of the Euclidean space on which such operations are carried out quite easily. Although the topological structure of a manifold does not allow us to evaluate directly the derivative of a real-valued function on a manifold we will be able to describe it indirectly in Sec. 2.3 by making use of local charts and well known differentiability in the Euclidean space. We further extend this description to define differentiable mappings between manifolds. In Sec. 2.4 we utilise differentiable mappings to define submersions, immersions and embeddings between manifolds and we discuss various approaches to generate submanifolds via those mappings. Differentiable curves embedded on manifolds are considered in Sec. 2.5. Sec. 2.6 is devoted to the construction of the tangent space of a manifold at a given point as the vector space of all tangent vectors at that point of all differentiable curves through that point which are constructed by employing local images of these curves in the Euclidean space. A more convenient vector space that is isomorphic to the tangent space is introduced as the space of linear operators determined as derivatives of a scalar function in the direction of tangent vectors. In Secs. 2.7 we define the differential of a differentiable mapping between two manifolds as a linear operator mapping a tangent space into another at the corresponding points of manifolds. We show in Sec. 2.8 that the fibre bundle generated by patching all tangent spaces at all points of the manifold can be equipped with a differentiable structure through which we can define a vector field on the manifold. We investigate properties of a mapping called *flow* generated by trajectories of a vector field, namely, by curves tangent to the vector field in Sec. 2.9. The Lie derivative that measures the variation of a vector field on a manifold with respect to another vector field is defined in Sec. 2.10. This derivative, which is also called the Lie product, is utilised to construct a Lie algebra on the tangent space. Finally, in Sec. 2.11 we define a distribution produced by choosing same dimensional subspaces of the tangent spaces at every points of the manifold. It is shown that these elementary fragments of vector subspaces attached to every points of the manifold can be patched together smoothly to form a submanifold if and only if the distribution is involutive, i.e., if its vectors constitute a Lie subalgebra. This is known as the Frobenius theorem.

2.2. DIFFERENTIABLE MANIFOLDS

Let M be a non-empty set. $\mathcal{P}(M)$ denotes the power set of M which is the collection of all subsets of M, the set M itself and the empty set $\emptyset$. Let $\mathfrak{M} \subseteq \mathcal{P}(M)$ be a class of subsets of M. Let us assume that the class $\mathfrak{M}$ satisfies the following *axioms*:

(i). M and $\emptyset$ belong to the class $\mathfrak{M}$.
(ii). The union of *any number* of members of $\mathfrak{M}$ (even uncountably many) belongs to the class $\mathfrak{M}$.
(iii). The intersection of *any finite number* of members of the class $\mathfrak{M}$ belongs to the class $\mathfrak{M}$.

Such a class $\mathfrak{M}$ is called a ***topology*** on the set M. The ordered pair $(M, \mathfrak{M})$ is called a ***topological space***. Unless it causes an ambiguity, a set M endowed with a topology will also be usually called a topological space M. However, we should remark that several topologies may be defined on the same set M generating different topological spaces. We usually name the elements of a topological space as its ***points***. The members of the topology $\mathfrak{M}$ will be called ***open sets*** of M. Therefore a set $U \subseteq M$ is open if and only if $U \in \mathfrak{M}$. If the complement V' of a subset $V \subseteq M$ with respect to M is open, that is, if $V' \in \mathfrak{M}$, then V is called a ***closed set***. Since $M' = \emptyset$ and $\emptyset' = M$, we conclude that the sets M and $\emptyset$ are *both open and closed sets*, simultaneously. Whether the topological space M contains subsets other than those two sets having this property is closely related to the topological concept of *connectedness*. We immediately see that the class of closed set will satisfy the following rules directly obtainable from the familiar de Morgan laws of the set theory: (i) X and $\emptyset$ are closed sets, (ii) the intersection of *any number* of closed sets (even uncountably many) is a closed set, (iii) the union of *any finite number* of closed sets is a closed set.

The ***relative topology*** on a subset $A \subseteq M$ is the class of subsets of A defined by $\mathfrak{M}_A = \{U_A = A \cap U : U \in \mathfrak{M}\}$. It is straightforward to show that $(A, \mathfrak{M}_A)$ is a topological subspace. Indeed, $\emptyset \in \mathfrak{M}$ and $M \in \mathfrak{M}$ implies that $\emptyset = A \cap \emptyset \in \mathfrak{M}_A$ and $A = A \cap M \in \mathfrak{M}_A$. Let us consider a family of subsets $\{V_\lambda \in \mathfrak{M}_A : \lambda \in \Lambda\}$ where Λ is an index set. Then for each $\lambda \in \Lambda$, there exists an open set $U_\lambda \in \mathfrak{M}$ such that $V_\lambda = A \cap U_\lambda$. We thus obtain for the arbitrary union $\bigcup_{\lambda \in \Lambda} V_\lambda = \bigcup_{\lambda \in \Lambda} (A \cap U_\lambda) = A \cap \big(\bigcup_{\lambda \in \Lambda} U_\lambda\big) \in \mathfrak{M}_A$. We now choose a *finite* index set $\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subseteq \Lambda$. Since $\bigcap_{i=1}^{n} U_{\lambda_i} \in \mathfrak{M}$, we eventually obtain $\bigcap_{i=1}^{n} V_{\lambda_i} = \bigcap_{i=1}^{n} (A \cap U_{\lambda_i}) = A \cap \big(\bigcap_{i=1}^{n} U_{\lambda_i}\big) \in \mathfrak{M}_A$. We thus conclude that the class $\mathfrak{M}_A$ complies with the axioms of topology. *It should be noted that the set $U_A \in \mathfrak{M}_A$ may not in general be an open set of* $\mathfrak{M}$. If only A itself is an open set of X, then open sets of relative topology coincide with the open sets of M. Evidently, the closed sets of the relative topology are of the form $A \cap U'_\lambda$.

A subset N_p of M is called a ***neighbourhood*** of the point p if there exists an open set U_p such that $p \in U_p \subseteq N_p$. An ***open neighbourhood*** of the point p is just an open set of M containing p. Let A be a subset of a topological space M. If a point $a \in A$ belongs to an open set contained in A, i.e., if there is a set $U \subseteq A$, $U \in \mathfrak{M}$ such that $a \in U$, then a is an ***interior point*** of the set A. In other words, *if the set A is a neighbourhood of the point $a \in A$, then a is an interior point of A*. We can thus propose at once that *the set $A \subseteq M$ is open if and only if A is a neighbourhood of each of its points.*

In fact, let us first assume that A is open and $a \in A$. Due to the obvious relation $a \in A \subseteq A$, the set A is a neighbourhood of the point a. Now let us suppose that A is a neighbourhood of each of its points. Therefore, for each $a \in A$, there exists an open set U_a such that $a \in U_a \subseteq A$. We next define the open set $V = \bigcup_{a \in A} U_a$. Since $U_a \subseteq A$ for each $a \in V$, we find that $V \subseteq A$. On the other hand, each point of the set A belongs to a set U_a and consequently to V. This implies that $A \subseteq V$. We thus obtain the result $A = V$. Hence the set A is open. □

Collection of all neighbourhoods of a point is called the system of neighbourhoods of that point. If each neighbourhood of a point p contains at least one member of a family of neighbourhoods $\{N_{p\lambda} : \lambda \in \Lambda\}$, where Λ is an index set, then this family is a ***fundamental system of neighbourhoods*** of p. A topological space is called a ***first countable space*** if each of its points has *a countable fundamental system of open neighbourhoods*. The set of all interior points of a set $A \subseteq M$ is called the ***interior*** of A and is denoted by $\mathring{A}$. It is easy to see that *the largest open set contained in A is its interior $\mathring{A}$*. It is rather straightforward to verify that $(A \cap B)^\circ = \mathring{A} \cap \mathring{B}$.

The ***closure*** of a subset $A \subseteq M$ is the intersection of all closed sets containing A. We denote the closure of a set A by $\overline{A}$. Since the intersection of any number of closed sets is also closed, we deduce that $\overline{A}$ is a closed set. Hence, the closure of a set A is then *the smallest* closed set containing A. We can then show the following proposition:

Let A be any non-empty subset of a topological space M. A point $p \in M$ belongs to the closure $\overline{A}$ if and only if the intersection of each neighbourhood of p with A is not empty.

We first consider a point $p \in \overline{A}$ and assume that there exists a particular open neighbourhood $U_p \in \mathfrak{M}$ of p such that $U_p \cap A = \emptyset$. We thus have $A \subseteq U_p'$. But, since U_p' is closed we conclude that $\overline{A} \subseteq U_p'$. Therefore, we reach to the contradiction that the point p belongs to both U_p and U_p'. Consequently, we ought to take $U_p \cap A \neq \emptyset$. Hence, every open neighbourhood of each point in the closure of the set A must intersect A. Now, conversely, we assume that the intersection of each open neighbourhood of a point $p \in M$ with A is not empty, but p does not belong to $\overline{A}$, that is, for all $U_p \in \mathfrak{M}$ we should have $U_p \cap A \neq \emptyset$, $p \notin \overline{A}$ so that the point p has to belong to the open set $\overline{A}'$. Consequently, there must exist an open neighbourhood U_0 of p such that $U_0 \subseteq \overline{A}'$. This open set U_0 cannot intersect A and this gives rise to a contradiction so that $p \in \overline{A}$. Hence we are led to define the closure of a set A as the set $\overline{A} = \{p \in M : U_p \cap A \neq \emptyset$ for all $U_p \in \mathfrak{M}\}$. □

It can easily be verified that $\overline{A \cup B} = \overline{A} \cup \overline{B}$ and if $A \subseteq B$, then one deduce at once that $\overline{A} \subseteq \overline{B}$.

The ***boundary of a subset*** $A \subseteq M$ is defined by $\partial A = \bar{A} - \mathring{A} = \bar{A} \cap (\mathring{A})'$. The boundary ∂A of a set A is always closed since it is described by the intersection of two closed sets.

Let A and B be two subsets of a topological space M. If $B \subseteq \overline{A}$, then we say that A is a ***dense set*** in B. On the other hand, if $B = \overline{A}$, then A is called an ***everywhere dense set*** in B. When $B = M$, a set A which is dense in M naturally has to satisfy the relation $M = \overline{A}$. Therefore, a set dense in M is always an everywhere dense set in M. A topological space M is called a ***separable space*** if it possesses a *countable* dense subset $A = \{p_1, p_2, \ldots p_n, \ldots\}$so that one gets $\overline{A} = M$.

A topological space M is called a ***Hausdorff space*** if each pair of its distinct points p_1, p_2 have *disjoint* neighbourhoods, that is, if $p_1, p_2 \in M$ such that $p_1 \neq p_2$, then there exist open sets U_1 and U_2 so that $p_1 \in U_1$, $p_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$ [after German mathematician Felix Hausdorff (1869-1942)].

Let M be a Hausdorff space. If $p \in M$, then the singleton $\{p\}$, i.e., the set of just the single point p is a closed set.

To observe this, let use take any point $q \in \{p\}' = M - \{p\}$. Since $q \neq p$, there are disjoint open sets $U_p, U_q \in \mathfrak{M}$ such that $U_p \cap U_q = \emptyset$. Therefore, the open set U_q does not contain the point p and we get $U_q \subset \{p\}'$ implying that the point q is an interior point of the set $\{p\}'$. Since all points of the set $\{p\}'$ are interior points, it is open and therefore the set $\{p\}$ *is closed.* □

A subclass $\mathfrak{N}$ of the power set $\mathcal{P}(M)$ is a ***basis*** for a topological space $(M, \mathfrak{M})$ (the term ***open basis*** will, in fact, be more appropriate) if every open set in the topology $\mathfrak{M}$ is expressible as a union of some sets in $\mathfrak{N}$. Elements of $\mathfrak{N}$ are called ***basic open sets***. If we are given a class of subsets $\mathfrak{N} \subset \mathcal{P}(M)$ satisfying naturally the condition $M = \cup\{N\}$ where $N \in \mathfrak{N}$, we cannot usually generate a topology on M by considering all unions of subsets in $\mathfrak{N}$ because the intersection axiom of the topology does not hold in general. It is rather straightforward to see that the necessary and sufficient condition for a class of subsets $\mathfrak{N}$ of a set M to constitute a basis for a topology are provided as follows:

A subclass $\mathfrak{N} \subset \mathcal{P}(M)$ with the condition $M = \cup\{N : N \in \mathfrak{N}\}$ is a basis for a topology on M if and only if for any two sets $N_1, N_2 \in \mathfrak{N}$ and any point $p \in N_1 \cap N_2$, there exists a set $N_3 \in \mathfrak{N}$ such that $p \in N_3 \subseteq N_1 \cap N_2$. □

For instance in a topology on $\mathbb{R}$ basic open sets are open intervals. It is shown in real analysis that every open sets in $\mathbb{R}$ is expressible as a *countable* union of open intervals. A topological space M possessing a *countable basis* is called a ***second countable space***. Such a topological space enjoys several pleasant and rather remarkable properties. For instance a *second countable space is a separable space.* This property is quite easy to show. Let $\mathfrak{N}$ be a countable basis for a topological space M. We choose a point $p_N \in N$ in each non-empty set $N \in \mathfrak{N}$ and then introduce the subset $D = \{p_N : N \in \mathfrak{N}\}$ of M. D is obviously a countable set. Since there is a member of the basis, and consequently, a point of D, in every neighbourhood of each point of M, the countable set D would be dense in X.

Compactness. A ***cover*** $\mathcal{A}$ of a set X is a collection of some subsets of X whose union is X, that is, $\mathcal{A} = \{U_\lambda \subseteq X : \lambda \in \Lambda\}$ where Λ is an index set is a cover of X if and only if $X = \underset{\lambda\in\Lambda}{\cup} U_\lambda$. If a subclass $\mathcal{B}$ of $\mathcal{A}$ is also a cover of X, then $\mathcal{B}$ is a ***subcover*** of X. A cover $\mathcal{A}$ is an ***open cover*** of a topological space M if all members of $\mathcal{A}$ are open sets. *If every open cover $\{U_\lambda \subseteq M : \lambda \in \Lambda\}$ of a topological space M has a finite subcover*, namely, if one is able to write $M = \overset{n}{\underset{i=1}{\cup}} U_{\lambda_i}, \lambda_i \in \Lambda$ where n is finite integer, then M is a ***compact topological space***. *Compactness of a subspace of M is naturally defined with respect to its relative topology*.

We can show that closed subspaces of compact topological spaces are also compact.

Let M be a topological space and $A \subset M$ be a closed subspace. We consider an arbitrary open cover $\{V_\lambda\}_{\lambda\in\Lambda}$ of A. We know that $V_\lambda = U_\lambda \cap A$ where $\{U_\lambda\}_{\lambda\in\Lambda}$ is a class of open sets in M. Since A' is open, the class $\{U_\lambda, A' : \lambda \in \Lambda\}$ is an open cover of the space M. Since M is compact this cover must have a finite subcover $\{U_{\lambda_i}, A' : \lambda_i \in \Lambda, i = 1, 2, \ldots, n\}$ so that one can write $A' \cup U_{\lambda_1} \cup \cdots \cup U_{\lambda_n} = M$. Since $M = A \cup A'$, we conclude that $A \subseteq U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}$ and finally $A = V_{\lambda_1} \cup \cdots \cup V_{\lambda_n}$. This means that A is compact. □

In Hausdorff spaces the converse of the above statement is also valid. *Let M be a Hausdorff space and let $A \subset M$ be a compact subspace. Then A is closed.*

In order to prove this proposition, we have to show that A' is an open set. We take a point $p \in A'$. Since M is a Hausdorff space, for any point $a \in A$, we can find disjoint open sets $U_{p,a}$ and U_a containing the points p and a, respectively. The class $\{U_a \cap A : a \in A\}$ is an open cover of A in relative topology. But A is compact, hence there is a finite set $\{a_1, \dots, a_n\} \subset A$ such that $A = \bigcup_{i=1}^{n} U_{a_i} \cap A \subseteq \bigcup_{i=1}^{n} U_{a_i}$. It is now clear that the finite intersection $U = \bigcap_{i=1}^{n} U_{p,a_i}$ is an open neighbourhood of the point p and $U \cap A = \emptyset$. We thus obtain $U \subset A'$. Hence the arbitrary point p is an interior point of A', i.e., A' is open and A is closed. We can now easily deduce the following corollary: *if M is a compact Hausdorff space, then a subspace is compact if and only if it is closed*. □

A subspace of a topological space M is called ***relatively compact*** if its closure is compact. A topological space each point of which admits a compact neighbourhood is called a ***locally compact space***. *If M is a locally compact Hausdorff space, we can replace the term "compact neighbourhood" by "relatively compact neighbourhood"*. Indeed, let the point $p \in M$ admit the compact neighbourhood N. Since M is a Hausdorff space, N is closed. On the other hand, the relation $\mathring{N} \subset N$ implies that $\overline{\mathring{N}} \subset N$. $\overline{\mathring{N}}$ is a closed subset of a compact set. Therefore, it is compact. Hence. p has an open neighbourhood with a compact closure.

A useful generalisation of compactness is ***paracompactness***. This concept was introduced in 1944 by French mathematician Jean Alexander Eugène Dieudonné (1906-1992). Let $\mathcal{A} = \{U_\lambda \subseteq M : \lambda \in \Lambda\}$ be a class of subsets of a space M. Another class of subsets $\mathcal{B} = \{V_\gamma \subseteq M : \gamma \in \Gamma\}$ is called a ***refinement*** of class $\mathcal{A}$ if and only if for any $V_\gamma \in \mathcal{B}$ there exists a $U_\lambda \in \mathcal{A}$ such that $V_\gamma \subseteq U_\lambda$. An open cover $\mathcal{A}$ of a topological space M is called ***locally finite*** if every point $p \in M$ has a neighbourhood that intersects only finitely many sets in the cover. In other words $\mathcal{A} = \{U_\lambda \subseteq M\}$ is locally finite if each point $p \in M$ has a neighbourhood $V(p)$ such that the set $\{\lambda \in \Lambda : V(p) \cap U_\lambda \neq \emptyset\}$ is finite. *M is a **paracompact space** if any open cover of M admits an open refinement that is locally finite*. It is obvious that every compact space is also paracompact.

It can be shown that a locally compact, second countable Hausdorff space M is paracompact.

Let $\{V_i : i \in \mathbb{N}\}$ where $\mathbb{N}$ denotes the set of natural numbers be a countable basis for M. We shall first form a countable basis with compact closure. By our assumption, there exists a relatively compact open set U_p containing a point $p \in M$. Since U_p is expressible as union of some basic open sets, there is a set V_{i_p} such that $p \in V_{i_p}$ and $V_{i_p} \subseteq U_p$ whence we obtain $\overline{V}_{i_p} \subseteq \overline{U}_p$. But $\overline{U}_p$ is compact. Being a closed subset of a compact set, $\overline{V}_{i_p}$

is also compact. Therefore $\{V_{i_p} : p \in M\} \subseteq \{V_i : i \in \mathbb{N}\}$ is a countable relatively compact basis. Let us now suppose that $\{U_i\}$ is such a basis. Next, we construct inductively a sequence of *nested* open sets $\{W_i\}$ with the following properties: (i) $\overline{W}_i$ is compact, $(ii)\, W_i \subset \overline{W}_i \subset W_{i+1}$, (iii) $M = \bigcup\limits_{i=1}^{\infty} W_i$. We further adopt the convention that $W_0 = \emptyset$. We take $W_1 = U_1$. Hence, $\overline{W}_1 = \bar{U}_1$ is compact. We now introduce the open set

$$W_k = U_1 \cup U_2 \cup \cdots \cup U_{j_k} = \bigcup_{i=1}^{j_k} U_i$$

Since $\overline{W}_k = \overline{\bigcup\limits_{i=1}^{j_k} U_i} = \bigcup\limits_{i=1}^{j_k} \bar{U}_i$ is a finite union of compact sets, it is also compact. So it must be covered by finitely many elements of the open cover $\{U_i\}$. We then take the index j_{k+1} as the least positive integer greater than the index j_k so that one is able to write

$$\overline{W}_k \subseteq \bigcup_{i=1}^{j_{k+1}} U_i.$$

We then define the next member of the sequence as

$$W_{k+1} = \bigcup_{i=1}^{j_{k+1}} U_i.$$

This completes the construction of the sequence $\{W_i\}$. The property (iii) is then satisfied automatically. Let $\{U_\lambda : \lambda \in \Lambda\}$ be an arbitrary open cover of M. The set $K_i = \overline{W}_i - W_{i-1} = \overline{W}_i \cap W'_{i-1}$ is compact since it is a closed subset of the compact set $\overline{W}_i$. We obviously get $K_1 = \overline{W}_1$. On the other hand, properties of the sequence imply that K_i is contained in open set $Z_i = W_{i+1} - \overline{W}_{i-2} = W_{i+1} \cap \overline{W}'_{i-2}$. For $i \geq 3$, we can choose a finite subcover of the open cover $\{U_\lambda \cap Z_i : \lambda \in \Lambda\}$ of the compact set K_i. For the compact set $K_2 = \overline{W}_2 - W_1$, we choose a finite subcover of the open cover $\{U_\lambda \cap W_3 : \lambda \in \Lambda\}$. Similarly, the compact set K_1 will be covered by a finite subcover of $\{U_\lambda \cap W_2 : \lambda \in \Lambda\}$. Because of the relation $W_i \subset \overline{W}_i$, we get $W_i - W_{i-1} \subset \overline{W}_i - W_{i-1} = K_i$. Since the sequence $\{W_i\}$ is nested, we obviously obtain

$$M = \bigcup_{i=1}^{\infty} W_i = \bigcup_{i=1}^{\infty} (W_i - W_{i-1}) \subset \bigcup_{i=1}^{\infty} K_i$$

implying that $M = \bigcup\limits_{i=1}^{\infty} K_i$ where each K_i is covered by finitely many members of the open cover $\{U_\lambda \cap Z_i : \lambda \in \Lambda\}$. It is straightforward to see that this open cover is a locally finite, countable refinement which consists of a countable union of finite unions. Hence, M is a paracompact space. □

Let us consider topological spaces $(M, \mathfrak{M})$ and $(N, \mathfrak{N})$. It is a simple exercise to see that we can endow the Cartesian product $M \times N$ with a topology by choosing its open sets as unions of *elementary open sets* $U \times V$ where $U \in \mathfrak{M}$, $V \in \mathfrak{N}$. Such a topology on $M \times N$ is called the ***product topology***. This definition may be, of course, extended to Cartesian product of any number of topological spaces. For instance, in $\mathbb{R}^n$ the elementary open sets are ***open n-rectangles*** obtained as Cartesian products $(a_1, b_1) \times \cdots \times (a_n, b_n)$ of open intervals in $\mathbb{R}$. It is easy to see that $\mathbb{R}^n$ *is a second countable topological space* because it has a countable basis that is the collection of all Cartesian products $\prod_{i=1}^{n}(a_i, b_i)$ where $(a_i, b_i) \in \mathbb{R}$ is an open interval with *rational* end points.

$(M, \mathfrak{M})$ and $(N, \mathfrak{N})$ are topological spaces. *The function $f : M \to N$ is **continuous at the point** $p_0 \in M$ if for each neighbourhood V of the image point $f(p_0) \in N$, there exists a neighbourhood U of the point p_0 such that $f(U) \subseteq V$.* Another completely equivalent definition may be given as follows: *the function f is continuous at a point p_0 if the inverse image $f^{-1}(V)$ of every neighbourhood V of the point $f(p_0)$ is a neighbourhood of the point p_0*. Indeed, if the set U is a neighbourhood of p_0 satisfying the relation $f(U) \subseteq V$, we immediately get $U \subseteq f^{-1}(f(U)) \subseteq f^{-1}(V)$. Conversely, suppose that the set $f^{-1}(V)$ is a neighbourhood of p_0. If we write $U = f^{-1}(V)$, we find that $f(U) = f(f^{-1}(V)) \subseteq V$.

A function $f : M \to N$ is ***continuous*** on M if it is continuous at every point of its domain. *We can easily show that f is a continuous function if and only if the inverse image of every open set in N is an open set in M, i.e., if $f^{-1}(V) \in \mathfrak{M}$ for all $V \in \mathfrak{N}$.*

Let f be a continuous function. Consider an arbitrary open set $V \in \mathfrak{N}$ and define the set $U = f^{-1}(V) \subseteq M$. Let p be a point of U. We obviously have $f(p) \in V$. Since V is an open set, $f(p)$ is an interior point of V. Thus, there exists an open set $V_{f(p)}$ such that $f(p) \in V_{f(p)} \subseteq V$. Due to the continuity of f, the set $f^{-1}(V_{f(p)}) \subseteq U$ is a neighbourhood of p. Hence, there exists an open set $U_p \in \mathfrak{M}$ such that $p \in U_p \subseteq U$. All points of U are, therefore, interior points, that is, U is an open set. Conversely, let us now assume that for all $V \in \mathfrak{N}$, we have $f^{-1}(V) \in \mathfrak{M}$. Consider an arbitrary point p in M and assume that $f(p) \in V \in \mathfrak{N}$. The set $U = f^{-1}(V)$ is an open neighbourhood of the point p. Consequently, f is continuous at all points of M. □

It is not too difficult to demonstrate that the following definitions for the continuity of functions are equivalent:

(a). *The function f is continuous.*
(b). *The inverse image of every open set is open.*
(c). *The inverse image of every closed set is closed.*
(d). *For every subset $B \subseteq N$, the relation $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ is satisfied.*
(e). *For every subset $A \subseteq M$, the relation $f(\overline{A}) \subseteq \overline{f(A)}$ is satisfied.*

It is evident from the definition of the continuity that the composition of continuous functions is also a continuous function.

One can easily demonstrate that images of compact sets are also compact under continuous functions. We thus have to prove that *if $f : M \to N$ is a continuous function from a compact space M into a topological space N, then the set $f(M) \subseteq N$ is a compact subspace*.

We assume that the class $\{V_\lambda\}_{\lambda\in\Lambda}$ is an arbitrary open cover of the range $f(M) \subseteq N$ in its relative topology. We know that its members are in the form $V_\lambda = U_\lambda \cap f(M)$ where U_λ are open sets in N. Obviously, the class $\{f^{-1}(V_\lambda)\}_{\lambda\in\Lambda}$ is a cover of M implying that $M = \bigcup_{\lambda\in\Lambda} f^{-1}(V_\lambda) = \bigcup_{\lambda\in\Lambda} f^{-1}\big(U_\lambda \cap f(M)\big) = \bigcup_{\lambda\in\Lambda} f^{-1}(U_\lambda) \cap f^{-1}\big(f(M)\big) = \bigcup_{\lambda\in\Lambda} f^{-1}(U_\lambda) \cap M = \bigcup_{\lambda\in\Lambda} f^{-1}(U_\lambda)$. The continuity of f requires that the class $\{f^{-1}(U_\lambda)\}_{\lambda\in\Lambda}$ is an open cover of M and must have a finite subcover since M is compact. We thus obtain $M = \bigcup_{i=1}^{n} f^{-1}(U_{\lambda_i}), \lambda_i \in \Lambda, i = 1, \dots, n$, and hence, we find that $f(M) = \bigcup_{i=1}^{n} f\big(f^{-1}(U_{\lambda_i})\big) \subseteq \bigcup_{i=1}^{n} U_{\lambda_i}$. The class $\{V_{\lambda_i} = U_{\lambda_i} \cap f(M)\}$ is a finite subcover of $f(M)$ in its relative topology since one can clearly write $f(M) = \bigcup_{i=1}^{n} V_{\lambda_i}$. Therefore, $f(M)$ is a compact subspace of N. □

We can then deduce the following corollary: *if a bijective function $f : M \to N$ from a compact space M into a Hausdorff space N is continuous, then the inverse function $f^{-1} : N \to M$ is also continuous.*

In order to prove that the function f^{-1} is continuous, it would be sufficient to show that the image $f(A)$ in N of an arbitrary closed set A in M is also closed. Since A is closed, it must be a compact subspace of M. Since f is a continuous function $f(A)$ will be a compact subspace of N. Hence $f(A)$ is closed. □

Since topologies are governed by open sets, it is evident that in order to establish a ***topological equivalence*** between two topological spaces, it would be sufficient to be able to transform open sets in one space to open sets in the other. This mapping must be bijective to ensure numerical equivalence. If $h : M \to N$ is a continuous bijective mapping, then the inverse images of open sets in N would be open in M. If the inverse mapping $h^{-1} : Y \to X$ is continuous as well, then the images of open sets in M will be open in N. A bijective mapping $h : M \to N$ between topological spaces $(M, \mathfrak{M})$ and $(N, \mathfrak{N})$ is called a ***homeomorphism*** if both h and h^{-1} are continuous. Such topological spaces M and N are said to be ***homeomorphic***. We thus conclude that two spaces are topologically equivalent if we can show that there exists a homeomorphism between them. If h is a homeomorphism, then we get $h(U) \in \mathfrak{N}$ for all $U \in \mathfrak{M}$ and, conversely, $h^{-1}(V) \in \mathfrak{M}$ for all $V \in \mathfrak{N}$. It can, therefore, be said that a homeomorphism is an ***open***, continuous and bijective mapping. A property which remains invariant under a homeomorphism is called a ***topological property***, namely, a topological property observed in a topological space remains unchanged in all homeomorphic images of this space. For instance, we see at once that Hausdorff property is a topological property. It is quite obvious that the inverse of a homeomorphism or a composition of two homeomorphisms are also homeomorphisms. It is not difficult to observe that the set of all homeomorphisms of a topological space onto itself equipped with a binary operation defined as the composition of two homeomorphisms constitute a group with respect to this operation.

In the light of the above statements we can conclude at once that *if the function $f : M \to N$ from a compact space M onto a Hausdorff space N is continuous and bijective, then the mapping f is a homeomorphism.* In this case, N must clearly be a compact space as well.

Let R be an equivalence relation on a topological space $(M, \mathfrak{M})$ [*see p*. 5]. We know that the set $[p]$ consisting of all points that are related to $p \in M$ through R is an equivalence class. Each point in the set $[p]$ generates the same equivalence class. Thus distinct equivalence classes are disjoint sets. They form a partition of the set M. The set $M/R = \{[p] : p \in M\}$ has already been called the *quotient set*. Therefore, to each point p in the set M there corresponds a unique equivalence class $[p]$ in the set M/R, that is, there is a function $\pi : M \to M/R$ such that $\pi(p) = [p]$. π is called a ***canonical*** or ***natural projection***. It is evident that the canonical mapping π is surjective, but it is also clear that it is not injective. We now define a class of subsets of M/R by

$$\mathfrak{M}_R = \{V \in \mathcal{P}(M/R) : \pi^{-1}(V) \in \mathfrak{M}\}.$$

It is easily seen that this class is a topology on M/R. The relations $\emptyset = \pi^{-1}(\emptyset) \in \mathfrak{M}$ and $M = \pi^{-1}(M/R) \in \mathfrak{M}$ mean that $\emptyset \in \mathfrak{M}_R$ and $M/R \in \mathfrak{M}_R$. Let us now consider a family of sets $\{V_\lambda : \lambda \in \Lambda\} \subseteq \mathfrak{M}_R$ where Λ is an index set. Our definition implies that $U_\lambda = \pi^{-1}(V_\lambda) \in \mathfrak{M}$ so that one can write

$$\pi^{-1}(\underset{\lambda\in\Lambda}{\cup} V_\lambda) = \underset{\lambda\in\Lambda}{\cup} \pi^{-1}(V_\lambda) = \underset{\lambda\in\Lambda}{\cup} U_\lambda \in \mathfrak{M}.$$

Thus one has $\underset{\lambda\in\Lambda}{\cup} V_\lambda \in \mathfrak{M}_R$. Let $\{V_{\lambda_i} : 1 \leq i \leq n\} \subseteq \mathfrak{M}_R$ be finite family. Because of the relation $\pi^{-1}(\overset{n}{\underset{i=1}{\cap}} V_i) = \overset{n}{\underset{i=1}{\cap}} \pi^{-1}(V_i) = \overset{n}{\underset{i=1}{\cap}} U_i \in \mathfrak{M}$, we obtain $\overset{n}{\underset{i=1}{\cap}} V_i \in \mathfrak{M}_R$. Hence, $\mathfrak{M}_R$ is a topology and the pair $(M/R, \mathfrak{M}_R)$ is a topological space. We call $\mathfrak{M}_R$ ***the quotient topology*** and $(M/R, \mathfrak{M}_R)$ ***the quotient space***. It is quite clear that through the topology so defined the canonical projection π is rendered continuous.

Certain topological spaces possess quite a useful property called *the partition of unity*.

Partition of Unity. Let M be a topological space and $\{V_i : i \in \mathcal{I}\}$, where $\mathcal{I}$ is an index set, be a locally finite open cover of M. Hence, we have $M = \underset{i\in\mathcal{I}}{\cup} V_i$ and every point $p \in M$ has an open neighbourhood U_p whose intersection with only finitely many members of the cover is not empty. If a family of continuous functions $f_i : M \to [0, 1]$ satisfies the conditions

$$(i).\ \textit{supp}\,(f_i) \subset V_i \text{ for each index } i,$$

$$(ii).\ \sum_{i\in\mathcal{I}} f_i(p) = 1 \text{ for each } p \in M$$

then the family of ordered pair $\{V_i, f_i\}$ is called a partition of unity. Here the ***support of a function*** $f : M \to \mathbb{R}$ is defined as the *closed set*

$$\textit{supp}\,(f) = \overline{\{p \in M : f(p) \neq 0\}} = \overline{f^{-1}(\mathbb{R} - \{0\})} \subseteq M.$$

Since the family $\{V_i\}$ is locally finite there are only finitely many, say N number of non-empty open sets $V_i \cap U_p$ containing a point p. Consequently, $f_i(p) \neq 0$ only for a finite N number of functions f_i so that at any point p the sum $\sum_{i\in\mathcal{I}} f_i(p)$ must contain only finitely many terms and one can write $\sum_{k=1}^{N(p)} f_{i_k}(p) = 1,\ \{i_1, \ldots, i_N\} \subset \mathcal{I}$. Naturally the number $N < \infty$ may be dependent on the position points of M.

Let M be a topological space on which there exists a partition of unity $\{V_i, f_i\}$ as defined above and let the family $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover of M. *If for each member V_i of locally finite open cover one can find an open set U_{λ_i} such that supp $(f_i) \subset U_{\lambda_i}$ then we say that the partition of unity $\{V_i, f_i\}$ is subordinate to the open cover $\{U_\lambda : \lambda \in \Lambda\}$.*

As we shall see later in dealing with integration on manifolds, the existence of a partition of unity on a topological space will prove to be very effective in reducing certain global properties to some local properties.

Connectedness. If a topological space $(M, \mathfrak{M})$ cannot be expressible as the union of two non-empty disjoint open sets, that is, if $M \neq U_1 \cup U_2$; $U_1, U_2 \in \mathfrak{M}$, $U_1 \cap U_2 = \emptyset$, we say that it is a ***connected space***. Conversely, if there exist such open sets U_1 and U_2 so that $M = U_1 \cup U_2$, then M is a ***disconnected space***. In a disconnected space we naturally have $U_1' = U_2$ and $U_2' = U_1$. Hence the sets U_1 and U_2 are both open and closed sets in topology $\mathfrak{M}$ whence we conclude that a topological space is connected if it cannot be expressed as the union of two disjoint closed sets. It is straightforward to see that a space M being connected means that only the sets $\emptyset$ and M are both open and closed. Indeed, if M possesses a proper subset A that is both open and closed, then its complement A' ought to be both open and closed. Since $M = A \cup A'$ and $A \cap A' = \emptyset$, M becomes expressible as the union of two disjoint open or closed sets. Hence, M is a disconnected space.

A ***connected subspace*** of a topological space M is a subspace $A \subset M$ that is connected with respect to its relative topology. According to this definition, a subset A is connected if it cannot be contained in the union of two open sets of M whose intersections with A are disjoint and non-empty.

It is almost straightforward to show that the image of a connected space under a continuous function between two topological spaces is also connected.

Another concept of connectedness which is not entirely equivalent to the one described above may be introduced by resorting to a more geometrical approach. Let M be a topological space and $\mathcal{I} = [0, 1] \subset \mathbb{R}$ in which the topology is determined by open intervals. A ***path***, or an ***arc*** on the space M is defined as the continuous mapping $\phi : \mathcal{I} \to M$. We say that ϕ joins the points p_1 and p_2 in M if $\phi(0) = p_1$ and $\phi(1) = p_2$. If $\phi(t) \in M$ for every $t \in [0, 1]$, then *the path ϕ stays in the space M*. If any two points in the space M can be joined by a path staying in M, then M is called a ***path-connected*** or an ***arc-connected*** space. If this property is valid for a subspace of M, then this subspace is path-connected. Such a space is schematically described in Fig. 2.2.1.

If M is path-connected, N is a topological space and $f : M \to N$ is a continuous mapping, then we immediately deduce from the fact that composition of continuous mappings is also continuous, the subspace $f(M)$ is path-connected as well. *If a topological space M is path-connected, then it is also connected.* However, the converse statement is generally not true.

When $\phi(0) = \phi(1) = p_1$, we say that the path is *closed*. If every closed path in the space M can be contracted continuously to a point inside the path, the space M is called ***simply connected***. Equivalently, we say that a connected topological space M is simply connected if a path connecting any two points of M can be continuously deformed into every other curve connecting these two points.

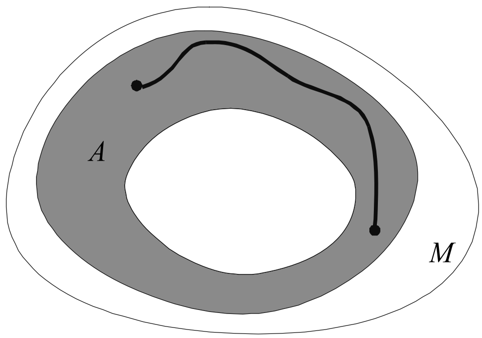

Fig. 2.2.1. A path-connected subspace.

Metric Spaces. A topology on a set M can be defined sometimes by means of a real-valued function. Let M be a non-empty set. Let us suppose that we can define a real-valued, non-negative function $d : M \times M \to \mathbb{R}^+$ on this set. We further impose the following conditions on the function d:

(i). *For each* $p_1, p_2 \in M$ *one has* $d(p_1, p_2) \geq 0$.
(ii). $d(p_1, p_2) = 0$ *if and only if* $p_1 = p_2$.
(iii). *For each* $p_1, p_2 \in M$ *one has* $d(p_1, p_2) = d(p_2, p_1)$.
(iv). *For each* $p_1, p_2, p_3 \in M$ *one has* $d(p_1, p_2) \leq d(p_1, p_3) + d(p_3, p_2)$.

The inequality (iv) above is known as *the triangle inequality*. We call such a function $d(p_1, p_2)$ a ***metric*** on the set M and we interpret its value as the ***distance*** between two points p_1 and p_2 of the set M. In fact, we can easily verify that the metric concept coincides entirely with the familiar distance concept in the Euclidean space. The pair (M, d) is called a ***metric space***. The ***open ball*** of radius r centred at the point $p \in M$ is defined as the set

$$B_r(p) = \{p_1 \in M : d(p, p_1) < r\} \subset M. \tag{2.2.1}$$

We can generate a topology on a metric space called ***metric topology*** *by noting that open balls constitute a basis for a topology*. Consider a class of subsets $\mathfrak{B}_d = \{B_r(p) : p \in M, r \geq 0\}$ of the set M. It is evident that $M = \cup \{B_r(p) : p \in M, r > 0\}$. $\emptyset \in \mathfrak{B}_d$ since $B_0(p) = \emptyset$. In order to show that the class $\mathfrak{B}_d$ is in fact a basis for a topology on M, all we have to do is to demonstrate that any point in the intersection of two open balls belongs to an open ball contained in that intersection. Let us consider two open balls centred at the points p_1 and p_2 with radii r_1 and r_2, respectively. If their intersection is empty, the criterion is automatically satisfied. Hence, we assume that the intersection of these open balls is not empty and take a point p in their intersection $B_{r_1}(p_1) \cap B_{r_2}(p_2)$ into consideration. Hence we can write $d(p_1, p) < r_1$ and $d(p_2, p) < r_2$. Let us now choose

$$r = \min \{r_1 - d(p_1, p), r_2 - d(p_2, p)\} > 0.$$

The open ball $B_r(p)$ is contained both in the sets $B_{r_1}(p_1)$ and $B_{r_2}(p_2)$. For an arbitrary point $q \in B_r(p)$ the triangle inequality yields $d(p_1, q) \leq d(p_1, p) + d(p, q) < r_1 - r + r = r_1$ implying that $q \in B_{r_1}(p_1)$. In the same fashion, we obtain this time $d(p_2, q) \leq d(p_2, p) + d(p, q) < r_2 - r + r = r_2$ and $q \in B_{r_2}(p_2)$. We thus find that $B_r(p) \subset B_{r_1}(p_1) \cap B_{r_2}(p_2)$. Consequently, the class $\mathfrak{B}_d$ constitutes a basis for a topology on M. Each open set of this topology is given by unions of some open balls, that is, if U is an open set, then it is expressible as $U = \bigcup_{p \in U} B_{r(p)}(p)$ for some $r(p)$. The set

$$B_r[p] = \{p_1 \in M : d(p, p_1) \leq r\} \subset M \tag{2.2.2}$$

is called a ***closed ball*** with centre $p \in M$ and radius r. It is easy to verify that $B_r[p]$ is a closed set. It can easily be observed that $\overline{B_r(p)} \subseteq B_r[p]$. Let us consider all open balls centred at a point whose radii are rational numbers. We immediately observe that these open balls constitute a countable fundamental system of neighbourhoods of that point. Therefore, *metric spaces are first countable spaces*.

Metric spaces has quite a distinctive property. *They are all Hausdorff spaces.* Indeed, if we consider two distinct points of a metric space M, we must have $d(p, q) = r_1 > 0$ whenever $p \neq q$. By choosing $r \leq r_1/2$, one easily demonstrates that it is always possible to find two open balls with radius $r > 0$ such that $B_r(p) \cap B_r(q) = \emptyset$.

Let us consider a sequence of points $\{p_n\} \subset M$. This sequence converges to a point $p \in M$, if there exists a natural number $N(\epsilon)$ for each $\epsilon > 0$ such that $d(p_n, p) < \epsilon$ whenever $n \geq N(\epsilon)$. The sequence $\{p_n\}$ is called a ***Cauchy sequence*** [French mathematician Augustin-Louis Cauchy (1789-1857)] if to each $\epsilon > 0$ there corresponds a natural number $N(\epsilon) \in \mathbb{N}$ such that $d(p_m, p_n) < \epsilon$ whenever $m, n \geq N$. If every Cauchy sequence in a metric space is convergent, then we say that this metric space is ***complete***. It can be shown that a *subspace of a complete metric space is complete if and only if it is closed* .

It can also be proven that metric spaces are paracompact spaces *though we have to omit its difficult proof because it is beyond our scope.*

Let A be a subset of a metric space. *The diameter of* A is defined by the non-negative number $D(A) = \sup_{p_1, p_2 \in A} d(p_1, p_2)$. If $D(A) < \infty$, then A is a *bounded set*. Obviously open and closed balls of radius r are bounded and their diameters are both $2r$.

The standard metric on the set of real numbers is $d(x, y) = |x - y|$. Let us now consider the set $\mathbb{R}^n$. If $x \in \mathbb{R}^n$, then $x = (x^1, x^2, \ldots x^n)$ is an ordered n-tuple of real numbers where $x^i \in \mathbb{R}$, $i = 1, 2, \cdots, n$. Next, we define the function

$$d(x, y) = \Big(\sum_{i=1}^{n} |x^i - y^i|^2\Big)^{1/2} \tag{2.2.3}$$

for a pair of points $x, y \in \mathbb{R}^n$. It is straightforward to observe that this function is actually a metric on $\mathbb{R}^n$. We name the set $\mathbb{R}^n$ equipped with this *standard metric* as the n-dimensional ***Euclidean space*** E_n. Since $\mathbb{R}^n$ formed by the Cartesian product of the real line n times, the real numbers $\{x^i\}$ determining a point $x \in E_n$ are called *Cartesian coordinates* of that point. The collection of all such numbers constitutes the *coordinate cover* of E_n. *The length* or *the norm* of an element $x \in E_n$ is given by

$$\|x\| = \Big(\sum_{i=1}^{n} |x^i|^2\Big)^{1/2} \tag{2.2.4}$$

so that we can write $d(x, y) = \|x - y\|$.

A ***norm*** on a complex vector space V defined over a field of scalars $\mathbb{F}$ is a real-valued, non-negative function $\| \cdot \| : V \to \mathbb{R}^+$ satisfying the following conditions:

(i). $\|v\| \geq 0$ *for all* $v \in V$ *and* $\|v\| = 0$ *if and only if* $v = \mathbf{0}$.
(ii). $\|v\| = |\alpha| \|v\|$ *for all* $v \in V$ *and* $\alpha \in \mathbb{F}$.
(iii). $\|u + v\| \leq \|u\| + \|v\|$ *for all* $u, v \in V$.

We say that a vector space equipped with a norm, i.e., the ordered pair $(V, \| \cdot \|)$ is a ***normed linear space*** or a ***normed vector space*** or simply a ***normed space***. By taking $\alpha = 0$ and $\alpha = -1$, we obtain $\|\mathbf{0}\| = 0$ and $\| - v\| = \|v\|$, respectively. (iii) is known as the *triangle inequality*. It is then rather easy to establish directly by induction that the following inequality holds for a number of vectors $v_1, v_2, \ldots, v_n \in V$:

$$\|v_1 + v_2 + \cdots + v_n\| \leq \|v_1\| + \|v_2\| + \cdots + \|v_n\|.$$

For any two vectors $u, v \in V$, we can write $\|u - v\| \geq \|u\| - \|v\|$ and $\|u - v\| = \|v - u\| \geq \|v\| - \|u\|$. We thus find

$$\|u - v\| \geq \big|\|u\| - \|v\|\big|$$

for all $u, v \in V$. These properties of the norm amply justify our interpreting the norm of a vector as its ***length***. By means of the norm, we can introduce a function $d : V \times V \to \mathbb{R}^+$ as follows:

$$d(u, v) = \|u - v\|.$$

Evidently, this function satisfies the conditions $d(u, v) \geq 0$; $d(u, v) = 0$ if and only if $u = v$ and $d(u, v) = d(v, u)$. Furthermore, one can write

$$d(u, v) = \|u - w + w - v\| \leq \|u - w\| + \|w - v\| = d(u, w) + d(w, v)$$

so that d holds the triangle inequality. Hence, we realise that the function d so defined is actually a metric on the vector space V. We call this metric generated by the norm, the ***natural metric*** on the normed space V. But, in addition to its commonly known properties, this metric satisfies the following equalities for all $u, v, w \in V$ and $\alpha \in \mathbb{F}$:

$$d(\alpha u, \alpha v) = |\alpha| d(u, v), \quad d(u + w, v + w) = d(u, v).$$

The last relation indicates the fact that the distance between two vectors does not change by their parallel translations.

