CHAPTER III

LIE GROUPS

3.1. SCOPE OF THE CHAPTER

This chapter is devoted to a concise exposition of Lie groups that help illuminate various structural peculiarities of mappings on manifolds. These groups are so named because it was M. S. Lie who has first studied family of continuous functions forming a group and recognised their effectiveness in revealing some very important and fundamental properties of differential equations. We first define in Sec. 3.2 a Lie group as a smooth manifold endowed with a group operation in which multiplication and inversion operations are supposed to be smooth functions. Some of the salient features of Lie groups are then briefly examined. Next, in Sec. 3.3 we discuss left and right translations generated by an element of the group that are diffeomorphisms mapping the manifold onto itself. Left- and right-invariant vector fields are introduced by means of differentials of these mappings and it is shown that they constitute Lie algebras. After that we briefly investigate in Sec. 3.4 the group homomorphism between Lie groups that preserve group operations. We then consider in Sec. 3.5 one-parameter subgroups of a Lie group that are homomorphisms between the commutative Lie group of real numbers and an abstract Lie group. We then discuss the exponential mapping that may help characterise such one-parameter subgroups. Afterwards in Sec. 3.6 the group of automorphisms mapping the Lie group onto itself and generated by elements of the Lie group itself is defined and it is shown that this group, which is called adjoint representation, is isomorphic to the Lie group. In Sec. 3.5 we examine some notable properties of Lie transformation groups that map a smooth manifold onto itself and form also a Lie group. Finally, Killing vector fields were introduced.

3.2. LIE GROUPS

We assume that a binary operation $* : G \times G \to G$ on a set G, which

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will be called briefly as a *product*, satisfy the following conditions:

(i). Operation is closed: $g_1 * g_2 \in G$ for all $g_1, g_2 \in G$.
(ii). Operation is associative: $g_1 *(g_2 * g_3) = (g_1 * g_2) * g_3$ for all $g_1, g_2, g_3 \in G$.
(iii). There is an identity element $e \in G$: $e * g = g * e = g$ for all $g \in G$.
(iv). For each $g \in G$ there is an inverse $g^{-1} \in G$: $g * g^{-1} = g^{-1} * g = e$.

Then $(G, *)$ is called a ***group***. It is easily observed that the identity element e and the inverse element g^{-1} of an element $g \in G$ are *uniquely specified*. *A* ***Lie group*** *G is also a smooth manifold and the mappings*

$$\sigma : G \times G \to G \quad \textit{and} \quad \iota : G \to G$$

defined by $\sigma(g_1, g_2) = g_1 * g_2$ and $\iota(g) = g^{-1}$ are smooth mappings.

These two last conditions can be combined into a single one imposing that *the mapping $\overline{\sigma} : G \times G \to G$ defined by the rule $\overline{\sigma}(g_1, g_2) = g_1 * g_2^{-1}$ is smooth*. To prove this proposition, let us first introduce the smooth mapping $\mathcal{I} : G \to G \times G$ by the simple rule $\mathcal{I}(g) = (e, g)$. We see that $\overline{\sigma} \circ \mathcal{I} = \iota$. Indeed, we find at once that $(\overline{\sigma} \circ \mathcal{I})(g) = \overline{\sigma}(e, g) = e * g^{-1} = g^{-1} = \iota(g)$ for all $g \in G$. Since ι is now written as the composition of two smooth mappings, it turns out to be a smooth mapping as well. Similarly, let us introduce the smooth mapping $\mathfrak{I} : G \times G \to G \times G$ through the relation

$$\mathfrak{I}(g_1, g_2) = (g_1, g_2^{-1}) = \big(g_1, \iota(g_2)\big)$$

from which it follows that $\overline{\sigma}\big(\mathfrak{I}(g_1, g_2)\big) = \overline{\sigma}(g_1, g_2^{-1}) = g_1 * g_2 = \sigma(g_1, g_2)$ for all $g_1, g_2 \in G$. Thus the mapping $\sigma = \overline{\sigma} \circ \mathfrak{I}$ is also smooth. If G is a finite m-dimensional manifold, then it is called an *m-parameter Lie group*.

Let $(G, *)$ and $(H, \diamond)$ be two Lie groups. The Cartesian product $G \times H$ of the manifolds G and H can easily be equipped with a group structure by defining the product of elements (g_1, h_1) and (g_2, h_2) of the product manifold $G \times H$ where $g_1, g_2 \in G$ and $h_1, h_2 \in H$ in the following fashion

$$(g_1, h_1) \bullet (g_2, h_2) = (g_1 * g_2, h_1 \diamond h_2) \in G \times H.$$

One checks readily that the binary operation $\bullet$ is a group operation since it is solely determined by group operations on the Lie groups G and H and smoothness requirements are clearly met. If G and H are m- and n-parameter Lie groups, respectively, then the product manifold $G \times H$ turns out to be an $(m + n)$-parameter Lie group. Such a group is called a ***direct product*** of groups G and H.

Let us now consider some examples to Lie groups.

Example 3.2.1. The smooth manifold $\mathbb{R}^n$ (*see* Example 2.2.1) is a commutative Lie group with respect to the operation of addition in $\mathbb{R}^n$. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then we have $\mathbf{y}^{-1} = -\mathbf{y}$ so that we obtain $\mathbf{x}*\mathbf{y}^{-1} = \mathbf{x} - \mathbf{y} = (x^1 - y^1, \dots, x^n - y^n)$. This is obviously a smooth function.

Example 3.2.2. Let us consider the manifold $GL(n, \mathbb{R})$ which we had introduced in Example 2.2.2 and we had already called the ***general linear group of degree n***. It is immediately seen that this manifold becomes also a non-commutative group with respect to the usual matrix multiplication. Let $\mathbf{A}, \mathbf{B} \in GL(n, \mathbb{R})$. With the coordinates $a^i_j,\ b^i_j \in \mathbb{R}, i, j = 1, \dots, n$ these matrices are represented by $\mathbf{A} = [a^i_j], \mathbf{B} = [b^i_j]$ and we know that the matrix $\mathbf{AB}^{-1}$ is expressed as follows

$$\mathbf{AB}^{-1} = [a^i_k {b^{-1}}^k_j] = [a^i_k\,(\text{cofactor } b^k_j)^{\mathrm{T}}/\det \mathbf{B}].$$

Nevertheless, this is a smooth function because it is obviously the ratio of two polynomials. Hence $GL(n, \mathbb{R})$ is a Lie group of dimension n^2.

Let us now define a subset of the general linear group given by

$$SL(n, \mathbb{R}) = \{\mathbf{A} \in GL(n, \mathbb{R}) : \det \mathbf{A} = 1\}$$

It is clear that this subset is also a group with respect to matrix multiplication. In view of Theorem 2.4.1, $SL(n, \mathbb{R})$ is a submanifold of dimension $n^2 - 1$ of the general linear group. Hence, it is a Lie group. This group is called the ***special linear group*** or the ***unimodular group***.

We now consider the following subset

$$O(n) = \{\mathbf{A} \in GL(n, \mathbb{R}) : \mathbf{AA}^{\mathrm{T}} = \mathbf{I}\}$$

of the group $GL(n, \mathbb{R})$ which is formed by orthogonal matrices. Since the product of two orthogonal matrices is again an orthogonal matrix, $O(n)$ is a group and Theorem 2.4.1 implies that it is a submanifold of $GL(n, \mathbb{R})$ with the dimension $n^2 - n(n+1)/2 = n(n-1)/2$. Thus, it is a Lie group. $O(n)$ is called the ***orthogonal group***. If $\mathbf{A} \in O(n)