It is now clear that a normed space is a Hausdorff space equipped with a metric topology induced by its natural metric. In this topology, *open* and *closed balls of radius r centred at a vector v* are of course defined, respectively, by

$$B_r(v) = \{u \in V : \|u - v\| < r\}, \;\; B_r[v] = \{u \in V : \|u - v\| \leq r\}.$$

The basis for this topology is the class $\{B_r(v) : \text{for all } v \in V \text{ and } r > 0\}$. We obviously have $B_0(v) = \emptyset$, $B_0[v] = \{v\}$. One immediately verifies that an open ball $B_r(v)$ is obtained by just simply translating all vectors in the open ball $B_r(\mathbf{0})$ of radius r centred at the zero vector $\mathbf{0}$ by the vector v. If M is a subset of V, the set $v + M = \{v + u : \text{for all } u \in M\}$ is said to be the ***translation*** of the set M by the vector v. We thus have $B_r(v) = v + B_r(\mathbf{0})$. The same property will also be valid for closed balls. Unlike general metric spaces, it can easily be demonstrated that one always obtains $\overline{B_r(v)} = B_r[v]$ in all normed spaces.

Let us consider a sequence of vectors $\{v_n\} \subset V$. This sequence converges to a vector $v \in V$ if there exists a natural number $N(\epsilon)$ for each $\epsilon > 0$ such that $\|v_n - v\| < \epsilon$ whenever $n \geq N(\epsilon)$. The sequence $\{v_n\}$ is a

Cauchy sequence if there exists a natural number $N(\epsilon)$ for each $\epsilon > 0$ such that $\|v_n - v_m\| < \epsilon$ for all $n, m \geq N(\epsilon)$. *If every Cauchy sequence relative to its natural metric of a normed space V is convergent,* then V is called a *complete* normed space. A complete normed space is named as a ***Banach space*** [after Polish mathematician Stefan Banach (1892-1945)].

An ***inner product*** on a complex vector space V is a scalar-valued function $(\,\cdot\,,\,\cdot\,) : V \times V \to \mathbb{F}$ satisfying the following rules:

(i) $(u, v) = \overline{(v, u)}$ *for all vectors* $u, v \in V$.
(ii) $(\alpha u, v) = \alpha(u, v)$ *for all vectors* $u, v \in V$ *and scalars* $\alpha \in \mathbb{F}$.
(iii) $(u + v, w) = (u, w) + (v, w)$ *for all vectors* $u, v, w \in V$.
(iv) $(u, u) > 0$ *for all non-zero vectors* $u \in V$.

An overbar here denotes the complex conjugate. We can easily extract from this definition some novel results:

(a). $(\mathbf{0}, v) = (0 \cdot u, v) = 0 \cdot (u, v) = 0$ and similarly $(u, \mathbf{0}) = 0$ from which we naturally deduce that $(\mathbf{0}, \mathbf{0}) = 0$.

(b). Since $(u, u) = \overline{(u, u)}$ in compliance with (i), one finds $(u, u) \in \mathbb{R}$ and the property (iv) becomes meaningful. If $(u, u) = 0$, we then obtain that $u = \mathbf{0}$.

(c). The inner product is linear in its first argument because of the properties (ii) and (iii). On the other hand, we can easily observe that

$$
\begin{aligned}
(u, v + w) &= \overline{(v + w, u)} = \overline{(v, u)} + \overline{(w, u)} = (u, v) + (u, w),\\
(u, \alpha v) &= \overline{(\alpha v, u)} = \overline{\alpha (v, u)} = \overline{\alpha}\;\overline{(v, u)} = \overline{\alpha}(u, v).
\end{aligned}
$$

Hence the inner product is additive in its second argument but is not homogeneous because of the fact that the conjugate of the scalar multiplier is involved. This situation is known as the ***conjugate linearity***. Thus, the inner product on a complex vector space is a ***sesquilinear*** (**$1\frac{1}{2}$- *linear***) function with respect to its two arguments.

(d). If $(u, w) = (v, w)$ or $(w, u) = (w, v)$ for all $w \in V$, then we find that $u = v$. We can indeed prove this by simply taking $w = u - v \in V$ in the relation $(u - v, w) = 0$.

For a real-valued inner product on a real vector space, the property (i) is reduced to the *symmetry* condition $(u, v) = (v, u)$. A real inner product is linear in its second argument too since $(u, \alpha v) = \alpha(u, v)$ for $\alpha \in \mathbb{R}$. Hence, an inner product on a real vector space is a ***bilinear*** function.

A linear vector space endowed with an inner product is called an ***inner product space***.

Inner product must hold an important relation which is called ***Cauchy-Bunyakowski-Schwarz's inequality*** or briefly the ***Schwarz inequality***

[German mathematician Karl Hermann Amandus Schwarz (1843-1921) and Russian mathematician Viktor Yakovlevich Bunyakowski (1804-1889) who had actually discovered this inequality that had appeared in one of his books published in 1859]. *Let H be an inner product space. The inequality $|(u,v)| \leq \sqrt{(u,u)(v,v)}$ holds for all non-zero vectors $u, v \in H$. The equality is valid if and only if the vectors u and v are linearly dependent.*

If one of the vectors in that inequality is zero, the relation holds trivially as $0 = 0$. For any two vectors $u, v \in H$ with $v \neq \mathbf{0}$ and any scalar number $\alpha \in \mathbb{F}$, we can write

$$0 \leq (u - \alpha v, u - \alpha v) = (u,u) - \alpha\overline{(u,v)} - \overline{\alpha}(u,v) + |\alpha|^2(v,v).$$

The right-hand side vanishes if and only if $u = \alpha v$, namely, if two vectors are linearly dependent. Let us next choose $\alpha = (u,v)/(v,v)$ to cast the above inequality into the form

$$(u,u) - \frac{|(u,v)|^2}{(v,v)} - \frac{|(u,v)|^2}{(v,v)} + \frac{|(u,v)|^2}{(v,v)} = (u,u) - \frac{|(u,v)|^2}{(v,v)} \geq 0$$

or

$$|(u,v)|^2 \leq (u,u)(v,v).$$

The square root of the above inequality yields the Schwarz inequality. □

The Schwarz inequality helps us to show that a norm is derivable from the inner product. *Let us define* $\|u\| = \sqrt{(u,u)}$. We immediately see from the definition that $\|u\| \geq 0$ for all $u \in V$ and $\|u\| = 0 \Leftrightarrow u = \mathbf{0}$. If $\alpha \in \mathbb{F}$, then we readily observe that $\|\alpha u\| = \sqrt{(\alpha u, \alpha u)} = \sqrt{|\alpha|^2(u,u)} = |\alpha|\|u\|$. Moreover, we easily obtain that $\|u+v\|^2 = (u+v, u+v) = (u,u) + (u,v) + \overline{(u,v)} + (v,v) = \|u\|^2 + \|v\|^2 + 2\Re(u,v)$. $\Re(u,v) \leq |\Re(u,v)| \leq |(u,v)|$ yields through Schwarz's inequality $\Re(u,v) \leq \|u\|\|v\|$. We thus obtain $\|u+v\|^2 \leq \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| = (\|u\| + \|v\|)^2$. Hence the triangle inequality

$$\|u+v\| \leq \|u\| + \|v\|$$

follows at once. □

By this definition of the norm, Schwarz's inequality is expressed as

$$|(u,v)| \leq \|u\|\|v\|.$$

The norm generated by an inner product imposes a restriction that any two vectors in an inner product space must obey the ***parallelogram law***. Let $u, v \in H$. We have

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2\Re(u,v), \|u-v\|^2 = \|u\|^2 + \|v\|^2 - 2\Re(u,v).$$

By adding those two expressions, we obtain

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

This relation reflects the fact that a well-known relation of the classical geometry, namely, the sum of the squares of two diagonals of a parallelogram being equal to the sum of the squares of all of its four sides is still valid in an arbitrary inner product space.

The natural norm induced by an inner product generates now a *natural metric* on the vector space V through the function

$$d(u,v) = \|u-v\| = \sqrt{(u-v,u-v)}.$$

An inner product space which is complete relative to its natural metric is called a ***Hilbert space*** [after German mathematician David Hilbert (1862-1943)]. It goes without saying that a Hilbert space is also a Banach space.

A quite a simple generalisation of the classical Heine-Borel theorem [German mathematician Heinrich Eduard Heine (1821-1881) and French mathematician Félix Édouard Justin Émile Borel (1871-1956)] of real analysis leads to the result that every subset of E_n that is closed and bounded is compact. Let us consider an open set of E_n and a point inside this set. Hence, there exists an open ball centred at this point and contained in the open set. On the other hand, there is a closed ball with the same centre inside this open ball that is closed and bounded. Therefore, E_n is a *locally compact* metric space. If there is no ambiguity, we prefer to employ henceforth the notation $\mathbb{R}^n$ instead of E_n to denote the Euclidean space that illustrates the formation of this space more clearly. $\mathbb{R}^n$ is also a complete metric space. It can be shown that the class of open balls generated by the metric (2.2.3) constitutes a *countable basis* for the metric topology on $\mathbb{R}^n$. Thus the *second countable metric space* $\mathbb{R}^n$ *is a paracompact topological space* according to the statement in p. 57.

Let (M,d) and (N,ρ) be two metric spaces and consider a function $f: M \to N$. The topological concept of continuity takes now a purely metrical form. We say that the function f is *continuous* at a point $p \in M$ if for each number $\epsilon > 0$ there corresponds a number $\delta(\epsilon;p) > 0$ such that $\rho\big(f(p), f(p_1)\big) < \epsilon$ for all points $p_1 \in M$ satisfying $d(p,p_1) < \delta$. If f is a continuous function and if we can find for each ϵ a number $\delta(\epsilon)$ that is independent of points p, then f is called a *uniformly continuous function.*

Manifold. A ***differentiable manifold*** is essentially a topological space. But it is also equipped with a particular structure that makes it possible to support differentiable mappings, vectors, tensors and exterior differential forms associated with those topological spaces.

Let us first consider a more general definition. An n-dimensional ***topological manifold*** *M is a Hausdorff space every point of which belongs to an open set that is homeomorphic to an open set of the Euclidean space $\mathbb{R}^n$. These open sets constitute an open cover of M.* Thus a topological manifold is *locally equivalent* to the Euclidean space $\mathbb{R}^n$.

It proves to be advantageous for a topological manifold to be a paracompact space if we wish to develop a workable theory of integration on manifolds. That is the reason why many authors prefer to assume that the principal ingredient of a topological manifold is a second countable, hence a separable, locally compact Hausdorff space. As we have mentioned earlier, the concept of manifold stems from the desire to make an abstraction of the classical notion of smooth surfaces in the Euclidean space, to endow a topological space with a local structure supporting differentiability and to be able patch together these local structure to cover the entire manifold. The above definition means that when we consider a point $p \in M$, there will be a connected open set $U \in \mathfrak{M}$ containing the point p and a homeomorphism $\varphi : U \to V \subseteq \mathbb{R}^n$. Thus, the function φ is bijective, and φ and φ^{-1} are continuous. Since the metric space $\mathbb{R}^n$ is a Hausdorff space, it is imperative that M has also the Hausdorff property in order this homeomorphism to exist. Obviously the set $V = \varphi(U) \subseteq \mathbb{R}^n$ is open, hence it is expressible as a union of some open balls in the Euclidean space $\mathbb{R}^n$. A ***chart*** on M is the pair (U, φ). n is the *dimension* of this chart. The open set U is the domain of the chart. Let us now write $\varphi(p) = \mathbf{x} = (x^1, x^2, \dots x^n) \in \mathbb{R}^n$ and we choose clearly continuous functions $g^i : \mathbb{R}^n \to \mathbb{R}, i = 1, 2, \ \dots, n$ by the rule $g^i(\mathbf{x}) = x^i$. We say that the real-valued continuous functions $\varphi^i = g^i \circ \varphi : \ U \to \mathbb{R}, \ i = 1, \ \dots, n$ are the ***coordinate functions*** of the chart (U, φ) providing the mapping $\varphi^i(p) = x^i$ whereas the real numbers $(x^1, x^2, \dots x^n)$ will be called the ***coordinates*** of the point $p \in M$ in the chart (U, φ) (Fig. 2.2.2). Thus a chart gives rise to a ***local system of coordinates*** on the manifold. If every point of a topological manifold M has an n-dimensional chart, we say that M is an n-dimensional manifold. When we want to emphasise its dimension we denote this manifold by M^n. The union of local coordinates systems in all charts covering M constitute the ***coordinate cover*** of the manifold M. If there is a point $p_0 \in U$ such that $\varphi(p_0) = \mathbf{0}$, then we say that the local coordinate system is *centred* at p_0.

Let us consider a function $f : M \to \mathbb{R}^n$. The function $f^i = g^i \circ f : M \to \mathbb{R}$ is called the ith *component function* of f.

The inverse mapping $\varphi^{-1} : V \to U$ is called a ***parametrisation*** of the open set U. The coordinates $x^1, x^2, \dots x^n$ are then called ***parameters*** of U.

The coordinate lines on M are the images of Cartesian coordinate lines on $\mathbb{R}^n$ under the mapping $\varphi^{-1} : \varphi(U) = V \to U$ (Fig. 2.2.2).

It is now clear that the manifold M behaves *locally* just like an open set of the Euclidean space $\mathbb{R}^n$ in the *vicinity* of the point $p \in M$. Since the Euclidean space is locally compact and homeomorphism preserves compactness, *a finite-dimensional topological manifold must also be locally compact*. In fact, let us consider a point $p \in M$ contained in a chart (U, φ). The point $\varphi(p)$ will be in an open neighbourhood in $\mathbb{R}^n$. Hence, it belongs to an open ball inside $\varphi(U)$. Since the closure of this open ball is contained in a closed ball that is both closed and bounded, then $\varphi(p)$ has a compact neighbourhood K. Because the function φ^{-1} is continuous, then the point p also must have the compact neighbourhood $\varphi^{-1}(K)$ in the open set U.

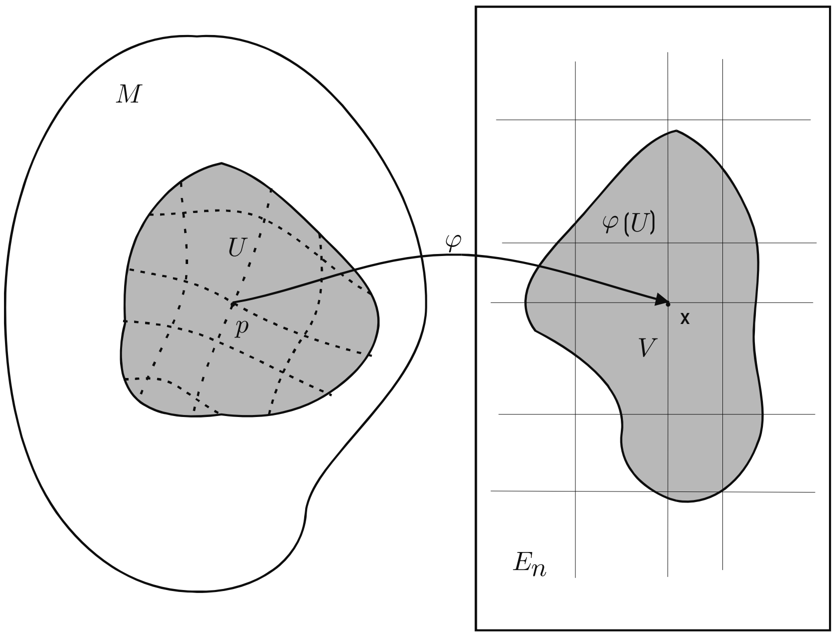

Fig. 2.2.2. A chart on the manifold M.

A $\boldsymbol{C^k}$***-atlas*** $\mathcal{A}$ on a topological manifold M is a family of charts $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in \mathcal{I}\}$ where $\mathcal{I}$ is an index set. Moreover this family must satisfy the following conditions: (i) all charts have the same dimension and the union of their domains constitute an open cover of the manifold, that is, $M = \bigcup_{\alpha\in\mathcal{I}} U_\alpha$, (ii) consider two different charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) of the atlas. Let us assume that $U_\alpha \cap U_\beta \neq \emptyset$. Images of the open intersection $U_\alpha \cap U_\beta$ under mappings φ_α and φ_β will usually be different open set in $\mathbb{R}^n$

(Fig. 2.2.3). On the overlapping domain $U_\alpha \cap U_\beta$ of the homeomorphisms φ_α and φ_β, we can define the following ***transition functions***:

$$\begin{aligned} \varphi_{\alpha\beta} &= \varphi_\beta \circ \varphi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}^n, \\ \varphi_{\alpha\beta}^{-1} &= \varphi_\alpha \circ \varphi_\beta^{-1} : \mathbb{R}^n \to \mathbb{R}^n. \end{aligned} \tag{2.2.5}$$

$\varphi_{\alpha\beta}$ is also a homeomorphism because it is the composition of two homeomorphisms. A better description of these homeomorphisms may be illustrated more clearly as

$$\begin{aligned} \varphi_{\alpha\beta} &: \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta), \\ \varphi_{\alpha\beta}^{-1} &: \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta). \end{aligned}$$

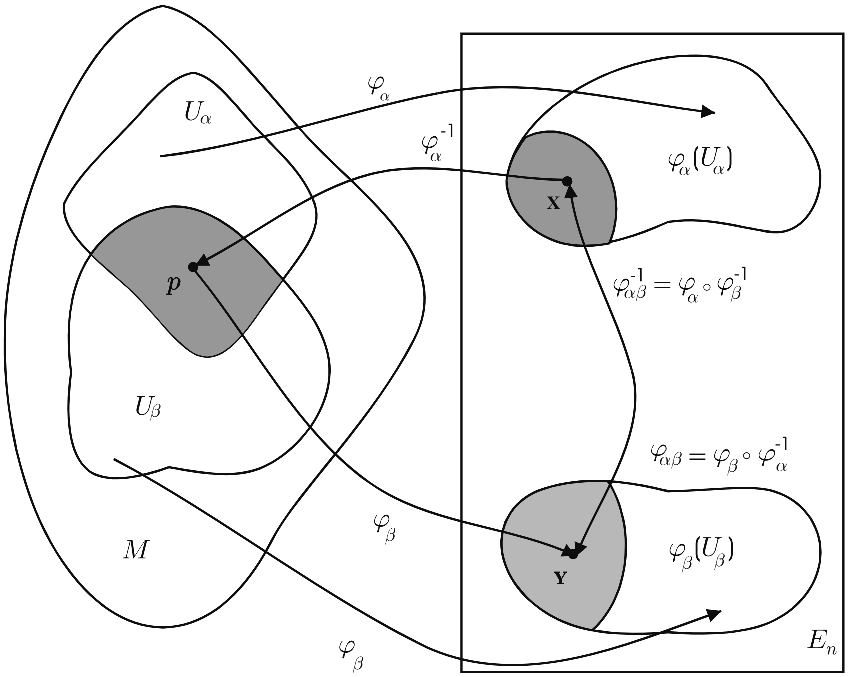

Fig. 2.2.3. Overlapping charts on a manifold M.

Let us denote the coordinates in charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) by $\{x^i\}$ and $\{y^i\}$, respectively. Then, the transition mapping $\varphi_{\alpha\beta}$ leads to a relation between images $\mathbf{x}$ and $\mathbf{y}$ of the same point $p \in M$ with respect to two overlapping charts in the form $\mathbf{y} = \varphi_{\alpha\beta}(\mathbf{x}) \in \mathbb{R}^n$ that can be expressed as

$$y^i = \phi^i_{\alpha\beta}(x^j),\ i,j = 1,2,\dots n;\ \mathbf{x} \in \varphi_\alpha(U_\alpha \cap U_\beta) \qquad (2.2.6)$$

Naturally, the mapping $\varphi^{-1}_{\alpha\beta}$ yields the inverse relation:

$$x^i = \psi^i_{\alpha\beta}(y^j),\ i,j = 1,2,\dots n;\ \mathbf{y} \in \varphi_\beta(U_\alpha \cap U_\beta). \qquad (2.2.7)$$

The foregoing relations corresponds clearly to a coordinate transformation on the open set $U_\alpha \cap U_\beta$. We know that partial derivatives are defined on $\mathbb{R}^n$. We say that the charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are ***C^k-compatible*** if the functions $\phi^i_{\alpha\beta}$ are continuously differentiable of order k or they are of class C^k. This of course means that they have continuous partial derivatives with respect to all variables x^j up to and including order k. Two charts are ***C^k-related*** if either they are C^k-compatible or $U_\alpha \cap U_\beta = \emptyset$. *A C^k-atlas is an atlas in which all charts are C^k-related.*

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two C^k-atlases. We say that they are ***C^k-compatible*** or ***equivalent*** atlases if and only if their union $\mathcal{A}_1 \cup \mathcal{A}_2$ is a C^k-atlas, in other words, if every chart in $\mathcal{A}_1$ is C^k-related to every chart in $\mathcal{A}_2$. It is easily seen that to be C^k-related gives rise to an equivalence relation on a family of atlases. In fact, it is obvious that this relation is reflexive and symmetric. In order to verify transitivity, let us consider three C^k-atlases $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and assume that $\mathcal{A}_1$, $\mathcal{A}_2$ are C^k-compatible and $\mathcal{A}_2$, $\mathcal{A}_3$ are C^k-compatible. $(U_1, \varphi_1) \in \mathcal{A}_1, (U_2, \varphi_2) \in \mathcal{A}_2, (U_3, \varphi_3) \in \mathcal{A}_3$ are three arbitrary charts. If $U_1 \cap U_3 = \emptyset$, then the charts (U_1, φ_1) and (U_3, φ_3) become trivially C^k-compatible. Thus, let us assume that the intersection $U_1 \cap U_3$ is not empty. Then the functions $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_2 \cap U_1 \cap U_3) \to \varphi_2(U_2 \cap U_1 \cap U_3)$ and $\varphi_3 \circ \varphi_2^{-1} : \varphi_2(U_2 \cap U_3 \cap U_1) \to \varphi_3(U_2 \cap U_3 \cap U_1)$ are of class C^k. On the other hand, we can write $\varphi_3 \circ \varphi_1^{-1} = (\varphi_3 \circ \varphi_2^{-1}) \circ (\varphi_2 \circ \varphi_1^{-1})$ so that this function is of class C^k. Hence the charts (U_1, φ_1) and (U_3, φ_3) are also C^k-compatible. We thus conclude that all C^k-atlases are partitioned into equivalence classes. The union of all atlases in an equivalence class will naturally be in this class. This means that every equivalence class contains exactly one *maximal atlas*.

*A C^k-**differentiable structure** on a topological manifold M is an equivalence class of C^k-atlases*. We can also say that *a C^k-**differentiable structure** on a topological manifold M is the choice of a maximal C^k-atlas*. *A **C^k-differentiable manifold** is a topological manifold equipped with a C^k-differentiable structure*.

If real-valued functions (2.2.6) and (2.2.7) with real variables have continuous derivatives of all orders, we obtain a C^∞-atlas and C^∞-differentiable manifold. A C^∞-differentiable manifold will also be called a ***smooth***

manifold. If the coordinate transformations are real analytical functions, that is, if they are expressible as convergent power series, then we get an ***analytical manifold*** or a C^{ω}-class manifold. It is evident that every C^{ω}-function is also a C^{∞}-function. But we know that the converse statement is generally not true. We can thus write symbolically $1 \leq k \leq m \leq \infty < \omega$. A C^m-differentiable structure $\mathcal{A}$ prescribed on a manifold M determines a unique C^k-differentiable structure on M for $k \leq m$. In order to see this it suffices to enlarge the set of admissible charts by adding all charts which are C^k-related with charts in $\mathcal{A}$ to the structure $\mathcal{A}$. Conversely, we can ask this question: when we are given a C^k-differentiable structure, is it possible to obtain a C^m-atlas for $m \geq k$ by discarding some charts? The answer to this question is provided by the following classical theorem whose proof we avoid to give because it is quite long and rather difficult.

Theorem 2.2.1 (Whitney's theorem). *Every C^k-structure with $k \geq 1$ on a second countable topological manifold is C^k-equivalent to a C^{ω}-structure.* □

This theorem means that if we locally make a coordinate transformation $y^i = f^i(x^j)$ of class C^k on an n-dimensional second countable topological manifold, there exist such functions $z^i = g^i(y^j)$ of C^k-class that the composition $z^i = g^i(f^j(x^m))$ is of C^{ω}-class, that is, they are analytical functions.

This theorem had been proven by Whitney[2]. That a C^0-manifold cannot be equivalent to a C^1-manifold can be shown through a more difficult theorem. According to the Whitney theorem we can choose all second countable or separable differentiable manifolds as analytical manifolds without loss of generality. However, it is not very comfortable to work with C^{ω}-functions as it is with C^{∞}-functions. Therefore, it will prove to be more advantageous to consider smooth manifolds. Henceforth, unless stated otherwise we take only *smooth manifolds* into consideration.

It is possible to extend above definitions to infinite dimensional manifolds. However, for this purpose we have the replace the Euclidean space by a Banach space, that is, by a complete normed space. In this case a chart $(U_\alpha, \varphi_\alpha)$ implies that the homeomorphism φ_α maps an open set U_α of the manifold M to an open subset V of a Banach space $\mathcal{V}$ such that $\varphi_\alpha(p) = v \in V$ where $p \in M$. The differentiable structure is now defined by Fréchet differentiability of the transition function $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : V \to V$ on the overlapping domain $U_\alpha \cap U_\beta$ of the homeomorphisms φ_α and φ_β. A brief definition of the Fréchet derivative is given below.

Let $\mathcal{U}$ and $\mathcal{V}$ be Banach spaces and let $T : \mathcal{U} \to \mathcal{V}$ be a possibly non-

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[2]Whitney, H., Differentiable manifolds, *Ann. of Math.* **37**, 645-680, 1936.

linear operator. Suppose that $\Omega = \mathcal{D}(T) \subseteq \mathcal{U}$ is an open set. If a *continuous linear operator* $T'(u) \in \mathcal{U} \to \mathcal{V}$ exists at a vector $u \in \Omega$ such that

$$\lim_{\|\Delta u\| \to 0} \frac{\|T(u + \Delta u) - T(u) - T'(u)\Delta u\|}{\|\Delta u\|} = 0$$

for all vectors $\Delta u \in \mathcal{U}$, then $T'(u)$ is called the ***Fréchet derivative of the operator $\boldsymbol{T}$ at a vector $\boldsymbol{u}$***. $T'(u)$ depend$s$ possibly nonlinearly on the vector u. This derivative was introduced by French mathematician Maurice René Fréchet (1878-1973) in 1925. The domain of the operator T' contains naturally all vectors in $\mathcal{U}$ at which the Fréchet derivative of T can be defined. The above definition amounts to say clearly that for each $\epsilon > 0$, there exists a number $\delta(\epsilon) > 0$ such that

$$\frac{\|T(u + \Delta u) - T(u) - T'(u)\Delta u\|}{\|\Delta u\|} < \epsilon$$

or

$$\|T(u + \Delta u) - T(u) - T'(u)\Delta u\| < \epsilon\|\Delta u\|$$

for all $\Delta u \in U$ satisfying the condition $\|\Delta u\| < \delta$. It is then straightforward to see that the following relation is valid:

$$T(u + w) - T(u) = T'(u)(w) + \omega(u; w), \quad \lim_{\|w\| \to 0} \frac{\|\omega(u; w)\|}{\|w\|} = 0.$$

We thus conclude that the existence of the Fréchet derivative at a vector u brings about the possibility of evaluating the vector $T(u + w) - T(u)$ approximately through a continuous linear operator for all vectors w with sufficiently small norms.

It is straightforward to see that the Fréchet derivative may also be expressible in the form

$$T'(u)(w_1) = \lim_{t \to 0} \frac{T(u + tw_1) - T(u)}{t}.$$

By following exactly the same procedure we have employed in evaluating the Fréchet derivative of T, we can of course define the Fréchet derivative of the operator $T'(u)$ as

$$T''(u)(w_1, w_2) = \lim_{t \to 0} \frac{T'(u + tw_2)(w_1) - T'(u)(w_1)}{t}$$

for all $w_1, w_2 \in \mathcal{U}$. If this derivative exists, then the operator $T''(u)$ is called the second Fréchet derivative of T at u. This operator must be linear in each vector w_1 and w_2. This approach permits us to define higher order derivatives as well. Let us suppose that the $(k-1)$th order Fréchet derivative $T^{(k-1)}(u)$ is known. Then the kth order Fréchet derivative can be similarly defined as follows

$$T^{(k)}(u)(w_1, w_2, \ldots, w_k) \\ = \lim_{t\to 0} \frac{T^{(k-1)}(u + tw_k)(w_1, \ldots, w_{k-1}) - T^{(k-1)}(u)(w_1, \ldots, w_{k-1})}{t}$$

for all ordered sets of vectors $w_1, w_2, \ldots, w_k \in \mathcal{U}$. Evidently, the operator $T^{(k)}(u) : \mathcal{U}^k \to \mathcal{V}$ is an k-linear function, that is, it is linear in each vector $w_i \in \mathcal{U}, i = 1, \ldots, k$. We can immediately extract from the definition that the operator $T^{(k)}(u)$ may be formally expressed in the following form

$$T^{(n)}(u)(\boldsymbol{w}) = \frac{\partial^n}{\partial t_1 \partial t_2 \cdots \partial t_n} T(u + t_1 w_1 + t_2 w_2 + \cdots + t_k w_k)\Big|_{t_1 = t_2 = \cdots = t_k = 0}$$

where $\boldsymbol{w} = (w_1, w_2, \ldots, w_k)$.

In this work, we shall always deal with finite-dimensional manifolds.

Open Submanifold. Let U be an *open subset* of a differentiable manifold M with a differentiable structure. We can define a differentiable structure on U by

$$\mathcal{A}_U = \{(U \cap U_\alpha, \varphi_\alpha|_{U \cap U_\alpha}) : (U_\alpha, \varphi_\alpha) \in \mathcal{A}\}.$$

since $U \cap U_\alpha$ are open sets covering U. It is clearly seen that the open set U endowed with this structure becomes itself a differentiable manifold called an ***open submanifold*** of M. Since the same homeomorphism is utilised, *this open submanifold has evidently the same dimension as the manifold M.*

Product Manifolds. Let us consider two differentiable manifolds M of dimension m and N of dimension n. We choose, respectively, atlases $\mathcal{A}_M = \{(U_\alpha, \varphi_\alpha) : \alpha \in \mathcal{I}\}$ and $\mathcal{A}_N = \{(V_\beta, \psi_\beta) : \beta \in \mathcal{J}\}$ from the differentiable structures of these manifolds. The set $M \times N$ of the Cartesian product of these manifolds can now be equipped with a structure of an $(m+n)$-dimensional differentiable manifold by choosing the topology on $M \times N$ as the *product topology* and by introducing an atlas in the form $\mathcal{A}_{MN} = \{(U_\alpha \times V_\beta, \omega_{\alpha\beta}) : (\alpha, \beta) \in \mathcal{I} \times \mathcal{J}\}$. Here, the mapping $\omega_{\alpha\beta}$ is identified by

$$\omega_{\alpha\beta} : U_\alpha \times V_\beta \to \varphi_\alpha(U_\alpha) \times \psi_\beta(V_\beta) \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}.$$

Thus, if $p \in U_\alpha \subset M$ and $q \in V_\beta \subset N$, then we have to write

$$\omega_{\alpha\beta}(p,q) = (\varphi_\alpha(p), \psi_\beta(q)) = (\mathbf{x}, \mathbf{y}) \in \varphi_\alpha(U_\alpha) \times \psi_\beta(V_\beta) \subset \mathbb{R}^{m+n}$$

where $\mathbf{x} = \varphi_\alpha(p)$ and $\mathbf{y} = \psi_\beta(q)$.

We now consider some samples of manifolds.

Example 2.2.1. Cartesian Spaces. The *standard* manifold structure on the Euclidean space $\mathbb{R}^n$ is prescribed by an atlas including a single chart $(\mathbb{R}^n, i_{\mathbb{R}})$ where $i_{\mathbb{R}} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity mapping. Coordinate functions φ^i, $i = 1, \ldots, n$ are just Cartesian coordinates $\{x^i : i = 1, \ldots, n\}$ of a point of $\mathbb{R}^n$. As a differentiable manifold $\mathbb{R}^n$ is called the ***affine space***.

The space $\mathbb{R}$ acquires a manifold structure with the single chart $(\mathbb{R}, \varphi_1)$ where $\varphi_1 : \mathbb{R} \to \mathbb{R}$ is given by $\varphi_1(x) = x$. Similarly if we replace φ_1 by $\varphi_2(x) = x^3$, then $\mathbb{R}$ becomes a manifold with the chart $(\mathbb{R}, \varphi_2)$. But these two atlases are not compatible, because the mapping $\varphi_{12} : \mathbb{R} \to \mathbb{R}$ given by $\varphi_{12}(x) = \varphi_1 \circ \varphi_2^{-1}(x) = x^{1/3}$ is not differentiable at the point $x = 0$. ■

We can observe at once that every open subset of $\mathbb{R}^n$ is an n-dimensional manifold. Furthermore, we can easily show that an *n-dimensional connected manifold is equivalent to an open submanifold of $\mathbb{R}^n$ if and only if its atlas contains only a single chart*. Indeed, if the entire manifold M is homeomorphic to a single open set of the space $\mathbb{R}^n$, then its atlas has only one chart. Conversely if the atlas of an n-dimensional manifold M has only one chart, then the entire space M is homeomorphic to a single open submanifold of $\mathbb{R}^n$.

Example 2.2.2. Finite-Dimensional Vector Spaces. Let V be an n-dimensional real vector space equipped with an arbitrary norm. We choose a set of basis vectors by $(e_1, e_2, \ldots, e_n)$. Then any vector $v \in V$ is expressed as $v = x^1 e_1 + x^2 e_2 + \cdots + x^n e_n$ where $x^i \in \mathbb{R}$, $i = 1, \ldots, n$. If we denote $x = (x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$, it becomes obvious that there is an isomorphism and hence a linear homeomorphism $\varphi : V \to \mathbb{R}^n$ such that $x = \varphi(v)$. It then follows that V is also an n-dimensional smooth manifold since $\mathbb{R}^n$ is a smooth manifold. Evidently this property is independent of the choice of the basis in V. As a concrete example to finite-dimensional vector spaces, let us consider the set of $m \times n$ matrices defined on real numbers. According to the rule of matrix addition and scalar multiplication, this set is an mn - dimensional vector space. Indeed, we can write any member of this set in the form $\mathbf{M} = a^{\alpha i}\mathbf{M}_{\alpha i}, \alpha = 1, \ldots, m, i = \ , \ldots, n$ where the matrix $\mathbf{M}_{\alpha i}$ has 1 in its row α and its column i while all other entries are 0. These mn linearly independent matrices $\mathbf{M}_{\alpha i}$ constitute a basis for this vector space. This vector space is isomorphic to the space $\mathbb{R}^{mn}$ whose points are identified by elements $(a^{11}, \ldots a^{1n}, \ldots, a^{m1}, \ldots, a^{mn})$ of matrices. Hence, such matrices constitute an mn-dimensional smooth manifold.

We denote the set of $n \times n$ real square matrices by $gl(n, \mathbb{R})$. $gl(n, \mathbb{R})$

is a smooth manifold of n^2-dimension. Let us consider the subset $GL(n,\mathbb{R})$ of regular matrices of $gl(n,\mathbb{R})$. This set is called the ***general linear group***. Let det : $gl(n,\mathbb{R}) \to \mathbb{R}$ be the determinant function. In this case the general linear group is expressed as the following set difference:

$$GL(n,\mathbb{R}) = gl(n,\mathbb{R}) - \det^{-1}\{0\}.$$

Since the determinant is a continuous function and the singleton $\{0\} \in \mathbb{R}$ is a closed set, then $\det^{-1}\{0\} \in gl(n,\mathbb{R})$ is a closed set. Thus $GL(n,\mathbb{R})$ is an open set, that is, it is an open submanifold of the manifold $gl(n,\mathbb{R})$.

We can obtain similar results on matrices defined on the field of complex numbers. But, a complex number is given by two real numbers. Consequently, the dimension of the real manifold to which space of matrices is homeomorphic becomes twice as much. For instance, the ***general linear group*** $GL(n,\mathbb{C})$ of regular $n \times n$ complex matrices is a smooth manifold of $2n^2$-dimension. ■

Example 2.2.3. The Sphere in $\mathbb{R}^3$. We consider a spherical surface of radius R in $\mathbb{R}^3$. Any point $P(x,y,z)$ of this 2-dimensional surface $\mathbb{S}^2$ can be written in the form

$$\begin{aligned} x &= R\sin\theta\cos\phi, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi, \\ y &= R\sin\theta\sin\phi, \\ z &= R\cos\theta \end{aligned}$$

by employing spherical coordinates (θ,ϕ). By defining $x^1 = \theta$ and $x^2 = \phi$ we can determine a function $\varphi_1 : \mathbb{S}^2 \to \mathbb{R}^2$ mapping $\mathbb{S}^2$ on the region $[0,\pi] \times [0,2\pi]$ of $\mathbb{R}^2$ (Fig. 2.2.4).

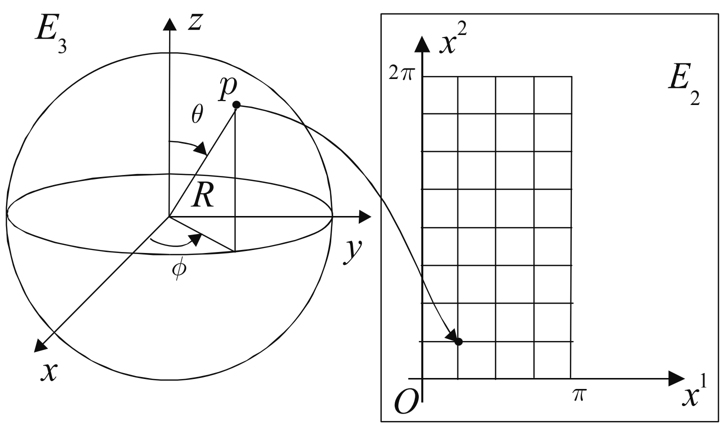

Fig. 2.2.4. 2-dimensional sphere.

It is straightforward to find the inverse function φ_1^{-1}:

$$x^1 = \arctan \frac{\sqrt{x^2+y^2}}{z}, \quad x^2 = \arctan \frac{y}{x}$$

Unfortunately, φ_1 is not a homeomorphism on the entire sphere. The poles $(0,0,R)$ and $(0,0,-R)$ of the sphere ($\theta = 0$ and $\theta = \pi$, respectively) are mapped onto sets $\{0\} \times [0,2\pi]$ and $\{\pi\} \times [0,2\pi]$ in E_2. Furthermore, the image of a point on the half-circle $\phi = 0$ are two points on the lines $x^2 = 0$ and $x^2 = 2\pi$. Hence, on this set φ_1 is not even a function. In order to make the mapping φ_1 a homeomorphism, we exclude from the set $\mathbb{S}^2$ the poles $(0,0,R)$, $(0,0,-R)$ and the half-circle $\phi = 0$ joining them. Thus we have to choose $\theta \in (0,\pi)$ and $\phi \in (0,2\pi)$ and to restrict φ_1^{-1} on the open set $(0,\pi) \times (0,2\pi)$ in $\mathbb{R}^2$, in other words, we have to take $0 < x^1 < \pi$, $0 < x^2 < 2\pi$. It is now evident that the set

$$U_1 = \mathbb{S}^2 - \{(0,0,R)\} \cup \{(0,0,-R)\} \cup \{\phi = 0\} \subset \mathbb{S}^2$$

is open since it is the homeomorphic image of the open set $(0,\pi) \times (0,2\pi)$. Consequently (U_1, φ_1) is a chart but it cannot cover the entire manifold $\mathbb{S}^2$. This result should be expected because the sphere $\mathbb{S}^2$ is a closed and bounded subset of the manifold $\mathbb{R}^3$. In order to find another chart, let us choose now the point $(0,R,0)$ as a pole of the sphere. As above, we write

$$\begin{aligned}
x &= R \sin y^1 \sin y^2, \quad 0 < y^1 < \pi, \ \ 0 < y^2 < 2\pi, \\
y &= R \cos y^1, \\
z &= R \sin y^1 \cos y^2.
\end{aligned}$$

These relations determine a mapping φ_2 and (U_2, φ_2) becomes a chart where U_2 is the open set obtained by deleting now the points $(0,R,0)$, $(0,-R,0)$ and the half-circle behind the sphere joining those two points from the manifold $\mathbb{S}^2$. It is obvious that $U_1 \cup U_2 = \mathbb{S}^2$, that is, $\{(U_1, \varphi_1)$ and $(U_2, \varphi_2)\}$ constitute an atlas on $\mathbb{S}^2$. In the images of overlapping charts in $\mathbb{R}^2$, we can easily obtain the following coordinate transformation:

$$\begin{aligned}
y^1 &= \arccos(\sin x^1 \sin x^2),\ y^2 = \arctan(\tan x^1 \cos x^2); \\
x^1 &= \arccos(\sin y^1 \cos y^2), x^2 = \arctan(\cot y^1 \sin y^2); \\
&\quad 0 < x^1 < \pi,\ 0 < x^2 < 2\pi; \\
&\quad 0 < y^1 < \pi,\ 0 < y^2 < 2\pi.
\end{aligned}$$

Since these functions are analytic, $\mathbb{S}^2$ is an analytical manifold. ■

Example 2.2.4. The Sphere in $\mathbb{R}^{n+1}$. Let us consider the n-dimensional spherical hypersurface $\mathbb{S}^n$ with radius R in $\mathbb{R}^{n+1}$. If we denote the

Cartesian coordinates in $\mathbb{R}^{n+1}$ by $(x_0, x_1, \ldots, x_n)$ then the set $\mathbb{S}^n$ is determined by the equation

$$x_0^2 + x_i x_i = R^2,\ i = 1, 2, \ldots, n.$$

We choose the pole k of $\mathbb{S}^n$ as the point $(R, 0, \ldots, 0)$. We specify the subspace E_n by the condition $x_0 = 0$. To describe the mapping $\varphi_1 : \mathbb{S}^n \to \mathbb{R}^n$, we impose that the image point $q = \varphi_1(p)$ of a point $p \in \mathbb{S}^n$ in $\mathbb{R}^n$ is the point of intersection of the straight line joining the pole k and the point p with the *hyperplane* $\mathbb{R}^n$ (Fig. 2.2.5). This mapping is known as ***stereographic projection***. If the coordinates of the point p are (x_0, x_i), then the relation $x_0 = \mp\sqrt{R^2 - x_i x_i}$ must be satisfied. Let the *unit* basis vectors in $\mathbb{R}^{n+1}$ be $\{e_0, e_i\}$. The vector e_0 is in the direction $\overrightarrow{Ok}$, while basis vectors in $\mathbb{R}^n$ are $e_i, i = 1, \ldots, n$. Let us denote the coordinates of the point $q \in \mathbb{R}^n$ at which the line joining the points k and p intersects the space $\mathbb{R}^n$ by $\mathbf{y} = (y_1, y_2, \ldots, y_n)$. Thus, we can write

$$Re_0 + \lambda\big[(x_0 - R)e_0 + x_i e_i\big] = y_i e_i$$

where λ is a real parameter. Then, it follows that

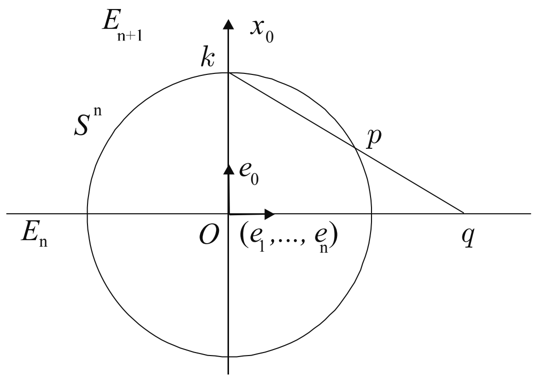

Fig. 2.2.5. Stereographic projection for an n-dimensional sphere.

$$\lambda = \frac{R}{R - x_0}, \quad y_i = \frac{Rx_i}{R - x_0}, \quad y_i y_i = R^2 \frac{R + x_0}{R - x_0}.$$

Hence, the points on $\mathbb{S}^n$ with the same elevation x_0 form now an $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. The radius of that sphere is of course given by $R\sqrt{(R + x_0)/(R - x_0)}$. It is greater than R if $0 < x_0 < R$ and less than R if $-R < x_0 < 0$. Let $\mathbf{x} = (x_0, x_1, \ldots, x_n) \in \mathbb{S}^n$, then the above relations prescribe a mapping $\varphi_1 : \mathbb{S}^n \to \mathbb{R}^n$ where $\varphi_1(\mathbf{x}) = \mathbf{y} \in \mathbb{R}^n$. The inverse mapping $\varphi_1^{-1} : \mathbb{R}^n \to \mathbb{S}^n$ is easily found to be

$$\mathbf{x} = \varphi_1^{-1}(\mathbf{y}), \quad x_0 = R\frac{y_iy_i - R^2}{y_jy_j + R^2}, \quad x_i = \frac{R - x_0}{R}y_i.$$

We can immediately observe that φ_1 is not a homeomorphism on the entire $\mathbb{S}^n$. Indeed, the pole k determined by $x_0 = R$, $x_i = 0$ is mapped on a "set of infinities" in $\mathbb{R}^n$ under φ_1. We can simply observe that φ_1 becomes a homeomorphism if we delete the single point k from $\mathbb{S}^n$. Thus (U_1, φ_1) is a chart where $U_1 = \mathbb{S}^n - \{Re_0\}$ is an open set. We can next introduce another chart by choosing the point $\{-Re_0\}$ as another pole of $\mathbb{S}^n$ and by defining the function $\varphi_2 : U_2 \to \mathbb{R}$ as follows

$$\mathbf{z} = \varphi_2(\mathbf{x}), \quad z_i = \frac{Rx_i}{R + x_0}, \quad z_iz_i = R^2\frac{R - x_0}{R + x_0}$$

where $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{R}^n$ and $U_2 \subset \mathbb{S}^n$ is the open set $\mathbb{S}^n - \{-Re_0\}$. The inverse mapping $\varphi_2^{-1} : \mathbb{R}^n \to U_2$ is easily provided by the following relations

$$x_0 = R\frac{R^2 - z_iz_i}{z_jz_j + R^2}, \quad x_i = \frac{R + x_0}{R}z_i.$$

Obviously (U_2, φ_2) is also a chart. Since $U_1 \cup U_2 = \mathbb{S}^n$, then we have the atlas $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$. In the region $(\mathbb{S}^n - \{Re_0\}) \cap (\mathbb{S}^n - \{-Re_0\})$ where the two charts overlap, the coordinate transformation $\varphi_2 \circ \varphi_1^{-1}$ is found to be

$$z_i = \frac{R - x_0}{R + x_0}y_i = \frac{R^2}{y_jy_j}y_i.$$

Thus $\mathbb{S}^n$ is an analytical manifold. ■

Example 2.2.5. Torus. We denote the surface of a 2-torus in $\mathbb{R}^3$ by $\mathbb{T}^2$. This surface is obtained, for instance, by rotating a circle with radius b whose distance of its centre from z-axis is a about that axis (Fig. 2.2.6). We can thus write $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 = (\mathbb{S}^1)^2$ as a product manifold. The manifold $\mathbb{S}^1$ represents a 1-dimensional sphere, namely, a circle. Thus, one-dimensional torus $\mathbb{T}^1$ is just the circle. In view of Example 2.2.3, the manifold $\mathbb{S}^1$ has an atlas with two charts homeomorphic to $\mathbb{R}^1$. In this case we expect that the product manifold $\mathbb{T}^2$ will have an atlas with four charts homeomorphic to open subsets of $\mathbb{R}^2$.

On the other hand, a torus may be determined parametrically in $\mathbb{R}^3$ by the following relations

$$x = (a + b\sin\theta)\cos\phi,$$
$$y = (a + b\sin\theta)\sin\phi,$$
$$z = b\cos\theta$$

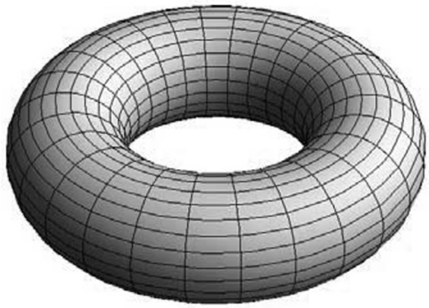

Fig. 2.2.6. 2-dimensional torus.

where the condition $b < a$ should be satisfied. The parameters ϕ and θ measure the angles along small and large circles. If we write $x^1 = \theta, x^2 = \phi$ these relations define a mapping $\varphi_1^{-1} : \mathbb{R}^2 \to \mathbb{T}^2$. But to render this mapping injective we have to restrict its domain to an open set in $\mathbb{R}^2$ prescribed by inequalities $0 < x^1 < 2\pi$, $0 < x^2 < 2\pi$. Let U_1 be the open set obtained by deleting from $\mathbb{T}^2$ the circle with radius a at the plane $z = b$ and the circle with radius b at the xz-plane centred at the point $x = a, z = 0$. (U_1, φ_1) then becomes a chart. We define a new mapping φ_2 by

$$x = \,-\,(a + b\sin y^1)\sin y^2,$$
$$y = (a + b\sin y^1)\cos y^2,$$
$$z = b\cos y^1.$$

Let U_2 be the open set obtained by deleting from $\mathbb{T}^2$ the circle with radius a at the plane $z = b$ and the circle with radius b at the yz-plane centred at the point $y = a, z = 0$. It is straightforward to see that (U_2, φ_2) is now a chart. The region $U_1 \cap U_2$ in which two charts overlap is the union of two open sets V_1 and V_2 that are *disconnected* where

$$V_1 = \varphi_1^{-1}\big((0, 2\pi) \times (\pi/2, 2\pi)\big),\;\; V_2 = \varphi_1^{-1}\big((0, 2\pi) \times (0, \pi/2)\big).$$

There are analytical coordinate transformations $y^1 = x^1, y^2 = x^2 - \frac{\pi}{2}$ on V_1

and $y^1 = x^1$, $y^2 = x^2 + \frac{3\pi}{2}$ on V_2. Finally, let us consider the mapping φ_3 given by

$$\begin{aligned} x &= (a + b\cos z^1)\cos z^2, \\ y &= (a + b\cos z^1)\sin z^2, \\ z &= -\,b\sin z^1. \end{aligned}$$

The open set U_3 is obtained by deleting from $\mathbb{T}^2$ the circle with radius $a+b$ at the plane $z=0$ and the circle with radius b at the xz-plane centred at the point $x=a, z=0$. (U_3,φ_3) is a chart. The region $U_1\cap U_3$ in which the charts (U_1,φ_1) and (U_3,φ_3) overlap is obviously the union of two open sets W_1 and W_2 that are *disconnected* where

$$W_1 = \varphi_1^{-1}\big((\pi/2, 2\pi)\times\big((0,2\pi)\big)\big),\ W_2 = \varphi_1^{-1}\big((0,\pi/2)\times(0,2\pi)\big)$$

There are analytical coordinate transformations $z^1 = x^1 - \frac{\pi}{2}$, $z^2 = x^2$ on W_1 and $z^1 = \ x^1 + \frac{3\pi}{2}$, $z^2 = x^2$ on W_2. The charts (U_2,φ_2) and (U_3,φ_3) overlap on $U_2\cap U_3$ which is the union of open sets Z_1, Z_2, Z_3 and Z_4. These sets are given by

$$\begin{aligned} Z_1 &= \varphi_2^{-1}\big((0,\pi/2)\times(0,3\pi/2)\big),\ Z_2 = \varphi_2^{-1}\big((0,\pi/2)\times(3\pi/2,2\pi)\big) \\ Z_3 &= \varphi_2^{-1}\big((\pi/2,2\pi)\times(0,3\pi/2)\big), Z_4 = \varphi_2^{-1}\big((\pi/2,2\pi)\times(3\pi/2,2\pi)\big). \end{aligned}$$

Analytical coordinate transformations on these four sets are determined by the following expressions, respectively

$$\begin{aligned} z^1 &= y^1 + \frac{3\pi}{2}, z^2 = y^2 + \frac{\pi}{2};\ \ z^1 = y^1 + \frac{3\pi}{2}, z^2 = y^2 - \frac{3\pi}{2}; \\ z^1 &= y^1 - \frac{\pi}{2},\ z^2 = y^2 + \frac{\pi}{2};\ \ z^1 = y^1 - \frac{\pi}{2}, z^2 = y^2 - \frac{3\pi}{2} \end{aligned}$$

Since $U_1\cup U_2\cup U_3 = \mathbb{T}^2$, we conclude that 2-torus has an analytical atlas with three charts $\{(U_1,\varphi_1),(U_2,\varphi_2),(U_3,\varphi_3)\}$.

An n-torus may be described in a similar fashion as a product manifold $\mathbb{T}^n = \mathbb{S}^1\times\mathbb{S}^1\times\cdots\times\mathbb{S}^1 = (\mathbb{S}^1)^n$. ■

Example 2.2.6. Klein Bottle $\mathbb{K}^2$. The Klein bottle is a 2-dimensional manifold in the space $\mathbb{R}^4$ [It was introduced in 1882 by German mathematician Felix Christian Klein (1849-1925)]. We denote the coordinates in $\mathbb{R}^4$ by (x,y,z,v). $\mathbb{S}^1$ is a circle with radius b at the xz-plane whose centre is the point $(a,0,0,0)$. We assume that $a>b$. Klein bottle is produced by the following process: while turning the centre C of that circle about O in the xy-plane by an angle ϕ, we rotate its plane in 4-dimensional space about the axis OC that remains perpendicular to the zv-plane by an angle $\phi/2$

(Fig. 2.2.7). It can be shown that this operation is tantamount to first forming a cylindrical surface by gluing two mutual edges together of a rectangular strip, then trying to glue one edge of this cylinder to the other after giving a half-twist with respect to the other one. In 3-dimensional space this operation cannot be realised without intersecting the surface. Therefore, Klein bottle can be considered as a manifold only in a 4-dimensional space. It cannot be embedded into $\mathbb{R}^3$ since in such a mapping self-intersections should not be permissible. However, it is possible to immerse this surface into 3-dimensional space if we allow self-intersections [for properties of these sort of mappings see Sec. 2.4]. These immersions are found to be unfortunately not unique. Two different immersions is depicted in Fig. 2.2.8.

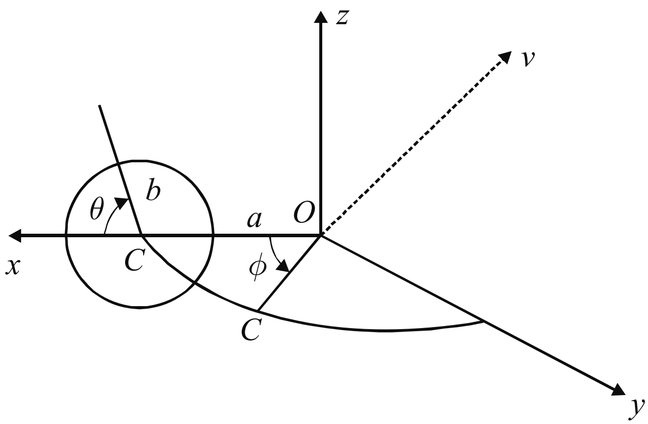

Fig. 2.2.7. Description of Klein bottle in 4-dimensional space.

It is now obvious that a point on Klein bottle is represented parametrically by equations

$$\begin{aligned}
x &= (a + b\cos\theta)\cos\phi, \\
y &= (a + b\cos\theta)\sin\phi, \\
z &= b\sin\theta\cos(\phi/2), \\
v &= b\sin\theta\sin(\phi/2), \quad 0 \le \theta \le 2\pi,\ 0 \le \phi \le 2\pi
\end{aligned}$$

When we eliminate these parameters, Klein bottle is given in Cartesian coordinates with the following relations

$$\begin{aligned}
y(z^2 - v^2) - 2xzv = 0 \\
x^2 + y^2 + z^2 + v^2 - 2a\sqrt{x^2 + y^2} + a^2 - b^2 = 0
\end{aligned}$$

With $x^1 = \theta$, $x^2 = \phi$, these relations determine a mapping $\varphi_1 : \mathbb{K}^2 \to \mathbb{R}^2$. However, in order to render the mapping $\varphi_1^{-1} : \mathbb{R}^2 \to \mathbb{K}^2$ injective, we have to restrict its domain in $\mathbb{R}^2$ to the open set determined by the inequalities $0 < x^1 < 2\pi$, $0 < x^2 < 2\pi$. Hence the domain of φ_1 is the open set U_1 obtained by deleting from $\mathbb{K}^2$ the circles $\phi = 0$ given by $x - a = b\cos\theta$, $z = b\sin\theta$ and $\theta = 0$ given by $x = (a+b)\cos\phi$, $y = (a+b)\sin\phi$. Thus, the inverse mapping φ_1^{-1} is found as follows when $z \neq 0$

$$\sin x^1 = \frac{\sqrt{z^2+v^2}}{b}, \ \cos x^1 = \frac{\sqrt{x^2+y^2}-a}{b},$$
$$x^2 = 2\arctan\frac{v}{z} = \arctan\frac{y}{x}.$$

If $z = 0$, we have either $x^1 = \pi$ or $x^2 = \pi$. Consequently, inverse mappings become, respectively

$$x^1 = \pi, \qquad x^2 = \arctan\frac{y}{x},$$
$$x^1 = \arctan\frac{v}{a-x}, \quad x^2 = \pi.$$

Hence (U_1, φ_1) is a chart. Let us now define a mapping φ_2 by relations

$$\begin{aligned} x &= -(a + b\cos y^1)\sin y^2, \\ y &= (a + b\cos y^1)\cos y^2, \\ z &= b\sin y^1 \cos\Big(\frac{y^2}{2} + \frac{\pi}{4}\Big), \\ v &= b\sin y^1 \sin\Big(\frac{y^2}{2} + \frac{\pi}{4}\Big),\ 0 < y^1 < 2\pi,\ 0 < y^2 < 2\pi \end{aligned}$$

where y^2 is representing now the angle in xy-plane measured from y-axis. We can easily observe that the mapping φ_2 is a homeomorphism on the open set U_2 obtained by deleting from $\mathbb{K}^2$ the circle with radius $a+b$ in xy-plane and the circle with radius b centred at $y = a$ and located on the bisecting plane of yz- and yv-planes. Hence, (U_2, φ_2) is a second chart and it contains the set $\{x^2 = 0\}$. We see that $U_1 \cap U_2 = V_1 \cup V_2$ where V_1 and V_2 are open disconnected sets given by

$$V_1 = \varphi_1^{-1}\big((0, 2\pi) \times (\pi/2, 2\pi)\big), \ \ V_2 = \varphi_1^{-1}\big((0, 2\pi) \times (0, \pi/2)\big)$$

The coordinate transformation on V_1 is $y^1 = x^1$, $y^2 = x^2 - \frac{\pi}{2}$ whereas that on V_2 is $y^1 = 2\pi - x^1$, $y^2 = x^2 + \frac{3\pi}{2}$. Finally, let us define a mapping φ_3 by the relations

$$\begin{aligned} x &= (a + b \sin z^1) \cos z^2, \\ y &= (a + b \sin z^1) \sin z^2, \\ z &= -\, b \cos z^1 \cos (z^2/2), \\ v &= -\, b \cos z^1 \sin (z^2/2),\ 0 < z^1 < 2\pi,\ 0 < z^2 < 2\pi \end{aligned}$$

where z^1 now denotes the angle translated $90°$. The open set U_3 is obtained by deleting from $\mathbb{K}^2$ the circle with radius b centred at the point $x = a$ in xz-plane and the circle with radius a in xy-plane and the circle with radius b in zv-plane both centred at the point O. It is obvious that (U_3, φ_3) is a chart and it contains the set $\{x^1 = 0\}$. We thus obtain $U_1 \cup U_2 \cup U_3 = \mathbb{K}^2$. In the same fashion one can show that coordinate transformations at the overlapping subsets of all these charts are simple analytical functions. Thus, Klein bottle $\mathbb{K}^2$ is an analytical manifold. ■

Example 2.2.7. Real Projective Spaces. Let us consider the space $\mathbb{R}^{n+1}$ whose origin $\mathbf{0} = (0, 0, \dots, 0)$ is deleted. A point of $\mathbb{R}^{n+1}$ is denoted by $\mathbf{x} = (x^1, x^2, \dots, x^{n+1})$. We define a relation R on the set $\mathbb{R}^{n+1} - \{\mathbf{0}\}$ by $\mathbf{x}R\mathbf{y}$ if and only $\mathbf{y} = \lambda\mathbf{x}$, $\lambda \in \mathbb{R} - \{0\}$, or $y^i = \lambda x^i, 1 \le i \le n+1$. It is straightforward to see that R is an equivalence relation. The n-dimensional real projective space $\mathbb{RP}^n$ is defined as the quotient space of the topological space $\mathbb{R}^{n+1} - \{\mathbf{0}\}$ with respect to this equivalence relation R: $\mathbb{RP}^n = (\mathbb{R}^{n+1} - \{\mathbf{0}\})/R$. It is clear that the elements of this space that are equivalence classes are straight lines through the origin $\mathbf{0}$ of $\mathbb{R}^{n+1}$. In this case, the canonical projection $\pi : \mathbb{R}^{n+1} - \{\mathbf{0}\} \to \mathbb{RP}^n$ [*see p.* 61] assigns to a non -zero point $\mathbf{x} \in \mathbb{R}^{n+1}$ the line through this point and the origin. Therefore, if we denote a point of the quotient space $\mathbb{RP}^n$ by the equivalence class $[\mathbf{x}] = [x^1, x^2, \dots, x^{n+1}]$, then for each $\lambda \in \mathbb{R}$, $\lambda \neq 0$ the equivalence class $[\lambda\mathbf{x}] = [\lambda x^1, \lambda x^2, \dots, \lambda x^{n+1}]$ specifies the same point, i.e., $[\lambda\mathbf{x}] = [\mathbf{x}]$. The numbers $x^1, x^2, \dots, x^{n+1}$ are called the *homogeneous coordinates* of the point $[\mathbf{x}]$. Employing those coordinates, we can represent the coordinates $\{\xi^1, \dots, \xi^n\}$ of a point in $\mathbb{R}^n$ by the ratios

$$\xi^1 = \frac{x^1}{x^{n+1}},\ \xi^2 = \frac{x^2}{x^{n+1}},\ \cdots\ ,\xi^n = \frac{x^n}{x^{n+1}},\ x^{n+1} \neq 0.$$

As corresponding to a point $[\mathbf{x}]$ in the projective space, these coordinates are uniquely determined. We now want to equip the projective space by the quotient topology [*see p.* 62]. Let us choose the sets $U_i, i = 1, 2, \dots, n+1$ in the projective space as follows

$$U_i = \{[\mathbf{x}] \in \mathbb{RP}^n : x^i \neq 0\}.$$

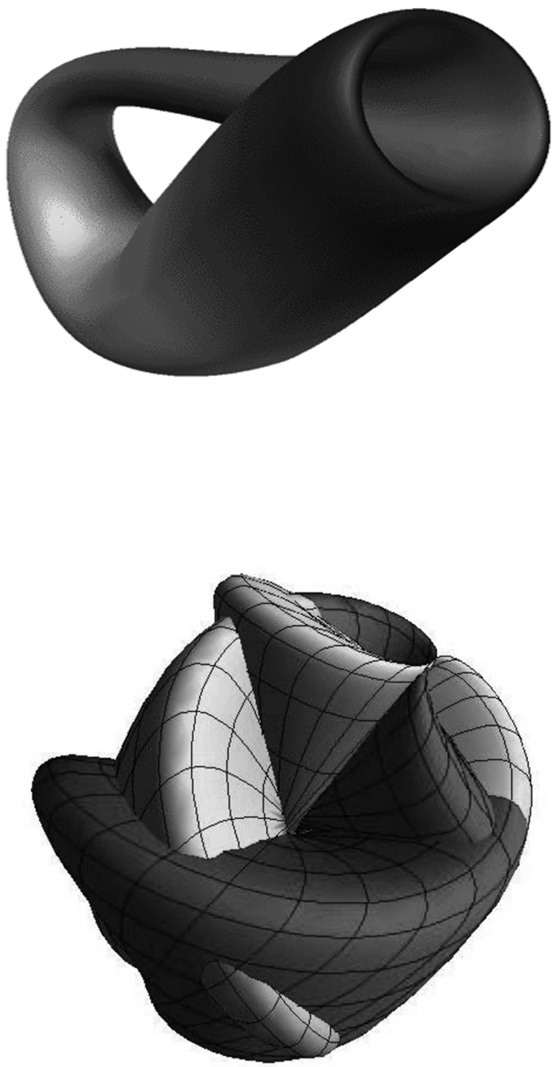

Fig. 2.2.8. Images of Klein bottle in $\mathbb{R}^3$ for two different immersions.

The set U_i consists clearly of the straight lines through the origin of the space $\mathbb{R}^{n+1}$ that do not belong to the n-dimensional subspace determined by the coordinates $(x^1, \dots, x^{i-1}, 0, x^{i+1}, \dots x^{n+1})$ except at the origin. Since the set

$$\pi^{-1}(U_i) = \{\mathbf{x} \in \mathbb{R}^{n+1} - \{\mathbf{0}\} : x^i \neq 0\} \subset \mathbb{R}^{n+1} - \{\mathbf{0}\}$$

is open, the set $U_i \subset \mathbb{RP}^n$ is also open in the quotient topology. Moreover, we see at once that $\bigcup_{i=1}^{n+1} U_i = \mathbb{RP}^n$. We define a mapping $\varphi_i : U_i \to \mathbb{R}^n$ by

$$\varphi_i([\mathbf{x}]) = \Big(\frac{x^1}{x^i}, \dots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \dots, \frac{x^{n+1}}{x^i}\Big), \quad [\mathbf{x}] \in U_i.$$

Evidently this mapping is a homeomorphism. Hence, (U_i, φ_i) is a chart and the collection $\{(U_i, \varphi_i) : i = 1, 2, \dots, n+1\}$ is an atlas for $\mathbb{RP}^n$. On the other hand, in the intersection $U_i \cap U_j$ where charts are overlapping the transition function is easily found to be

$$\begin{aligned}\varphi_j \circ \varphi_i^{-1}\Big(\frac{x^1}{x^i}, \dots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \dots, \frac{x^{n+1}}{x^i}\Big) &= \Big(\frac{x^1}{x^j}, \dots, \frac{x^{j-1}}{x^j}, \frac{x^{j+1}}{x^j}, \dots, \frac{x^{n+1}}{x^j}\Big)\\ &= \frac{x^i}{x^j}\Big(\frac{x^1}{x^i}, \dots, \frac{x^{j-1}}{x^i}, \frac{x^{j+1}}{x^i}, \dots, \frac{x^{n+1}}{x^i}\Big),\ x^i \neq 0, x^j \neq 0.\end{aligned}$$

Since transitions functions are analytic, we conclude that $\mathbb{RP}^n$ is an analytical manifold.

The interest of mathematicians to the real projective plane $\mathbb{RP}^2$ goes rather back in history. It has been observe that this 2-dimensional manifold can be embedded smoothly into $\mathbb{R}^4$. Werner Boy [1879-1914] who was a student of Hilbert had shown in 1901 that this surface can also be immersed in $\mathbb{R}^3$ if it is allowed for the surface to intersect itself. A quite an interesting parametrisation of *Boy's surface* was discovered by American mathematicians Robert B. Kusner and Robert L. Bryant (1953): we define the functions

$$\begin{aligned} g_1 &= -\frac{3}{2}\Im\frac{\zeta(1-\zeta^4)}{\zeta^6+\sqrt{5}\,\zeta^3-1}, \qquad g_2 = -\frac{3}{2}\Re\frac{\zeta(1+\zeta^4)}{\zeta^6+\sqrt{5}\,\zeta^3-1}\\ g_3 &= \Im\frac{1+\zeta^4}{\zeta^6+\sqrt{5}\,\zeta^3-1}-\frac{1}{2}, \qquad g = g_1^2+g_2^2+g_3^2\end{aligned}$$

where $\zeta = u + iv$ is a complex variable subject to the restriction $|\zeta| \leq 1$ and $\Re$ and $\Im$ denote the real and imaginary parts of a complex number, respectively. Then the Cartesian coordinates of a point on the surface is parametrically given by

$$x(u,v) = \frac{g_1}{g}, \quad y(u,v) = \frac{g_2}{g}, \quad z(u,v) = \frac{g_3}{g}.$$

Boy's surface is depicted in Fig. 2.2.9.

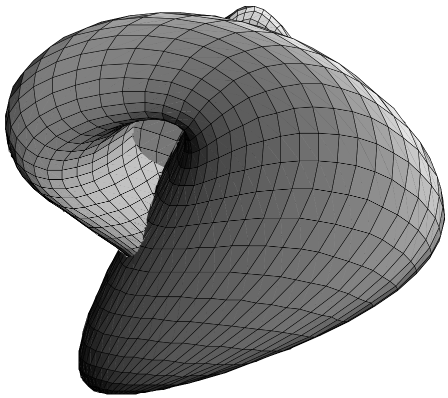

Fig. 2.2.9. Image of $\mathbb{RP}^2$ in 3-dimensional space (Boy's surface). ■

Manifolds with Boundary. In order to define a topological manifold with boundary we need a slightly more generalised concept. Let M_1 be a topological space that is an n-dimensional differentiable manifold. We consider a *closed* subset M of M_1. When M has a boundary ∂M we cannot generate a differentiable structure on the topological subspace M in the usual way because a point $p \in \partial M$ does not have an open neighbourhood remaining entirely inside M that is homeomorphic to an open set of $\mathbb{R}^n$. In order to solve this problem, we propose to consider the following subspace $\mathbb{H}^n$ of $\mathbb{R}^n$:

$$\mathbb{H}^n = \{\mathbf{x} = (x^1, x^2, \ldots x^n) \in \mathbb{R}^n : x^n \geq 0\}.$$

The hyperplane $\mathbb{R}^{n-1}$ defined by the relation $x^n = 0$ is the boundary of this *closed half-space*. We know that open sets of the subspace $\mathbb{H}^n$ in the relative topology are intersections of standard open sets in $\mathbb{R}^n$ with $\mathbb{H}^n$. Let

$V \subset \mathbb{H}^n$ be an open set defined this way (Fig. 2.2.10). We denote ***the interior*** of the set V by $\text{Int}\, V = V \cap \{\mathbf{x} \in \mathbb{R}^n : x^n > 0\}$ and its ***boundary*** by $\partial V = V \cap \{\mathbf{x} \in \mathbb{R}^n : x^n = 0\}$. It is clear that $V = \text{Int}\, V \cup \partial V$. We immediately observe that ∂V is not the topological boundary of the set V given on p. 55. Actually, ∂V is the intersection of the topological boundary with the boundary $x^n = 0$ of $\mathbb{H}^n$. If this intersection is empty, then V has no boundary according to this definition although the topological boundary may exist in the form $\overline{V} \cap (\mathring{V})'$.

The ***interior*** of M denoted by $\text{Int}\, M$ is the set of points of M that have open neighbourhoods homeomorphic to open subsets of $\mathbb{R}^n$. The ***boundary*** ∂M of M is the complement of $\text{Int}\, M$ with respect to M. The points on ∂M are mapped by homeomorphism to the points on the boundary $x^n = 0$ of $\mathbb{H}^n$. We now define a differentiable structure on M by an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in \mathcal{I}\}$ where U_α are open sets in relative topology on M and $\varphi_\alpha : U_\alpha \to V_\alpha$ are homeomorphisms. V_α is an open subset of $\mathbb{H}^n$. Naturally domains of charts will obey the rules (i)-(iii) mentioned on p. 53. We can now express the boundary ∂M and the interior $\text{Int}\, M$ of the manifold M by the relations (*see* Fig. 2.2.11)

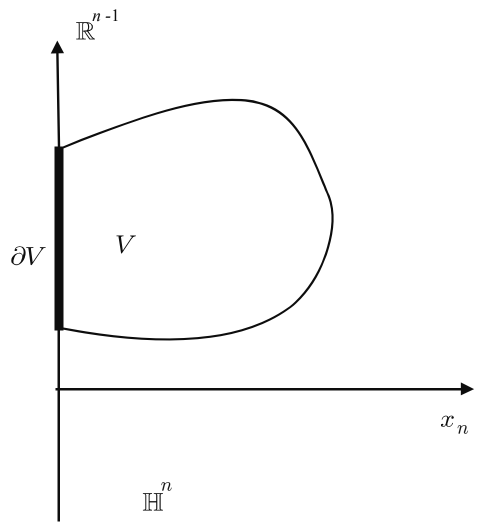

Fig. 2.2.10. An open set in $\mathbb{H}^n$.

$$\partial M = \bigcup_{\alpha \in \mathcal{I}} \varphi_\alpha^{-1}\big(\partial\,(\varphi_\alpha(U_\alpha))\big), \quad \text{Int}\, M = \bigcup_{\alpha \in \mathcal{I}} \varphi_\alpha^{-1}\big(\text{Int}\,(\varphi_\alpha(U_\alpha))\big).$$

If a point $p \in \partial M$ belongs to a chart (U, φ), then its parametrisation is obviously in the form

$$\varphi(p) = (x^1, x^2, \ldots, x^{n-1}, 0).$$

It is clear that Int M *is a n-dimensional manifold without boundary*. We shall show in the sequel that *the boundary ∂M of M is an $(n-1)$-dimensional manifold without boundary*. But we first prove the following lemma.

Lemma 2.2.1. *The position of a point on the boundary of the manifold M is independent of the parametrisation used.*

Let us consider two charts (U_1, φ_1) and (U_2, φ_2) containing a point $p \in \partial M$. We suppose that $\varphi_1(p) = \mathbf{x}_1 = (x^1, x^2, \ldots, x^{n-1}, 0)$ and $\varphi_2(p) = \mathbf{x}_2 = (x^1, x^2, \ldots, x^{n-1}, x^n), x^n > 0$. The transition mapping

$$\varphi_{12}^{-1} = \varphi_1 \circ \varphi_2^{-1} : \varphi_2(U_1 \cap U_2) \to \varphi_1(U_1 \cap U_2)$$

is a homeomorphism on $\mathbb{H}^n$. On the other hand, we assumed that the point $\mathbf{x}_2 \in \mathbb{H}^n$ is an interior point of $\mathbb{R}^n$. Hence, this point has an open neighbourhood $V_{\mathbf{x}_2} \subseteq \varphi_2(U_1 \cap U_2)$ in $\mathbb{R}^n$ that does not intersect the boundary $x^n = 0$. The function φ_{12}^{-1} transforms this open neighbourhood into the open neighbourhood $V_{\mathbf{x}_1} = \varphi_{12}^{-1}(V_{\mathbf{x}_2})$ of $\mathbf{x}_1$ in $\mathbb{R}^n$ (Fig. 2.2.12). But this set contains the points in the form $\{(x^1, x^2, \ldots, x^{n-1}, x^n) : x^n < 0\}$ that does not belong to $\mathbb{H}^n$. This is of course a contradiction. □

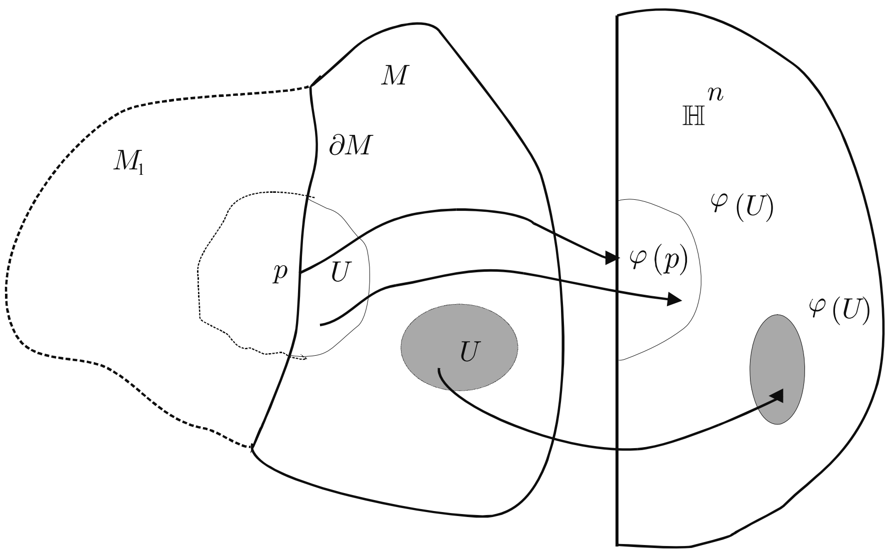

Fig. 2.2.11. A manifold with boundary.

Theorem 2.2.2. *The boundary of an n-dimensional differentiable manifold with boundary is an $(n-1)$-dimensional differentiable manifold.*

Let ∂M be the boundary of the manifold M. If a chart $(U_\alpha, \varphi_\alpha)$ of an atlas $\mathcal{A}$ contains a boundary point $p \in \partial M$, we can then write $\varphi_\alpha(\overline{U}_\alpha) = \varphi_\alpha(U_\alpha) \cap \mathbb{R}^{n-1}$ where we now define $\overline{U}_\alpha = U_\alpha \cap \partial M$ and $\mathbb{R}^{n-1} = \{(x^1, x^2, \ldots, x^{n-1}, x^n) \in \mathbb{R}^n : x^n = 0\}$. The set $\varphi_\alpha(\overline{U}_\alpha)$ is an open set in $\mathbb{R}^{n-1}$ in the relative topology. We denote the restriction of φ_α to the set $\overline{U}_\alpha$ by $\varphi_\alpha|_{\overline{U}_\alpha} = \overline{\varphi}_\alpha : \overline{U}_\alpha \subseteq \partial M \to V_\alpha \subseteq \mathbb{R}^{n-1}$. Evidently, $\overline{\varphi}_\alpha$ is also a homeomorphism. Therefore, the pair $(\overline{U}_\alpha, \overline{\varphi}_\alpha)$ is a chart on ∂M. Since the family $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in \mathcal{I}\}$ is an atlas on M, it is quite clear that the family $\overline{\mathcal{A}} = \{(\overline{U}_\alpha, \overline{\varphi}_\alpha) : \alpha \in \mathcal{I}\}$ becomes an atlas on ∂M. If this atlas has overlapping charts at a boundary point, these charts will be compatible in view of Lemma 2.2.1. Thus the atlas $\overline{\mathcal{A}}$ gives rise to a differentiable structure on ∂M. Hence the topological space ∂M is an $(n-1)$-dimensional differentiable manifold. □

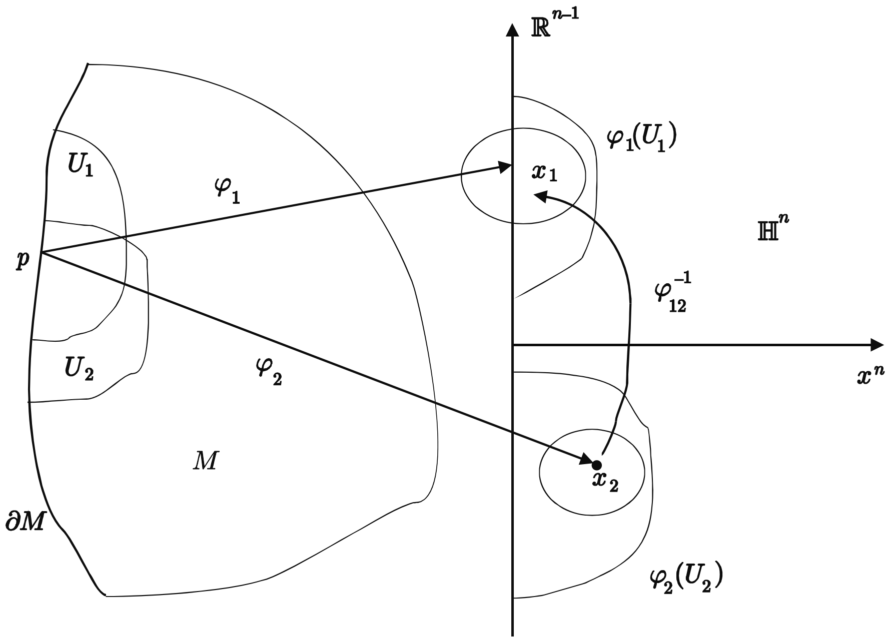

Fig. 2.2.12. A point on the boundary of a manifold.

2.3. DIFFERENTIABLE MAPPINGS

We consider a mapping $f : M \to \mathbb{R}$ on an m-dimensional differentiable manifold $(M, \mathcal{A})$, that is, $f(p) \in \mathbb{R}$ if $p \in M$. Let us assume that the point p is contained in the chart $(U_\alpha, \varphi_\alpha) \in \mathcal{A}$. Then, we can write $f(p) = f\big(\varphi_\alpha^{-1}(\mathbf{x})\big) = (f \circ \varphi_\alpha^{-1})(\mathbf{x})$. If we define a real-valued function of m real variables by $f'_\alpha = f \circ \varphi_\alpha^{-1} : \mathbb{R}^m \to \mathbb{R}$ on the open set $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^m$, then the equality $f(p) = f'_\alpha(\mathbf{x})$ becomes valid provided that the condition $\mathbf{x} = \varphi_\alpha(p)$ is satisfied (Fig. 2.3.1). If the function $f'_\alpha(x^1, x^2, \dots, x^m)$ is of class C^r at the point $\mathbf{x} \in \varphi_\alpha(U_\alpha)$, we say that the function f is ***differentiable*** and a ***C^r-function*** at the point $p \in M$ and we usually write $f \in C^r(M, \mathbb{R})$ or just $f \in C^r(M)$. Let us note that $r \le k$ if the atlas on M is of C^k-class. When we use only the adjectives ***differentiable*** or ***smooth***, we will always mean a function of C^∞-class. If a function f is differentiable at every point of the manifold M, then it is a function differentiable on M. We denote the set of all differentiable functions on M by $C^\infty(M)$ or merely by $C(M)$. We had seen that the set $C^r(M)$ can be equipped with a vector space structure [*see* Example 1.2.2], i.e., we can write $\alpha f + \beta g \in C^r(M)$ where $\alpha, \beta \in \mathbb{R}$. We can also define a product of vectors $f, g \in C^r(M)$ by utilising the familiar rules of multiplication in $\mathbb{R}$ as $(fg)(p) = f(p)g(p)$ at each point $p \in M$ so that we have $fg \in C^r(M)$. Hence, these sets are actually algebras. Of course $C(M)$ is also an algebra.

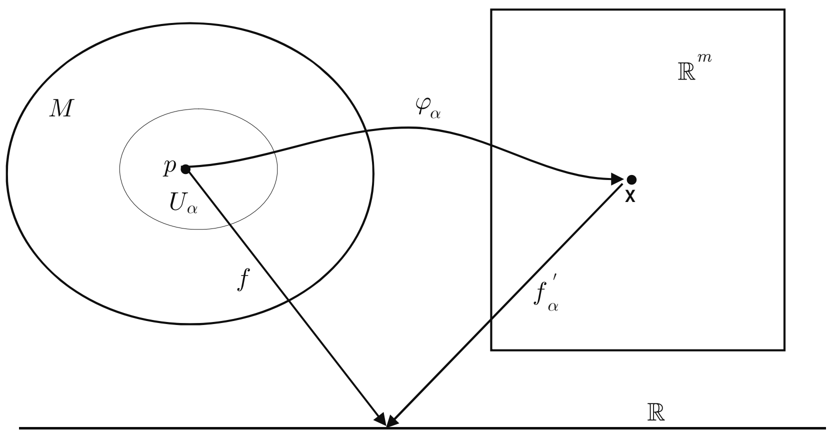

Fig. 2.3.1. A differentiable function f.

We can easily prove that the differentiability of a function $f : M \to \mathbb{R}$ is independent of the chosen atlas among compatible atlases. Let us consider another atlas $\mathcal{B}$ on M and assume that the point $p \in M$ belongs also to the chart $(V_\beta, \psi_\beta) \in \mathcal{B}$. We can thus write

$$f(p) = f'_\alpha(\mathbf{x}) = f'_\beta(\mathbf{y});\;\; p \in U_\alpha \cap V_\beta,\;\; \mathbf{x} = \varphi_\alpha(p),\;\; \mathbf{y} = \psi_\beta(p)$$

where we have of course defined $f'_\beta = f \circ \psi_\beta^{-1}$. Therefore, we obtain

$$f'_\beta = (f'_\alpha \circ \varphi_\alpha) \circ \psi_\beta^{-1} = f'_\alpha \circ (\varphi_\alpha \circ \psi_\beta^{-1}).$$

Because atlases are compatible, we conclude that if f'_α is differentiable, then the function f'_β must also be differentiable since it is expressed as a composition of differentiable functions. By definition, the partial derivative of a function f at a point $p \in M$ with respect to a coordinate x^i in an open set of $\mathbb{R}^m$ determined by a chart $(U_\alpha, \varphi_\alpha)$ containing the point p will be written at the point $\mathbf{x} = \varphi_\alpha(p)$ as

$$D_i f(p) = \frac{\partial f'_\alpha(\mathbf{x})}{\partial x^i}, \;\; i = 1, 2, \ldots, m.$$

Higher order derivatives will be represented in the same fashion.

Since a differentiable manifold is actually a topological space, the existence of the partition of unity on this manifold can be discussed. The partition of unity $\{V_i, f_i\}$ on a topological space was discussed on p. 62. But, here we further impose the condition that the functions $f_i : M \to [0, 1]$ are to be *smooth*.

It can be shown that if the manifold M is paracompact as a topological space, then for each atlas $\mathcal{A} = \{(U_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$ there exists a partition of unity subordinate to the open cover $\{U_\lambda : \lambda \in \Lambda\}$.

To prove this proposition in its most generality is beyond the scope of this work. Instead, we shall try to manage it for a paracompact space that is Hausdorff, locally compact and second countable [*see* p. 57]. These properties, however, are enjoyed by many differentiable manifolds encountered in applications. To this end, we start first by demonstrating the existence of a smooth function $\phi : \mathbb{R}^m \to \mathbb{R}$ which is equal to 1 on the closed cube $C[1]$ and is 0 on the complement of the open cube $C(2)$. The *open cube* $C(r)$ with sides of length $2r$ about the origin of $\mathbb{R}^m$ is defined as the subset

$$C(r) = \{\mathbf{x} \in \mathbb{R}^m : |x^i| < r, \, i = 1, \ldots, n\} \subset \mathbb{R}^m$$

where $\mathbf{x} = (x^1, x^2, \ldots, x^m)$ while the *closed cube* is the subset

$$C[r] = \{x \in \mathbb{R}^m : |x^i| \le r, \, i = 1, \ldots, n\} \subset \mathbb{R}^m.$$

Let us consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} e^{-1/t} & t > 0, \\ 0 & t \le 0 \end{cases}$$

which is non-negative, smooth and positive for $t > 0$. Then, we introduce the function

$$g(t) = \frac{f(t)}{f(t) + f(1-t)}$$

depicted in Fig. 2.3.2.

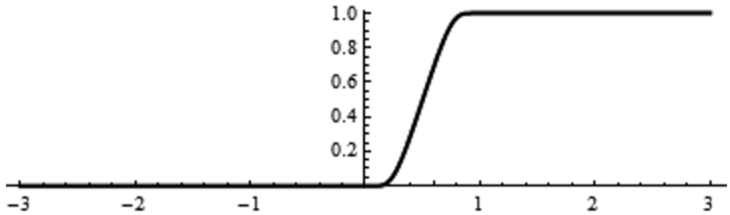

Fig. 2.3.2. The function $g(t)$.

This function is non-negative, smooth, and it is equal to 1 for $t \geq 1$ and to zero for $t \leq 0$.

Next, we construct the function

$$h(t) = g(t+2)g(t-2)$$

shown in Fig. 2.3.3. $h(t)$ is a smooth non-negative function which is equal to 1 on the closed interval $[-1,1]$ and to zero on the complement of the open interval $(-2,2)$.

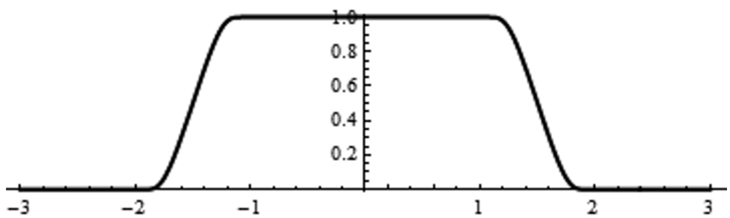

Fig. 2.3.3. The function $h(t)$.

We now define a function $\phi : \mathbb{R}^m \to \mathbb{R}$ by the product

$$\phi(\mathrm{x}) = (h \circ g^1)(\mathbf{x}) \cdots (h \circ g^m)(\mathbf{x}) = h(x^1) \cdots h(x^m)$$

where g^i were defined on p. 71. Obviously, this function is equal to 1 on the closed cube $C[1]$ and to zero on the complement of the open cube $C(2)$.

We now consider the relatively compact open cover $\{W_i\}$ of M introduced on p. 58. For a point $p \in M$, let i_p be the largest integer such that $p \in M - \overline{W}_{i_p} = (\overline{W}_{i_p})'$. Suppose that for an index $\lambda_p \in \Lambda$ one has $p \in U_{\lambda_p}$.

By definition, we also have $p \in (W_{i_p+1} - \overline{W}_{i_p-1}) = Z_{i_p}$. We consider an open set V in the intersection of the open set of the chart to which the point p belongs with the open set $U_{\lambda_p} \cap Z_{i_p}$. We shall assume that (V, φ) where $V \subseteq U_{\lambda_p} \cap Z_{i_p}$ is a coordinate system centred at the point p chosen in such a way that $\varphi(V) \in \mathbb{R}^m$ contains the closed cube $C[2]$. Next, we define the function $\psi_p : M \to \mathbb{R}$ by

$$\psi_p = \begin{cases} \phi \circ \varphi & \textit{if } p \in V \\ 0 & \textit{otherwise.} \end{cases}$$

Obviously ψ_p is a smooth function on the manifold M. The continuity of ψ_p implies that it is equal to 1 on some open neighbourhood $V_p = \psi_p^{-1}\big(C(1)\big)$ in V and it has a compact support given by $\psi_p^{-1}\big(C[2]\big) \subset V$. We know that $U_\lambda \cap Z_i$ is an open cover for the compact set $K_i = \overline{W}_i - W_{i-1} \subset Z_i$ Thus, for each $i \geq 1$, we can find a finite set of points p_j so that the open sets $U_{\lambda_{p_j}} \cap Z_{i_{p_j}}$ form a finite cover of K_i. Hence, for each i we have a finite family of sets V_{p_j} on which ψ_{p_j} take the value 1 and their supports forms a locally finite family of compact subsets of M. Hence, the set of functions $\{\psi_p\}$ is actually a countable union of finite sets. Therefore they can be enumerated as $\{\psi_i : i \in \mathbb{N}\}$. Thus the function

$$\psi = \sum_{i=1}^{\infty} \psi_i$$

is a well defined smooth function on M and at each point p all but a finitely many functions in this series do vanish. Therefore, we have $\psi(p) > 0$ at each point $p \in M$. Let us now define the functions $f_i : M \to [0,1]$ as

$$f_i = \frac{\psi_i}{\psi}$$

Hence, the countable family of functions $0 \leq f_i \leq 1$ constitute a partition of unity subordinate to the open cover $\{U_\lambda\}$ with compact supports. □

As we shall see later, this property will prove to be quite significant when we try to define the integration over manifolds.

Example 2.3.1. In the manifold $\mathbb{S}^1$, a partition of unity subordinate to the open cover $\{(0, 2\pi), (-\pi, \pi)\}$ is clearly $\{\sin^2\frac{\theta}{2}, \cos^2\frac{\theta}{2}\}$. ■

We shall now give two seemingly different definitions of the *differentiability of a mapping between two differentiable manifolds*. We shall then show that they are actually equivalent.

(i). We consider two differentiable manifolds M and N with dimensions m and n and a continuous mapping $\phi : M \to N$. This mapping will assign a point $q \in N$ to a point $p \in M$ by the relation $q = \phi(p)$. Due to the continuity of ϕ, to each open neighbourhood V of the point q there corresponds an open neighbourhood $U = \phi^{-1}(V)$ of the point p. It is evident that the set inclusion relation $\phi(U) = \phi(\phi^{-1}(V)) \subseteq V$ will be satisfied. Let $g : N \to \mathbb{R}$ be a differentiable function defined on the open set V. We can then define a function on the open set U in the manifold M whose value at the point $p \in M$ is given by the relation $f(p) = g(q) = g(\phi(p))$. Thus each function $g : N \to \mathbb{R}$ defined on V generates a function $f : M \to \mathbb{R}$ defined on U because $\phi(U) \subseteq V$. We can denote the functional relation between them by $f = g \circ \phi = \phi^* g$. The function $\phi^* g$ is called the ***pull-back*** or ***reciprocal image*** of the function g. *If for every differentiable function g defined on N, the function $f = \phi^* g$ is differentiable on M, that is, if for all $g \in C(N)$ one obtains $\phi^* g \in C(M)$, then the mapping $\phi : M \to N$ will be called a* ***differentiable mapping***. Consequently, a differentiable mapping $\phi : M \to N$ produces a mapping $\phi^* : C(N) \to C(M)$ between algebras $C(N)$ and $C(M)$. The mapping ϕ^* is called the ***dual mapping*** or ***pull-back mapping*** of ϕ. *When ϕ is a homeomorphism and both ϕ and its inverse $\phi^{-1} : N \to M$ are differentiable, then we shall say that the mapping ϕ is a* ***diffeomorphism***. If we establish a diffeomorphism between two manifolds, they are called ***diffeomorphic manifolds***. Evidently, diffeomorphic manifolds are equivalent as far as their topological and differentiability properties are concerned.

It follows from the definition of pull-back mappings that

$$\phi^*(g_1 + g_2) = \phi^* g_1 + \phi^* g_2,\ \phi^*(g_1 g_2) = (\phi^* g_1)(\phi^* g_2).$$

where $g_1, g_2 \in C(N)$. Hence, we deduce that *the pull-back mapping is an algebra homomorphism*.

If M are N differentiable C^k-manifolds and if there corresponds a $\phi^* g \in C^r(M)$ function for each function $g \in C^r(N)$ for an $r \leq k$, we say that the mapping $\phi : M \to N$ is ***C^r-differentiable***. If ϕ is a homeomorphism and both ϕ and ϕ^{-1} are C^r-differentiable, then we say that ϕ is a ***C^r-diffeomorphism***.

(ii). Let $\phi : M \to N$ be a continuous mapping. This mapping will assign to each point $p \in M$ a point $\phi(p) = q \in N$. These points are located in local charts (U, φ) and (V, ψ), respectively and we can write $\phi(U) \subseteq V$ due to the continuity of ϕ. We denote the local coordinates in those charts by $\mathbf{x} = (x^1, \dots, x^m)$ and $\mathbf{y} = (y^1, \dots, y^n)$, respectively. Hence one writes $\mathbf{x} = \varphi(p) \in \mathbb{R}^m$ and $\mathbf{y} = \psi(q) \in \mathbb{R}^n$. We define by using the transformation $\mathbf{y} = \psi\big(\phi(\varphi^{-1}(\mathbf{x})\big)$, a composite mapping

$$\Phi = \psi \circ \phi \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^m \to \psi(V) \subseteq \mathbb{R}^n$$

so that we can express this relation by $\mathbf{y} = \Phi(\mathbf{x})$ or $y^i = \Phi^i(x^1, \dots, x^m)$, $i = 1, \dots, n$ (Fig. 2.3.4). If the functions Φ^i have continuous derivatives of all orders at the point $\mathbf{x} = \varphi(p)$, namely, if $\Phi \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$, then we say that ϕ is a *differentiable* or a *smooth* mapping (if Φ is continuously differentiable of order r, then ϕ is a C^r*-differentiable* mapping). If this property is valid for every chart of an atlas, then ϕ *is a differentiable mapping on the manifold* M. If ϕ is a diffeomorphism, then $\phi^{-1} : N \to M$ exists and is differentiable. In this case, it is straightforward to see that ϕ^{-1} is locally represented by a function $\Psi \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ given by the inverse relation $\mathbf{x} = \Psi(\mathbf{y})$ such that $\Psi = \varphi \circ \phi^{-1} \circ \psi^{-1}$.

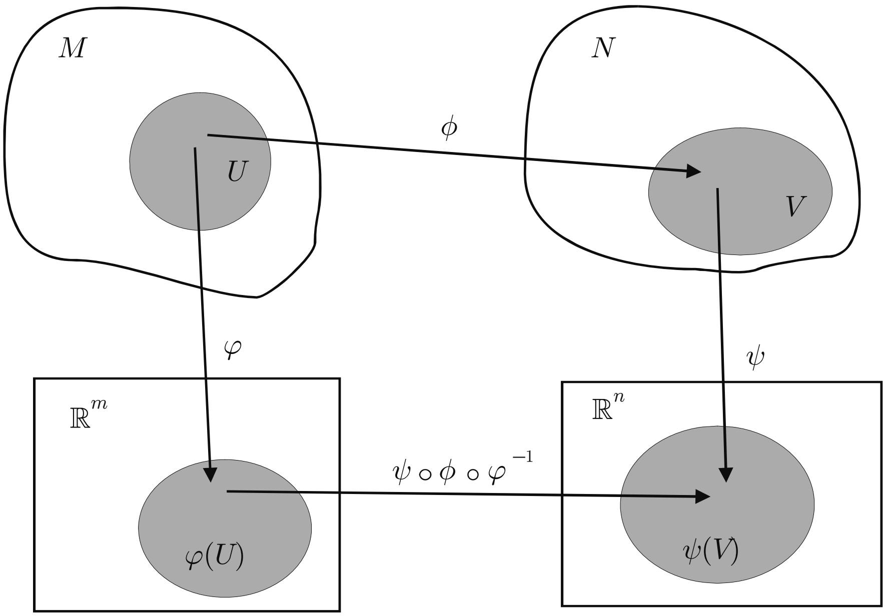

Fig. 2.3.4. A differentiable mapping ϕ.

We shall now try to prove the equivalence of these two definitions:

(i)* ⇒ *(ii) : We assume that ϕ is differentiable. Let (U, φ) and (V, ψ) be charts enclosing the points $p \in M$ and $\phi(p) = q \in N$, respectively. We define the continuous functions $g_M^i : \mathbb{R}^m \to \mathbb{R}$ and $g_N^i : \mathbb{R}^n \to \mathbb{R}$ by $g_M^i(\mathbf{x}) = x^i$ and $g_N^i(\mathbf{y}) = y^i$. The coordinate functions in those charts are then $\varphi^i = g_M^i \circ \varphi : U \to \mathbb{R}$, $\varphi^i(p) = x^i$, $i = 1, 2, \dots, m$ and $\psi^i = g_N^i \circ \psi : V \to \mathbb{R}$, $\psi^i(q) = \psi^i(\phi(p)) = y^i, i = 1, 2, \dots, n$. Since the function ψ^i is clearly differentiable and the set relation $\phi(U) \subseteq V$ is satisfied, the function $\psi^i \circ \phi : U \to \mathbb{R}$ is also differentiable due to ***(i)***. Since we can write

$\psi^i \circ \phi \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ or $g_N^i \circ (\psi \circ \phi \circ \varphi^{-1}) : \varphi(U) \to \mathbb{R}$, we find finally that the function $\Phi = \psi \circ \phi \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is differentiable at the arbitrary point $p \in M$, i.e., $\Phi \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$. □

(ii)* ⇒ *(i) : We assume again that ϕ is differentiable and we consider an arbitrary function $g : V \to \mathbb{R}$ which is differentiable at a point $q \in N$. Hence the function $g_\psi = g \circ \psi^{-1} : \psi(V) \subseteq \mathbb{R}^n \to \mathbb{R}$ will also be differentiable at the point $\mathbf{y} = \psi(q)$. We can thus write

$$g(q) = g \circ \phi(p) = g_\psi \circ \psi \circ \phi(p) = g_\psi \circ (\psi \circ \phi \circ \varphi^{-1})(\mathbf{x}) = g_\psi \circ \Phi(\mathbf{x}).$$

We have assumed that the function $\Phi = \psi \circ \phi \circ \varphi^{-1}$ is differentiable, that is, $\Phi \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$. By noting that composition of differentiable real-valued functions is also differentiable, we arrive at the result that the function $\phi^* g = g \circ \phi : U \to \mathbb{R}$ must be differentiable. Furthermore, if we write the above equality in the form

$$\phi^* g \circ \varphi^{-1}(\mathbf{x}) = g_\psi \circ \Phi(\mathbf{x}) = \Phi^* g_\psi(\mathbf{x})$$

we obtain the following relation on a chosen chart (U, φ)

$$(\phi^* g)_\varphi = \Phi^* g_\psi$$

for each $g \in C(N)$. The pull-back function Φ^* and $(\phi^* g)_\varphi$ are of the form $\Phi^* : C(\mathbb{R}^n) \to C(\mathbb{R}^m)$ and $(\phi^* g)_\varphi : \varphi(U) \subseteq \mathbb{R}^m \to \mathbb{R}$. □

Let M_1, M_2, M_3 be differentiable manifolds. Assume that mappings $\phi_1 : M_1 \to M_2$, $\phi_2 : M_2 \to M_3$ are continuous. If their composition exists, then one has $\phi = \phi_2 \circ \phi_1 : M_1 \to M_3$. For any $g \in C(M_3)$, we obtain

$$\phi^* g = (\phi_2 \circ \phi_1)^* g = g \circ \phi_2 \circ \phi_1 = (\phi_2^* g) \circ \phi_1 = \phi_1^*(\phi_2^* g).$$

Because this relation must be valid for every $g \in C(M_3)$ we arrive at the following rule of composition

$$(\phi_2 \circ \phi_1)^* = \phi_1^* \circ \phi_2^*. \tag{2.3.1}$$

This result can of course be extended to an arbitrary number of compositions. Let us now take into account the *identity mapping* $i_M : M \to M$ on the differentiable manifold M. We thus find $i_M(p) = p$ for each $p \in M$. In this case, we obtain $i_M^* g = g \circ i_M = g$ for each $g \in C(M)$. Consequently, we reach to the identity mapping on $C(M)$:

$$i_M^* = i_{C(M)} \tag{2.3.2}$$

Example 2.3.2. Consider the manifold $\mathbb{R}$ with the standard chart $(\mathbb{R}, i_\mathbb{R})$. The function $\phi : \mathbb{R} \to \mathbb{R}$ prescribed by $y = \phi(x) = x^\alpha$, $\alpha > 1$ is a differentiable homeomorphism, but it is not a diffeomorphism. Because the inverse mapping $x = \phi^{-1}(y) = y^{1/\alpha}$ cannot be differentiated at the point $y = 0$. We define now a new differentiable structure on $\mathbb{R}$ by another chart $(\mathbb{R}, \psi = \phi^{-1})$. Let $\mathbb{R}_\phi$ denote the manifold $\mathbb{R}$ equipped by this structure. Hence, for each $y \in \mathbb{R}$ one has $\psi(y) = y^{1/\alpha}$. the local representation of the mapping $\phi : \mathbb{R} \to \mathbb{R}_\phi$ is now given by $\phi^{-1} \circ \phi \circ i_{\mathbb{R}}^{-1} = i_{\mathbb{R}}$, whereas that of the inverse mapping $\phi^{-1} : \mathbb{R}_\phi \to \mathbb{R}$ becomes $i_{\mathbb{R}} \circ \phi^{-1} \circ (\phi^{-1})^{-1} = i_{\mathbb{R}}$. This amounts to say that $\phi : \mathbb{R} \to \mathbb{R}_\phi$ is a diffeomorphism. ■

Example 2.3.3. The manifold $\mathbb{S}^2 = \{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ in $\mathbb{R}^3$ will now be considered. We know that this sphere can only be homeomorphic to the plane $\mathbb{R}^2$ by employing two charts of its atlas and two differentiable functions ϕ_1, ϕ_2 given below:

$$\mathbf{y} = \phi_1(\mathbf{x}),\ y_i = \frac{x_i}{1 - x_3};\ \mathbf{y} = \phi_2(\mathbf{x}),\ y_i = \frac{x_i}{1 + x_3};\ \mathbf{y} \in \mathbb{R}^2, \mathbf{x} \in \mathbb{S}^2$$

[*see pp.* 81-82]. Thus, we cannot find a single diffeomorphism $\phi : \mathbb{S}^2 \to \mathbb{R}^2$. Hence, the sphere cannot be diffeomorphic to the plane. On the other hand, when we choose the ellipsoidal surface as another manifold given by $M = \left\{\mathbf{y} \in \mathbb{R}^3 : \frac{y_1^2}{a^2} + \frac{y_2^2}{b^2} + \frac{y_3^2}{c^2} = 1\right\}$, the mapping $\phi : \mathbb{S}^2 \to M$ defined by

$$y_1 = ax_1,\ \ y_2 = bx_2,\ \ y_3 = cx_3$$

is evidently a diffeomorphism. Thus the sphere and the ellipsoid are diffeomorphic manifolds. ■

Example 2.3.4. Let us consider the unit circle $\mathbb{S}^1$ in $\mathbb{R}^2$ and the projective space $\mathbb{RP}^1$. These manifolds will be represented as follows: $\mathbb{S}^1 = \{e^{i\theta}\}$ and $\mathbb{RP}^1 = \left\{\xi = \frac{x_2}{x_1} : \mathbf{x} \in \mathbb{R}^2 - \{\mathbf{0}\}\right\}$. It is easily observed that the single mapping $\phi : S^2 \to \mathbb{RP}^1$ determined by $\xi = \tan\theta$ is a diffeomorphism between two charts (U_1 and U_2) of the projective space $\mathbb{RP}^1$ [*see p.* 87] and two charts of the circle $\mathbb{S}^1$. Hence, these manifolds are diffeomorphic. ■

Example 2.3.5. We define the mapping $\phi : (a, b) \to \mathbb{R}$ by the relation

$$x = \phi(\xi) = \frac{(b-a)(2\xi - a - b)}{4(\xi - a)(b - \xi)},\ \ \xi \in (a, b).$$

The inverse of this function is obtained as

$$\xi = \phi^{-1}(x) = \frac{b^2 - a^2 - 4abx}{\sqrt{(b-a)^2(1 + 4x^2)} - 2(a+b)x + b - a}$$

if we note that ξ must belong to the open interval (a,b). The functions ϕ and ϕ^{-1} are continuous and differentiable former on (a,b) while the latter on $(-\infty,\infty)$. Thus ϕ is a diffeomorphism. This means that every open interval in $\mathbb{R}$ is diffeomorphic to $\mathbb{R}$ itself. ■

Example 2.3.6. A mapping $\phi:\mathbb{T}^2\to\mathbb{R}^3$ between differentiable manifolds $\mathbb{T}^2$ and $\mathbb{R}^3$ can be defined as follows [*see p.* 82]

$$\phi(\theta,\phi)=\big((a+b\sin\theta)\cos\phi,(a+b\sin\theta)\sin\phi,b\cos\theta\big)=(x,y,z).$$

This mapping is clearly differentiable and smooth. The image of the manifold $\mathbb{T}^2$ in $\mathbb{R}^3$ under the mapping ϕ is the surface

$$x^2+y^2+z^2-2a\sqrt{x^2+y^2}+a^2-b^2=0$$

obtained by eliminating parameters θ and ϕ. ■

Let $\phi:M\to N$ be a smooth mapping from the m-dimensional manifold M to the n-dimensional manifold N. We consider points $p\in M$ and $q=\phi(p)\in N$ in the local charts (U,φ) and (V,ψ), respectively. Then the mapping ϕ is represented by the function $\Phi:\varphi(U)\subseteq\mathbb{R}^m\to\psi(V)\subseteq\mathbb{R}^n$ that can be written as $\mathbf{y}=\Phi(\mathbf{x})$ or $y^i=\Phi^i(x^1,\dots,x^m)$, $i=1,\dots,n$ in terms of local coordinates. We know that Φ^i are smooth functions. *The rank of the mapping ϕ at the point p* is defined as the rank of the following $n\times m$ Jacobian matrix [German mathematician Carl Gustav Jacob Jacobi (1804-1851)]

$$\mathbf{J}(\phi)=\Big[\frac{\partial\Phi^i}{\partial x^j}\Big]=\begin{bmatrix}\dfrac{\partial\Phi^1}{\partial x^1} & \dfrac{\partial\Phi^1}{\partial x^2} & \cdots & \dfrac{\partial\Phi^1}{\partial x^m}\\ \dfrac{\partial\Phi^2}{\partial x^1} & \dfrac{\partial\Phi^2}{\partial x^2} & \cdots & \dfrac{\partial\Phi^2}{\partial x^m}\\ \vdots & \vdots & & \vdots\\ \dfrac{\partial\Phi^n}{\partial x^1} & \dfrac{\partial\Phi^n}{\partial x^2} & \cdots & \dfrac{\partial\Phi^n}{\partial x^m}\end{bmatrix}.$$

If the rank of ϕ at a point $p\in M$ admits its greatest value, that is, if it is equal to min $\{m,n\}$, then we say that its rank is *maximal* at that point. If the rank of ϕ is maximal every point $p\in S$ of a subset $S\subseteq M$, then its rank is maximal on S.

Theorem 2.3.1. *Let the rank of a mapping $\phi:M\to N$ be maximal at a point $p\in M$. Consider the chart (U,φ) at the point p and the chart (V,ψ) at the point $\phi(p)$ such that $\phi(U)\subseteq V$. Then the local coordinates $(x^1,x^2,\dots,x^m)$ in the neighbourhood of the point* $\mathbf{x}=\varphi(p)$ *and $(y^1,y^2,\dots,y^m,\dots,y^n)$ in the neighbourhood of the point* $\mathbf{y}=\psi(\phi(p))=\Phi(\mathbf{x})$ *can be so chosen that the local representation $\Phi=\psi\circ\phi\circ\varphi^{-1}$ of ϕ*

admits the following forms

$$\mathbf{y} = (x^1, x^2, \ldots, x^n) \ \text{if} \ n \leq m,$$
$$\mathbf{y} = (x^1, x^2, \ldots, x^m, 0, \ldots, 0) \ \text{if} \ n > m.$$

In terms of arbitrary coordinates in charts, consider the representation $\eta^i = \Phi^i(\xi^1, \ldots, \xi^m), i = 1, \ldots, n$. $n \times m$ Jacobian matrix is $[\partial\eta^i/\partial\xi^j]$.

If $n \leq m$, the rank of this matrix is n. Let us rearrange the variables in such a way that the determinant of the square matrix $[\partial\eta^i/\partial\xi^j]$, $i, j = 1, \ldots, n$ does not vanish. Then according to the well known implicit function theorem, the equations $x^i = \Phi^i(\xi^1, \ldots, \xi^n, \xi^{n+1}, \ldots, \xi^m)$ have uniquely determined smooth solutions $\xi^i = \Psi^i(x^1, \ldots, x^n, \xi^{n+1}, \ldots, \xi^m), 1 \leq i \leq n$ in a sufficiently small neighbourhood. If we now write $\xi^{n+1} = x^{n+1}, \ldots, \xi^m = x^m$, the new local coordinates in a neighbourhood of the point p become $(x^1, \ldots, x^n, x^{n+1}, \ldots, x^m)$. Thus, the local coordinates in the neighbourhood of the image point $\phi(p)$ takes the form $\boldsymbol{\eta} = \mathbf{y} = (x^1, \ldots, x^n)$.

If $n > m$, the rank of the Jacobian matrix is m. Let us now rearrange the variables in such a way that a $m \times m$ square submatrix $[\partial\eta^i/\partial\xi^j]$, $i, j = 1, \ldots, m$ of the Jacobian matrix has a non-zero determinant. We now choose the new local coordinates in a neighbourhood of the point q as $x^i = \Phi^i(\xi^1, \ldots, \xi^m), \ i = 1, \ldots, m$. Then, we can uniquely determine smooth solutions $\xi^i = \Psi^i(x^1, \ldots, x^m), i = 1, \ldots, m$. Thus, we can define the new local coordinates in a neighbourhood of the point $\phi(p)$ by $\eta^i = y^i = x^i$, $i = 1, \ldots, m$ and $y^i = \eta^i - \Phi^i(\xi^1, \ldots, \xi^m) = \eta^i - \Omega^i(y^1, \ldots, y^m), \ i = m+1, \ldots, n$ where $\Omega = \Phi \circ \Psi$. However, because of the initial relations $\eta^i = \Phi^i(\xi^1, \ldots, \xi^m)$, $i = m+1, \ldots, n$, we immediately see that we are led to $\mathbf{y} = (x^1, \ldots, x^m, 0, \ldots, 0)$. □

2.4. SUBMANIFOLDS

Let $\phi : M \to N$ be a smooth mapping between manifolds M^m and N^n. If $m \geq n$ and the rank of ϕ at every point $p \in M$ is n, then the mapping ϕ is called a ***submersion***. In this case, Theorem 2.3.1 indicates that the local representation Φ of ϕ is simply expressible as follows

$$y^1 = x^1, \ y^2 = x^2, \ \ldots \ , \ y^n = x^n$$

with an appropriate choice of coordinates.

Example 2.4.1. The mapping $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ is given by the relations

$$y^1 = x^2 - x^3, \ y^2 = x^1.$$

Jacobian matrix of this mapping is

$$\mathbf{J} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

and its rank is 2 everywhere. Thus ϕ is a submersion. ■

Example 2.4.2. Let $U \subseteq \mathbb{R}^3$ be an open set. Hence U is a 3-dimensional differentiable manifold. Jacobian matrix of a mapping $\phi : U \to \mathbb{R}$ is of course given by

$$\mathbf{J} = \left[\frac{\partial \phi}{\partial x^1} \quad \frac{\partial \phi}{\partial x^2} \quad \frac{\partial \phi}{\partial x^3} \right].$$

If ϕ has at least one non-vanishing partial derivative at each point of U, then the rank of this matrix is 1. In this case ϕ is a submersion. As an example let us choose the open set $U = \{\mathbf{x} \in \mathbb{R}^3 : (x^1)^2 + (x^2)^2 + (x^3)^2 > 0\}$ and the mapping given by $\phi(\mathbf{x}) = (x^1)^2 + (x^2)^2 + (x^3)^2$. The Jacobian matrix of this mapping is $\mathbf{J} = 2[\, x^1 \quad x^2 \quad x^3 \,]$ whose entries cannot be all zero in U. Thus ϕ is a submersion. On the other hand, the Jacobian matrix for the mapping $\phi_1(\mathbf{x}) = x^1 x^2 x^3$ is $\mathbf{J} = [\, x^2 x^3 \quad x^1 x^3 \quad x^1 x^2 \,]$. All entries of this matrix may vanish at some points of U (for instance, at $x^1 = x^2 = 0$, $x^3 \neq 0$). At such kind of points the rank of $\mathbf{J}$ is 0. Hence, the mapping $\phi_1 : U \to \mathbb{R}$ is not a submersion. ■

Let $\phi : M \to N$ be a smooth mapping. If $n \geq m$ and the rank of ϕ at every point $p \in M$ is m, then the mapping ϕ is called an ***immersion***. Again, Theorem 2.3.1 implies that the local representation Φ of ϕ is expressible now in the form

$$y^1 = x^1, y^2 = x^2, \ldots, y^m = x^m, y^{m+1} = 0, \ldots y^n = 0$$

with an appropriate choice of coordinates.

Example 2.4.3. The mapping $\phi : \mathbb{R} \to \mathbb{R}^2$ is defined by the relations $y^1 = \cos x^1, y^2 = \sin x^1$. Obviously, this mapping wraps the entire real axis $\mathbb{R}$ on the unit circle $\mathbb{S}^1$. The Jacobian matrix of this mapping becomes $\mathbf{J} = [\, -\sin x^1 \quad \cos x^1 \,]$. The rank of this matrix is 1 everywhere on $\mathbb{R}$. Hence ϕ is an immersion. Since all the points $x^1_n = x^1 + 2n\pi, n \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers, are mapped on the same point $\mathbf{y} = (y^1, y^2)$ the mapping ϕ is obviously not injective. ■

Example 2.4.4. The mapping $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ revolves the plane curve $x^1 = f(x^2)$ infinitely many times about x^2-axis. $f(x^2) > 0$ is a smooth function. This mapping can be prescribed by the relations

$$y^1 = f(u) \cos v, \;\; y^2 = f(u) \sin v, \;\; y^3 = u; \;\; u, v \in \mathbb{R}$$

where we wrote $x^1 = u, x^2 = v$. The Jacobian matrix is then given by

$$\mathbf{J} = \begin{bmatrix} f'(u)\cos v & -f(u)\sin v \\ f'(u)\sin v & f(u)\cos v \\ 1 & 0 \end{bmatrix}.$$

Since $f(u) > 0$, the rank of this matrix is 2 everywhere. Thus ϕ is an immersion. Clearly, it is not injective. ■

Example 2.4.5. We consider the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. The circle $\mathbb{S}^1$ may be represented by complex numbers with constant modulus in the complex plane $\mathbb{C}$. Therefore, we can write

$$\mathbb{T}^2 = \{(z_1, z_2) : z_1, z_2 \in \mathbb{C}, |z_1| = a, |z_2 - a| = b, b < a\} \subset \mathbb{C}^2.$$

We define a mapping $\phi : \mathbb{R} \to \mathbb{T}^2$ by the relations $z_1 = ae^{it}, z_2 - a = be^{irt}$ where r is a *rational number*. We can observe at once that this mapping is an immersion and it produces a closed curve on the torus. In fact, if choose the integer m and n such that $n = mr$ we reach to the same points

$$e^{it+2\pi mi} = z_1,\ be^{irt+2\pi ni} = z_2 - a$$

at all points $t_m = t + 2\pi m$. This means that we reach to the same point on the torus after having revolved m times the point z_1 and n times the point z_2 about O. This immersion is clearly not injective. ■

If $\phi : M \to N$ is an *injective immersion* and if the surjective, consequently, bijective mapping $\phi : M \to \phi(M) \subseteq N$ is a homeomorphism with respect to the relative topology on $\phi(M) \subseteq N$ generated by the topology on the manifold N, then the mapping ϕ is called an ***embedding***.

If the set M^m is a topological subspace of the manifold N^n and the *inclusion mapping* $\mathcal{I} : M \to N$ defined by $\mathcal{I}(p) = p \in N$ for each $p \in M$ is an embedding, then the subpace M is called a ***submanifold*** of dimension $m \leq n$ of the manifold N. Indeed, we can readily generate a differentiable structure on M by making use of the differentiable structure on the manifold N. Let us consider a point $p \in M \subseteq N$. This point is located in a chart (U, φ) of the atlas on N. $U' = U \cap M$ is an open set of M in the relative topology. The mapping $\varphi' = \varphi \circ \mathcal{I} : U \cap M \to \mathbb{R}^n$ is a homeomorphism because it is the composition of two homeomorphisms. Let us denote the set of coordinates of the point p in N by **x** and the set of coordinates in M by **y**. As is well known, we write the expression $\mathbf{y} = \mathfrak{I}(\mathbf{x})$ where we define

$$\mathfrak{I} = \varphi' \circ \mathcal{I} \circ \varphi^{-1}.$$

Since the rank of $\mathcal{I}$ is $m < n$ and rank remains invariant under composition of homeomorphisms, the rank of the mapping $\mathfrak{I} : \mathbb{R}^n \to \mathbb{R}^n$ is also $m < n$.

This means that an appropriate choice of coordinates leads to local coordinates $\mathbf{y} = (x^1, \ldots, x^m, 0, \ldots, 0) \in \mathbb{R}^m$ [Theorem 2.3.1]. Hence, one has $\varphi' : U' \to \mathbb{R}^m$. Consequently, the topological subspace M is an m-dimensional differentiable submanifold. Let us now denote $\varphi'(p) = \boldsymbol{\xi} \in \mathbb{R}^m$, $\boldsymbol{\xi} = (\xi^1, \ldots, \xi^m)$ for a point $p \in U' \subseteq M$. Then the equality $p = \varphi^{-1}(\mathbf{x}) = \varphi'^{-1}(\boldsymbol{\xi})$ yields the coordinate transformation $\mathbf{x} = (\varphi \circ \varphi'^{-1})(\boldsymbol{\xi}) = \psi(\boldsymbol{\xi})$ where the mapping $\psi : \mathbb{R}^m \to \mathbb{R}^n$ is expressed by $x^i = \psi^i(\xi^1, \ldots, \xi^m)$, $i = 1, \ldots, n$. These relations describe fully the submanifold M. Evidently, the rank of the matrix $\left[\partial x^i / \partial \xi^\alpha\right], \alpha = 1, \ldots, m$ should be m.

If $\phi : M^m \to N^n$ is an embedding, then the subspace $\phi(M) \subseteq N$ is an m-dimensional submanifold of the manifold N.

We take a point $p \in M$ into account and let $q = \phi(p) \in \phi(M) \subseteq N$. Because ϕ is a homeomorphism on its range $\phi(M)$, there exists a chart (U, φ) enclosing the point p of the manifold M and a chart (V, ψ) enclosing the point q of the manifold N such that the open set $\phi(U)$ is contained in the open set V. The rank of the function $\Phi = \psi \circ \phi \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ which is the local representation of the mapping ϕ is equal to the rank m of the embedding ϕ since φ and ψ are homeomorphisms. Hence, we can rewrite $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ and on the open set $V' = \phi(U) \cap V = \phi(U)$ in the relative topology we have $\psi : V' \to \mathbb{R}^m$. Thus the subspace $\phi(M) \subseteq N$ is an m-dimensional differentiable submanifold of the manifold N. In such a case we sometimes prefer to regard the manifold M as a submanifold of N even if they are actually different manifolds. □

Let the mapping $\phi : M^m \to N^n$ be a submersion. Thus the condition $m \geq n$ will hold and the rank of ϕ will become n. If $Q \subseteq N$ is a submanifold, then the subspace $P = \phi^{-1}(Q) \subseteq M$ is either a submanifold of M or it is empty.

Let us assume that $P = \phi^{-1}(Q)$ is not empty so that $Q \cap \mathcal{R}(\phi) \neq \emptyset$. Since ϕ is a submersion, we can choose the local coordinates $\mathbf{x} = \varphi(p)$ and $\mathbf{y} = \psi(q)$ of the points $p \in P$ and $q = \phi(p) \in Q$ in local charts (U, φ) and (V, ψ) in the form $x^1, x^2, \ldots, x^n, x^{n+1}, \ldots, x^m$ and $y^1 = x^1, y^2 = x^2, \ldots, y^n = x^n$. If the dimension of the submanifold Q is r with $1 \leq r \leq n$, then one can find a coordinate transformation $\mathbf{z} = F(\mathbf{y})$, or $z^1 = F^1(y^1, \ldots, y^n)$, $\ldots, z^n = F^n(y^1, \ldots, y^n)$ such that the local coordinates of the point q can be prescribed by imposing the conditions $z^{r+1} = \cdots = z^n = 0$. We now choose the local coordinates of the point p as follows:

$$\begin{aligned} w^1 &= F^1(x^1, \ldots, x^n), \ldots, w^n = F^n(x^1, \ldots, x^n), \\ w^{n+1} &= x^{n+1}, \ldots, w^m = x^m. \end{aligned}$$

Therefore, the local representation $\mathbf{z} = \Phi(\mathbf{w})$ of the mapping ϕ becomes

$z^1 = w^1, \ldots, z^n = w^n$. But the submanifold Q is determined by the conditions $z^{r+1} = \cdots = z^n = 0$. This implies that the subspace $\phi^{-1}(Q)$ in the vicinity of the point p is described by coordinates $(w^1, \ldots, w^r, 0, \ldots, 0, w^{n+1}, \ldots, w^m)$. This is tantamount to say that $\phi^{-1}(Q)$ is an $(m - n + r)$-dimensional submanifold. □

Example 2.4.6. As we have seen before, any open set of a manifold M is an open submanifold [*see p.* 77]. ■

Example 2.4.7. Let us consider a smooth function $\phi : \mathbb{R}^m \to \mathbb{R}$. We further suppose that at a point $\mathbf{x} \in \mathbb{R}^m$, at least one of the partial derivatives $\partial\phi/\partial x^i, i = 1, \ldots, n$ does not vanish. Thus the mapping ϕ is a submersion of rank 1. Since we can trivially observe that the singleton $\{0\} \subset \mathbb{R}$ is a 0-dimensional submanifold of the 1-dimensional manifold $\mathbb{R}$, then the subspace $\phi^{-1}(\{0\}) \subset \mathbb{R}^m$, that is, the set $M = \{\mathbf{x} \in \mathbb{R}^m : \phi(\mathbf{x}) = 0\}$ is an $(m - 1)$-dimensional submanifold. ■

Example 2.4.8. The function $\phi : (0, \infty) \subset \mathbb{R} \to \mathbb{R}^2$ is given by

$$\phi(t) = \Big(\phi^1(t) = t\cos\frac{1}{t},\ \phi^2(t) = t\sin\frac{1}{t}\Big) \in \mathbb{R}^2.$$

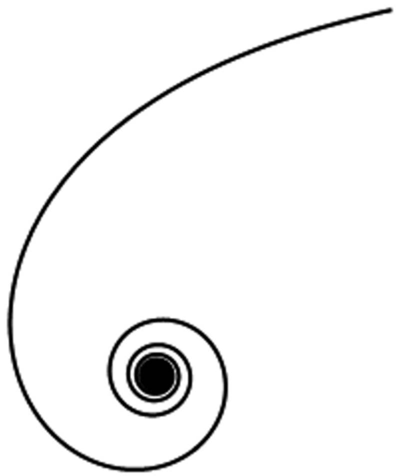

Fig. 2.4.1. Spiral in $\mathbb{R}^2$.

Hence the range $C = \phi\big((0, \infty)\big)$ of the mapping ϕ is a spiral around the point $\mathbf{0}$ in $\mathbb{R}^2$ depicted in Fig. 2.4.1. We obtain $\phi^1(t) \to \infty$ and $\phi^2(t) \to 1$ as $t \to \infty$. We can easily note that this mapping is injective and its rank is 1. Thus it is an injective immersion. The relative topology on C is defined in the usual way by means of open sets $\{C \cap V\}$ where V is an open set in $\mathbb{R}^2$. With respect to these topologies, the mappings ϕ and $\phi^{-1} : C \to (0, \infty)$

are both continuous. Hence ϕ is a homeomorphism, thus it is an embedding. Consequently C is a 1-dimensional submanifold in $\mathbb{R}^2$. ■

Submanifolds can also be determined by means of a set of equations.

Theorem 2.4.1. *We define a subset M of an n-dimensional differentiable manifold N by means of differentiable functions $f^\alpha : N \to \mathbb{R}, \alpha = 1, \dots, m$ where $m \leq n$ as follows*

$$M = \{p \in N : f^\alpha(p) = 0, \alpha = 1, \dots, m\} \subseteq N.$$

We further assume that the rank of the function $f : N \to \mathbb{R}^m$ prescribed by $f(p) = \big(f^1(p), \dots f^m(p)\big)$ is m at each point $p \in M$. In this case M proves to be a submanifold of dimension $n - m$.

Let (U, φ) be a chart containing a point $p \in M$ and let the local coordinates be $\varphi(p) = (x^1, \dots, x^n)$. Since the rank of the mapping f is m on the set M, the matrix $[\partial(f^\alpha \circ \varphi^{-1})/\partial x^i]$ has at least one $m \times m$ square submatrix whose determinant does not vanish. We may rename the variables if necessary so that this square matrix is specified by $[\partial(f^\alpha \circ \varphi^{-1})/\partial x^i], \alpha = 1, \dots, m; i = 1, \dots, m$. Hence, we can perform the following coordinate transformation

$$x'^\alpha = (f^\alpha \circ \varphi^{-1})(\mathbf{x}),\ x'^{m+j} = x^j;\ \alpha = 1, \dots, m, j = 1, \dots, n - m$$

in an open neighbourhood $U' \subseteq U$ of the point p. Thus, the local chart (U', φ') containing the point $p \in M$ yields

$$\varphi'(U' \cap M) = \{0, \dots, 0, x'^{m+1}, \dots, x'^n\}.$$

Since similar charts would exist at every point of M, this set is an $(n - m)$-dimensional submanifold. It is clear that such a submanifold may be also prescribed by a family of differentiable functions $f^\alpha(p) = c^\alpha$ where c^α's are constants. This will help us to define a family of submanifolds. □

By utilising this theorem we can readily demonstrate that $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$ is a submanifold of $\mathbb{R}^n$. The sphere with a radius R is the subset

$$\mathbb{S}^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) = \sum_{i=1}^n (x^i)^2 - R^2 = 0\}.$$

The rank of the function $f : \mathbb{R}^n \to \mathbb{R}$ is 1 at every point $\mathbf{x} \in \mathbb{S}^{n-1}$. Hence, $\mathbb{S}^{n-1}$ is an $(n-1)$-dimensional submanifold. On the other hand, the cone

$$C^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) = (x^1)^2 - \sum_{i=2}^n (x^i)^2 = 0\}$$

is not a submanifold of $\mathbb{R}^n$ because the rank of f is 0 at the point $\mathbf{x} = \mathbf{0}$, while it is 1 at all other points. Therefore, if only we delete the point $\mathbf{0}$, then the punctured cone becomes an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$.

If the mapping $\phi : M \to N$ is solely an injective immersion, then the subspace $\phi(M) \subseteq N$ is called an ***immersed manifold***. Unless the mapping ϕ is a homeomorphism on its range, an immersed manifold is obviously not a submanifold.

Example 2.4.9. Let us define the mapping $\phi : \mathbb{R} \to \mathbb{T}^2$ by the relations $z_1 = ae^{it}$ and $z_2 - a = be^{i\alpha t}$ [*see* Example 2.4.5]. Here α is now an irrational number. Hence, we find $t_1 = t_2$ when $\phi(t_1) = \phi(t_2)$. Thus ϕ is injective and its rank is 1. Consequently, it is an injective immersion and $\phi(\mathbb{R})$ becomes an immersed manifold. We can easily show that the set $M = \phi(\mathbb{R})$ is dense in $\mathbb{T}^2$. The mapping ϕ winds the line $\mathbb{R}$ around the torus $\mathbb{T}^2$ without ever traversing the same point on the torus again. In order to prove that the set M is dense in $\mathbb{T}^2$, we have to show that we can find a point in M that is as close as we wish to a given point in $\mathbb{T}^2$. Let us consider an arbitrary point $(ae^{i\omega}, a + be^{i\theta}) \in \mathbb{T}^2$ where $\omega, \theta \in \mathbb{R}$. The distance between the selected point in $\mathbb{T}^2$ and a point in M is given by

$$\begin{aligned}|ae^{i\omega} - ae^{it}| + |a + be^{i\theta} - a - be^{i\alpha t}| &= a|e^{i(\omega - t)} - 1| + b|e^{i(\theta - \alpha t)} - 1| \\ &= a\sqrt{2\big(1 - \cos(\omega - t)\big)} + b\sqrt{2\big(1 - \cos(\theta - \alpha t)\big)} \\ &= 2a\Big|\sin\frac{\omega - t}{2}\Big| + 2b\Big|\sin\frac{\theta - \alpha t}{2}\Big|.\end{aligned}$$

Rational numbers are dense in real numbers. Therefore, for each $\epsilon > 0$ and real numbers ω, θ, t, we can find integers p_1, q_1, m and p_2, q_2, n such that the inequalities

$$\Big|\frac{\omega - t}{4\pi} - \frac{p_1}{q_1} - m\Big| < \epsilon, \quad \Big|\frac{\theta - \alpha t}{4\pi} - \frac{p_2}{q_2} - n\Big| < \epsilon$$

are satisfied. The integers m and n are so chosen that we ought to have $|p_1/q_1| < 1$ and $|p_2/q_2| < 1$. If we now write $t_1 = t + 4\pi(p_1/q_1)$ and $t_2 = \alpha t + 4\pi(p_2/q_2)$, then the foregoing inequalities take the form

$$|\omega - t_1 - 4\pi m| < 4\pi\epsilon, \quad |\theta - t_2 - 4\pi n| < 4\pi\epsilon.$$

By introducing $t_3 = \max(t_1, t_2)$, these inequalities may be transformed into

$$|\omega - t_3 - 4\pi m| < 4\pi\epsilon, \quad |\theta - t_3 - 4\pi n| < 4\pi\epsilon.$$

Hence, for given real numbers ω, θ we can find a real number t_3 so that one obtains

$$\begin{aligned}2a\Big|\sin\frac{\omega-t}{2}\Big|&+2b\Big|\sin\frac{\theta-\alpha t}{2}\Big|\\&=2a\Big|\sin\Big(\frac{\omega-t_3}{2}-2\pi m\Big)\Big|+2b\Big|\sin\Big(\frac{\theta-t_3}{2}-2\pi n\Big)\Big|\\&<2a|\sin 2\pi\epsilon|+2b|\sin 2\pi\epsilon|<4\pi\epsilon(a+b)\end{aligned}$$

It is easy to see that the immersed manifold M is not a submanifold. In fact, under the mapping ϕ the line $\mathbb{R}$ intersects an open set in $\mathbb{T}^2$ infinitely many times. Therefore, an open set in the relative topology on M is the union of infinitely many pieces. Thus it is unbounded. This implies that the image of a bounded open set in $\mathbb{R}$ is unbounded. Hence the mapping ϕ is not continuous with respect to the relative topology, that is, it is not a homeomorphism on its range. ■

2.5. DIFFERENTIABLE CURVES

A *differentiable curve* C on an m-dimensional differentiable manifold M is defined through a differentiable (C^∞) mapping $\gamma:\mathcal{I}\to M$ where $\mathcal{I} = (a,b)\subseteq\mathbb{R}$ is an open interval on the real line. Thus, a point p of the curve $C=\gamma(\mathcal{I})\subset M$ is given by $p=\gamma(t)$, $t\in\mathcal{I}$. The interval must be open in order to secure differentiability at neighbourhoods of endpoints. If the curve is defined on a closed interval $[a,b]$, then we shall have to assume that the mapping γ admits a C^∞ extension $\overline{\gamma}:(a-\epsilon,b+\epsilon)\to M$ for a number $\epsilon>0$ so that

$$\overline{\gamma}(t)=\gamma(t),\ \ t\in[a,b].$$

To realise the local representation of any point $p=\gamma(t)$ of the curve, it suffices to consider a chart (U,φ) enclosing the point $p\in M$. The locus of the points $\mathbf{x}(t)=\varphi\big(\gamma(t)\big)\subset\mathbb{R}^m$ is the local representation of a part of the curve C in the open set $\varphi(U)\subseteq\mathbb{R}^m$. Naturally, when we move on the curve C local representations may change together with charts taken into consideration. By employing the coordinate functions $\varphi^i=g^i\circ\varphi:U\to\mathbb{R}$, $i=1,\cdots,m$ [*see p.* 71] the parametric representation of the curve C in the open set $\varphi(U)$ is provided by functions $x^i=\varphi^i\big(\gamma(t)\big)=\gamma^i(t)$ in local coordinates where we have defined the mappings $\gamma^i=\varphi^i\circ\gamma:\mathcal{I}\subseteq\mathbb{R}\to\mathbb{R}$, $i=1,\cdots,m$. Since γ is a differentiable mapping, the functions $\gamma^i(t)$ have clearly derivatives of all orders with respect to t. If at every point on the curve, at least one of the first order derivatives does not vanish, then the rank of the mapping γ is 1. In this case, γ becomes an immersion. But the curve may intersect itself, thus we cannot claim that this immersion is injective (Fig. 2.5.1).

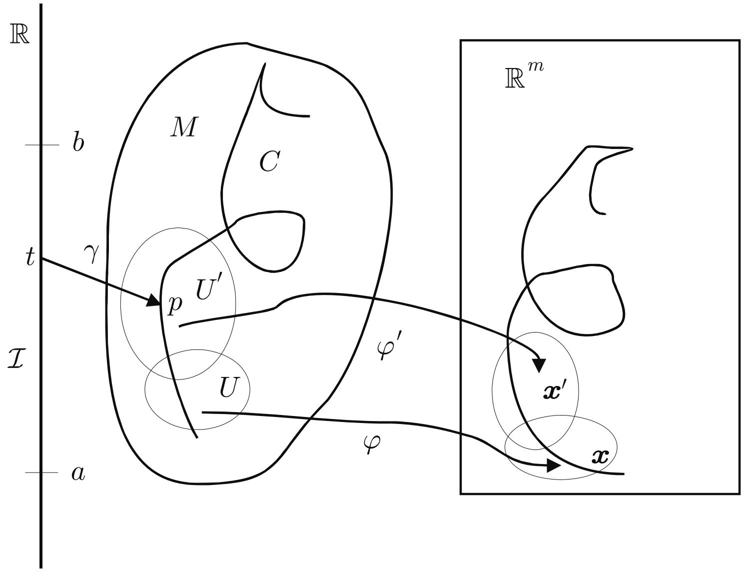

Fig. 2.5.1. A curve on a differentiable manifold.

If the curve C is defined on a closed interval $\mathcal{I} = [a, b]$, we call the points $p_a = \gamma(a)$ and $p_b = \gamma(b)$ the *initial point* and the *end point* of the curve, respectively. We get a *closed curve* if $\gamma(a) = \gamma(b)$. A *simple closed curve* is a closed curve defined on $[a, b]$, however, γ must be an injective mapping on the half-open interval $[a, b)$.

Example 2.5.1. A mapping $\boldsymbol{\gamma} : [0, 2\pi] \to \mathbb{R}^2$ is prescribed by functions $x^1 = \cos t, x^2 = t \sin 2t$. The closed curve in $\mathbb{R}^2$ generated by this mapping is shown in Fig. 2.5.2. We observe that this curve intersects itself. Therefore γ is not an injective mapping. Moreover, it has a corner point.

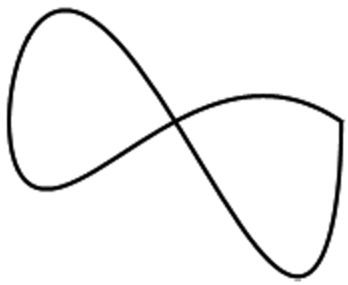

Fig. 2.5.2. A closed curve. ■

2.6. VECTORS. TANGENT SPACES

Our aim in defining tangent vectors and the tangent space formed by these vectors at a point p on a differentiable manifold is essentially twofold: (i) to extend the concept of directional derivative of a differentiable function with which we are quite familiar in the Euclidean space to differentiable manifolds, (ii) to be able to specify differentiability properties of various quantities at the vicinity of the point p as independent of local coordinates and to approximate the manifold locally by a linear vector space. A differentiable manifold does generally not possess the structure of a vector space. Thus vector spaces cannot be incorporated globally into such a manifold. Hence, we shall try to manage this task locally. Our first endeavour will be to find a tangible way that help define tangent vectors at a point p of a finite-dimensional manifold. To this end, we take all curves through the point p on the manifold into account and we specify all vectors at this point on the manifold by means of tangent vectors at the image point of curves obtained by making use of the local representations of these curves in the Euclidean space. Thus, all curves that are tangent to one another at the point p will generate the same vector. We now define a relation on the set of all curves through the point p of the manifold as *being tangent at the point p.* We can readily verify that this is an equivalence relation. Indeed, we see immediately that this relation is *reflexive* (each curve is tangent to itself), *symmetric* (if the curve C_1 is tangent to C_2, then C_2 is tangent to C_1 as well) and *transitive* (if C_1 is tangent to C_2 and C_2 to C_3, then the curve C_1 is obviously tangent to the curve C_3). Hence, all curves through the point p are partitioned into disjoint equivalence classes. All curves in an equivalence class are tangent to one another at the point p, therefore they possess the same tangent vector. We can thus try to identify tangent vectors at a point p of the manifold with equivalence classes of curves through this point. We define the set of equivalence classes, namely, the quotient set as the ***tangent space*** at the point p. We shall now attempt to provide these somewhat abstract ideas with a fully concrete content.

Let us consider a point p on the manifold M^m and a curve C through this point specified by the mapping $\gamma : \mathcal{I} \to M$. We so choose the parameter t of the curve as $p = \gamma(0)$. We know that in the classical analysis, the tangent vector to the curve C at the point p is found by means of differentiation with respect to the parameter. However, it is not possible to apply the usual differentiation operation on a general manifold. Thus we opt to transfer this operation on $\mathbb{R}^m$ by employing a local chart. Let (U, φ) be a chart containing the point p. In terms of local coordinates provided by this chart, local representation C' of the curve C in $\mathbb{R}^m$ is determined parametrically through the differentiable functions $\gamma^i : \mathcal{I} \to \mathbb{R}$ as follows:

$$\overline{x}^i = \gamma^i(t),\ i = 1, \dots, m. \tag{2.6.1}$$

The local coordinate of the point p is supposed to be $x^i = \gamma^i(0)$. (2.6.1) can now be collectively written as

$$\overline{\mathbf{x}} = \boldsymbol{\gamma}(t) = \gamma^i(t)\mathbf{e}_i$$

where the vectors $\mathbf{e}_i = (0, \dots, 0, \underset{i}{1}, 0, \dots 0)$, $i = 1, \dots, m$ are standard basis vectors for the vector space $\mathbb{R}^m$. As is well known, a tangent vector to the curve C' at a point is specified by its components $\overline{v}^i$ defined by

$$\overline{\mathbf{v}}(t) = \frac{d\overline{\mathbf{x}}}{dt} = \overline{v}^i(t)\mathbf{e}_i,\ \ \overline{v}^i(t) = \frac{d\overline{x}^i}{dt} = \frac{d\gamma^i}{dt}.$$

Thus, the tangent vector to the curve C' at the point $\mathbf{x} = \varphi(p) \in \mathbb{R}^m$ is given by

$$\mathbf{v} = v^i\mathbf{e}_i,\quad v^i = \frac{d\gamma^i}{dt}\bigg|_{t=0},\quad i = 1, \dots, m. \tag{2.6.2}$$

Since $\gamma^i(t)$ are all smooth functions they can be expanded into a Maclaurin series about the point $t = 0$ [after Scottish mathematician Colin Maclaurin (1698-1746)]. Thus we can write

$$\overline{x}^i = \gamma^i(t) = \gamma^i(0) + \frac{d\gamma^i}{dt}\bigg|_{t=0} t + \frac{1}{2}\frac{d^2\gamma^i}{dt^2}\bigg|_{t=0} t^2 + \cdots = x^i + v^i t + o(t).$$

where the *Landau symbol* $o(t)$ [after German mathematician Edmund Georg Hermann Landau (1877-1938)] represents all functions f satisfying the relation $f(t)/t \to 0$ as $t \to 0$. Another curve through the point $\varphi(p)$ can be represented in a similar fashion by expressions

$$\widetilde{\gamma}^{\,i}(t) = x^i + \widetilde{v}^{\,i}t + o(t),\quad \widetilde{v}^{\,i} = \frac{d\widetilde{\gamma}^{\,i}}{dt}\bigg|_{t=0}.$$

Therefore the *difference* between those two curves is found to be

$$\widetilde{\gamma}^{\,i}(t) - \gamma^i(t) = (\widetilde{v}^{\,i} - v^i)t + o(t).$$

If those two curves are tangent to one another at the point $\varphi(p)$ and have a common tangent vector, then one obtains $\widetilde{v}^{\,i} = v^i$. This, of course, leads to $\widetilde{\gamma}^{\,i}(t) - \gamma^i(t) = o(t)$. Hence, the closeness of two such curves is of second order. It is clear that a relation so defined is an equivalence relation. (2.6.2) implies that tangent vectors at a point **x** of $\mathbb{R}^m$ constitute an m-dimensional linear vector space. This vector space is called the *tangent space* at the point **x** of $\mathbb{R}^m$ and is denoted by $T_{\mathbf{x}}(\mathbb{R}^m)$. We see at once that the tangent space $T_{\mathbf{x}}(\mathbb{R}^m)$ and $\mathbb{R}^m$ are isomorphic. The isomorphism $\mathbb{R}^m \to T_{\mathbf{x}}(\mathbb{R}^m)$ is provided by the linear mapping that assigns a vector $\mathbf{v} = v^i\mathbf{e}_i \in T_{\mathbf{x}}(\mathbb{R}^m)$ to an ordered m-tuple $(v^1, \dots, v^m) \in \mathbb{R}^m$.

The above approach makes it possible to identify curves tangent to one another at a point p on M as images of curves tangent to one another at the point $\varphi(p)$ in the open set $\varphi(U)$ under the homeomorphism φ^{-1}. We interpret an equivalence class of curves so formed as a tangent vector at a point $p \in M$ to the manifold M. However, since M is generally not endowed with a vector space structure we cannot emplace such vectors into the manifold in the usual sense. In order to achieve this, we have to develop a new but equivalent concept. For this purpose, the classical notion of directional derivative of a function turns out to be very helpful.

We had denoted the set of smooth functions $f : M \to \mathbb{R}$ on a manifold M by $C(M)$. We have seen that this set is an algebra [*see p.* 94]. Henceforth we denote this algebra by $\Lambda^0(M)$.

Let a point $p \in M^m$ be contained in the chart (U, φ). In a neighbourhood of the image point $\mathbf{x} = \varphi(p) \in \varphi(U) \subseteq \mathbb{R}^m$ we define an operator $V'_{\mathbf{x}} : \Lambda^0(\mathbb{R}^m) \to \mathbb{R}$ at that point as follows: this operator will assign a real number to each smooth function $f' \in \Lambda^0(\mathbb{R}^m)$ in association with a given vector $\mathbf{v}(\mathbf{x}) = v^i(\mathbf{x})\mathbf{e}_i$ at that point or, in other words, with a curve C' tangent to this vector at **x** by the rule

$$V'_{\mathbf{x}}(f') = \frac{df'\big(\boldsymbol{\gamma}(t)\big)}{dt}\bigg|_{t=0} = \Big(\frac{d\gamma^i(0)}{dt}\frac{\partial}{\partial x^i}\Big)f' = v^i(\mathbf{x})\frac{\partial f'(\mathbf{x})}{\partial x^i}. \quad (2.6.3)$$

We know that $V'_{\mathbf{x}}(f')$ is the directional derivative of the function f' at the point **x** along the curve C', or in the direction of the vector **v**. Hence the operator $V'_{\mathbf{x}}$ at the point **x** can be defined in the following way

$$V'_{\mathbf{x}} = v^i\frac{\partial}{\partial x^i} = \frac{d}{dt}\bigg|_{t=0}. \quad (2.6.4)$$

If there is no ambiguity, we can dispense with the subscript denoting with which point the operator is associated. It is clear from the definition that for every functions $f', g' \in \Lambda^0(\mathbb{R}^n)$ and number $\alpha \in \mathbb{R}$, we can write

$$V'(f' + g') = V'(f') + V'(g'), \quad V'(\alpha f') = \alpha V'(f').$$

Thus V' is a *linear operator* on $\mathbb{R}$. It is also evident that there corresponds a unique operator to each vector $\mathbf{v}$. It is straightforward to see that the set of all these linear operators constitutes a linear vector space. Consider the operators $V_1' = v_1^i \partial/\partial x^i$ and $V_2' = v_2^i \partial/\partial x^i$. We find that

$$\alpha_1 V_1' + \alpha_2 V_2' = (\alpha_1 v_1^i + \alpha_2 v_2^i)\frac{\partial}{\partial x^i} = v^i \frac{\partial}{\partial x^i} = V'$$

for every $\alpha_1, \alpha_2 \in \mathbb{R}$. The mapping $\mathbf{v} \to V'$ between two linear vector spaces is an isomorphism. Indeed, this mapping is linear, because we have $\mathbf{v}_1 + \mathbf{v}_2 \to V_1' + V_2'$, $\alpha\mathbf{v} \to \alpha V'$. This mapping is surjective because each operator V' is generated by a vector $\mathbf{v}$. Let us now suppose that the same operator is associated with two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Consequently, for *every* function $f' \in \Lambda^0(\mathbb{R}^n)$ one writes

$$V'(f') = v_1^i \frac{\partial f'}{\partial x^i} = v_2^i \frac{\partial f'}{\partial x^i}.$$

When we choose the function $f' = x^j$, we obtain $V'(x^j) = v_1^i \delta_i^j = v_2^i \delta_i^j$ and $v_1^j = v_2^j$ for $j = 1, \dots, m$ or $\mathbf{v}_1 = \mathbf{v}_2$. Thus the mapping is injective, hence bijective. In this case the linear vector spaced formed by operators V' is also m-dimensional. Practically, two isomorphic vector spaces can be considered as the same as far as their algebraic properties are concerned. Therefore, instead of the tangent space $T_{\mathbf{x}}(\mathbb{R}^m)$ at a point $\mathbf{x}$ we can take into consideration the isomorphic vector space formed by the operators $V_{\mathbf{x}}'$ at that point.

Let us next consider a curve C on the manifold M through the point $p \in M$ that is determined by a mapping $\gamma : I \to M$, $\gamma(0) = p$. We shall now try to designate similarly an operator V representing the tangent vector of the curve at the point p as a derivative along the curve C. Let us assume that the point p is contained in a chart (U, φ). For each function $f \in \Lambda^0(U)$, we introduce the following operator at the point p

$$V_p(f) = \frac{df\big(\gamma(t)\big)}{dt}\bigg|_{t=0} = \frac{d(f \circ \gamma)}{dt}\bigg|_{t=0}. \tag{2.6.5}$$

We determine the function $f' \in \Lambda^0(\mathbb{R}^m)$ such that $f'(\mathbf{x}) = f(p)$ at the point $\mathbf{x} = \varphi(p) \in \mathbb{R}^m$. Hence, this function is given by $f' = f \circ \varphi^{-1}$ and using the relation $f = f' \circ \varphi$, we obtain

$$V_p(f) = \frac{d(f \circ \gamma)}{dt}\bigg|_{t=0} = \frac{d(f' \circ \varphi \circ \gamma)}{dt}\bigg|_{t=0} = \frac{df'\big(\boldsymbol{\gamma}(t)\big)}{dt}\bigg|_{t=0}$$

Therefore, we can write below the defining rule for the operator V_p:

$$V_p(f) = V'_{\mathbf{x}}(f'), \quad \mathbf{x} = \varphi(p). \tag{2.6.6}$$

Thus, the action of the operator V at the point p on a function f is uniquely determined by the components $v^i = dx^i(t)/dt$ of the tangent vector to the curve $C' = \varphi(C)$ at the point **x** with local coordinates x^i as follows:

$$V(f) = V_p(f) = \frac{df'}{dt}\bigg|_{t=0} = v^i \frac{\partial f'}{\partial x^i} = v^i \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}. \tag{2.6.7}$$

(2.6.7) now amply justifies the interpretation that $V(f)$ is the derivative of the function f at a point p along a curve through this point whose tangent vector there is specified by the operator V. We can immediately conclude from the foregoing relations that if the equality $V_1(f) = V_2(f)$ holds for every function $f \in \Lambda^0(U)$, then two curves whose tangent vectors at the point $p \in M$ are given by V_1 and V_2 are tangent to one another at p. Indeed, if we insert coordinate functions $\varphi^j \in \Lambda^0(U), j = 1, \dots, m$ satisfying $\varphi^j(p) = x^j$ into (2.6.7), we find

$$v_1^i \frac{\partial x^j}{\partial x^i} = v_2^i \frac{\partial x^j}{\partial x^i} \text{ and } v_1^i \delta_i^j = v_2^i \delta_i^j$$

leading to $v_1^j = v_2^j, j = 1, \dots, m$. V is a linear operator on $\mathbb{R}$. The relations

$$\begin{aligned} V\big((f+g)(p)\big) &= V\big(f(p) + g(p)\big) = V'\big(f'(\mathbf{x}) + g'(\mathbf{x})\big) \\ &= V'\big(f'(\mathbf{x})\big) + V'\big(g'(\mathbf{x})\big) = V\big(f(p)\big) + V\big(g(p)\big) \\ V\big((\alpha f)(p)\big) &= V\big(\alpha f(p)\big) = V'\big(\alpha f'(\mathbf{x})\big) = \alpha V'\big(f'(\mathbf{x})\big) = \alpha V\big(f(p)\big) \end{aligned}$$

imply that $V(f+g) = V(f) + V(g)$ and $V(\alpha f) = \alpha V(f)$. Furthermore, the linear operator V meets the rule given first by German mathematician and philosopher Gottfried Wilhelm von Leibniz (1646-1716):

$$\begin{aligned} V\big((fg)(p)\big) &= V\big(f(p)g(p)\big) = V'\big(f'(\mathbf{x})g'(\mathbf{x})\big) = g'(\mathbf{x})V'\big(f'(\mathbf{x})\big) \\ &\quad + f'(\mathbf{x})V'\big(g'(\mathbf{x})\big) = g(p)V\big(f(p)\big) + f(p)V\big(g(p)\big) \end{aligned}$$

whence we obtain $V_p(fg) = gV_p(f) + fV_p(g)$ at a point. A linear operator satisfying this *Leibniz rule* on an algebra is called a ***derivation***. When we take notice that the action of the operator V on a function f is specified by (2.6.6), we opt for denoting this operator at the point p by

$$V_p = \frac{d}{dt}\bigg|_{t=0} = v^i \frac{\partial}{\partial x^i} \tag{2.6.8}$$

with a somewhat slight abuse of notation. As we have mentioned before, the quantity $V_p(f)$ measures the variation in a function $f \in \Lambda^0(M)$ at a point $p \in M$ along a curve C or, in other words, along an equivalence class generated by C, at that point. Let us consider a curve in $\mathbb{R}^m$ defined by

$$\boldsymbol{\gamma}^{\boldsymbol{i}}(t) = (0, \dots, 0, x^i + t, 0, \dots, 0).$$

This curve is obviously the coordinate line in Cartesian coordinates through the point $(0, \dots, 0, x^i, 0, \dots, 0)$ in $\mathbb{R}^m$. We thus obtain

$$\mathbf{v}_\mathbf{x} = (0, \dots, 0, 1, 0, \dots, 0).$$

We now define a *coordinate line* on M through the point p by the curve $C^i = \varphi^{-1}\big(\boldsymbol{\gamma}^{\boldsymbol{i}}(t)\big)$. We then conclude that the operator $\partial/\partial x^i$ helps measure the variation of a function along a coordinate line at the point p.

It is clear that all linear operators V at a point $p \in M$ forms a linear vector space. Due to the relation (2.6.8), this vector space is evidently isomorphic to the tangent space $T_\mathbf{x}(\mathbb{R}^m)$ at the point $\mathbf{x} = \varphi(p)$. Hence, its dimension is m. We call this vector space the ***tangent space*** to the manifold M at the point p and denote it by $T_p(M)$. We also regard the operators V_p as tangent vectors to the manifold M at the point p (Fig.. 2.6.1).

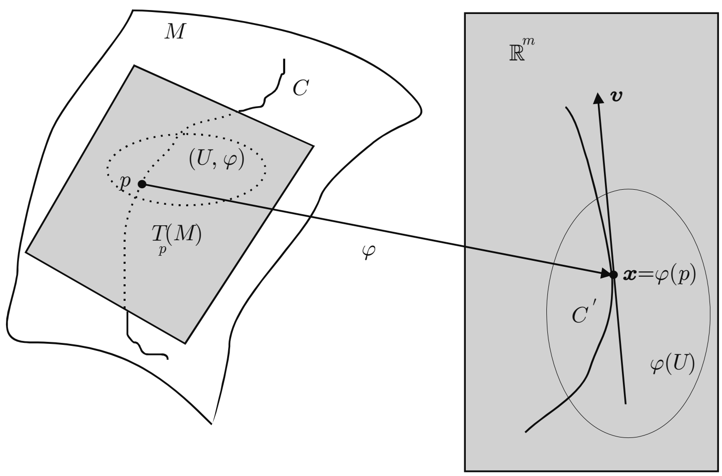

Fig. 2.6.1. Tangent space.

While having defined a vector V at a point $p \in M$ by means of the relation (2.6.8), we utilised the local coordinates provided by a chosen chart at that point. In order that this definition makes sense, we have to prove that the vector, or the operator, V is actually independent of the chosen chart.

Let us take into account two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) enclosing the point p. We denote the corresponding local coordinates by $\mathbf{x}_\alpha$ and $\mathbf{x}_\beta$, respectively. The function $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to \varphi_\beta(U_\beta)$ on the open set $U_\alpha \cap U_\beta$ gives rise to a coordinate transformation $\mathbf{x}_\beta = \varphi_{\alpha\beta}(\mathbf{x}_\alpha)$ (It is obvious that the summation convention will not be valid now on Greek indices). We have then two representations of a curve $C \subset M$ in $\mathbb{R}^m$ through the point p that is determined by the mapping $\gamma : I \to M$:

$$\boldsymbol{\gamma}_\alpha(t) = \varphi_\alpha\big(\gamma(t)\big), \ \ \boldsymbol{\gamma}_\beta(t) = \varphi_\beta\big(\gamma(t)\big).$$

But, in the vicinity of the point p, these two representations are related by

$$\boldsymbol{\gamma}_\beta(t) = \varphi_{\alpha\beta}\big(\boldsymbol{\gamma}_\alpha(t)\big)$$

whence the chain rule leads to

$$\frac{d\gamma^i_\beta}{dt} = \frac{\partial x^i_\beta}{\partial x^j_\alpha}\frac{d\gamma^j_\alpha}{dt}.$$

Thus, at $t = 0$, the components of the tangent vector in two different coordinate systems are connected by the relations

$$v^i_\beta = \frac{\partial x^i_\beta}{\partial x^j_\alpha} v^j_\alpha. \tag{2.6.9}$$

We usually call elements of the tangent space as ***contravariant vectors*** due to this rule of transformation. When we consider a function $f \in \Lambda^0(M)$, it will now have two local representations: $f(p) = f'_\alpha(\mathbf{x}_\alpha) = f'_\beta(\mathbf{x}_\beta)$. We can thus write

$$\begin{aligned} V(f) &= v^i_\beta \frac{\partial f'_\beta(\mathbf{x}_\beta)}{\partial x^i_\beta} = v^i_\beta \frac{\partial f'_\alpha(\mathbf{x}_\alpha)}{\partial x^i_\beta} = \frac{\partial x^i_\beta}{\partial x^j_\alpha} v^j_\alpha \frac{\partial f'_\alpha(\mathbf{x}_\alpha)}{\partial x^k_\alpha}\frac{\partial x^k_\alpha}{\partial x^i_\beta} \\ &= v^j_\alpha \frac{\partial f'_\alpha(\mathbf{x}_\alpha)}{\partial x^k_\alpha}\,\delta^k_j = v^j_\alpha \frac{\partial f'_\alpha(\mathbf{x}_\alpha)}{\partial x^j_\alpha} \end{aligned}$$

which shows that the vector V is expressed in the same form in both charts. Hence, the definition (2.6.8) does not depend on the chosen chart.

Theorem 2.6.1. *m-dimensional tangent space $T_p(M)$ at a point p of an m-dimensional differentiable manifold M has basis vectors, or operators, $\partial/\partial x^i$, $i = 1, \dots, m$ determined by a choice of a local chart.*

Since the vector space $T_p(M)$ is m-dimensional, the set of vectors $\{\dfrac{\partial}{\partial x^i}\}$, where $\{x^i\}$ are local coordinates, must be linearly independent in order to constitute a basis. Let us write

$$V_0 = c^i \frac{\partial}{\partial x^i} = 0$$

where c^i, $i = 1, \dots, m$ are arbitrary constants. Therefore, we ought to get $V_0(f) = 0$ for *every* function $f \in \Lambda^0(M)$. Then, if we introduce the coordinate functions $\varphi^j \in \Lambda^0(M), j = 1, \dots, n$ into that expression, we find that

$$c^i \frac{\partial x^j}{\partial x^i} = c^i \delta_i^j = c^j = 0, \;\; j = 1, \dots, m.$$

Consequently, the set $\{\partial/\partial x^i\}$ is linearly independent. □

The set $\{\partial/\partial x^i\}$ at the point p is called the ***natural basis*** or ***coordinate basis*** of the tangent space $T_p(M)$. The local coordinates generating this basis will sometimes be called ***natural coordinates***. Let

$$V = v^i \frac{\partial}{\partial x^i}$$

be a tangent vector at the point p. We then obtain for a coordinate function

$$V(\varphi^k) = v^i \frac{\partial x^k}{\partial x^i} = v^i \delta_i^k = v^k. \tag{2.6.10}$$

Thus, we can write

$$V = V(\varphi^i) \frac{\partial}{\partial x^i}. \tag{2.6.11}$$

Evidently, there is an isomorphism between $T_p(M)$ and $\mathbb{R}^m$ provided by the mapping $(v^1, \dots, v^m) \to V_p$.

So far we have defined a tangent space $T_p(M)$ associated with each point of the manifold that contains all "vectors" tangent to the manifold at that point. We can construct a ***vector field*** by a set of vectors formed by choosing a vector $V_p \in T_p(M)$ at each point p of the manifold. We can denote a vector field by $V(p), p \in M$. A vector of the field at a point p can then be enounced as

$$V(p) = v^i(\mathbf{x}) \frac{\partial}{\partial x^i}, \quad \mathbf{x} = \varphi(p) \tag{2.6.12}$$

by employing a chart (U, φ). We have to note that as the point p moves on the manifold, the vector field might be represented by different local coordinates originated from different charts. When we say that the coordinate cover of the manifold M is given by $(x^1, \dots, x^m)$, we actually mean the union of such coordinate systems that might be different in charts covering the manifold. If the functions $v^i(\mathbf{x})$ are all smooth, then we say that V *is a smooth vector field*. When V is a smooth vector field, we deduce that it has the form $V : \Lambda^0(M) \to \Lambda^0(M)$ as a linear operator.

2.7. DIFFERENTIAL OF A MAP BETWEEN MANIFOLDS

Let M^m and N^n be two differentiable manifolds and $\phi : M \to N$ be a differentiable mapping. We know that to each smooth function $g \in \Lambda^0(N)$ there corresponds a smooth function $f = \phi^* g \in \Lambda^0(M)$ [*see* p. 98]. The mapping $\phi^* : \Lambda^0(N) \to \Lambda^0(M)$ is generated by ϕ in the form $\phi^* g = g \circ \phi$ for all $g \in \Lambda^0(N)$. We now try to find a mapping $\phi_* : T_p(M) \to T_{\phi(p)}(N)$ in conjunction with the mapping ϕ that transforms the equivalence class of curves that are tangent at a point $p \in M$ into an equivalence class of curves that are tangent at the point $q = \phi(p) \in N$. Let us now choose a vector $V \in T_p(M)$ and determine a vector $V^* \in T_{\phi(p)}(N)$ such that the equality

$$V(\phi^* g) = V(g \circ \phi) = V^*(g) \tag{2.7.1}$$

is to be satisfied for *all* functions $g \in \Lambda^0(N)$. We can also express this relation for all $g \in \Lambda^0(N)$ as follows:

$$(\phi_* V)(g) = V(\phi^* g), \quad \phi_* : T_p(M) \to T_{\phi(p)}(N) \tag{2.7.2}$$

where $V^* = \phi_* V$. The mapping ϕ_*, which will also be denoted occasionally by $d\phi$, is called the ***differential*** of the mapping ϕ at the point p.

Let us assume that a curve C on a manifold M is specified by a mapping $\gamma : \mathcal{I} \to M$. We also suppose that $0 \in \mathcal{I}$ and $p = \gamma(0)$. The image C^* of the curve C in the manifold N under the mapping ϕ is given by the mapping $\gamma^* = \phi \circ \gamma : \mathcal{I} \to N$. We consider a vector V that is tangent to the curve C at the point p. For any function $g \in \Lambda^0(N)$, we can write

$$\begin{aligned} V(g \circ \phi) &= \frac{d\big((g \circ \phi) \circ \gamma\big)}{dt}\bigg|_{t=0} = \frac{d\big(g \circ (\phi \circ \gamma)\big)}{dt}\bigg|_{t=0} \\ &= \frac{d(g \circ \gamma^*)}{dt}\bigg|_{t=0} = V^*(g). \end{aligned} \tag{2.7.3}$$

Here we make use of the associativity of the composition. We deduce from the relation (2.7.3) that the vector V^* is tangent to the image curve $C^* = \phi(C)$ at the point $\phi(p) \in N$.

ϕ_* is a linear operator on real numbers. In fact, if we consider a real number α and vectors $V_1, V_2 \in T_p(M)$, we see that ϕ_* obeys the rules

$$\begin{aligned}\phi_*(V_1+V_2)(g) &= (V_1+V_2)(\phi^*g) = V_1(\phi^*g)+V_2(\phi^*g)\\ &= \phi_*V_1(g)+\phi_*V_2(g) = (\phi_*V_1+\phi_*V_2)(g)\\ \phi_*(\alpha V)(g) &= \alpha V(\phi^*g) = \alpha\phi_*(V)(g)\end{aligned}$$

for all functions $g \in \Lambda^0(N)$. That proves the linearity of ϕ_* at the point p:

$$\begin{aligned}&\phi_*(V_1+V_2) = \phi_*V_1+\phi_*V_2,\\ &\phi_*(\alpha V) = \alpha\phi_*V.\end{aligned}$$

We now manage to endow the operator ϕ_* so defined in the above with a more concrete structure by utilising local charts in manifolds M and N. Let us assume that the point $p \in M$ belongs to a chart (U, φ), and the point $q = \phi(p) \in N$ belongs to a chart (V, ψ). We denote the local coordinates by $\mathbf{x} = \varphi(p)$, $\mathbf{y} = \psi(q) = (\psi\circ\phi)(p) = (\psi\circ\phi\circ\varphi^{-1})(\mathbf{x}) = \Phi(\mathbf{x})$ from which we can deduce that $\phi = \psi^{-1}\circ\Phi\circ\varphi$. Thus, the local coordinates of corresponding points under the mapping ϕ are functionally related by $y^\alpha = \Phi^\alpha(x^1, \dots, x^m)$, $\alpha = 1, \dots, n$. By means of functions

$$\begin{aligned}(g\circ\phi)' &= g\circ\phi\circ\varphi^{-1} \in \Lambda^0(\mathbb{R}^m),\\ g' &= g\circ\psi^{-1} \in \Lambda^0(\mathbb{R}^n)\end{aligned}$$

where $g \in \Lambda^0(N)$, we find that $(g\circ\phi)' = g'\circ\psi\circ\phi\circ\varphi^{-1} = g'\circ\Phi$. Thus, for every function $g' \in \Lambda^0(\mathbb{R}^n)$, the expression (2.7.1) takes the form

$$v^{*\alpha}\frac{\partial g'(\mathbf{y})}{\partial y^\alpha} = v^i\frac{\partial g'\big(\Phi(\mathbf{x})\big)}{\partial x^i} = v^i\frac{\partial g'}{\partial y^\alpha}\frac{\partial\Phi^\alpha}{\partial x^i}.$$

which leads to the relation

$$V^* = \phi_*V = v^{*\alpha}\frac{\partial}{\partial y^\alpha} = v^i\frac{\partial\Phi^\alpha}{\partial x^i}\frac{\partial}{\partial y^\alpha} \in T_{\phi(p)}(N) \tag{2.7.4}$$

where $V = v^i\partial/\partial x^i \in T_p(M)$. Consequently, we deduce that the mapping $\phi_* : T_p(M) \to T_q(N)$ transforms a vector at the point $p \in M$ with components v^i in local coordinates to a vector at the point $q = \phi(p) \in N$ with components

$$v^{*\alpha}\big(\phi(p)\big) = \Big(\frac{\partial\Phi^\alpha}{\partial x^i}v^i\Big)(p) \tag{2.7.5}$$

in local coordinates. This transformation is governed by the Jacobian matrix $\mathbf{J}(\phi) = [\partial\Phi^\alpha/\partial x^i]$ of the mapping ϕ. If only the mapping ϕ has an inverse $\phi^{-1} : N \to M$, then the relation (2.7.5) is expressible as dependent of the point $q \in N$ so that one will then be able to write

$$v^{*\alpha}(q) = \Big[\Big(\frac{\partial \Phi^\alpha}{\partial x^i} v^i\Big) \circ \phi^{-1}\Big](q).$$

If such is the case, one readily observes that the following relation is valid

$$\phi_*(fV_1 + gV_2) = \big((\phi^{-1})^* f\big)(\phi_* V_1) + \big((\phi^{-1})^* g\big)(\phi_* V_2)$$

for any $f, g \in \Lambda^0(M)$ and $V_1, V_2 \in T_p(M)$.

A basis vector

$$\frac{\partial}{\partial x^i} = \delta_i^j \frac{\partial}{\partial x^j}$$

in $T_p(M)$ is transformed in view of (2.7.4) by the operator ϕ_* to a vector

$$\phi_*\Big(\frac{\partial}{\partial x^i}\Big) = \delta_i^j \frac{\partial \Phi^\alpha}{\partial x^j} \frac{\partial}{\partial y^\alpha} = \frac{\partial \Phi^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha} \tag{2.7.6}$$

in $T_{\phi(p)}(N)$. Therefore, the matrix representing the linear operator ϕ_* with respect to *natural bases* at the points p and q is the Jacobian matrix $\mathbf{J}(\phi)$. Obviously, the rank of the matrix $\mathbf{J}(\phi)$ at a point $p \in M$ gives the number of linearly independent vectors in the tangent space $T_{\phi(p)}(N)$. If the linear operator $\phi_* = d\phi$ at the point $p \in M$ is surjective, then the rank of $\mathbf{J}(\phi)$ is n. If ϕ_* is injective, the rank of $\mathbf{J}(\phi)$ is m. In that case, ϕ is a submersion if ϕ_* is surjective at every point $p \in M$, whereas ϕ is an immersion if ϕ_* is injective everywhere. When $m = n$ and $\det \mathbf{J}(\phi) \neq 0$, then ϕ_* is an isomorphism and there is an inverse $(\phi_*)^{-1} : T_q(N) \to T_{\phi^{-1}(q)}(M)$ at the point $q \in N$ which is clearly represented with respect to natural bases by the inverse matrix $\mathbf{J}^{-1}$. This means that the equation $\mathbf{y} = \Phi(\mathbf{x})$ has a differentiable inverse $\mathbf{x} = \Phi^{-1}(\mathbf{y})$ in a neighbourhood of the point q in accordance with the celebrated inverse mapping theorem. We can now introduce the mapping $\psi = \varphi^{-1} \circ \Phi^{-1} \circ \varphi' : N \to M$. Then we immediately obtain the composition $\phi \circ \psi = \varphi'^{-1} \circ \Phi \circ \varphi \circ \varphi^{-1} \circ \Phi^{-1} \circ \varphi' = i_N$. Similarly, we come up with $\psi \circ \phi = i_M$ implying that $\psi = \phi^{-1}$ and ψ is differentiable. We thus conclude that *the mapping ϕ becomes a local diffeomorphism at the point $p \in M$ if ϕ_* is an isomorphism at p.*

A point $q \in N$ is called a ***regular value*** of the smooth mapping ϕ if $d\phi : T_p(M) \to T_q(N)$ is surjective at every point p such that $q = \phi(p)$. A point $p \in M$ is then called a ***regular point*** of ϕ if $d\phi : T_p(M) \to T_{\phi(p)}(N)$ is surjective. A point $q \in N$ that is not a regular value is called a ***critical value*** of ϕ. If q is such a point, then the rank of $\mathbf{J}(\phi)$ at points p satisfying $q = \phi(p)$ is less than n. A point $p \in M$ is then called a ***critical point*** of ϕ if

$d\phi$ is not surjective at that point. An important theorem known as the ***Sard theorem*** [after American mathematician Arthur Sard (1909-1980)] states that for second countable manifolds critical values constitute a *null* subset (a set of measure zero) of the manifold N.

Let an m-dimensional smooth manifold M be ***second countable****, and consequently,* ***separable****. It can be demonstrated that such a manifold can be immersed in at most $2m$-dimensional Euclidean space $\mathbb{R}^{2m}$ ($\mathbb{R}^{2m-1}$ if $m > 1$), or it can be embedded in at most $(2m+1)$-dimensional Euclidean space $\mathbb{R}^{2m+1}$ ($\mathbb{R}^{2m}$ if M is not an analytical manifold). These results are known as* ***Whitney's theorems***. We confine ourselves only to say a little bit about the proof. We assume that an m-dimensional manifold M has transversal self-intersections. The main idea of the proof is the possibility of removing self-intersections by embedding the space $\mathbb{R}^m$ into a higher dimensional Euclidean space. Whitney has shown that one can construct an immersion $\phi : M^m \to \mathbb{R}^{2m}$ by removing all self-intersections or double-points and then resorting to the Sard theorem. Since M is locally homeomorphic to $\mathbb{R}^m$, Whitney has introduced a local immersion $\psi_m : \mathbb{R}^m \to \mathbb{R}^{2m}$ that is approximately linear outside of the unit ball containing a single double-point. He has further assumed that the local chart is so parametrised by $(u_1, u_2, \dots, u_m) \in \mathbb{R}^m$ that the double point is given by

$$\mathbf{x}(1, 0, \dots, 0) = \mathbf{x}(-1, 0, \dots, 0).$$

Then we easily verify that the local mapping defined by

$$\psi_m(u_1, u_2, \dots, u_m) = \Big(\frac{1}{u}, u_1 - \frac{2u_1}{u}, \frac{u_1 u_2}{u}, u_2, \cdots, \frac{u_1 u_m}{u}\Big) \in \mathbb{R}^{2m}$$

where $u = (1 + u_1^2)(1 + u_2^2)\cdots(1 + u_m^2)$ is an immersion for all $m \geq 1$ removing the double-point. In fact, we observe that

$$\psi_m(1, 0, \dots, 0) = (1, -1, 0, \cdots, 0),$$
$$\psi_m(-1, 0, \dots, 0) = (1, 1, 0, \cdots, 0).$$

Actually, it can be verified that ψ_m is an embedding except for the double-point. Furthermore, if the norm $\|\mathbf{x}(u_1, u_2, \dots, u_m)\|$ is large, then ψ_m becomes approximately the linear embedding

$$\psi_m(u_1, u_2, \dots, u_m) \approx (0, u_1, 0, u_2, \cdots, 0, u_m).$$

Let M_1, M_2, M_3 be three differentiable manifolds and $\phi_1 : M_1 \to M_2$, $\phi_2 : M_2 \to M_3$ be two differentiable mappings. Let us consider the composition $\phi_2 \circ \phi_1 : M_1 \to M_3$. For every $h \in \Lambda^0(M_3)$ and $V \in T_p(M_1)$, we can write

$$(\phi_2 \circ \phi_1)_* V(h) = V(h \circ \phi_2 \circ \phi_1) = V\big((h \circ \phi_2) \circ \phi_1\big) = (\phi_1)_* V(h \circ \phi_2)$$
$$= \big[(\phi_2)_*\big((\phi_1)_* V\big)\big](h) = \big((\phi_2)_* \circ (\phi_1)_*\big)V(h).$$

We thus conclude that

$$(\phi_2 \circ \phi_1)_* = (\phi_2)_* \circ (\phi_1)_* \text{ or } d(\phi_2 \circ \phi_1) = d\phi_2 \circ d\phi_1. \tag{2.7.7}$$

This is known as the ***chain rule***. Let us note that the relation (2.7.7) actually implies that

$$d(\phi_2 \circ \phi_1)(p) = d\phi_2\big(\phi_1(p)\big) \circ d\phi_1(p)$$

at a point p of M.

Let $i_M : M \to M$ be the identity mapping so that we have $i_M(p) = p$ for all $p \in M$. Accordingly one has $di_M : T_p(M) \to T_p(M)$. Since

$$di_M V(f) = V(f \circ i_M) = V(f)$$

for all $f \in \Lambda^0(M)$, we obtain $di_M V = V$ and finally $di_M = i_{T_p(M)}$. $i_{T_p(M)}$ is the identity operator on the vector space $T_p(M)$.

Let the mapping $\phi : M \to N$ be a diffeomorphism so that the mapping $\phi^{-1} : N \to M$ also exists and differentiable. Hence, we get $\phi^{-1} \circ \phi = i_M$, $\phi \circ \phi^{-1} = i_N$ and differentials of these mappings yield in view of (2.7.7)

$$d(\phi^{-1} \circ \phi) = d\phi^{-1} \circ d\phi = i_{T_p(M)},$$
$$d(\phi \circ \phi^{-1}) = d\phi \circ d\phi^{-1} = i_{T_{\phi(p)}(N)}.$$

We thus infer that $d\phi^{-1} = (d\phi)^{-1}$. This result implies that *the linear operator $d\phi$ between tangent spaces $T_p(M)$ and $T_{\phi(p)}(N)$ is an isomorphism since it is a regular operator if ϕ is a diffeomorphism*. If we recall the statement made in p. 122 we can obviously say that *a differentiable mapping $\phi : M \to N$ is a local diffeomorphism at a point $p \in M$ if and only if the linear operator $d\phi(p) : T_p(M) \to T_{\phi(p)}(N)$ that is the differential of ϕ is an isomorphism*. Of course, this statement will make sense if only if tangent spaces $T_p(M)$ and $T_{\phi(p)}(N)$ have the same dimension.

While defining the differential of a mapping between manifolds M and N, we come up with a rather special situation if one of these manifolds is $\mathbb{R}$. Let us first take $M = \mathcal{I}$ where $\mathcal{I} \subseteq \mathbb{R}$ is an open interval and define a curve C on the manifold N by the differentiable mapping $\gamma : \mathcal{I} \to N$. Therefore, the differential of the mapping γ at a point $t \in \mathcal{I}$ is a linear operator $d\gamma = \gamma_* : T_t(\mathcal{I}) \to T_p(N)$ where $p = \gamma(t)$. Since the tangent vector in $\mathbb{R}$ is of the form d/dt, the tangent vector to the curve C at the point $p = \gamma(t) \in N$ is given by

$$V = \gamma_* \Big(\frac{d}{dt}\Big), \ t \in \mathcal{I}$$

In view of (2.7.3), the tangent vector, say, at a point $\gamma(0) \in N$ will satisfy the relation

$$V(g) = \frac{d(g \circ \gamma)}{dt}\bigg|_{t=0}, \ \forall g \in \Lambda^0(N).$$

If we make use of the equality (2.7.4) and notice that the chart on M is simply $(\mathcal{I}, i_{\mathbb{R}})$, we obtain the tangent vector V in terms of local coordinates $y^\alpha(t) = \varphi'^\alpha\big(\gamma(t)\big)$, $\alpha = 1, \dots, n$ as

$$V = \gamma_* \Big(\frac{d}{dt}\Big) = \frac{dy^\alpha}{dt}\frac{\partial}{\partial y^\alpha}. \tag{2.7.8}$$

Let us now take $N = \mathbb{R}$ and let $\phi : M \to \mathbb{R}$ be a differentiable mapping. The chart on N is now $(\mathbb{R}, i_{\mathbb{R}})$ so it follows that $\Phi = i_{\mathbb{R}} \circ \phi \circ \varphi^{-1} = \phi \circ \varphi^{-1}$. Thus (2.7.4) yields for a vector $V \in T_p(M), p \in M$

$$V^* = \phi_* V = v^i \frac{\partial \Phi}{\partial x^i}\frac{d}{dt} = V(\phi)\frac{d}{dt}, \ \ t \in \mathbb{R}. \tag{2.7.9}$$

Since the tangent space $T_t(\mathbb{R})$ is isomorphic to the linear vector space $\mathbb{R}$, we can take as a basis vector $d/dt \mapsto 1$ and write $\phi_* : T_p(M) \to \mathbb{R}$ so that we obtain $\phi_* V = V(\phi)$. Thus this interpretation allows us to say that the number $d\phi(p)V = \phi_*(p)V$ gives the derivative of the function ϕ at the point $p \in M$ in the direction of V. In this case the operator ϕ_* assigns a real number to every vector in the tangent space $T_p(M)$. Hence, the linear operator $\phi_* = d\phi$ turns out to be actually a linear functional on $T_p(M)$ and, consequently, it can be regarded as an element of the dual space $T_p^*(M)$. Let us now consider the vector $V = \partial/\partial x^i$ whose components are simply $v^i = 1$, $v^j = 0$, $j \neq i$. We thus conclude that

$$d\phi\Big(\frac{\partial}{\partial x^i}\Big) = \frac{\partial \Phi}{\partial x^i} = \frac{\partial(\phi \circ \varphi^{-1})}{\partial x^i}.$$

We now insert the coordinate function $\phi = \varphi^j$ into the foregoing general expression. Since $\varphi^j(p) = x^j$, we obtain

$$d\varphi^j\Big(\frac{\partial}{\partial x^i}\Big) = dx^j\Big(\frac{\partial}{\partial x^i}\Big) = \frac{\partial x^j}{\partial x^i} = \delta^j_i.$$

This means that the elements $\{dx^1, \dots, dx^m\}$ constitute a *reciprocal basis* for the dual vector space $T_p^*(M)$. This results in

$$d\phi = \alpha_i dx^i,\ \ \alpha_i = d\phi\Big(\frac{\partial}{\partial x^i}\Big) = \frac{\partial \Phi}{\partial x^i}\ \text{ and }\ d\phi = \frac{\partial \Phi}{\partial x^i} dx^i$$

which coincides with the classical definition of differential of a function. If we consider a vector $V = v^i \partial/\partial x^i$, then we find that $dx^i(V) = v^i$ and $d\phi(V) = v^i \partial\Phi/\partial x^i = V(\phi)$.

Finally, let us consider the mapping $\varphi : U \to \mathbb{R}^m$ of a chart (U, φ) at a point $p \in M$. Since $\varphi(p) = \mathbf{x}$, we get $\varphi_* = d\varphi : T_p(M) \to T_{\mathbf{x}}(\mathbb{R}^m) \simeq \mathbb{R}^m$. Consequently, we can write

$$d\varphi V(f) = V(f \circ \varphi)$$

for any $f \in \Lambda^0(\mathbb{R}^m)$ and $V \in T_p(U)$. On the other hand, due to (2.6.7) and (2.6.6) we obtain

$$d\varphi V(f) = V(f \circ \varphi) = v^i \frac{\partial(f \circ \varphi \circ \varphi^{-1})}{\partial x^i} = v^i \frac{\partial f}{\partial x^i} = V'(f),$$

so we find that $d\varphi V = V'$. Thus, the operator $d\varphi$ assigns an element $\mathbf{v} = (v^1, \dots, v^m) \in \mathbb{R}^m$ to a vector $V \in T_p(U)$. It is straightforward to verify that the operator $d\varphi : T_p(U) \to \mathbb{R}^m$ is an isomorphism.

We now take into account the inverse mapping $\varphi^{-1} : \mathbb{R}^m \to U$. Then we get $d\varphi^{-1} : \mathbb{R}^m \to T_p(U)$ and we obtain $d\varphi^{-1} V'(f) = V'(f \circ \varphi^{-1})$ $= V(f)$ for all $f \in \Lambda^0(M)$ and $V' \in T_p(\mathbb{R}^m)$ yielding the relation $d\varphi^{-1} V'$ $= V$. We thus obtain $d\varphi^{-1} = (d\varphi)^{-1}$. Hence, we conclude that the mapping φ of a chart is a diffeomorphism.

2.8. VECTOR FIELDS. TANGENT BUNDLE

We have seen that we can construct a vector field on a manifold M^m by associating a vector $V(p)$ in the tangent space $T_p(M)$ to each point $p \in M$. If we choose the natural basis in each tangent space $T_p(M)$ a vector field is now expressible as

$$V(p) = v^i(\mathbf{x}) \frac{\partial}{\partial x^i}, \quad \mathbf{x} = \varphi_\alpha(p),\ p \in U_\alpha \tag{2.8.1}$$

where $(U_\alpha, \varphi_\alpha)$ is a chart and $\cup U_\alpha = M$. We know that if $v^i : \mathbb{R}^m \to \mathbb{R}$, $i = 1, \dots, m$ are all smooth functions, $V(p)$ is called a smooth vector field. Evidently, a smooth vector field is built by a linear combination of natural basis vectors with functions chosen from the set $C^\infty(\mathbb{R}^m)$. It is known that the set $C^\infty(\mathbb{R}^m)$ is a commutative ring. If we consider a non-zero function its inverse $1/f(\mathbf{x})$ does not exist at points satisfying the equation $f(\mathbf{x}) = 0$. Therefore, although the vector field V designates an element of a vector space at every point $p \in M$, it constitutes actually a ***module*** $\mathfrak{V}(M)$ on the manifold M. In fact, sum of two vector fields and multiplication of a vector field by a smooth function are again vector fields. It goes without saying that $\mathfrak{V}(M)$ reduces to a linear vector space on real numbers.

Let us define a set $T(M)$ as the union of *disjoint* tangent spaces at all points of a manifold M:

$$T(M) = \bigcup_{p\in M} T_p(M) = \{(p, V) : p \in M, V \in T_p(M)\}. \qquad (2.8.2)$$

It is obvious that this set is produced as the union of sets each of which is obtained by attaching to each point $p \in M$ the linear vector space $T_p(M)$ at that point. We shall now try to equip the set $T(M)$ with a differentiable structure of $2m$-dimension. The differentiable manifold $T(M)$ so structured will be called the ***tangent bundle*** of the manifold M. The set M is named as the ***base*** and tangent spaces as the ***fibres*** of the fibre bundle. The ***natural projection***

$$\pi : T(M) \to M, \quad \pi(p, V) = p, \;\; V \in T_p(M) \qquad (2.8.3)$$

projects every vector in a tangent space to its base point $p \in M$ to which a particular fibre is attached. It is clear that we can write $T_p(M) = \pi^{-1}(\{p\})$. Moreover, let us consider the set $\mathcal{V} = \pi^{-1}(U) \subset T(M)$ corresponding to an open set $U \in \mathfrak{M}$ where $\mathfrak{M}$ is the topology on M. Because of the properties of the set function π^{-1} we can write obviously

$$\bigcup_{U\in\mathfrak{M}} \mathcal{V} = \bigcup_{U\in\mathfrak{M}} \pi^{-1}(U) = \pi^{-1}\Big(\bigcup_{U\in\mathfrak{M}} U\Big) = \pi^{-1}(M) = T(M),$$
$$\emptyset = \pi^{-1}(\emptyset).$$

Furthermore, if Λ is an index set, we have the relations

$$\bigcup_{\lambda\in\Lambda} \mathcal{V}_\lambda = \bigcup_{\lambda\in\Lambda} \pi^{-1}(U_\lambda) = \pi^{-1}\Big(\bigcup_{\lambda\in\Lambda} U_\lambda\Big), U_\lambda \in \mathfrak{M}$$
$$\bigcap_{i=1}^{n} \mathcal{V}_{\lambda_i} = \bigcap_{i=1}^{n} \pi^{-1}(U_{\lambda_i}) = \pi^{-1}\Big(\bigcap_{i=1}^{n} U_{\lambda_i}\Big), \lambda_i \in \Lambda, U_{\lambda_i} \in \mathfrak{M}.$$

Therefore the class $\mathfrak{T} = \{\mathcal{V} = \pi^{-1}(U) : U \in \mathfrak{M}\}$ is a topology on the set $T(M)$ and $\mathcal{V}$ is an open set in $\mathfrak{T}$. It is clear that the projection π is continuous in this topology. The structure of the tangent bundle is schematically depicted in Fig. 2.8.1.

We suppose that an atlas on the manifold M is given by the family of charts $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in \mathcal{I}\}$. The set

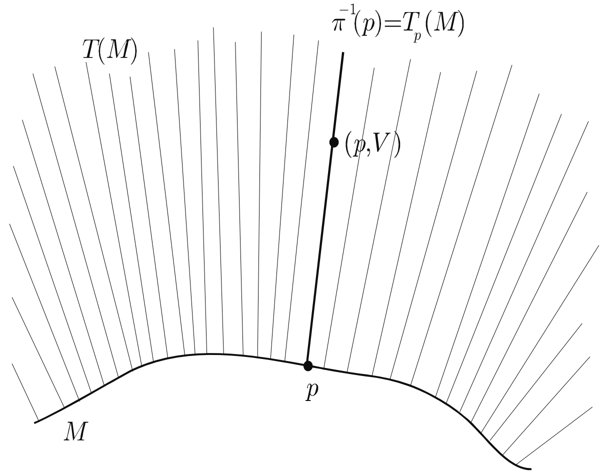

Fig. 2.8.1. Tangent bundle.

$$\mathcal{V}_\alpha = T(U_\alpha) = \pi^{-1}(U_\alpha) \subseteq T(M)$$

will be open in the topology $\mathfrak{T}$. Let us consider a point $(p, V) \in T(M)$. The point $p \in M$ will be located inside a chart $(U_\alpha, \varphi_\alpha)$ of the manifold M and the point (p, V) will be in the open set $\mathcal{V}_\alpha = \pi^{-1}(U_\alpha)$. Hence, in terms of local coordinates $\mathbf{x} = (x^1, \dots, x^m) \in \mathbb{R}^m$ in the open set $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^m$, a vector $V \in \mathcal{V}_\alpha$ is expressible as

$$V = v^i \frac{\partial}{\partial x^i}, \ \mathbf{v} = (v^1, \dots, v^m) \in \mathbb{R}^m.$$

We define the mapping $\psi_\alpha : \mathcal{V}_\alpha \to \varphi_\alpha(U_{\underline{\alpha}}) \times \mathbb{R}^m \subseteq \mathbb{R}^{2m}$ in such a way that, for all points $(p, V) \in \mathcal{V}_\alpha$ we get

$$\begin{aligned} \psi_\alpha(p, V) &= \big(\varphi_\alpha(p), d\varphi_\alpha V\big) \\ &= (x^1, \dots, x^m, v^1, \dots, v^m) \in \mathbb{R}^{2m}. \end{aligned} \tag{2.8.4}$$

It is clear that the mapping ψ_α is a homeomorphism. We shall now demonstrate that the family $\{\big(\mathcal{V}_\alpha = \pi^{-1}(U_\alpha), \psi_\alpha\big) : \alpha \in \mathcal{I}\}$ constitutes an atlas on the topological space $T(M)$. We know that $T(M) = \bigcup_{\alpha \in \mathcal{I}} \mathcal{V}_\alpha$. Let us now consider two charts $\big(\mathcal{V}_\alpha, \psi_\alpha\big)$ and $\big(\mathcal{V}_\beta, \psi_\beta\big)$ (the summation convention will of course be suspended on Greek indices). We have to prove that the transition mapping

$$\psi_{\alpha\beta} = \psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(\mathcal{V}_\alpha \cap \mathcal{V}_\beta) \subseteq \mathbb{R}^{2m} \to \psi_\beta(\mathcal{V}_\alpha \cap \mathcal{V}_\beta) \subseteq \mathbb{R}^{2m}$$

is smooth. It follows from the relation

$$\begin{aligned}(p, V) &= \psi_\alpha^{-1}(\mathbf{x}, \mathbf{v}) = \big(\varphi_\alpha^{-1}(\mathbf{x}), (d\varphi_\alpha)^{-1}(\mathbf{v})\big) \\ &= \big(\varphi_\alpha^{-1}(\mathbf{x}), d\varphi_\alpha^{-1}(\mathbf{v})\big)\end{aligned}$$

that

$$\begin{aligned}\psi_{\alpha\beta}(\mathbf{x}, \mathbf{v}) &= \big(\varphi_\beta \circ \varphi_\alpha^{-1}(\mathbf{x}), d\varphi_\beta \circ d\varphi_\alpha^{-1}(\mathbf{v})\big) \\ &= \big(\varphi_\beta \circ \varphi_\alpha^{-1}(\mathbf{x}), d(\varphi_\beta \circ \varphi_\alpha^{-1})(\mathbf{v})\big).\end{aligned}$$

Since $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \mathbb{R}^m \to \mathbb{R}^m$ is smooth, the differential mapping $d\varphi_{\alpha\beta}$ is also smooth. Thus $T(M)$ acquires a structure of a $2m$-dimensional differentiable manifold with the atlas $\{(\pi^{-1}(U_\alpha), \psi_\alpha) : \alpha \in \mathcal{I}\}$. Local coordinates of this manifold is given by $(x^1, \dots, x^m, v^1, \dots, v^m)$.

Due to the relation (2.6.9), the linear operator $d\varphi_{\alpha\beta}$ is represented by the matrix

$$d\varphi_{\alpha\beta} = \mathbf{K}_{\alpha\beta} = \left[\frac{\partial x_\beta^i}{\partial x_\alpha^j}\right]. \tag{2.8.5}$$

Hence $d\varphi_{\alpha\beta} : T_\mathbf{x}(\mathbb{R}^m) \to T_\mathbf{x}(\mathbb{R}^m)$ is an *automorphism*, *an isomorphism mapping a vector space onto itself* at a point $p \in M$. We know that we can take $T_\mathbf{x}(\mathbb{R}^m) = \mathbb{R}^m$. Therefore, denoting the *general linear group* formed by $m \times m$ regular matrices on fibres $\mathbb{R}^m$ by $GL(m, \mathbb{R})$, we infer that

$$d\varphi_{\alpha\beta} \in GL(m, \mathbb{R}).$$

$GL(m, \mathbb{R})$, or one of its subgroups $G \subseteq GL(m, \mathbb{R})$, is called the ***structural group*** of the tangent bundle. This group ascertains the global character of $T(M)$ and helps us distinguish different bundles defined over the same base space. Then we deduce that in an intersection $\mathcal{V}_\alpha \cap \mathcal{V}_\beta$ on the fibre bundle, the coordinate transformation is determined through the relations

$$\mathbf{x}_\beta = \varphi_{\alpha\beta}(\mathbf{x}_\alpha), \;\; \mathbf{v}_\beta = \mathbf{K}_{\alpha\beta}\,\mathbf{v}_\alpha, \;\; \mathbf{K}_{\alpha\beta} \in G.$$

If the bundle $T(M)$ is diffeomorphic to the product manifold $M \times \mathbb{R}^m$, it is then called *a globally trivial bundle*. Since every tangent bundle is locally represented as $U \times \mathbb{R}^m$, it is *locally trivial*. Whether this property is also valid globally depends on the structural group. If the tangent bundle is trivial, we can always choose points (p, V_0) in $M \times \mathbb{R}^m$ for all points $p \in M$ where V_0 is a *constant* vector. Hence, the inverse mapping creates a vector field in $T(M)$ that vanishes nowhere on M. This means that *the tangent bundle cannot be trivial if it is not possible to find a vector field that* *vanishes nowhere on the manifolds.*

A ***local cross section*** of the tangent bundle is the smooth mapping $\sigma : U \to T(U)$ where $U \subseteq M$ is an open set. σ must possess the property $\pi \circ \sigma = i_U$, that is, one has $\pi(\sigma(p)) = p$ for all $p \in U$. If $U = M$, then σ is called a ***global cross section***. The mapping σ will clearly assigns a vector to each point of an open submanifold of M, or M itself. Hence, it prescribes a vector field (Fig. 2.8.2).

Example 2.8.1. As the base manifold, let us choose the circle. It is straightforward to observe that one finds easily a vector field that vanishes nowhere on $\mathbb{S}^1$. Therefore $T(\mathbb{S}^1)$ is a trivial bundle and it can be represented as $\mathbb{S}^1 \times \mathbb{R}$. As a matter of fact if we choose fibres as shown in Fig. 2.8.3(a), then the tangent bundle becomes the Cartesian product of $\mathbb{S}^1$ and $\mathbb{R}$. Since $\mathbb{S}^1$ is designated by a single coordinate, the transformation of coordinates at a point p in overlapping charts are given by $x_\beta = \varphi_{\alpha\beta}(x_\alpha)$, $v_\beta = K_{\alpha\beta}\, v_\alpha$

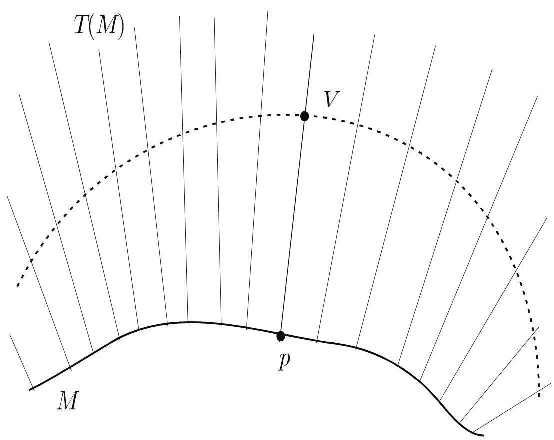

Fig. 2.8.2. Vector field as a cross section.

where the constant $K_{\alpha\beta}$ is the value of $d\varphi_{\alpha\beta}/dx_\alpha$ at p. This number is a member of the multiplication group on $\mathbb{R}$ which is also the structural group of the bundle. In order to find a simple representation let us cut the circle at a point p, unwrap the bundle and make it lie on $\mathbb{R}^2$. To assemble the bundle again all we have to do is to identify p with p', u with u' and v with v'. In this case, the transition mapping in overlapping charts is simply found as the identity mapping $(p, v) \to (p, v)$ and the structural group of the tangent bundle becomes just $\{1\}$. However, we can reassemble the bundle to form the ***Möbius band*** if we identify u with v' and v with u' by twisting the strip. In this case the tangent bundle is no longer trivial. Transition mapping in some overlapping charts is again given by $(p, v) \to (p, v)$ whereas in some others by $(p, v) \to (p, -v)$ and the structural group of the bundle is now $\{1, -1\}$.

Let us consider a Möbius band whose middle circle is situated at the plane $z = 0$, centred at the origin with radius R and its half-width is w. Its parametric equations are given by

$$x = [R + v\cos(u/2)]\cos u, y = [R + v\cos(u/2)]\sin u, z = v\sin(u/2)$$

where $0 \le u < 2\pi$ and $-w \le v \le w$. Indeed for $u = 0$ we get $x = R + v$, $y = z = 0$ while for $u = 2\pi$ we obtain $x = R - v, y = z = 0$. Thus we obtain the description described in Fig.. 2.8.3(b).

Möbius band, or strip, is named after German mathematician August Ferdinand Möbius (1790-1868) who had introduced it on September 1858. Strictly speaking, the band had already been found a little bit earlier by German mathematician Johann Benedict Listing (1808-1882) on July 1858.

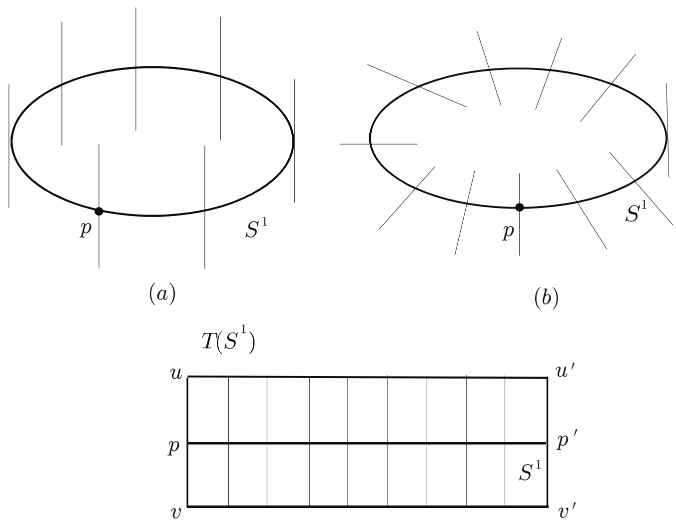

Fig. 2.8.3. Fibre bundles: (a) circle, (b) Möbius band.

Therefore, it would have been more appropriate to call it as *Listing band*. Möbius band is perhaps the most prominent example to *one-sided* and *one-edged surfaces*. In fact, when we start moving on the surface we pass eventually under the surface without crossing the edge. The representation of Möbius band in $\mathbb{R}^3$ is depicted in Fig. 2.8.4.

Let $\phi : M \to N$ be a differentiable mapping between two differentiable manifolds. The differential of ϕ can now be written as an operator between tangent bundles as $\phi_* = d\phi : T(M) \to T(N)$. However, we have to keep in mind that the linear operator $d\phi$ transforms pointwise the vector space $T_p(M)$ into the vector space $T_{\phi(p)}(N)$ and its action can only be embodied through the local charts at points p and $\phi(p)$. Let a smooth vector field in the tangent bundle $T(M)$ be V. Then we define $V^* = d\phi(V)$ by the following relation again

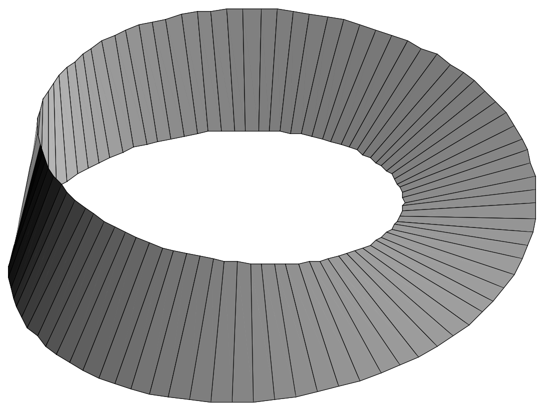

Fig. 2.8.4. Möbius band. ■

$$d\phi(V)(g)\big(\phi(p)\big) = V(g \circ \phi)(p), \quad p \in M$$

for all $g \in \Lambda^0(N)$. This implies that the diagram

$$\begin{array}{ccc} T(M) & \xrightarrow{d\phi} & T(N) \\ \pi_M \downarrow & & \downarrow \pi_N \\ M & \xrightarrow{\phi} & N \end{array}$$

is commutative, that is, $\phi \circ \pi_M = \pi_N \circ d\phi$ where $\pi_M : T(M) \to M$ and $\pi_N : T(N) \to N$ are natural projections,

If only the mapping ϕ has an inverse, then one can write

$$V^*(g)(q) = \big[V(g \circ \phi)\big] \circ \phi^{-1}(q), \quad q \in N. \qquad (2.8.6)$$

Thus only for invertible mappings, their differentials are able to assign a vector $V^*(q)$ at every point $q \in N$. In other words, if ϕ^{-1} does not exist, then the image of a vector field on M is generally not a vector field on N. *If ϕ is a diffeomorphism, then the image of a smooth vector field becomes also a smooth vector field . If ϕ^{-1} exists but not smooth, then the image is a vector field but it may not be smooth.*

2.9. FLOWS OVER MANIFOLDS

Let M^m be a smooth manifold and let $V \in T(M)$ be a given vector field. A differentiable curve described by the smooth mapping $\gamma : \mathcal{I} \to M$, $\mathcal{I} = (a, b) \subseteq \mathbb{R}$ will be called an ***integral curve*** of the vector field V, if it is tangent to the field V, i.e., if the relation

$$\gamma_*\Big(\frac{d}{dt}\Big) = V|_{\gamma(\mathcal{I})}$$

is satisfied. In dynamical system, this curve is also called a ***trajectory*** or an ***orbit***. This relation is symbolically expressed as follows:

$$\frac{d\gamma(t)}{dt} = V\big(\gamma(t)\big), \quad t \in (a, b). \tag{2.9.1}$$

We know that the image of this curve in $\mathbb{R}^m$ is determined by expressions

$$x^i = \gamma^i(t) = \varphi^i\big(\gamma(t)\big) \in \mathbb{R}, \;\; i = 1, \dots, m$$

in local coordinates.

Theorem 2.9.1. *Let a vector field V on a differentiable manifold M^m be given by*

$$V(p) = v^i(\mathbf{x})\frac{\partial}{\partial x^i}, \quad p = \varphi^{-1}(\mathbf{x}).$$

where (U, φ) is the chart to which $p \in M$ belongs. A curve $\gamma : \mathcal{I} \to M$ is an integral curve of the vector field V if and only if the coordinate functions $x^i(t)$ are solutions of the following system of local ordinary differential equations in $\mathbb{R}^m$

$$\frac{dx^i}{dt} = v^i\big(\mathbf{x}(t)\big), \;\; i = 1, \dots, m. \tag{2.9.2}$$

Indeed, if we take into consideration the relation (2.7.8), we can transform (2.9.1) into the form

$$\frac{dx^i}{dt}\frac{\partial}{\partial x^i} = v^i\big(\mathbf{x}(t)\big)\frac{\partial}{\partial x^i}$$

which can be satisfied if and only if the differential equations

$$\frac{dx^i}{dt} = v^i\big(\mathbf{x}(t)\big), \ \ i = 1, \dots, m$$

are held. □

We see that in order to find integral curves of a vector field on a manifold M^m we need to generate curves in $\mathbb{R}^m$ as solutions of differential equations (2.9.2) and then carry them on M by making use of local charts.

Let V be a smooth vector field on M. Hence, all components $v^i(\mathbf{x})$ are smooth functions. If M is also a smooth manifold, then the functions $V^i(p) = v^i(\varphi(p))$ will be smooth, either. Next, we consider a point $p_0 \in M$ and a chart (U, φ) enclosing this point. It is known from the theory of system of ordinary differential equations that [*see* Coddington and Levinson (1955), *p.* 22, Theorem 7.1] *for each point* $\mathbf{x}_0 = \varphi(p_0) \in \mathbb{R}^m$ *there exist an open set* $\mathcal{U}(\mathbf{x}_0) \subseteq \mathbb{R}^m$ *containing this point and an open interval* $\mathcal{I}(\mathbf{x}_0) \subseteq \mathbb{R}$ *so that for all* $\mathbf{x} \in \mathcal{U}(\mathbf{x}_0)$ *and* $t \in \mathcal{I}(\mathbf{x}_0)$ *the following system of ordinary differential equations*

$$\frac{d\phi^i}{dt} = v^i(\boldsymbol{\phi}) \tag{2.9.3}$$

has a unique solution $\boldsymbol{\phi}(t;\mathbf{x})$ *satisfying the initial condition* $\boldsymbol{\phi}|_{t=t_0} = \mathbf{x}_0$ *where* $t_0 \in \mathcal{I}(\mathbf{x}_0)$ *and* $\boldsymbol{\phi}$ *is a vector-valued smooth function of variables* t *and* $\mathbf{x} = (x^1, \dots, x^m)$. If $0 \in \mathcal{I}(\mathbf{x}_0)$, then we usually choose $t_0 = 0$. Thus the function $\boldsymbol{\phi}(t;\mathbf{x})$ designate a curve in $\mathbb{R}^m$ through the point $\overline{\mathbf{x}}(t_0) = \mathbf{x}$ whose equation is parametrically given by

$$\overline{\mathbf{x}}(t) = \boldsymbol{\phi}(t;\mathbf{x}) \in \mathcal{U}(\mathbf{x}_0), \ \ \boldsymbol{\phi}(t_0;\mathbf{x}) = \mathbf{x}$$

where $\mathbf{x} \in \mathcal{U}(\mathbf{x}_0)$ and $t \in \mathcal{I}(\mathbf{x}_0)$. If we fix t and write $\boldsymbol{\phi}_t(\mathbf{x}) = \boldsymbol{\phi}(t;\mathbf{x})$, then $\boldsymbol{\phi}_t : \mathcal{U}(\mathbf{x}_0) \to \mathbb{R}^m$ denotes a *family of smooth mappings* depending on the parameter $t \in \mathcal{I}(\mathbf{x}_0)$. For a fixed t, each point $\mathbf{x} \in \mathcal{U}(\mathbf{x}_0)$ is transported along the integral curve of the vector field V to the point $\boldsymbol{\phi}_t(\mathbf{x}) \in \mathbb{R}^m$ determined by this value of the parameter t. *Because of the uniqueness of the solutions such curves cannot intersect*. An open neighbourhood $U_{p_0} = \varphi^{-1}\big(\mathcal{U}(\mathbf{x}_0)\big) \subseteq M$ is associated with each point $p_0 = \varphi^{-1}(\mathbf{x}_0) \in M$ and an integral curve through a point $p \in U_{p_0}$ is characterised by

$$\phi(t;p) = \varphi^{-1} \circ \boldsymbol{\phi}\big(t;\varphi(p)\big), \ t \in \mathcal{I}_{p_0} = \mathcal{I}\big(\varphi(p_0)\big) \subseteq \mathbb{R}. \tag{2.9.4}$$

This function must of course satisfy the relation $\phi(t_0;p) = p$. Points on this curve are found by the transformation

$$\overline{p}(t) = \phi(t; p) = \phi_t(p), \;\; t \in \mathcal{I}_{p_0}.$$

We have to point out that the definition of ϕ by (2.9.4) is valid only for points p and $\overline{p}(t)$ situated in the same chart of the manifold. A new solution is required for a different chart. Therefore, the family of *local* smooth mappings $\{\phi_t : U_{p_0} \to M, t \in \mathcal{I}_{p_0}\}$ transports each point $p \in U_{p_0}$ of the manifold M along an integral curve of the vector field V through this point to the point $\phi_t(p) \in M$. Thus, in essence, the mapping ϕ should be written in the form

$$\phi : U_{p_0} \times \mathcal{I}_{p_0} \to M$$

where the set $U_{p_0} \times \mathcal{I}_{p_0}$ is an open subset of $(m+1)$-dimensional smooth manifold $M \times \mathbb{R}$. Hence, it is an $(m+1)$-dimensional smooth open submanifold. Let us now consider open neighbourhoods U_{p_λ} defined as above covering the manifold M so that $M = \bigcup_{\lambda \in \Lambda} U_{p_\lambda}$. Next, we define the interval $\mathcal{I} = \bigcap_{\lambda \in \Lambda} \mathcal{I}_{p_\lambda} \subseteq \mathbb{R}$. Whenever $\mathcal{I}$ is not empty, ϕ_t becomes a global mapping for all $t \in \mathcal{I}$ so that one is able to write $\phi_t : M \to M$. If M is a compact manifold, then it would be covered by finitely many open sets of the above family. In this case, $\mathcal{I}$ becomes, of course, the intersection of finitely many open intervals. Hence, it cannot be empty. Such a family of mappings generated by a vector field on the manifold is called the ***flow*** of that vector field. If $\mathcal{I} = \mathbb{R}$, then we say that $V \in T(M)$ is a ***complete vector field***. It can be shown that if the vector field is bounded, that is, if there exists a constant $K > 0$ such that $\sum_{i=1}^{m} |v^i(\mathbf{x})| \le K$ for all $\mathbf{x} \in \mathbb{R}^m$, then the solution of the system of differential equations (2.9.2) will be valid on the entire real axis [*see* Cronin (1980), *p*. 53]. When M is taken as a compact manifold, then all continuous functions defined on M ought to be bounded. Consequently, *smooth vector fields defined on compact manifolds are always complete*.

We now shall try to better understand the structure of the mapping ϕ_t. The functions $\phi^i(t; \mathbf{x})$ are to satisfy

$$\frac{d\phi^i}{dt} = v^i(\boldsymbol{\phi}), \;\; i = 1, \dots, m,$$
$$\phi^i(0; \mathbf{x}) = x^i.$$

We have assumed without loss of generality that $0 \in \mathcal{I}$. Since functions ϕ^i are smooth, they can be expanded into a Maclaurin series in a sufficiently small neighbourhood of the point $t = 0$:

$$\overline{x}^i(t) = \phi^i(t;\mathbf{x}) = \phi^i(0;\mathbf{x}) + \frac{d\phi^i}{dt}\bigg|_{t=0} t + \frac{1}{2!}\frac{d^2\phi^i}{dt^2}\bigg|_{t=0} t^2 + \cdots$$
$$+ \frac{1}{n!}\frac{d^n\phi^i}{dt^n}\bigg|_{t=0} t^n + \cdots.$$

We can evaluate the coefficients of this series at $t = 0$ by using the foregoing ordinary differential equations. As a matter of fact, if we note that we can write

$$\frac{d\phi^i}{dt} = v^i = v^j\delta^i_j = v^j\frac{\partial\phi^i}{\partial\phi^j} = \Big(v^j\frac{\partial}{\partial\phi^j}\Big)\phi^i$$

we easily obtain the following sequence

$$\frac{d^2\phi^i}{dt^2} = \frac{\partial v^i}{\partial\phi^j}\frac{d\phi^j}{dt} = \frac{\partial v^i}{\partial\phi^j}v^j = \Big(v^j\frac{\partial}{\partial\phi^j}\Big)v^i = \Big(v^j\frac{\partial}{\partial\phi^j}\Big)^2\phi^i$$
$$\frac{d^3\phi^i}{dt^3} = \frac{\partial}{\partial\phi^k}\Big(v^j\frac{\partial v^i}{\partial\phi^j}\Big)\frac{d\phi^k}{dt} = \Big(v^j\frac{\partial}{\partial\phi^j}\Big)^2 v^i = \Big(v^j\frac{\partial}{\partial\phi^j}\Big)^3\phi^i$$
$$\vdots$$
$$\frac{d^n\phi^i}{dt^n} = \Big(v^j\frac{\partial}{\partial\phi^j}\Big)^{n-1}v^i = \Big(v^j\frac{\partial}{\partial\phi^j}\Big)^n\phi^i$$
$$\vdots$$

We know that the vector field $V' \in T(\mathbb{R}^m)$ representing the vector field $V \in T(M)$ in local coordinates is denoted by

$$V'(\mathbf{x}) = v^j(\mathbf{x})\frac{\partial}{\partial x^j}.$$

Thus, after having evaluated the foregoing relations at the point $t = 0$, we arrive at the following result:

$$\frac{d\phi^i}{dt}\bigg|_{t=0} = v^i(\mathbf{x}) = V'(x^i)$$
$$\frac{d^2\phi^i}{dt^2}\bigg|_{t=0} = V'^2(x^i)$$
$$\frac{d^3\phi^i}{dt^3}\bigg|_{t=0} = V'^3(x^i)$$
$$\vdots$$

$$\left.\frac{d^n\phi^i}{dt^n}\right|_{t=0} = V'^n(x^i)$$
$$\vdots$$

The operator V'^n denotes n times composition $V' \circ V' \circ \cdots \circ V'$ of the operator $V' : \Lambda^0(\mathbb{R}^m) \to \Lambda^0(\mathbb{R}^m)$ by itself. Hence the Taylor series above [English mathematician Brook Taylor (1685-1731)] can be cast into the following series

$$\begin{aligned}\overline{x}^i(t) = \phi^i(t;\mathbf{x}) &= x^i + tV'(x^i) + \frac{t^2}{2!}V'^2(x^i) + \cdots + \frac{t^n}{n!}V'^n(x^i) + \cdots \\ &= \Big(I + tV' + \frac{t^2}{2!}V'^2 + \cdots + \frac{t^n}{n!}V'^n + \cdots\Big)(x^i).\end{aligned}$$

We shall now define the ***exponential operator*** by the absolutely convergent operator series

$$\exp(tV') = e^{tV'} = \sum_{n=0}^{\infty}\frac{t^n}{n!}V'^n \tag{2.9.5}$$

where we have adopted the convention $V'^0 = I$. Thus, we attain at the formula

$$\phi^i(t;\mathbf{x}) = \exp\Big(tv^j(\mathbf{x})\frac{\partial}{\partial x^j}\Big)(x^i) = e^{tV'}(x^i)$$

or

$$\overline{\mathbf{x}}(t) = \boldsymbol{\phi}(t;\mathbf{x}) = e^{tV'}(\mathbf{x}) \tag{2.9.6}$$

where the operator $e^{tV'} : \mathbb{R}^m \to \mathbb{R}^m$ is introduced by

$$e^{tV'}(\mathbf{x}) = \big(e^{tV'}(x^1), e^{tV'}(x^2), \ldots, e^{tV'}(x^m)\big) \in \mathbb{R}^m.$$

If the operators V_1', V_2' are commutative, namely, if they satisfy the relation $V_1' \circ V_2' = V_2' \circ V_1'$, we find that

$$e^{V_1'+V_2'} = e^{V_1'} \circ e^{V_2'} = e^{V_2'} \circ e^{V_1'}.$$

In effect, if these operators commute the classical binomial expansion yields

$$(V_1' + V_2')^n = \sum_{k=0}^{n}\binom{n}{k}V_1'^kV_2'^{n-k} = \sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!}V_1'^kV_2'^{n-k}.$$

We thus obtain

$$\begin{aligned} e^{V_1'+V_2'} &= \sum_{n=0}^{\infty}\frac{(V_1'+V_2')^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{V_1'^k V_2'^{n-k}}{k!\,(n-k)!} \\ &= \sum_{n=0}^{\infty}\frac{V_1'^n}{n!}\sum_{m=0}^{\infty}\frac{V_2'^m}{m!} = e^{V_1'}e^{V_2'}. \end{aligned}$$

Since the vector addition is a commutative operation, we infer at once the commutativity of exponential operators. It then follows from (2.9.6) that

$$\boldsymbol{\phi}(t+s;\mathbf{x}) = e^{(t+s)V'}(\mathbf{x}) = e^{tV'}\circ e^{sV'}(\mathbf{x}) = \boldsymbol{\phi}\big(t;\boldsymbol{\phi}(s;\mathbf{x})\big). \quad (2.9.7)$$

Next, we employ the expression (2.9.4) by assuming that $t, s, t+s \in \mathcal{I}_{p_0}$ to reach to the relation

$$\begin{aligned} \phi(t+s;p) &= \varphi^{-1}\circ\boldsymbol{\phi}\big(t+s;\varphi(p)\big) = \varphi^{-1}\circ\boldsymbol{\phi}\big(t;\boldsymbol{\phi}\big(s;\varphi(p)\big)\big) \\ &= \varphi^{-1}\circ\boldsymbol{\phi}\big(t;\varphi(\phi\big(s;p)\big)\big) = \phi\big(t;\phi(s;p)\big). \end{aligned}$$

This relation is actually independent of the chart in question. Indeed, according to the definition of the integral curve, both curves $t\mapsto\phi(t+s;p)$ and $t\mapsto\phi\big(t;\phi(s;p)\big)$ satisfy the same differential equations. The initial conditions at $t=0$ are also the same: $\phi(s;p) = \phi\big(0;\phi(s;p)\big) = \phi(s;p)$.

Hence, the uniqueness of solutions leads also to the conclusion

$$\phi(t+s;p) = \phi\big(t;\phi(s;p)\big). \quad (2.9.8)$$

Consequently, we can write

$$\phi_{t+s}(p) = \phi_t\big(\phi_s(p)\big) = \phi_t\circ\phi_s(p)$$

for all $p\in U_{p_0}$ whenever $t, s, t+s\in\mathcal{I}_{p_0}$. If the interval $\mathcal{I}$ is not empty, then (2.9.8) becomes valid for all $p\in M$ and the *global transformation operator* $\phi_t : M\to M$ satisfies the relation

$$\phi_{t+s} = \phi_t\circ\phi_s \quad (2.9.9)$$

if $t, s, t+s\in\mathcal{I}$. This implies that the composition of smooth functions ϕ_t and ϕ_s is again a smooth function provided that the parameters t and s are sufficiently small if $\mathcal{I}\neq\mathbb{R}$. Furthermore, if we take $s = -t$, then we find $\phi_0 = \phi_t\circ\phi_{-t} = i_M$ implying that $(\phi_t)^{-1} = \phi_{-t}$. Hence, the inverse mapping ϕ_t^{-1} is also smooth. This amounts to say that $\{\phi_t : t\in\mathcal{I}\}$ is a family of diffeomorphisms. It is clear that *this set constitutes a group under the operation of composition of mappings*. However, since the group structure prevails only for limited values of the parameter t including 0, this group is called **1-*parameter group of local diffeomorphisms*** of the manifold M. If $\mathcal{I} = \mathbb{R}$, the group is named **1-*parameter group of diffeomorphisms***. The flow ϕ_t is represented by a family of curves that are tangent to a given vector field V at every point of the manifold M. These curves are obtained as images of solutions of the set of differential equations (2.9.2) on M by means of charts. Due to the uniqueness of solutions of equations (2.9.2), the curves of this family cannot intersect except at the *critical points* satisfying the condition $V(p) = 0$ and they fill the manifold. Such a family of curves is called a ***congruence.***

The vector field V that help determine the flow is sometimes called an ***infinitesimal generator*** of the flow.

The flow $\phi(t;p)$ can be endowed with an appearance which makes its group structure more pronounced. Provided that the points $\overline{\mathbf{x}}(t)$ and $\mathbf{x} \in \mathbb{R}^m$ belong to the same chart, we then deduce from the relation $\overline{\mathbf{x}}(t) = e^{tV'}(\mathbf{x})$ that

$$\overline{p}(t) = \phi(t;p) = \varphi^{-1}(\overline{\mathbf{x}}(t)) = (\varphi^{-1} \circ e^{tV'} \circ \varphi)(p).$$

We can now locally define an ***exponential mapping*** $e^{tV} : M \to M$ by

$$e^{tV} = \varphi^{-1} \circ e^{tV'} \circ \varphi. \tag{2.9.10}$$

It is straightforward to demonstrate that this operator possesses the following properties:

$$\begin{aligned} e^{(t+s)V} &= \varphi^{-1} \circ e^{(t+s)V'} \circ \varphi = \varphi^{-1} \circ e^{tV'} \circ e^{sV'} \circ \varphi \\ &= \varphi^{-1} \circ e^{tV'} \circ \varphi \circ \varphi^{-1} \circ e^{sV'} \circ \varphi = e^{tV} \circ e^{sV}, \\ e^{tV} \circ e^{-tV} &= e^{0V} = \varphi^{-1} \circ i_{\mathbb{R}^m} \circ \varphi = i_M. \end{aligned}$$

These properties justify our calling e^{tV} as the exponential mapping and our using the familiar notation. Moreover, for two commutative operators V_1 and V_2 we again obtain

$$\begin{aligned} e^{(V_1+V_2)} &= \varphi^{-1} \circ e^{(V_1'+V_2')} \circ \varphi = \varphi^{-1} \circ e^{V_1'} \circ \varphi \circ \varphi^{-1} \circ e^{V_2'} \circ \varphi \\ &= e^{V_1} \circ e^{V_2} = e^{V_2} \circ e^{V_1}. \end{aligned}$$

We can now express the flow generated by the vector field V on the manifold M also in the form

$$\phi(t;p) = \phi_t(p) = e^{tV}(p). \tag{2.9.11}$$

Naturally, as the parameter t varies, (2.9.11) might tangibly be specified only through different charts.

Let us now take a function $f \in \Lambda^0(M)$ into consideration. We know that the function $f' \in \Lambda^0(\mathbb{R}^m)$ is related to f by the equality $f(p) = f'(\mathbf{x})$, that is, we get $f' = f \circ \varphi^{-1}$. Our task is to evaluate the value of the smooth function f' at the point $\overline{\mathbf{x}}(t)$. To this end, let us consider the expansion

$$f'\big(\overline{\mathbf{x}}(t)\big) = f'\big(e^{tV'}(\mathbf{x})\big) = f'|_{t=0} + t\frac{df'}{dt}\bigg|_{t=0} + \cdots + \frac{t^n}{n!}\frac{d^n f'}{dt^n}\bigg|_{t=0} + \cdots.$$

Introducing the relations

$$\begin{aligned} f'|_{t=0} &= f'(\mathbf{x}), \quad \frac{df'}{dt}\bigg|_{t=0} = \frac{\partial f'}{\partial \overline{x}^i}\frac{d\overline{x}^i}{dt}\bigg|_{t=0} = v^i(\mathbf{x})\frac{\partial f'}{\partial x^i} = V'\big(f'(\mathbf{x})\big) \\ \frac{d^2 f'}{dt^2}\bigg|_{t=0} &= \frac{\partial \overline{V}'(f')}{\partial \overline{x}^j}\frac{d\overline{x}^j}{dt}\bigg|_{t=0} = V'^2\big(f'(\mathbf{x})\big), \\ &\ \vdots \\ \frac{d^n f'}{dt^n}\bigg|_{t=0} &= \frac{\partial \overline{V}'^{(n-1)}(f')}{\partial \overline{x}^j}\frac{d\overline{x}^j}{dt}\bigg|_{t=0} = V'^n\big(f'(\mathbf{x})\big), \ldots \end{aligned}$$

into that expression we arrive at

$$\begin{aligned} f'\big(e^{tV'}(\mathbf{x})\big) &= f'(\mathbf{x}) + tV'\big(f'(\mathbf{x})\big) + \cdots + \frac{t^n}{n!}V'^n\big(f'(\mathbf{x})\big) + \cdots \\ &= \Big(f' + tV'(f') + \cdots + \frac{t^n}{n!}V'^n(f') + \cdots\Big)(\mathbf{x}) \end{aligned}$$

from which we conclude that

$$\begin{aligned} f'\big(\overline{\mathbf{x}}(t)\big) = f'\big(e^{tV'}(\mathbf{x})\big) &= \sum_{n=0}^{\infty}\frac{t^n}{n!}V'^n(f')(\mathbf{x}) \\ &= e^{tV'(f')}(\mathbf{x}) = e^{tV'}f'(\mathbf{x}) \end{aligned} \tag{2.9.12}$$

On the other hand, if the equalities $f(p) = f'(\mathbf{x}), V^n(f)(p) = V'^n(f')(\mathbf{x})$, $n = 1, 2, \ldots$ are utilised in the expression

$$f\big(\overline{p}(t)\big) = f\big(e^{tV}(p)\big) = f'\big(e^{tV'}(\mathbf{x})\big) = \sum_{n=0}^{\infty}\frac{t^n}{n!}V'^n(f')(\mathbf{x})$$

we find that

$$f\big(\overline{p}(t)\big) = f\big(e^{tV}(p)\big) = \sum_{n=0}^{\infty}\frac{t^n}{n!}V^n(f)(p) \tag{2.9.13}$$

$$= e^{tV(f)}(p) = e^{tV} f(p)$$

Since $V^n(f)(p) \in \mathbb{R}$, the foregoing expression makes sense. If a function f satisfies the equality $f\big(\overline{p}(t)\big) = f(p)$ for every point $p \in M$, we say that it is ***invariant*** under the flow. It immediately follows from (2.9.13) that *the necessary and sufficient condition for a function f to remain invariant under the flow generated by a vector field V is*

$$V(f) = 0. \tag{2.9.14}$$

Next, we consider a vector field $V \in T(M)$, its local representation $V' \in T(\mathbb{R}^m)$, and the integral curve through a point $\mathbf{x} \in \mathbb{R}^m$. In view of (2.9.6), we can write $\overline{\mathbf{x}}(t) = e^{tV'}(\mathbf{x})$. Hence, we obtain

$$\begin{aligned}\overline{v}^i(\overline{\mathbf{x}}) &= \frac{d\overline{x}^i}{dt} = \frac{d}{dt}\big[x^i + tV'(x^i) + \frac{t^2}{2!}V'^2(x^i) + \cdots + \frac{t^n}{n!}V'^n(x^i) + \cdots\big] \\ &= V'(x^i) + tV'^2(x^i) + \cdots + \frac{t^{n-1}}{(n-1)!}V'^n(x^i) + \cdots \\ &= \big[I + tV' + \frac{t^2}{2!}V'^2 + \cdots + \frac{t^{n-1}}{(n-1)!}V'^{n-1} + \cdots\big]V'(x^i) \\ &= e^{tV'}\big(V'(x^i)\big) = e^{tV'}v^i(\mathbf{x}) = v^i\big(e^{tV'}(\mathbf{x})\big) = v^i(\overline{\mathbf{x}}).\end{aligned}$$

Here, in the last line we used (2.9.12). We thus conclude that

$$\frac{d}{dt}e^{tV'}(\mathbf{x}) = V'\big(e^{tV'}(\mathbf{x})\big) \ \text{ and } \ \frac{d}{dt}e^{tV'}(\mathbf{x})\bigg|_{t=0} = V'(\mathbf{x}). \tag{2.9.15}$$

To summarise, one notes that a flow generated by a vector field V on a manifold M is determined as a solution of symbolic differential equation

$$\frac{d\phi_t(p)}{dt} = V\big(\phi_t(p)\big), \quad \phi_0(p) = p, \ p \in M$$

(operation of differentiation can only be realised by means of charts) in the form $\phi_t(p) = e^{tV}(p)$. We can also write

$$\frac{d}{dt}e^{tV}(p) = V\big(e^{tV}(p)\big) \ \text{ and } \ \frac{d}{dt}e^{tV}(p)\bigg|_{t=0} = V(p) \tag{2.9.16}$$

in accordance with relations (2.9.15).

Let us now consider a function $f \in \Lambda^0(M)$ and try to specify its variation along the flow generated by a vector field V. If the local representation of this function is $f' = f \circ \varphi^{-1} \in \Lambda^0(\mathbb{R}^m)$ subordinate to a chart, we can then write

$$\frac{df'\big(\overline{\mathbf{x}}(t)\big)}{dt} = \frac{df'\big(e^{tV'}(\mathbf{x})\big)}{dt} = \overline{v}^i(\overline{\mathbf{x}})\frac{\partial f'}{\partial \overline{x}^i} = V'(f')|_{\overline{\mathbf{x}}(t)} = V'(f')|_{e^{tV'}(\mathbf{x})}.$$

Since $V(f) = V'(f')$, the above relation leads to

$$\frac{df\big(e^{tV}(p)\big)}{dt} = V(f)|_{e^{tV}(p)} \quad \text{and} \quad \frac{df\big(e^{tV}(p)\big)}{dt}\bigg|_{t=0} = V(f)(p).$$

Let $\psi : M \to N$ be a *diffeomorphism* between two differentiable manifolds. We denote the flow brought out by a vector field V on M by the relation $\overline{p}(t) = e^{tV}(p)$. We get $f = g \circ \psi \in \Lambda^0(M)$ for any $g \in \Lambda^0(N)$. If $q = \psi(p)$ we obtain

$$\begin{aligned} g\big[\psi\big(e^{tV}(p)\big)\big] &= (g \circ \psi)\big(e^{tV}(p)\big) = f\big(e^{tV}(p)\big) = \sum_{n=0}^{\infty}\frac{t^n}{n!}V^n(f)(p) \\ &= \sum_{n=0}^{\infty}\frac{t^n}{n!}V^n(g \circ \psi)(p) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\big[V^n(g \circ \psi) \circ \psi^{-1}\big](q) \\ &= \sum_{n=0}^{\infty}\frac{t^n}{n!}V^{*n}(g)(q) = g\big(e^{tV^*}(q)\big), \qquad V^* = \psi_*(V) \end{aligned}$$

after having employed (2.9.13) and (2.8.6). Since this relation is in effect for every smooth function g, we infer that

$$\psi\big(e^{tV}(p)\big) = e^{t\,d\psi(V)}\psi(p) = e^{t\,\psi_*(V)}\psi(p). \tag{2.9.17}$$

This simply means that *a diffeomorphism between manifolds M and N transforms a flow on M onto a flow on N*.

2.10. LIE DERIVATIVE

Let us assume that we are given two vector fields $U, V \in T(M)$ on a manifold M and the U- and V-congruences generated by those fields are determined. We consider a curve of V-congruence through a point $p \in M$ for the value of the parameter $t = 0$, the point of which corresponding to the value t is the point $q \in M$. U-congruence has now two curves through the points p and q. This situation is depicted schematically in Fig. 2.10.1.

Hence, we can write $q = e^{tV}(p)$. Our aim is to establish a procedure that is able to measure the variation in vectors of the vector field U while one moves along a V-curve. In order to realise this, we have to suggest a

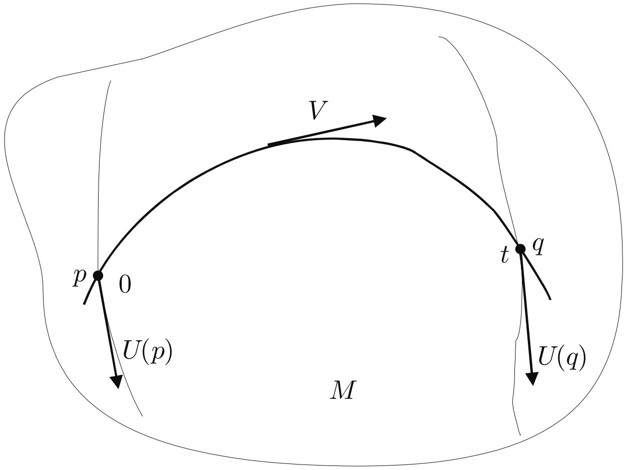

Fig. 2.10.1. Two congruences on a manifold.

scheme that makes it possible to compare vectors $U(p) \in T_p(M)$ and $U(q) \in T_q(M)$ which reside on disjoint vector spaces. In other words, we have to transport the vector $U(q)$ without changing its properties into the tangent space $T_p(M)$. To this end, we introduce a vector $U^* \in T_p(M)$ depending on the parameter t of the V-curve by the following relation

$$U^*(p;t) = (e^{-tV})_* U\big(e^{tV}(p)\big) = (e^{-tV})_*(e^{tV})^* U(p) \qquad (2.10.1)$$

where the linear operator $(\phi_t^{-1})_* = (e^{-tV})_* : T_q(M) \to T_p(M)$ is the differential of the *inverse flow* ϕ_t^{-1} at the point q and places the vector $U(q)$ into the tangent space at the point p. The operator $(e^{tV})^*$ is defined as usual by $(e^{tV})^* U = U \circ e^{tV}$. The vectors $U^*(p;t)$ and $U(p)$ now lie in the same tangent space. Therefore, their difference can now be calculated without any difficulty. We shall next define the ***Lie derivative*** of a vector field U with respect to the vector field V at the point p by the following limiting process

$$£_V U = \lim_{t\to 0} \frac{U^*(p;t) - U(p)}{t} = \lim_{t\to 0} \frac{(e^{-tV})_*(e^{tV})^* - I}{t} U(p). \qquad (2.10.2)$$

[Although it is always referred to the name of Norwegian mathematician Marius Sophus Lie (1842-1899), this concept was first introduced in 1931 by Polish mathematician Władisław Ślebodziński (1884-1972). However, the term *Lie derivative* was coined by Dutch mathematician David van Dantzig (1900-1959) in 1932]. Thus the *Lie derivative operator* can be expressed in the form

$$
£_V = \lim_{t\to 0} \frac{(e^{-tV})_*(e^{tV})^* - I}{t}. \tag{2.10.3}
$$

In the second step, we attempt to evaluate concretely the Lie derivative of a vector field U with respect to a vector field V by resorting to local charts at points p and q. Let the local coordinates at the point p be $(x^1, \dots, x^m)$ and those at the point $q = e^{tV}(p)$ be $(\overline{x}^1, \dots, \overline{x}^m)$. Thus we can write

$$
U(p) = u^i(\mathbf{x}) \frac{\partial}{\partial x^i}, \quad U(q) = u^i(\overline{\mathbf{x}}) \frac{\partial}{\partial \overline{x}^i}.
$$

In accordance with the relations (2.7.4) and (2.7.5), we obtain

$$
U^*(p;t) = u^{*j}\big(\overline{\mathbf{x}}(t)\big) \frac{\partial}{\partial x^j}, \quad u^{*j}\big(\overline{\mathbf{x}}(t)\big) = u^i\big(\overline{\mathbf{x}}(t)\big) \frac{\partial x^j}{\partial \overline{x}^i}.
$$

For very small values of the parameter t, the expression $\overline{\mathbf{x}}(t) = e^{tV'}(\mathbf{x})$ can be approximated by

$$
\overline{x}^i(t) = x^i + tv^i(\mathbf{x}) + o(t)
$$

where the vector field V is represented as $V = v^i \partial/\partial x^i$. Hence, we are led to a matrix whose elements are given by

$$
\frac{\partial \overline{x}^i}{\partial x^j} = \delta^i_j + t \frac{\partial v^i}{\partial x^j} + o(t).
$$

It follows from the chain rule of differentiation that the inverse of this matrix is prescribed by $[\partial x^i / \partial \overline{x}^j]$ from which we find approximately

$$
\frac{\partial x^i}{\partial \overline{x}^j} = \delta^i_j - t \frac{\partial v^i}{\partial x^j} + o(t).
$$

Indeed, it is straightforward to verify that

$$
\begin{aligned}
\delta^i_j = \frac{\partial \overline{x}^i}{\partial x^k} \frac{\partial x^k}{\partial \overline{x}^j} &= \Big(\delta^i_k + t \frac{\partial v^i}{\partial x^k}\Big)\Big(\delta^k_j - t \frac{\partial v^k}{\partial x^j}\Big) + o(t) \\
&= \delta^i_j + t\Big(\frac{\partial v^i}{\partial x^j} - \frac{\partial v^i}{\partial x^j}\Big) + o(t) = \delta^i_j + o(t).
\end{aligned}
$$

Furthermore, the Taylor series around the point $\mathbf{x}$ yields

$$
u^i\big(\overline{\mathbf{x}}(t)\big) = u^i\big(\mathbf{x} + t\mathbf{v}(\mathbf{x}) + \boldsymbol{o}(t)\big) = u^i(\mathbf{x}) + t \frac{\partial u^i}{\partial x^k} v^k + o(t).
$$

We thus find

$$\begin{aligned} u^{*j} &= \Big(u^i + t\frac{\partial u^i}{\partial x^k}v^k + o(t)\Big)\Big(\delta_i^j - t\frac{\partial v^j}{\partial x^i} + o(t)\Big) \\ &= u^j + t\Big(\frac{\partial u^j}{\partial x^k}v^k - \frac{\partial v^j}{\partial x^i}u^i\Big) + o(t) \end{aligned}$$

and obtain finally

$$U^*(p;t) - U(p) = t\Big(v^i\frac{\partial u^j}{\partial x^i} - u^i\frac{\partial v^j}{\partial x^i}\Big)\frac{\partial}{\partial x^j} + o(t).$$

Since $\lim\limits_{t\to 0} o(t)/t = 0$, we conclude that

$$£_V U = w^i\frac{\partial}{\partial x^i} = w^i\partial_i = W, \quad w^i = v^j u^i_{,j} - u^j v^i_{,j} \tag{2.10.4}$$

where we employed the abbreviations $(\,\cdot\,)_{,i} = \partial(\,\cdot\,)/\partial x^i$ and $\partial_i = \partial/\partial x^i$. We observe that *the Lie derivative of a vector field U with respect to a vector field V at every point p is again a vector in the tangent space $T_p(M)$ and the vector field $£_V U$ so created measures the rate of change of the vector U at every point in the manifold along the congruence generated by the vector field V*.

We readily obtain from (2.10.4) the following results

$$£_{\partial_j} U = \frac{\partial u^i}{\partial x^j}\frac{\partial}{\partial x^i}, \quad £_V\Big(\frac{\partial}{\partial x^j}\Big) = -\frac{\partial v^i}{\partial x^j}\frac{\partial}{\partial x^i}. \tag{2.10.5}$$

We can attribute another meaning to the Lie derivative evoking algebraic connotations. We take two vector fields $U, V \in T(M)$ into account on a manifold M. For any function $f \in \Lambda^0(M)$, we get $V(f) \in \Lambda^0(M)$ and also $U\big(V(f)\big) \in \Lambda^0(M)$ so that we can write

$$U\big(V(f)\big) = u^j\frac{\partial}{\partial x^j}\Big(v^i\frac{\partial f}{\partial x^i}\Big) = u^j(v^i f_{,i})_{,j} = u^j v^i_{,j} f_{,i} + u^j v^i f_{,ij}.$$

In a similar way, we arrive at

$$V\big(U(f)\big) = v^j u^i_{,j} f_{,i} + u^i v^j f_{,ij}.$$

Hence, we find that

$$V\big(U(f)\big) - U\big(V(f)\big) = (v^j u^i_{,j} - u^j v^i_{,j})f_{,i} + (u^i v^j - u^j v^i)f_{,ij}.$$

Second order derivatives $f_{,ij}$ are symmetric with respect to the indices i, j due to the well known relation $f_{,ij} = f_{,ji}$ whereas their coefficients are antisymmetric with respect to the same indices. This means that the second sum in the foregoing relation turns out to be zero. As a result, for every function $f \in \Lambda^0(M)$, the following equality holds

$$(VU - UV)(f) = W(f)$$

where the vector field W is given by (2.10.4). This of course implies that

$$VU - UV = W \in T(M).$$

We now define the ***commutator*** of two vector fields as

$$[V', U'] = [V, U] = VU - UV. \tag{2.10.6}$$

This is tantamount to say that the Lie derivative of a vector field U with respect to the vector field V is expressible as

$$£_V U = [V, U]. \tag{2.10.7}$$

It clearly follows from the definition that the commutation rule $[V, U] = -[U, V]$ is valid. Therefore, Lie derivatives of two vector fields with respect to one another are related by

$$£_V U = -£_U V. \tag{2.10.8}$$

The commutator $[V, U]$ is also called ***Lie bracket*** or ***Lie product***. *Lie product is antisymmetric and one naturally has* $[V, V] = 0$. It might be instructive to evaluate the Lie derivative given by (2.10.4) this time by means of the commutator:

$$\begin{aligned}
[V, U] &= [v^i \partial_i, u^j \partial_j] = v^i \partial_i (u^j \partial_j) - u^j \partial_j (v^i \partial_i) \\
&= v^i (\partial_i u^j) \partial_j - u^j (\partial_j v^i) \partial_i + (v^i u^j - u^j v^i) \partial_{ij} \\
&= (v^j \partial_j u^i - u^j \partial_j v^i) \partial_i = £_V U.
\end{aligned}$$

Let us now take $V = \partial_i, U = \partial_j$. It is then immediately seen that for all indices i, j, we find

$$[\partial_i, \partial_j] = 0. \tag{2.10.9}$$

Thus Lie derivatives of all natural basis vectors, produced by local charts, with respect to one another vanish.

Another geometrical meaning can easily be attributed to the Lie bracket, namely, the Lie derivative. Suppose that we are given two vector fields and U- and V-congruences generated by them on a manifold M are determined. Let these families of curves are parametrised by t_1 and t_2, respectively. We consider U- and V-curves through a point $p \in M$. Let the points

$q, r \in M$ be determined along respective flows for the values t_1 and t_2 of parameters as

$$q = e^{t_1 U}(p), \quad r = e^{t_2 V}(p).$$

Let us also consider the points $\overline{q} = e^{t_2 V}(q), \overline{r} = e^{t_1 U}(r)$ along flows. In this case, we write

$$\overline{q} = e^{t_2 V} \circ e^{t_1 U}(p), \quad \overline{r} = e^{t_1 U} \circ e^{t_2 V}(p). \tag{2.10.10}$$

We denote images of these points in $\mathbb{R}^m$ by $\mathbf{x}(p), \mathbf{x}(\overline{q}), \mathbf{x}(\overline{r})$. We obtain from relations (2.9.6) the expression

$$\mathbf{x}(\overline{q}) - \mathbf{x}(\overline{r}) = (e^{t_2 V'} e^{t_1 U'} - e^{t_1 U'} e^{t_2 V'})\mathbf{x}(p) = [e^{t_2 V'}, e^{t_1 U'}]\mathbf{x}$$

that can be thought as measuring the "difference" between the points $\overline{q}$ and $\overline{r}$ where we wrote $\mathbf{x}(p) = \mathbf{x}$. On choosing t_1, t_2 sufficiently small, the expansions of exponential mappings yields

$$\begin{aligned}
\mathbf{x}(\overline{q}) - \mathbf{x}(\overline{r}) &= \big\{(I + t_2 V' + \tfrac{1}{2}t_2^2 V'^2 + \cdots)(I + t_1 U' + \tfrac{1}{2}t_1^2 U'^2 + \cdots) \\
&\quad - (I + t_1 U' + \tfrac{1}{2}t_1^2 U'^2 + \cdots)(I + t_2 V' + \tfrac{1}{2}t_2^2 V'^2 + \cdots)\big\}\mathbf{x} \\
&= \big\{I + t_1 U' + t_2 V' + t_1 t_2 V'U' + \tfrac{1}{2}t_2^2 V'^2 + \tfrac{1}{2}t_1^2 U'^2 + \cdots \\
&\quad - I - t_1 U' - t_2 V' - t_1 t_2 U'V' - \tfrac{1}{2}t_1^2 U'^2 - \tfrac{1}{2}t_2^2 V'^2 - \cdots\big\}\mathbf{x} \\
&= \big\{t_1 t_2 (V'U' - U'V') + o(t_1^2, t_2^2, t_1 t_2)\big\}\mathbf{x}.
\end{aligned}$$

Next, we take the parameters t_1, t_2 of the order ϵ where ϵ is a small number. We thus conclude that

$$\begin{aligned}
\mathbf{x}(\overline{q}) - \mathbf{x}(\overline{r}) &= \epsilon^2 [V, U]\big(\mathbf{x}(p)\big) + \boldsymbol{o}(\epsilon^2) \\
&= \epsilon^2 £_V U\big(\mathbf{x}(p)\big) + \boldsymbol{o}(\epsilon^2).
\end{aligned} \tag{2.10.11}$$

In view of (2.10.4), this expression can be cast into the shape

$$x^i(\overline{q}) - x^i(\overline{r}) = t_1 t_2 w^j \frac{\partial x^i}{\partial x^j} + o(\epsilon^2) = t_1 t_2 w^i + o(\epsilon^2) \sim \epsilon^2 w^i + o(\epsilon^2)$$

in terms of components. It is seen that even if we consider rather close points p, q, r, the points $\overline{q}$ and $\overline{r}$ formed as above do not coincide in general. But the difference is of second order and its magnitude is governed by the Lie bracket at the point p (Fig. 2.10.2).

If vector fields U, V commute, then we have $VU = UV$ and $[V, U] = 0$. We know in this case that $e^{t_2 V} \circ e^{t_1 U} = e^{t_1 U} \circ e^{t_2 V} = e^{t_1 U + t_2 V}$. Hence (2.10.10) yields exactly $\overline{q} = \overline{r}$. In other words, the congruence curves

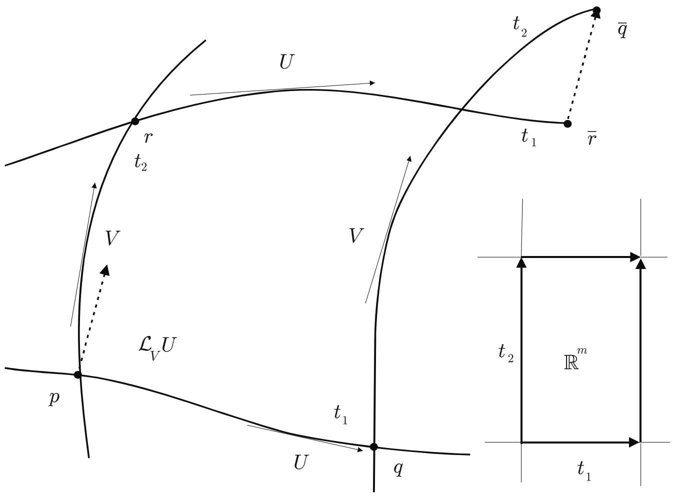

Fig. 2.10.2. The geometrical meaning of the Lie derivative.

through the points q and r intersects at the point $\overline{q}$ for the parameter values t_1 and t_2. This amounts to say that the U- and V- congruences play the part of two families of coordinate lines on M because t_1 and t_2 can now be regarded as two Cartesian coordinates in $\mathbb{R}^m$.

Conversely, we can immediately deduce from (2.10.11) that if we get $[e^{tV}, e^{tU}] = 0$ for all t and **x**, then we must have $[V, U] = 0$.

It follows directly from the relation (2.10.4) that the Lie product is distributive:

$$[V_1 + V_2, U] = [V_1, U] + [V_2, U], \;\; [V, U_1 + U_2] = [V, U_1] + [V, U_2]$$

Therefore, for all vector fields U, V and U_1, U_2, V_1, V_2 we can write

$$\pounds_{V_1+V_2} U = \pounds_{V_1} U + \pounds_{V_2} U, \;\; \pounds_V (U_1 + U_2) = \pounds_V U_1 + \pounds_V U_2$$

whence we reach to the operator equality

$$\pounds_{V_1+V_2} = \pounds_{V_1} + \pounds_{V_2}. \qquad (2.10.12)$$

Moreover, Lie product satisfies the ***Jacobi identity***. $U, V, W \in T(M)$ are arbitrary three vector fields. Then, the following identity holds

$$J = \big[U, [V, W]\big] + \big[V, [W, U]\big] + \big[W, [U, V]\big] = 0. \quad (2.10.13)$$

To verify this, let us begin with the calculation of the first term:

$$
\begin{aligned}
[U,[V,W]] &= \Big\{u^j(v^k w^i_{,k} - w^k v^i_{,k})_{,j} - (v^k w^j_{,k} - w^k v^j_{,k})u^i_{,j}\Big\}\frac{\partial}{\partial x^i} \\
&= \Big\{u^j v^k_{,j} w^i_{,k} + u^j v^k w^i_{,jk} - u^j v^i_{,k} w^k_{,j} \\
&\qquad - u^j v^i_{,jk} w^k - u^i_{,j} v^k w^j_{,k} + u^i_{,j} v^j_{,k} w^k\Big\}\frac{\partial}{\partial x^i}
\end{aligned}
$$

After having evaluated the other terms in (2.10.13) in a similar fashion, we consider their sum and by eliminating terms cancelling each other we reach to the result

$$
J = \Big\{u^i_{,jk}(v^j w^k - v^k w^j) + v^i_{,jk}(w^j u^k - w^k u^j) + w^i_{,jk}(u^j v^k - u^k v^j)\Big\}\frac{\partial}{\partial x^i}.
$$

However, in the above sums, the terms within parentheses are antisymmetric whereas mixed derivatives are symmetric with respect to relevant indices so that we finally obtain $J = 0$. This result can also be found by resorting to commutators. If we write (2.10.13) explicitly, we obtain

$$
\begin{aligned}
&U[V,W] - [V,W]U + V[W,U] - [W,U]V + W[U,V] - [U,V]W \\
&\quad = UVW - UWV - VWU + WVU + VWU - VUW - WUV \\
&\qquad + UWV + WUV - WVU - UVW + VUW = 0.
\end{aligned}
$$

We can now equip the module $\mathfrak{V}(M)$ that consists of all vector fields on a manifold M with a closed binary operation provided by the Lie bracket that assigns a vector field to every pair of vector fields. This way $\mathfrak{V}(M)$ acquires an algebraic structure. With this structure that is *anticommutative* but *not associative* as is clearly implied by the Jacobi identity (2.10.13), we can now venture to say, with a slight abuse of the term, that the module $\mathfrak{V}(M)$ has become a ***Lie algebra***. In fact, a Lie algebra is usually defined on a vector space. But $\mathfrak{V}(M)$ is a vector space only on the field of real numbers. Thus, strictly speaking, a Lie algebra can be formed on a real vector space by defining the product of two vectors as the Lie bracket. In this case, the Lie product turns out to be a bilinear operations so that for real numbers α_1, α_2, we can write

$$
\begin{aligned}
&[\alpha_1 U_1 + \alpha_2 U_2, V] = \alpha_1[U_1, V] + \alpha_2[U_2, V] \\
&[U, \alpha_1 V_1 + \alpha_2 V_2,] = \alpha_1[U, V_1] + \alpha_2[U, V_2].
\end{aligned}
$$

Obviously, tangent spaces $T_p(M)$ at every point $p \in M$ are Lie algebras in the true sense of the word.

We shall now attempt to measure the change in a vector field U along a V-curve through the point p. We can transport all vectors U in different tangent spaces at every point of the curve into the tangent space $T_p(M)$ utilising the mapping (2.10.1). Since we can add vectors in the same tangent space, the rate of change of the vector field U can be measured directly by the derivative

$$\frac{dU^*(p;t)}{dt} = \lim_{\tau\to 0}\frac{U^*(p;t+\tau) - U^*(p;t)}{\tau}. \tag{2.10.14}$$

We know that the diffeomorphism $e^{tV} : M \to M$ generated by a vector field V on the manifold M will satisfy $e^{(t+\tau)V} = e^{tV} \circ e^{\tau V} = e^{\tau V} \circ e^{tV}$. It then follows from the rule (2.7.7) concerning the composition of differentials that one can write

$$\begin{aligned} U^*(p;t+\tau) &= (e^{-(t+\tau)V})_*(e^{(t+\tau)V})^*U(p) \\ &= (e^{-\tau V})_* \circ (e^{-tV})_* \circ U \circ e^{\tau V} \circ e^{tV}(p) \\ &= (e^{-\tau V})_* \circ (e^{-tV})_* \circ (e^{tV})^* \circ \big(U \circ e^{\tau V}\big)(p) \\ &= (e^{-\tau V})_* \circ U^*(p;t) \circ e^{\tau V}(p) \\ &= (e^{-\tau V})_* \circ (e^{\tau V})^* U^*(p;t). \end{aligned}$$

Hence, the derivative (2.10.14) is expressible in the form

$$\frac{dU^*(p;t)}{dt} = \lim_{\tau\to 0}\frac{(e^{-\tau V})_*(e^{\tau V})^* - I}{\tau}U^*(p;t).$$

If we recall the relation (2.10.3), we conclude that

$$\frac{dU^*(p;t)}{dt} = \pounds_V U^*(p;t). \tag{2.10.15}$$

This is a differential equation satisfied by the operator U^* with the initial condition $U^*(p;0) = U(p)$. The solution of this equation is formally expanded into a Maclaurin series around $t = 0$ as follows

$$\begin{aligned} U^*(p;t) = U^*(p;0) + \frac{dU^*(p;t)}{dt}\bigg|_{t=0} t + \frac{1}{2}\frac{d^2U^*(p;t)}{dt^2}\bigg|_{t=0} t^2 + \cdots \\ + \frac{1}{n!}\frac{d^nU^*(p;t)}{dt^n}\bigg|_{t=0} t^n + \cdots. \end{aligned}$$

Since the operator $\pounds_V$ does not depend on the parameter t, we find that

$$\frac{dU^*(p;t)}{dt}\bigg|_{t=0} = \pounds_V U^*(p;0) = \pounds_V U(p),$$

$$\left.\frac{d^2U^*(p;t)}{dt^2}\right|_{t=0} = \pounds_V \left.\frac{dU^*(p;t)}{dt}\right|_{t=0} = \pounds_V^2 U(p), .$$
$$\vdots$$

where $U^*(p;0) = U(p)$. We thus arrive at the formal operator series

$$U^*(p;t) = U(p) + t\pounds_V U(p) + \frac{t^2}{2}\pounds_V^2 U(p) + \cdots + \frac{t^n}{n!}\pounds_V^n U(p) + \cdots.$$

We now define the exponential operator $e^{t\pounds_V}$ in the usual way as the absolutely convergent series

$$e^{t\pounds_V} = I + t\pounds_V + \frac{t^2}{2!}\pounds_V^2 + \cdots + \frac{t^n}{n!}\pounds_V^n + \cdots$$

whence we are led to the result

$$U^*(p;t) = e^{t\pounds_V} U(p) \in T_p(M), \quad p \in M. \tag{2.10.16}$$

We deduce from his relation an important property of vector fields. If $\pounds_V U = [V, U] = 0$, then we get $U^*(p;t) = U(p)$ implying that the vector field U does not change on V-congruence. In other words, the vector field U remains ***invariant*** with respect to the vector field V. On the other hand, if $\pounds_V U = 0$, then we have $\pounds_U V = 0$ due to (2.10.8). Therefore, we understand that *if the field U is invariant with respect to the field V, then the field V becomes necessarily invariant with respect to the field U.*

We can now write the Jacobi identity in the form

$$\pounds_U \pounds_V W + \pounds_V \pounds_W U + \pounds_W \pounds_U V = 0.$$

Then properties of Lie derivative allows us to transform this relation into

$$(\pounds_U \pounds_V - \pounds_V \pounds_U)W = \pounds_{\pounds_U V} W$$

or $[\pounds_U, \pounds_V]W = \pounds_{[U,V]}W$. Since this equality must hold for every vector field W, we arrive at the following rather elegant result between two Lie derivative operators

$$[\pounds_U, \pounds_V] = \pounds_{[U,V]}. \tag{2.10.17}$$

In exactly same way, we can define the Lie derivative of a function $f \in \Lambda^0(M)$ as follows

$$\pounds_V f = \lim_{t\to 0} \frac{f(e^{tV}p) - f(p)}{t}.$$

On using charts, we have $f(p) = f'(\mathbf{x})$, $f(e^{tV}p) = f'\big(\overline{\mathbf{x}}(t)\big)$. We can now approximately write $\overline{x}^i(t) = x^i + tv^i(\mathbf{x}) + o(t)$. Hence, Taylor series about the point **x** yields

$$f'\big(\overline{\mathbf{x}}(t)\big) - f'(\mathbf{x}) = f'(\mathbf{x}) + tv^i(\mathbf{x})\frac{\partial f}{\partial x^i} + o(t) - f'(\mathbf{x}) = tV'(f') + o(t)$$

and we finally obtain

$$\pounds_V f = V(f) = v^i \frac{\partial f}{\partial x^i} \tag{2.10.18}$$

Thus the Lie derivative of a function f is nothing but the directional derivative of f along the vector V. If $\pounds_V f = 0$, then the function f remains constant on every curve of V-congruence. *Naturally this constant may be different on each curve of the congruence*.

Suppose now that we are given two vector fields $U, V \in T(M)$ and two smooth functions $f, g \in \Lambda^0(M)$. For any function $h \in \Lambda^0(M)$, we can write

$$\begin{aligned}[fU, gV](h) &= fU\big(gV(h)\big) - gV\big(fU(h)\big) = fU(g)V(h) + fgUV(h)\\ &\quad - gV(f)U(h) - gfVU(h) = \big[fg\,[U,V] + fU(g)V - gV(f)U\big](h).\end{aligned}$$

where we have taken into account that vector fields are actually derivations. We thus obtain

$$[fU, gV] = fg\,[U,V] + fU(g)V - gV(f)U \tag{2.10.19}$$

or equivalently

$$\pounds_{fU}(gV) = fg\,\pounds_U V + f\pounds_U(g)V - g\pounds_V(f)U. \tag{2.10.20}$$

Let $\phi : M \to N$ be a differentiable mapping between manifolds M and N. We know that the differential of ϕ at a point $p \in M$ is the linear operator $\phi_* : T_p(M) \to T_{\phi(p)}(N)$. Consider two vector fields U and V on the manifold M. The Lie bracket of these vector fields at p is given by the vector

$$\begin{gathered}[U,V] = w^i \frac{\partial}{\partial x^i} \in T_p(M),\\ w^i = u^j \frac{\partial v^i}{\partial x^j} - v^j \frac{\partial u^i}{\partial x^j}\end{gathered}$$

in the local coordinates. In view of (2.7.4), the vector $\phi_*[U,V] \in T_{\phi(p)}(M)$ is expressed in the form

$$\phi_*[U,V] = w^{*\alpha}\frac{\partial}{\partial y^\alpha} = w^i\frac{\partial \Phi^\alpha}{\partial x^i}\frac{\partial}{\partial y^\alpha}.$$

Here, $\mathbf{y} = (y^1,\dots,y^n)$ are the local coordinates at the point $\phi(p) \in N$ and are related to the local coordinates $\mathbf{x} = (x^1,\dots,x^m)$ at the point $p \in M$ by a functional relation $\mathbf{y} = \Phi(\mathbf{x})$ or functions $y^\alpha = \Phi^\alpha(x^1,\dots,x^m), \alpha = 1,\dots,n$ associated with the mapping ϕ. Let us now explicitly evaluate components $w^{*\alpha}$:

$$\begin{aligned} w^{*\alpha} &= w^i\frac{\partial \Phi^\alpha}{\partial x^i} = \Big(u^j\frac{\partial v^i}{\partial x^j} - v^j\frac{\partial u^i}{\partial x^j}\Big)\frac{\partial \Phi^\alpha}{\partial x^i} \\ &= u^j\frac{\partial}{\partial x^j}\Big(v^i\frac{\partial \Phi^\alpha}{\partial x^i}\Big) - v^j\frac{\partial}{\partial x^j}\Big(u^i\frac{\partial \Phi^\alpha}{\partial x^i}\Big) + (v^ju^i - u^jv^i)\frac{\partial^2\Phi^\alpha}{\partial x^i\partial x^j} \\ &= u^j\frac{\partial v^{*\alpha}}{\partial x^j} - v^j\frac{\partial u^{*\alpha}}{\partial x^j} = u^j\frac{\partial v^{*\alpha}}{\partial y^\beta}\frac{\partial \Phi^\beta}{\partial x^j} - v^j\frac{\partial u^{*\alpha}}{\partial y^\beta}\frac{\partial \Phi^\beta}{\partial x^j} \\ &= u^{*\beta}\frac{\partial v^{*\alpha}}{\partial y^\beta} - v^{*\beta}\frac{\partial u^{*\alpha}}{\partial y^\beta} = [\phi_*U,\phi_*V]^\alpha. \end{aligned}$$

We thus conclude that

$$\phi_*[U,V] = [\phi_*U,\phi_*V] \tag{2.10.21}$$

or $[U,V]^* = [U^*,V^*]$, or $\phi_*(\pounds_U V) = \pounds_{\phi_*U}(\phi_*V)$.

2.11. DISTRIBUTIONS. THE FROBENIUS THEOREM

Let M be an m-dimensional differentiable manifold. Let us consider a subspace $\mathcal{D}_p = \mathcal{T}_p(M) \subset T_p(M)$ of dimension $k < m$ of the tangent space $T_p(M)$ at every point $p \in M$. We may constitute a *tangent subbundle* by union of disjoint subspaces $\mathcal{T}_p(M)$:

$$\mathcal{T}(M) = \bigcup_{p\in M}\mathcal{T}_p(M) = \{(p,V) : p \in M, V \in \mathcal{T}_p(M)\} \subset T(M) \tag{2.11.1}$$

This subbundle is called a ***k-dimensional distribution***. We denote it by $\mathcal{D} = \mathcal{T}(M)$. Thus a k-dimensional distribution really attaches to every point of the manifold a k-dimensional subspace of the tangent space at that point. In order to construct such a distribution, all we have to do is to select k linearly independent vector fields. If vector fields $U_\alpha,\ \alpha = 1,\dots,k$ are linearly independent, then the relation

$$a^\alpha(p)U_\alpha(p) = 0$$

with $a^\alpha \in \Lambda^0(M)$ can be satisfied if and only if $a^\alpha(p) = 0$ for $\alpha = 1, \dots, k$. Such vector fields U_α constitute a *basis* of the distribution.

A distribution $\mathcal{D}$ is called an ***involutive distribution*** *if for every vector fields $U, V \in \mathcal{D}$ one has $[U, V] \in \mathcal{D}$, namely, if $\mathcal{D}$ is closed under the Lie product*. It is clear that all Lie brackets remain in $\mathcal{D}$ if and only if it is possible to find functions $c^\alpha \in \Lambda^0(M)$ such that

$$[U, V] = c^\alpha(p)U_\alpha$$

for all $U, V \in \mathcal{D}$. Since basis vectors U_α are also in $\mathcal{D}$, a necessary condition for the distribution $\mathcal{D}$ to be involutive is that the relations

$$[U_\alpha, U_\beta] = c^\gamma_{\alpha\beta}(p)U_\gamma \tag{2.11.2}$$

should be satisfied for some functions $c^\gamma_{\alpha\beta} \in \Lambda^0(M)$. One can readily shows that this condition is also sufficient. Let us consider vectors $U = \lambda^\alpha U_\alpha$ and $V = \mu^\alpha U_\alpha$. It follows from (2.10.19) that

$$\begin{aligned}[U, V] = [\lambda^\alpha U_\alpha, \mu^\beta U_\beta] &= \lambda^\alpha \mu^\beta [U_\alpha, U_\beta] + \lambda^\alpha U_\alpha(\mu^\beta)U_\beta \\ &- \mu^\beta U_\beta(\lambda^\alpha)U_\alpha = \big\{c^\gamma_{\alpha\beta}\lambda^\alpha\mu^\beta + \lambda^\alpha U_\alpha(\mu^\gamma) - \mu^\alpha U_\alpha(\lambda^\gamma)\big\}U_\gamma \\ &= c^\gamma(p)U_\gamma \in \mathcal{D}.\end{aligned}$$

Due to the antisymmetry of Lie brackets, the coefficients $c^\gamma_{\alpha\beta}$ must be antisymmetric with respect to the subscripts:

$$c^\gamma_{\alpha\beta} = -\, c^\gamma_{\beta\alpha}. \tag{2.11.3}$$

Moreover, Lie brackets of vectors in $\mathcal{D}$ ought to satisfy the Jacobi identity. For basis vectors U_α, this identity is reduced to the form

$$\big[U_\alpha, [U_\beta, U_\gamma]\big] + \big[U_\beta, [U_\gamma, U_\alpha]\big] + \big[U_\gamma, [U_\alpha, U_\beta]\big] = 0.$$

On using (2.11.2), this identity yields

$$\begin{aligned}[U_\alpha, c^\delta_{\beta\gamma}U_\delta] + [U_\beta, c^\delta_{\gamma\alpha}U_\delta] + [U_\gamma, c^\delta_{\alpha\beta}U_\delta] &= c^\delta_{\beta\gamma}[U_\alpha, U_\delta] + c^\delta_{\gamma\alpha}[U_\beta, U_\delta] \\ + c^\delta_{\alpha\beta}[U_\gamma, U_\delta] &+ U_\alpha(c^\delta_{\beta\gamma})U_\delta + U_\beta(c^\delta_{\gamma\alpha})U_\delta + U_\gamma(c^\delta_{\alpha\beta})U_\delta \\ = \big\{c^\delta_{\beta\gamma}c^\lambda_{\alpha\delta} + c^\delta_{\gamma\alpha}c^\lambda_{\beta\delta} + c^\delta_{\alpha\beta}c^\lambda_{\gamma\delta} &+ U_\alpha(c^\lambda_{\beta\gamma}) + U_\beta(c^\lambda_{\gamma\alpha}) + U_\gamma(c^\lambda_{\alpha\beta})\big\}U_\lambda = 0.\end{aligned}$$

Since vectors U_λ are linearly independent, we deduce that the coefficients $c^\gamma_{\alpha\beta}$ must satisfy the following relations

$$c^\delta_{\beta\gamma}c^\lambda_{\alpha\delta} + c^\delta_{\gamma\alpha}c^\lambda_{\beta\delta} + c^\delta_{\alpha\beta}c^\lambda_{\gamma\delta} + U_\alpha(c^\lambda_{\beta\gamma}) + U_\beta(c^\lambda_{\gamma\alpha}) + U_\gamma(c^\lambda_{\alpha\beta}) = 0 \tag{2.11.4}$$

for all values of indices $\alpha, \beta, \gamma, \lambda = 1, \dots, k$. Because of the symmetry properties of these coefficients the number of independent relations in (2.11.4) is considerably smaller.

We have discussed in Sec. 2.4 some techniques to specify a submanifold of a given manifold M. We now propose another method to achieve that purpose. Let S be a k-dimensional submanifold of M determined by the relations $x^i = x^i(u^\alpha), i = 1, \dots, m$ and $\alpha = 1, \dots, k$. Then at a point $p \in S$ there will be a k-dimensional tangent space $T_p(S)$. But p is a point of M as well and all vectors at that point belong also to $T_p(M)$. Hence, we can write $T_p(S) \subset T_p(M)$, i.e., $T_p(S)$ is a subspace of $T_p(M)$. Since the inclusion map $\mathcal{I} : S \to M$ is an embedding, its differential $d\mathcal{I} : T(S) \to T(M)$ is an injective linear operator. Because $\mathcal{I} : S \to \mathcal{I}(S)$ is an identity mapping, we can write $d\mathcal{I}(V) = V, V \in T_p(S)$. Thus, if we consider a vector V in the tangent space of S at a point p, its components in tangent spaces $T_p(S)$ and $T_p(M)$ are related by

$$V = v^i \frac{\partial}{\partial x^i} = v^\alpha \frac{\partial}{\partial u^\alpha} = v^\alpha \frac{\partial x^i}{\partial u^\alpha} \frac{\partial}{\partial x^i} \text{ or } v^i = \frac{\partial x^i}{\partial u^\alpha} v^\alpha. \quad (2.11.5)$$

Let $U, V \in T_p(S)$. Due to (2.10.21) we obtain

$$d\mathcal{I}([U, V]) = [d\mathcal{I}(U), d\mathcal{I}(V)] = [U, V]$$

that results in $[U, V] \in T_p(S)$. This means that as long as $S \subseteq M$ is a submanifold, Lie products of vectors in $T_p(S)$ stay in $T_p(S)$. Therefore, such a subspace $T_p(S)$ is a Lie subalgebra of the Lie algebra $T_p(M)$.

Now, conversely, let us suppose that we are given k linearly independent vector fields on the manifold M. In other words, we choose a k-dimensional subspace of the tangent space at every point of the manifold. We then take congruences that are integral curves of those vector fields. Therefore, we can construct a local piece of the manifold which is tangent to a linear vector space formed by the chosen k vectors at every point of the manifold M. Next we have to ask the following question: under which conditions these small pieces of manifolds can be patched together *smoothly* in order to produce a smooth hypersurface forming a submanifold? This question can be quite easily answered qualitatively. When moving on an integral curve of a vector field, the variations of other vector fields are measured by Lie derivatives. In order that these integral curves stay on the hypersurface, Lie derivatives of vector fields must lie in the chosen subspace.

Let us consider a k-dimensional distribution $\mathcal{D}$ on a manifold M. If the tangent space at every point $p \in S$ of a k-dimensional submanifold $S \subseteq M$

is identical with the subspace $\mathcal{D}_p$ of the tangent space $T_p(M)$ of M, that is, if we have

$$d\mathcal{I}(T_p(S)) = \mathcal{D}_{p=\mathcal{I}(p)}, \;\; \forall p \in S \qquad (2.11.6)$$

where $\mathcal{I} : S \to M$ is the embedding mapping determining the submanifold S, then S is called an ***integral submanifold*** of the distribution $\mathcal{D}$. Sometimes instead of (2.11.6), we may prefer the weaker condition $d\mathcal{I}(T_p(S)) \subseteq \mathcal{D}_{p=i(p)}$ at each point $p \in S$. In this case the dimension of S may be less than k. *If a k-dimensional distribution $\mathcal{D}$ possesses a k-dimensional integral submanifold through every point $p \in M$, then $\mathcal{D}$ is called a* ***completely integrable distribution***. A fundamental theorem concerning such distributions is provided by German mathematician Ferdinand Georg Frobenius (1849-1917).

Theorem 2.11.1 (The Frobenius Theorem). *A distribution $\mathcal{D}$ on a manifold is completely integrable if and only it is involutive.*

If we assume that the distribution $\mathcal{D}$ is completely integrable, then there exists an integral submanifold S through every point $p \in M$ and at that point the subspace $\mathcal{D}_p \subset T_p(M)$ corresponds to the tangent space of S. Therefore, for each $U, V \in \mathcal{D}_p$ one finds $[U, V] \in \mathcal{D}_p$, namely, $\mathcal{D}$ is involutive.

For the proof of the converse statement, we consider a k-dimensional involutive distribution $\mathcal{D}$ on an m-dimensional manifold M. This distribution is specified by $k \leq m$ linearly independent vector fields $U_\alpha, \alpha = 1, 2, \ldots, k$ in the m-dimensional tangent bundle $T(M)$. Since $\mathcal{D}$ is an involutive distribution, there exist smooth functions $c_{\alpha\beta}^{\gamma} \in \Lambda^0(M)$ satisfying the relations $[U_\alpha, U_\beta] = c_{\alpha\beta}^{\gamma}(p) U_\gamma$ and verifying the conditions (2.11.3) and (2.11.4). Let us choose a new set of linearly independent vector fields by means of the transformation

$$V_\alpha(p) = A_\alpha^\beta(p) U_\beta(p), \;\; \alpha, \beta = 1, \ldots, k \qquad (2.11.7)$$

where $A_\alpha^\beta(p) \in \Lambda^0(M)$. The only restriction imposed on $k \times k$ matrix $\mathbf{A} = [A_\beta^\alpha]$ is that $\det \mathbf{A}(p) \neq 0$ at each point $p \in S$. Thus, we can write

$$\begin{aligned}[V_\gamma, V_\delta] = [A_\gamma^\alpha U_\alpha, A_\delta^\beta U_\beta] &= A_\gamma^\alpha A_\delta^\beta [U_\alpha, U_\beta] + A_\gamma^\alpha U_\alpha(A_\delta^\beta) U_\beta \\ &- A_\delta^\beta U_\beta(A_\gamma^\alpha) U_\alpha = \big[c_{\alpha\beta}^{\mu} A_\gamma^\alpha A_\delta^\beta + A_\gamma^\alpha U_\alpha(A_\delta^\mu) - A_\delta^\alpha U_\alpha(A_\gamma^\mu)\big] U_\mu\end{aligned}$$

Let us denote the inverse matrix by $\mathbf{A}^{-1} = \mathbf{B} = [B_\beta^\alpha]$, namely, the relations

$$A_\gamma^\alpha B_\beta^\gamma = B_\gamma^\alpha A_\beta^\gamma = \delta_\beta^\alpha$$

will hold. Hence, (2.11.7) gives

$$U_\alpha(p) = B_\alpha^\beta(p) V_\beta(p).$$

The commutators of vectors V_α then become

$$\begin{aligned}[V_\gamma, V_\delta] &= \big[c_{\alpha\beta}^\mu A_\gamma^\alpha A_\delta^\beta + A_\gamma^\alpha U_\alpha(A_\delta^\mu) - A_\delta^\alpha U_\alpha(A_\gamma^\mu)\big] B_\mu^\lambda V_\lambda \\ &= C_{\gamma\delta}^\lambda V_\lambda\end{aligned} \tag{2.11.8}$$

as it should be expected. We thus find that $[V_\gamma, V_\delta] \in \mathcal{D}$. Here, the functions $C_{\gamma\delta}^\lambda \in \Lambda^0(M)$ are given by

$$\begin{aligned}C_{\gamma\delta}^\lambda &= B_\mu^\lambda\big[c_{\alpha\beta}^\mu A_\gamma^\alpha A_\delta^\beta + A_\gamma^\alpha U_\alpha(A_\delta^\mu) - A_\delta^\alpha U_\alpha(A_\gamma^\mu)\big] \\ &= B_\mu^\lambda\big[c_{\alpha\beta}^\mu A_\gamma^\alpha A_\delta^\beta + V_\gamma(A_\delta^\mu) - V_\delta(A_\gamma^\mu)\big].\end{aligned}$$

The vector fields U_α are prescribed by

$$\begin{aligned}U_\alpha = u_\alpha^i(\mathbf{x})\frac{\partial}{\partial x^i}, \quad \mathbf{x} = \varphi(p), \\ i = 1, \dots, m; \quad \alpha = 1, \dots, k\end{aligned}$$

in a chart (U, φ) containing the point p. Therefore, the vector fields V_α are given by

$$V_\alpha = A_\alpha^\beta u_\beta^i \frac{\partial}{\partial x^i}. \tag{2.11.9}$$

Since k number of vectors U_α are linearly independent, the rank of the rectangular matrix

$$[u_\alpha^i] = \begin{bmatrix} u_1^1 & u_1^2 & \cdots & u_1^k & \cdots & u_1^m \\ u_2^1 & u_2^2 & \cdots & u_2^k & \cdots & u_2^m \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ u_k^1 & u_k^2 & \cdots & u_k^k & \cdots & u_k^m \end{bmatrix}$$

is k. We rename the coordinates x^i if necessary to arrange this matrix in such a way that $[u_\beta^\alpha]$ can be chosen as the $k \times k$ square matrix with non-vanishing determinant. Then (2.11.9) can be written as follows

$$V_\alpha = A_\alpha^\beta u_\beta^\gamma \frac{\partial}{\partial x^\gamma} + A_\alpha^\beta u_\beta^a \frac{\partial}{\partial x^a}, \; a = k+1, \dots, m. \tag{2.11.10}$$

So far the matrix $\mathbf{A}$ was arbitrary. We now select it as the inverse of the matrix $[u_\beta^\alpha]$:

$$A^\beta_\alpha u^\gamma_\beta = \delta^\gamma_\alpha, \quad A^\alpha_\beta = \overset{-1}{u}{}^\alpha_\beta$$

where the smooth functions $\overset{-1}{u}{}^\alpha_\beta \in \Lambda^0(M)$ are elements of the inverse matrix $[u^\alpha_\beta]^{-1}$. With this choice the structure of the expressions (2.11.10) reduces to a much simpler form

$$V_\alpha = \frac{\partial}{\partial x^\alpha} + v^a_\alpha \frac{\partial}{\partial x^a} \tag{2.11.11}$$

where we have introduced the functions v^a_α by

$$v^a_\alpha = u^a_\beta \overset{-1}{u}{}^\beta_\alpha. \tag{2.11.12}$$

On recalling that $[\partial_i, \partial_j] = 0$, we readily find

$$\begin{aligned}
[V_\alpha, V_\beta] = [\partial_\alpha + v^a_\alpha \partial_a, \partial_\beta + v^b_\beta\, \partial_b] &= \partial_\alpha \partial_\beta + v^b_{\beta,\alpha} \partial_b + v^b_\beta\, \partial_\alpha \partial_b \\
&+ v^a_\alpha \partial_a \partial_\beta + v^a_\alpha v^b_{\beta,a} \partial_a + v^a_\alpha v^b_\beta\, \partial_a \partial_b - \partial_\beta \partial_\alpha - v^a_{\alpha,\beta}\, \partial_a \\
&- v^a_\alpha\, \partial_\beta \partial_a - v^b_\beta\, \partial_b \partial_\alpha - v^b_\beta v^a_{\alpha,b} \partial_a - v^b_\beta v^a_\alpha \partial_b \partial_a \\
&= \{(v^a_{\beta,\alpha} + v^b_\alpha v^a_{\beta,b}) - (v^a_{\alpha,\beta} + v^b_\beta v^a_{\alpha,b})\} \partial_a
\end{aligned}$$

or

$$[V_\alpha, V_\beta] = \{V_\alpha(v^a_\beta) - V_\beta(v^a_\alpha)\} \frac{\partial}{\partial x^a}. \tag{2.11.13}$$

Next we insert (2.11.11) into (2.11.8) and rearrange the terms to obtain

$$[V_\alpha, V_\beta] = C^\gamma_{\alpha\beta} V_\gamma = C^\gamma_{\alpha\beta} \frac{\partial}{\partial x^\gamma} + C^\gamma_{\alpha\beta} v^a_\gamma \frac{\partial}{\partial x^a}.$$

If we compare this expression with (2.11.13) we deduce that all coefficient functions $C^\gamma_{\alpha\beta}$ must vanish. Hence, we conclude that

$$[V_\alpha, V_\beta] = 0. \tag{2.11.14}$$

Furthermore, (2.11.13) then implies that the following conditions should also be satisfied

$$V_\alpha(v^a_\beta) = V_\beta(v^a_\alpha). \tag{2.11.15}$$

(2.11.14) means that in an involutive distribution $\mathcal{D}$ one is always able to find k linearly independent vector fields V_α generating this distribution that commute with respect to the Lie product. Consequently, congruences produced by those vector fields constitute a k-dimensional net of coordinate lines at the vicinity of each point of the manifold M. In other words, they form an integral manifold. □

Making use of the information provided by the above theorem, we can determine in a concrete way the integral manifold of an involutive distribution S. Let $(\xi^1, \xi^2, \dots, \xi^k) \in \mathbb{R}^k$ denote the local coordinates that give rise to *natural basis* $\{V_\alpha\}$ of the tangent bundle $T(S)$. So we can write

$$V_\alpha = \frac{\partial}{\partial \xi^\alpha} = \frac{\partial}{\partial x^\alpha} + v^a_\alpha \frac{\partial}{\partial x^a},\ \alpha = 1, \dots, k; a = k+1, \dots, m.$$

Therefore, one has

$$V_\beta(x^i) = \frac{\partial x^i}{\partial \xi^\beta} = \frac{\partial x^i}{\partial x^\beta} + v^a_\beta \frac{\partial x^i}{\partial x^a} = \delta^i_\beta + v^a_\beta\, \delta^i_a$$

whence we conclude that

$$\frac{\partial x^\alpha}{\partial \xi^\beta} = \delta^\alpha_\beta, \quad \frac{\partial x^a}{\partial \xi^\alpha} = v^a_\alpha(\mathbf{x}). \tag{2.11.16}$$

Solutions of equations $(2.11.16)_1$ are trivially found as

$$x^\alpha = \xi^\alpha + c^\alpha, \quad \alpha = 1, \dots, k. \tag{2.11.17}$$

Since equations $(2.11.16)_2$ are generally non-linear, it is usually much more difficult to obtain their solutions. Utilising (2.11.17), we can put these equations into the form

$$\begin{aligned} \frac{\partial x^a}{\partial \xi^\alpha} &= v^a_\alpha(x^1, \dots, x^k, x^{k+1}, \dots, x^m) \\ &= v^a_\alpha(\xi^1 + c^1, \dots, \xi^k + c^k, x^{k+1}, \dots, x^m), \\ a &= k+1, \dots, m. \end{aligned}$$

Let us then calculate derivatives of equations $(2.11.16)_2$ with respect to variables ξ^β:

$$\frac{\partial^2 x^a}{\partial \xi^\alpha \partial \xi^\beta} = \frac{\partial v^a_\alpha}{\partial \xi^\beta} = \frac{\partial v^a_\alpha}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial \xi^\beta} + \frac{\partial v^a_\alpha}{\partial x^b}\frac{\partial x^b}{\partial \xi^\beta} = \frac{\partial v^a_\alpha}{\partial x^\beta} + v^b_\beta \frac{\partial v^a_\alpha}{\partial x^b} = V_\beta(v^a_\alpha).$$

This implies that equations $(2.11.16)_2$ can only be solved if the compatibility conditions $V_\alpha(v^a_\beta) = V_\beta(v^a_\alpha)$, that are naturally brought about by the symmetries of second order derivatives, are satisfied. However, these are none other than conditions (2.11.15) that must be elicited by functions v^a_α. Thus, equations $(2.11.16)_2$ can be integrated in principle and the set of coordinates $\{x^a\}$ are expressible in terms of variables ξ^α as below:

$$x^a = f^a(\xi^1, \xi^2, \ldots, \xi^k) + c^a$$

where $a = k+1, \ldots, m$ and c^a are $m-k$ arbitrary constants. Let us now define the new local coordinates by means of the relations

$$y^\alpha = \xi^\alpha,\; y^a = x^a - f^a(\xi^1, \xi^2, \ldots, \xi^k)$$

where $\alpha = 1, \ldots, k$ and $a = k+1, \ldots, m$. In the light of the above developments, we can thus rephrase the Frobenius theorem as follows: *Let $\mathcal{D}$ be a k-dimensional involutive distribution on an m-dimensional manifold. Then there exists local coordinates $y^i, 1 \le i \le m$ such that the vector fields $\partial/\partial y^1 = \partial/\partial\xi^1, \ldots, \partial/\partial y^k = \partial/\partial\xi^k$ constitute a local basis of the distribution $\mathcal{D}$ and submanifolds determined by $y^a =$ constant$, k+1 \le a \le m$ are integral manifolds of $\mathcal{D}$.*

It is now seen that a k-dimensional involutive distribution on an m-dimensional manifold M generates a k-dimensional smooth integral manifold through each point $p \in M$. Therefore, the manifold M can be reconstructed as the union of a family of k-dimensional submanifolds stacked on top of one another. Such a case is called a k-dimensional ***foliation*** of the class C^∞ on the manifold M. Each submanifold is known as a ***leaf*** of the foliation.

Example 2.11.1. Let $M = \mathbb{R}^3$ with a coordinate cover x, y, z. We define a 2-dimensional distribution $\mathcal{D}$ by the vector fields

$$U_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$
$$U_2 = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$$

where we take $x^1 = x, x^2 = y, x^3 = z$. It is easily verified that these vector fields are linearly independent if $y \neq 0$. In fact, we write with $f, g \in \Lambda^0(M)$

$$fU_1 + gU_2 = -yf\partial_x + (xf - zg)\partial_y + yg\partial_z = 0.$$

This relation is satisfied if and only if $f = g = 0$ when $y \neq 0$. On the other hand, the commutator of these vector fields becomes

$$[U_1, U_2] = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z} = \frac{z}{y}U_1 + \frac{x}{y}U_2 \in \mathcal{D}.$$

Thus $\mathcal{D}$ is an involutive distribution. Let us first determine the congruences produced by vector fields U_1 and U_2. The solutions of the following simple ordinary differential equations associated with vector fields U_1 and U_2, respectively

$$U_1 : \frac{d\overline{x}}{dt} = -\overline{y}, \ \frac{d\overline{y}}{dt} = \overline{x}, \quad \frac{d\overline{z}}{dt} = 0; \ \overline{\mathbf{x}}(0) = \mathbf{x}$$
$$U_2 : \frac{d\overline{x}}{ds} = 0, \quad \frac{d\overline{y}}{ds} = -\overline{z}, \ \frac{d\overline{z}}{ds} = \overline{y}; \ \overline{\mathbf{x}}(0) = \mathbf{x}$$

yield the U_1-congruence by the following parametric equations

$$\overline{x}(t) = x\cos t - y\sin t, \ \overline{y}(t) = x\sin t + y\cos t, \ \overline{z}(t) = z,$$

and U_2-congruence by equations

$$\overline{x}(s) = x, \ \overline{y}(s) = y\cos s - z\sin s, \ \overline{z}(s) = y\sin s + z\cos s.$$

It is immediately seen that both equations satisfy

$$\overline{x}(t)^2 + \overline{y}(t)^2 + \overline{z}(t)^2 = \overline{x}(s)^2 + \overline{y}(s)^2 + \overline{z}(s)^2 = x^2 + y^2 + z^2.$$

Hence, the 2-dimensional integral manifold through the point $\mathbf{x} = (x, y, z)$ is a sphere whose radius is equal to the distance of this point from the origin 0. But these congruences cannot form a coordinate net on the sphere. Indeed, let us move along U_1 integral curve through the point $\mathbf{x}$ to the point $\mathbf{x}_1$ by the parameter t, then along U_2 integral curve from the point $\mathbf{x}$ to the point $\mathbf{x}_2$ by the parameter s. We find that

$$x_1 = x\cos t - y\sin t, \ y_1 = x\sin t + y\cos t, \ z_1 = z$$
$$x_2 = x, \ y_2 = y\cos s - z\sin s, \ z_2 = y\sin s + z\cos s.$$

Next, we go along U_1 integral curve from $\mathbf{x}_2$ to the point $\mathbf{x}_3$ by t, and along U_2 integral curve from $\mathbf{x}_1$ to the point $\mathbf{x}_4$ by s. We obtain

$$x_3 = x_2\cos t - y_2\sin t, \ y_3 = x_2\sin t + y_2\cos t, \ z_3 = z_2$$
$$x_4 = x_1, \ y_4 = y_1\cos s - z_1\sin s, \ z_4 = y_1\sin s + z_1\cos s$$

or

$$x_3 = x\cos t - y\sin t\cos s + z\sin t\sin s, \qquad x_4 = x\cos t - y\sin t$$
$$y_3 = y\cos t\cos s - z\cos t\sin s, y_4 = x\sin t\cos s + y\cos t\cos s - z\sin s$$
$$z_3 = z\cos s + y\sin s, \qquad z_4 = x\sin t\sin s + y\cos t\sin s + z\cos s$$

It is evident that $\mathbf{x}_3 \neq \mathbf{x}_4$. For instance, for $t = s = \pi/2$ we have $\mathbf{x}_3 = (z, x, y)$, $\mathbf{x}_4 = (-y, -z, x)$. In this case, it would be necessary to produce two commutative vector fields generating the distribution $\mathcal{D}$. We write

$$V_1 = A_1^1 U_1 + A_1^2 U_2 = -A_1^1 y\frac{\partial}{\partial x} + (A_1^1 x - A_1^2 z)\frac{\partial}{\partial y} + A_1^2 y\frac{\partial}{\partial z}$$

$$V_2 = A_2^1 U_1 + A_2^2 U_2 = -A_2^1 y \frac{\partial}{\partial x} + (A_2^1 x - A_2^2 z) \frac{\partial}{\partial y} + A_2^2 y \frac{\partial}{\partial z}$$

and choose

$$A_1^1 = -1/y,\ A_1^2 = 0,\ A_2^1 = 0,\ A_2^2 = 1/y$$

with $\det \mathbf{A} = -1/y^2 \neq 0$. We thus obtain vectors

$$V_1 = \frac{\partial}{\partial x} - \frac{x}{y}\frac{\partial}{\partial y},\ \ V_2 = \frac{\partial}{\partial z} - \frac{z}{y}\frac{\partial}{\partial y}.$$

We see at once that $[V_1, V_2] = 0$. The congruences generated by these vector fields are found as solutions of ordinary differential equations

$$\frac{d\overline{x}}{dt} = 1,\ \frac{d\overline{y}}{dt} = -\frac{\overline{x}}{\overline{y}},\ \frac{d\overline{z}}{dt} = 0;\ \ \overline{\mathbf{x}}(0) = \mathbf{x},$$
$$\frac{d\overline{x}}{ds} = 0,\ \frac{d\overline{y}}{ds} = -\frac{\overline{z}}{\overline{y}},\ \frac{d\overline{z}}{ds} = 1;\ \ \overline{\mathbf{x}}(0) = \mathbf{x}.$$

These are respectively

$$\overline{x}(t) = x + t,\ \ \overline{y}(t)^2 = y^2 - 2xt - t^2,\ \ \overline{z}(t) = z,$$
$$\overline{x}(s) = x,\ \overline{y}(s)^2 = y^2 - 2zs - s^2,\ \overline{z}(s) = z + s.$$

As above, we now determine again the points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ and $\mathbf{x}_4$ starting from a point $\mathbf{x}$:

$$x_1 = x + t,\quad y_1^2 = y^2 - 2xt - t^2,\quad z_1 = z,$$
$$x_2 = x,\quad y_2^2 = y^2 - 2zs - s^2,\quad z_2 = z + s,$$
$$x_3 = x_2 + t,\ y_3^2 = y_2^2 - 2x_2 t - t^2,\ z_3 = z_2,$$
$$x_4 = x_1,\ y_4^2 = y_1^2 - 2z_1 s - s^2,\ z_4 = z_1 + s.$$

A short calculation then leads to

$$x_3 = x_4 = x + t, y_3^2 = y_4^2 = y^2 - 2xt - 2zs - t^2 - s^2, z_3 = z_4 = z + s.$$

Consequently V_1- and V_2-congruences form a 2-dimensional coordinate net on the sphere.

Let us now parametrise the integral manifolds by variables ξ and η via the general scheme that was given above. We thus write

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x} - \frac{x}{y}\frac{\partial}{\partial y},\ \ \frac{\partial}{\partial \eta} = \frac{\partial}{\partial z} - \frac{z}{y}\frac{\partial}{\partial y}$$

to obtain

$$\frac{\partial x}{\partial \xi} = 1,\ \frac{\partial x}{\partial \eta} = 0;\ \ \frac{\partial y}{\partial \xi} = -\frac{x}{y},\ \frac{\partial y}{\partial \eta} = -\frac{z}{y};\ \ \frac{\partial z}{\partial \xi} = 0,\ \frac{\partial z}{\partial \eta} = 1$$

the integration of which yields

$$\begin{aligned}
&x = \xi + c_1(\eta),\ \ c_1'(\eta) = 0 \ \text{ and }\ x = \xi + c_1,\\
&z = c_2(\eta),\ \ c_2'(\eta) = 1 \ \text{ and }\ z = \eta + c_2,\\
&\frac{\partial y^2}{\partial \xi} = -2(\xi + c_1), y^2 = -\xi^2 - 2c_1\xi + c_3(\eta),\ \frac{\partial c_3}{\partial \eta} = -2(\eta + c_2),\\
&c_3(\eta) = -\eta^2 - 2c_2\eta + c_3 \ \text{ and }\ y^2 = -\xi^2 - \eta^2 - 2c_1\xi - 2c_2\eta + c_3
\end{aligned}$$

where c_1, c_2 and c_3 are arbitrary constants. We define the new coordinates (ξ, η, r) by

$$\xi,\ \eta,\ r^2 = y^2 + \xi^2 + \eta^2 + 2c_1\xi + 2c_2\eta - c_3.$$

If we eliminate variables ξ and η in the expression for r^2 we find

$$r^2 = x^2 + y^2 + z^2 + c_1^2 + c_2^2 - c_3.$$

Hence $r =$ *constant* corresponds to a spherical integral manifold. ■

Let $f : M \to \mathbb{R}$ be a smooth function. The differential of this function is the linear operator $f_* = df : T(M) \to \mathbb{R}$, or a linear functional defined by the relation $f_*(V) = V(f)$ [*see p.* 126]. The vectors in the null space $\mathcal{N}(f_*)$ of the operator f_* that is a subbundle of $T(M)$ satisfy the condition $f_*(V) = V(f) = 0$. If $U, V \in \mathcal{N}(f_*)$, then we have $f_*(U) = f_*(V) = 0$ so that due to (2.10.21), we find $f_*[U, V] = [f_*U, f_*V] = [0, 0] = 0$ and thus $[U, V] \in \mathcal{N}(f_*)$. Hence, the distribution $\mathcal{N}(f_*)$ induced by the function f is involutive.

We next consider a k-dimensional distribution $\mathcal{D}$ of the tangent bundle $T(M)$. We know that this distribution is determined by k linearly independent vector fields U_α. A function $f : M \to \mathbb{R}$ is annihilated by the distribution $\mathcal{D}$ if the relations $U_\alpha(f) = 0$, $\alpha = 1, \dots, k$ are met. In this case, we obtain $U(f) = 0$ for all vector fields $U \in \mathcal{D}$. This of course implies that such a distribution must satisfy $\mathcal{D} \subseteq \mathcal{N}(f_*)$. Let us then consider the equalities $U_\alpha(f) = 0$, $U_\beta(f) = 0$ with $\alpha \neq \beta$. Utilising these relations, we arrive at the result

$$[U_\alpha, U_\beta](f) = U_\alpha\big(U_\beta(f)\big) - U_\beta\big(U_\alpha(f)\big) = 0. \tag{2.11.18}$$

If $[U_\alpha, U_\beta] \notin \mathcal{D}$ for $\alpha \neq \beta$ where $\alpha, \beta \in \{1, \dots, k\}$, then relations (2.11.18) provide additional conditions needed for the function f to be annihilated by the distribution $\mathcal{D}$. On the other hand, if the distribution $\mathcal{D}$ is involutive, the conditions (2.11.18) will be satisfied automatically:

$$[U_\alpha, U_\beta](f) = c^\gamma_{\alpha\beta} U_\gamma(f) = 0.$$

In this case, the relations $U_\alpha(f) = 0$ would be sufficient to determine functions annihilated by $\mathcal{D}$. In an involutive distribution, we can always choose *normal basis vectors* satisfying the conditions $[V_\alpha, V_\beta] = 0$ instead of arbitrary basis vectors $U_\alpha = u^i_\alpha(\mathbf{x})\,\partial/\partial x^i$. The vectors $V_\alpha, \alpha = 1, \dots, k$ are given by (2.11.11). Therefore,, we can take the equivalent relations $V_\alpha(f) = 0$ in lieu of $U_\alpha(f) = 0$. Thus, in order to determine functions annihilated by an involutive distribution $\mathcal{D}$, we have to solve the following set of first order partial differential equations

$$v^i_\alpha(\mathbf{x})\,\frac{\partial f}{\partial x^i} = 0,\ \ \alpha = 1, \dots, k. \tag{2.11.19}$$

The components $v^i_\alpha = \delta^i_\alpha + v^a_\alpha \delta^i_a$, where $a = k+1, \dots m$ are given by (2.11.12). These equations can be solved by the usual method of characteristics. We start with the first equation. Its characteristics are obtained as usual by solving the set of autonomous ordinary differential equations below

$$\frac{dx^1}{v^1_1(\mathbf{x})} = \frac{dx^2}{v^2_1(\mathbf{x})} = \cdots = \frac{dx^m}{v^m_1(\mathbf{x})}. \tag{2.11.20}$$

Evidently, characteristics are nothing but the integral curves of the vector field V_1 that are found by integrating the ordinary differential equations

$$\frac{dx^i}{dt} = v^i_1(\mathbf{x}).$$

It is well known that the solution of equations (2.11.20) is expressible in the form

$$g^1(\mathbf{x}) = c^1,\ g^2(\mathbf{x}) = c^2,\ \dots\ ,\ g^{m-1}(\mathbf{x}) = c^{m-1} \tag{2.11.21}$$

where $g^1, g^2, \dots, g^{m-1}$ are given smooth functions and c^1, $c^2, \dots, c^{m-1}$ are arbitrary constants. It follows from (2.11.21) that

$$0 = \frac{\partial g^r}{\partial x^i}\frac{dx^i}{dt} = v^i_1 \frac{\partial g^r}{\partial x^i} = V_1(g^r)$$
$$r = 1, 2, \dots, m-1.$$

We can thus see that the following equations

$$v_1^1 \frac{\partial f}{\partial x^1} + v_1^2 \frac{\partial f}{\partial x^2} + \cdots + v_1^m \frac{\partial f}{\partial x^m} = 0,$$
$$v_1^1 \frac{\partial g^1}{\partial x^1} + v_1^2 \frac{\partial g^1}{\partial x^2} + \cdots + v_1^m \frac{\partial g^1}{\partial x^m} = 0,$$
$$v_1^1 \frac{\partial g^2}{\partial x^1} + v_1^2 \frac{\partial g^2}{\partial x^2} + \cdots + v_1^m \frac{\partial g^2}{\partial x^m} = 0,$$
$$\vdots$$
$$v_1^1 \frac{\partial g^{m-1}}{\partial x^1} + v_1^2 \frac{\partial g^{m-1}}{\partial x^2} + \cdots + v_1^m \frac{\partial g^{m-1}}{\partial x^m} = 0$$

are to be held. Since $V_1 \neq 0$, this homogeneous set of linear equations in terms of m coefficient functions v_1^i can have a nontrivial solution if and only if the determinant of the coefficient functions vanishes:

$$\begin{vmatrix} \dfrac{\partial f}{\partial x^1} & \dfrac{\partial f}{\partial x^2} & \cdots & \dfrac{\partial f}{\partial x^m} \\ \dfrac{\partial g^1}{\partial x^1} & \dfrac{\partial g^1}{\partial x^2} & \cdots & \dfrac{\partial g^1}{\partial x^m} \\ \dfrac{\partial g^2}{\partial x^1} & \dfrac{\partial g^2}{\partial x^2} & \cdots & \dfrac{\partial g^2}{\partial x^m} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial g^{m-1}}{\partial x^1} & \dfrac{\partial g^{m-1}}{\partial x^2} & \cdots & \dfrac{\partial g^{m-1}}{\partial x^m} \end{vmatrix} = \frac{\partial(f, g^1, \dots, g^{m-1})}{\partial(x^1, x^2, \dots, x^m)} = 0.$$

This means that the function f is not independent of functions $g^1, \dots, g^{m-1}$. We thus conclude that

$$f = F(g^1, \dots, g^{m-1}). \tag{2.11.22}$$

Let us now take the equation $V_2(f) = 0$ into account. Inserting (2.11.22) into this equation, we obtain

$$0 = v_2^i \frac{\partial f}{\partial x^i} = v_2^i \frac{\partial F}{\partial g^r} \frac{\partial g^r}{\partial x^i} = V_2(g^r) \frac{\partial F}{\partial g^r}.$$

On the other hand, commutativity of vectors V_α results in

$$V_1\big(V_2(g^r)\big) = V_2\big(V_1(g^r)\big) = V_2(0) = 0.$$

Hence, functions $V_2(g^r)$ are solutions of the equation

$$V_1(g) = 0.$$

Thus, we must write

$$V_2(g^r) = h^r(g^1, \dots, g^{m-1}).$$

Consequently, we find that

$$V_2(f) = h^r(g^1, \dots, g^{m-1}) \frac{\partial F}{\partial g^r} = 0.$$

The solution of this differential equation is similarly expressed as

$$F = \mathcal{F}(m^1, m^2, \dots, m^{m-2})$$

where the functions

$$m^s = m^s(g^1, \dots, g^{m-1}),$$
$$s = 1, 2, \dots, m-2$$

are determined just as in the previous step. If we continue this way, we observe that every function annihilated by a k-dimensional involutive distribution is represented in the form

$$f = \mathfrak{F}(\mathfrak{g}^1, \mathfrak{g}^2, \dots, \mathfrak{g}^{m-k}). \tag{2.11.23}$$

$m-k$ functions $\mathfrak{g}^1, \mathfrak{g}^2, \dots, \mathfrak{g}^{m-k}$ are definite functions of variables $x^1, x^2, \dots, x^m$ obtained through all the foregoing steps. These functions constitute a set of *maximal solutions* if they are functionally independent, that is, if the following Jacobian with an appropriate ordering of local coordinates does not vanish

$$\frac{\partial(x^1, x^2, \dots, x^k, \mathfrak{g}^1, \mathfrak{g}^2, \dots, \mathfrak{g}^{m-k})}{\partial(x^1, x^2, \dots, x^m)} =$$

$$= \begin{vmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ \dfrac{\partial \mathfrak{g}^1}{\partial x^1} & \dfrac{\partial \mathfrak{g}^1}{\partial x^2} & \cdots & \dfrac{\partial \mathfrak{g}^1}{\partial x^k} & \cdots & \dfrac{\partial \mathfrak{g}^1}{\partial x^n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial \mathfrak{g}^{n-k}}{\partial x^1} & \dfrac{\partial \mathfrak{g}^{n-k}}{\partial x^2} & \cdots & \dfrac{\partial \mathfrak{g}^{n-k}}{\partial x^k} & \cdots & \dfrac{\partial \mathfrak{g}^{n-k}}{\partial x^n} \end{vmatrix} \neq 0.$$

Such functions $\mathfrak{g}^I$, $I = 1, \dots, m-k$ are named as the *first integrals* or *integral functions* of the distribution $\mathcal{D}$. Since we must have $V_\alpha(f) = 0$ for every function in the form (2.11.23), we find that

$$0 = V_\alpha(f) = v^i_\alpha \frac{\partial f}{\partial \mathfrak{g}^J} \frac{\partial \mathfrak{g}^J}{\partial x^i} = V_\alpha(\mathfrak{g}^J) \frac{\partial f}{\partial \mathfrak{g}^J}.$$

If we select $f = \mathfrak{g}^I$, we then obtain

$$V_\alpha(\mathfrak{g}^J)\, \delta^I_J = V_\alpha(\mathfrak{g}^I) = 0,\ \alpha = 1, \dots, k,\ \ I = 1, \dots, m-k.$$

Hence, for each vector $V \in \mathcal{D}$ one gets

$$d\mathfrak{g}^I(V) = \mathfrak{g}^I_*(V) = V(\mathfrak{g}^I) = 0,\ \ I = 1, \dots, m-k. \qquad (2.11.24)$$

Let us now define a subset $\mathcal{M}$ of the differentiable manifold M with the help of local charts as follows

$$\mathcal{M} = \{p \in M : \mathfrak{g}^1(p) = c^1, \mathfrak{g}^2(p) = c^2, \dots, \mathfrak{g}^{m-k}(p) = c^{m-k}\}$$

where $c^1, c^2, \dots, c^{m-k}$ are arbitrary constants. Because of Theorem 2.4.1, we understand that $\mathcal{M}$ is a submanifold. We generate a family of submanifolds, namely, a foliation of the manifold M by giving different values to these constants. If we take into consideration the relations (2.11.24), it becomes clear that the distribution is now specified by

$$\mathcal{D} = \{V \in T(M) : d\mathfrak{g}^1(V) = 0, d\mathfrak{g}^2(V) = 0, \dots, d\mathfrak{g}^{m-k}(V) = 0\}.$$

Hence the family $\mathcal{M}$ are actually integral manifolds of the involutive distribution $\mathcal{D}$. The linear operators $d\mathfrak{g}^I$ are now expressible as

$$\begin{aligned} dx^\alpha &= dx^\alpha, \\ d\mathfrak{g}^I &= \frac{\partial \mathfrak{g}^I}{\partial x^\alpha}\, dx^\alpha + \frac{\partial \mathfrak{g}^I}{\partial x^a}\, dx^a,\ \ I = 1, \dots, m-k \end{aligned}$$

where $\alpha = 1, \dots, k,\ a = k+1, \dots, m$. Since we have assumed that the Jacobian defined above does not vanish, then the operators $(dx^\alpha, d\mathfrak{g}^I)$ are linearly independent. Let us now reconsider Example 2.11.1. We know that the normalised basis vectors are

$$V_1 = \frac{\partial}{\partial x} - \frac{x}{y}\frac{\partial}{\partial y},\ \ V_2 = \frac{\partial}{\partial z} - \frac{z}{y}\frac{\partial}{\partial y}$$

Hence, for a function $f = f(x, y, z)$, the solution of the equation

$$V_1(f) = \frac{\partial f}{\partial x} - \frac{x}{y}\frac{\partial f}{\partial y} = 0$$

is obtainable through characteristic equations

$$\frac{dx}{1} = -\frac{y\,dy}{x} = \frac{dz}{0}$$

whose integrals are given as

$$g^1 = x^2 + y^2 = c^1, \ \ g^2 = z = c^2.$$

We thus find $f = F(g^1, g^2)$. Since

$$V_2(g^1) = -2z = -2g^2, \quad V_2(g^2) = 1$$

the function F must satisfy

$$-2g^2\frac{\partial F}{\partial g^1} + \frac{\partial F}{\partial g^2} = 0.$$

Solution of the ordinary differential equation

$$-dg^1/2g^2 = dg^2/1$$

is $\mathfrak{g}^1 = g^1 + (g^2)^2 = x^2 + y^2 + z^2 = C^1$. Therefore, we arrive at the result $f = F(\mathfrak{g}^1) = F(x^2 + y^2 + z^2)$. Thus, integral manifolds, or leaves, of that 2-dimensional involutive distribution are spheres centred at the origin 0.

II. EXERCISES

2.1. Show that a *discreet topology* can be generated on a set M by choosing every point in M as an open set and this topological space has the structure of a 0-dimensional manifold.

2.2. The standard topology on $\mathbb{R}^2$ is given as unions of open rectangles $(a,b) \times (c,d)$. Discuss whether the mapping $f : [0, 2\pi) \to \mathbb{S}^1$ defined by the rule $f(t) = (\cos t, \sin t)$ is bijective, continuous and it is a homeomorphism with respect to relative topologies on $[0, 2\pi)$ and $\mathbb{S}^1$.

2.3. Two differentiable structures on $\mathbb{R}$ are provided by atlases $\varphi_1(x) = x$ and $\varphi_2(x) = x^3$. We know that these atlases are not compatible [*see* Example 2.2.1]. Yet show that they are diffeomorphic.

2.4. An equivalence relation $\sim$ is defined on the set $S = \{(x,y) \in \mathbb{R}^2 : y = \pm 1\}$ by $x \neq 0, (x,1) \sim (x, -1)$. Show that the quotient space $M = S/\sim$ is a locally Euclidean and second countable space, but not a Hausdorff space (This example is known as *straight line with two centres*).

2.5. $\mathbb{S}^2$ is the sphere given by the equation $x^2 + y^2 + z^2 = 1$. Let us consider its open upper hemisphere $U_z^+ = \{\mathbf{x} \in \mathbb{S}^2 : z > 0\}$, the open set $V_z = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and the mapping $\varphi_z^+ : U_z^+ \to V_z$ determined by $\varphi_z^+(x, y, \sqrt{1 - x^2 - y^2}) = (x, y)$. Similarly, on the open lower hemisphere $U_z^- = \{\mathbf{x} \in \mathbb{S}^2 : z < 0\}$, we define the mapping $\varphi_z^- : U_z^- \to V_z$ by the relation $\varphi_z^-(x, y, -\sqrt{1-x^2-y^2}) = (x, y)$. Show that the pairs (U_z^+, φ_z^+) and (U_z^-, φ_z^-) are charts. Prove that we obtain an atlas with six charts when we add to these two those charts (U_x^+, φ_x^+), (U_x^-, φ_x^-) and (U_y^+, φ_y^+), (U_y^-, φ_y^-) involving left, right and front, rear hemispheres constructed in the similar fashion.

2.6. Let $U \subset \mathbb{R}^2$ be an open set and $f : U \to \mathbb{R}$ be a smooth mapping. Show that the *graph* $\{\mathbf{x}, f(\mathbf{x})\}$ of this function is a 2-dimensional submanifold of $\mathbb{R}^3$.

2.7. Let $M_i, i = 1, 2, \ldots, n$ be differentiable manifolds. We take submanifolds $N_i \subset M_i$ into account. Show that the Cartesian product $N_1 \times N_2 \times \cdots \times N_n$ is a submanifold of the product manifold $M_1 \times M_2 \times \cdots \times M_n$.

2.8. Discuss whether the following curves defined by mappings $\phi_i : \mathbb{R} \to \mathbb{R}^2$ are immersion or embedding:

$$\begin{aligned}
&\phi_1(t) = (t^2-1, t^3-t), \quad 1 < t < \infty,\\
&\phi_2(t) = \Big(\frac{t+1}{2t}\cos t, \frac{t+1}{2t}\sin t\Big),\\
&\phi_3(t) = (2\cos t, \sin 2t), \quad \phi_4(t) = \big(2\cos(2\arctan t), \sin(4\arctan t)\big),\\
&\phi_5(t) = (at - b\sin t, a - b\cos t), \ \ a, b \in \mathbb{R},\\
&\phi_6(t) = \big(2\sin(at+b), c\cos dt\big), \ \ a, b, c, d \in \mathbb{R}.
\end{aligned}$$

2.9. Discuss whether the following mappings $\phi_1 : \mathbb{R}^2 \to \mathbb{R}^3$ and $\phi_2 : \mathbb{R}^2 \to \mathbb{R}^4$ are immersions or submanifolds:

$$\begin{aligned}
&\phi_1(u, v) = (R\sin u\cos v, R\sin u\sin v, R\cos u),\\
&\phi_2(u, v) = \big((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u\cos(v/2), b\sin u\cos(v/2).
\end{aligned}$$

2.10. Discuss whether the mappings $\phi_1 : (0, \infty)^2 \to \mathbb{R}^3$, $\phi_2 : (0, \infty)^3 \to \mathbb{R}^3$ and $\phi_3 : (0, \infty)^2 \to \mathbb{R}^2$ defined below are immersions or submersions

$$\begin{aligned}
&\phi_1(u, v) = \Big(u, u^2, v^2/u\Big), \qquad \phi_2(u, v, w) = (uvw, uv, w),\\
&\phi_3(u, v, w) = (vw - u, v - uw).
\end{aligned}$$

2.11. The mapping $\phi : \mathbb{R}^3 \to \mathbb{R}^4$ is given by $\phi(u, v, w) = (u^2 - v^2, uv, uw, vw)$. Show that the restriction $\phi|_{\mathbb{S}^2}$ of this mapping satisfies the relation $\phi|_{\mathbb{S}^2}(p) = \phi|_{\mathbb{S}^2}(-p)$ for all $p \in \mathbb{S}^2$. Let us define the mapping $\psi : \mathbb{RP}^2 \to \mathbb{R}^4$ by $\psi(\{p, -p\}) = \phi|_{\mathbb{S}^2}(p)$. Show that the mapping ψ is an embedding.

2.12. Let us consider the manifold $\mathbb{R}^6$ with the coordinate cover $(x_1, x_2, x_3, x_4, x_5, x_6)$. We define the following subsets:

$$\begin{aligned}
&M = \{\mathbf{x} \in \mathbb{R}^6 : x_1^2 - x_2^2 - x_3^2 = 1, x_4^2 - x_5^2 - x_6^2 = 1\} \subset \mathbb{R}^6,\\
&N = \{\mathbf{x} \in \mathbb{R}^6 : x_2^2 + x_3^2 = 1, x_5^2 + x_6^2 = 1\} \subset \mathbb{R}^6,\\
&P = \{\mathbf{x} \in \mathbb{R}^6 : x_1^2 - x_2^2 + x_3^2 - x_4^2 \leq 0\} \subset \mathbb{R}^6
\end{aligned}$$

Investigate whether (a) M, N, P are submanifolds of $\mathbb{R}^6$, (b) the set $M \cap N$ is a submanifold of $\mathbb{R}^6$, M, N, (c) P is a submanifold of N with boundary.

2.13. Show that the composition of two immersions is an immersion and the composition of two embeddings is an embedding.

2.14. Show that the subset $GL^+(n,\mathbb{R})$ of matrices with positive determinants and the set $SL(n,\mathbb{R})$ of matrices whose determinants are 1 constitute submanifolds of the manifold $GL(n,\mathbb{R})$.

2.15. Let us denote by $s(n,\mathbb{R})$ the subset of symmetric matrices of the manifold $gl(n,\mathbb{R})$. We define a mapping $\phi : gl(n,\mathbb{R}) \to s(n,\mathbb{R})$ by the rule $\phi(\mathbf{A}) = \mathbf{A}\mathbf{A}^{\mathrm{T}}$. Let $\mathbf{I}_n$ be $n \times n$ identity matrix. Then show that the mapping ϕ is a submersion on the subset $\phi^{-1}(\mathbf{I}_n) \subset gl(n,\mathbb{R})$ and it constitutes a submanifold of the subset $O(n,\mathbb{R})$ of *orthogonal matrices* $\mathbf{A} \in gl(n,\mathbb{R})$ satisfying the condition $\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{I}_n$.

2.16. Let us take a fixed vector $\mathbf{v}_0 \in \mathbb{R}^n$ into consideration and define a mapping $f : GL^+(n,\mathbb{R}) \to \mathbb{R}^n$ by the relation $f(\mathbf{A}) = \mathbf{A}\mathbf{v}_0$. We naturally assume that $\mathbf{v}_0 \neq \mathbf{0}$. Show that this mapping and its restriction $g = f|_{SO(n,\mathbb{R})}$ to the set of orthogonal matrices $SO(n,\mathbb{R})$ with unit determinants are submersions. Show further that inverse mapping $M = g^{-1}(\{\mathbf{v}_0\}) = \{\mathbf{A} \in SO(n,\mathbb{R}) : \mathbf{A}\mathbf{v}_0 = \mathbf{v}_0\}$ of the set $\{\mathbf{v}_0\}$ under g is a submanifold of the manifold $SO(n,\mathbb{R})$ that is isomorphic to the manifold $SO(n-1,\mathbb{R})$.

2.17. Let $\phi : M \to N$ be an injective immersion between two smooth manifolds. Show that the mapping ϕ is a submersion when M is a compact manifold.

2.18. Let $\phi : M \to N$ be an immersion. If $M_1 \subset M$ is a submanifold, then show that the restriction $\phi|_{M_1}$ is also an immersion.

2.19. We define the mapping $f : GL^+(n,\mathbb{R}) \to GL^+(n+m,\mathbb{R})$ in the form

$$f(\mathbf{A}) = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix}, \quad \mathbf{B} \in SO(m,\mathbb{R}).$$

Show that the restriction $f|_{SO(n,\mathbb{R})}$ is an embedding into $SO(n+m,\mathbb{R})$.

2.20. The mapping $\phi : \mathbb{R} \to \mathbb{R}^2$ is defined by $\phi(t) = (t, t^2)$. Determine the image $\phi_* U$ of the vector $U = d/dt$.

2.21. The curve $\gamma : \mathbb{R} \to \mathbb{S}^2$ is given by the relations

$$\gamma(t) = \big(x(t), y(t), z(t)\big) = \Big(\cos t \sin\big(t + \frac{\pi}{3}\big), \sin t \sin\big(t + \frac{\pi}{3}\big), \cos\big(t + \frac{\pi}{3}\big)\Big).$$

Let V denote the vector tangent to this curve at the point $t = 0$. Determine images of the point $\gamma(0)$ and the vector V under the stereographic projection.

2.22. We define a cylinder by $\mathbb{S}^1 \times \mathbb{R} = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$. Its coordinate cover can be taken as (ϕ, z) in polar coordinates. On using spherical coordinates we introduce a mapping $\Phi : \mathbb{S}^2 \to \mathbb{S}^1 \times \mathbb{R}$ by the relation $\Phi(\phi,\theta) = (\phi, \sin\theta)$. Evaluate the differential $d\Phi = \Phi_*$.

2.23. $U, V \in T(M)$ are two vector fields. Their flows are denoted by ϕ_t and ψ_s, respectively. Show that $\phi_t \circ \psi_s = \psi_s \circ \phi_t$ if and only if $[U, V] = 0$.

2.24. The vector field $U \in T(\mathbb{R}^2)$ is given by $U = \partial_x + \partial_y$. Find the flow generated by this vector field and show that this vector field is complete. Does this vector field retain its completeness when it is defined on the manifold $M = \mathbb{R}^2 - \{\mathbf{0}\}$?

2.25. The vector field $U \in T(\mathbb{R}^2 - \{\mathbf{0}\})$ is given by $U = -y\,\partial_x + x\,\partial_y$. Find the flow generated by this vector field and check whether it is a complete vector field.

2.26. Find the integral curves of the vector field $(1 + x^2)\partial_x \in T(\mathbb{R})$ and check whether it is a complete vector field.

2.27. The vector fields $U_1, U_2, U_3 \in T(\mathbb{R}^3)$ are given by

$$U_1 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z},$$
$$U_2 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x},$$
$$U_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

Show that $[U_1, U_2] = U_3$, $[U_2, U_3] = U_1$ and $[U_3, U_1] = U_2$.

2.28. $f \in \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. We define the vector field $U_f \in T(\mathbb{R}^2)$ by

$$U_f = \frac{\partial f}{\partial y}\frac{\partial}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial}{\partial y}.$$

Show that the set formed by such kind of vector fields is closed under the Lie product.

2.29. Let ϕ_t and ψ_t be flows of vector fields $U, V \in T(M)$, respectively. We consider the curve

$$\gamma(t) = \psi_{-\sqrt{t}} \circ \phi_{-\sqrt{t}} \circ \psi_{\sqrt{t}} \circ \phi_{\sqrt{t}}(p)$$

through the point $p \in M$. We assume that $t \in [0, \epsilon\,]$ for a sufficiently small $\epsilon > 0$. Let $f : M \to \mathbb{R}$ be a smooth function. Show that we can write

$$[U, V](f)|_p = \lim_{t \to 0} \frac{f\big(\gamma(t)\big) - f\big(\gamma(0)\big)}{t}$$

and we get $\gamma'(0) = [U, V]$. Verify this property in $T(\mathbb{R}^3)$ for vector fields $U = \partial/\partial y$ and $V = \partial/\partial x + y\,\partial/\partial z$.

2.30. Let $\Phi : M \to N$ be a diffeomorphism. We denote flows generated by vector fields $U \in T(M)$ and $V \in T(N)$ by $\phi_t : M \to M$ and $\psi_t : N \to N$, respectively. We say that vector fields U and V are Φ-*related* if the relation $\Phi_* U = V$, or more explicitly $\Phi_* U(p) = V\big(\Phi(p)\big)$ for all $p \in M$, is satisfied. Show that U and V are Φ-related if and only if $\Phi \circ \phi_t = \psi_t \circ \Phi$. If we take $\Phi = \phi_t$, this relation is satisfied identically so that we find $(\phi_t)_* U = U$. This means that vector fields are conserved under their own flows.

2.31. Let $\Phi : M \to N$ be a diffeomorphism and $U \in T(M)$. Suppose that at every points $p_1, p_2 \in M$ satisfying the condition $\Phi(p_1) = \Phi(p_2)$ we have $\Phi_* U(p_1)$

$= \Phi_* U(p_2) \in T(N)$. Is there a vector field $V \in T(N)$ that is Φ-related with the vector field U?

2.32. Let $\Phi : M \to N$ be a diffeomorphism, $U_1, U_2 \in T(M)$ and $V_1, V_2 \in T(N)$. If vector fields U_1 and V_1, U_2 and V_2 are Φ-related, then show that Lie products $[U_1, U_2]$ and $[V_1, V_2]$ are also Φ-related.

2.33. Let $\Phi : M \to N$ be a diffeomorphism. We assume that vector fields $U \in T(M)$ and $V \in T(N)$ are Φ-related. Show that

$$\pounds_U(\Phi^* g) = \Phi^* \pounds_V(g) = \Phi^* \pounds_{\Phi_* U}(g)$$

for a function g $\in \Lambda^0(N)$.

2.34. Let us consider $f \in \Lambda^0(M)$ and $U \in T(M)$. The flow generated by the vector field U is $\phi_t : M \to M$. Show that the function $\phi_t^* f = f \circ \phi_t$ satisfies the following differential equation

$$\frac{d(\phi_t^* f)}{dt} = \phi_t^* \pounds_U f$$

along the flow.

2.35. The function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ satisfies the following partial differential equation and initial condition

$$\frac{\partial f(\mathbf{x}, t)}{\partial t} = u^i(\mathbf{x}) \frac{\partial f(\mathbf{x}, t)}{\partial x^i}, \quad f(\mathbf{x}, 0) = g(\mathbf{x})$$

where $\mathbf{x} = (x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$. If the vector field $U = u^i(\mathbf{x})\partial/\partial x^i$ is complete and its flow is $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$, then show that the function $f(\mathbf{x}, t) = g\big(\phi_t(\mathbf{x})\big)$ is the solution.

2.36. Find the solution of initial value problem given below:

$$\frac{\partial f}{\partial t} = 2\frac{\partial f}{\partial x}, \quad f(x, 0) = \sin x.$$

2.37. Find the solution of initial value problem given below:

$$\frac{\partial f}{\partial t} = (x + y)\Big(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\Big), \quad f(x, y, 0) = xy.$$

2.38. Find the solution of initial value problem given below:

$$\frac{\partial f}{\partial t} = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}, \quad f(x, y, 0) = x + y.$$

2.39. We consider the vector fields $U, V \in T(M)$. $\phi_t : M \to M$ is the flow of the vector field U. Show that the following relation is valid:

$$\frac{d}{dt}(\phi_t^{-1})_* V = (\phi_t^{-1})_*(\pounds_U V)$$

2.40. Vector fields $U, V \in T(\mathbb{R}^n)$ depending also on a parameter t are given as follows:

$$U = u^i(\mathbf{x}, t)\frac{\partial}{\partial x^i}, \quad V = v^i(\mathbf{x}, t)\,\frac{\partial}{\partial x^i}.$$

We assume that the functions $v^i(\mathbf{x}, t)$ are satisfying the initial value problem

$$\frac{\partial v^i}{\partial t} = u^j \frac{\partial v^i}{\partial x^j} - \frac{\partial u^i}{\partial x^j} v^j, \quad v^i(\mathbf{x}, 0) = g^i(\mathbf{x})$$

for prescribed functions $u^i(\mathbf{x}, t)$. If

$$G = g^i(\mathbf{x})\frac{\partial}{\partial x^i}$$

and ϕ_t is the flow generated by the vector field U, then show that the vector $V = (\phi_t^{-1})_* G$ represents the solution of the initial value problem.

2.41. Find the solution of initial value problem given below:

$$\begin{aligned}
\frac{\partial v^1}{\partial t} &= (x+y)\frac{\partial v^1}{\partial x} - (x+y)\frac{\partial v^1}{\partial y} - v^1 - v^2, \;\; v^1(x,y,0) = y \\
\frac{\partial v^2}{\partial t} &= (x+y)\frac{\partial v^2}{\partial x} - (x+y)\frac{\partial v^2}{\partial y} + v^1 + v^2, \;\; v^2(x,y,0) = \sin x
\end{aligned}$$

2.42. Find the solution of initial value problem given below:

$$\begin{aligned}
\frac{\partial v^1}{\partial t} &= y\frac{\partial v^1}{\partial x} + x\frac{\partial v^1}{\partial y} - v^2, \;\; v^1(x,y,0) = x^2 \\
\frac{\partial v^2}{\partial t} &= y\frac{\partial v^2}{\partial x} + x\frac{\partial v^2}{\partial y} - v^1, \;\; v^2(x,y,0) = y
\end{aligned}$$

2.43. M is an m-dimensional smooth manifold. A k-dimensional involutive distribution $\mathcal{D} \subset T(M)$ is specified by linearly independent vector fields $U_\alpha \in T(M), \alpha = 1, \dots, k$ satisfying the conditions $[U_\alpha, U_\beta] = c^\gamma_{\alpha\beta}(p)U_\gamma$. Smooth functions $F_\alpha : M \times \mathbb{R} \to \mathbb{R}$ are denoted by $F_\alpha(p, t), p \in M, \alpha = 1, \dots, k$. We consider the differential equation

$$U_\alpha(f) = F_\alpha(\mathbf{x}, f), \;\; \mathbf{x} = (x^1, \dots, x^m), \;\; \alpha = 1, \dots, k$$

where $f : M \to \mathbb{R}$. Show that the solution $f(\mathbf{x})$ of this system of differential equations may only exists if the functions F_α satisfy the relations

$$\Big(U_\alpha + F_\alpha \frac{\partial}{\partial f}\Big)(F_\beta) - \Big(U_\beta + F_\beta \frac{\partial}{\partial f}\Big)(F_\alpha) = c^\gamma_{\alpha\beta} F_\gamma, \; \alpha, \beta = 1, \dots, k.$$

Show further that the solution is found as the solution of the following differential equations

$$\Big(U_\alpha + F_\alpha \frac{\partial}{\partial f}\Big)\mathcal{F} = 0, \quad \alpha = 1, \dots, k$$

when the above relations are satisfied.

2.44. We consider the manifold $M = \mathbb{R}^3 - \{\mathbf{0}\}$. Show that the vector fields $V^1 = z\partial_y - y\partial_z$, $V^2 = x\partial_z - z\partial_x$ and $V^3 = y\partial_x - x\partial_y$ in $T(M)$ give rise to a 2-dimensional involutive distribution. Determine its integral manifold.

2.45. Show that the distribution generated by vector fields $V^1 = \partial_y + x\partial_z$ and $V^2 = \partial_x + y\partial_t$ in $T(\mathbb{R}^4)$ does not possess a 2-dimensional integral manifold.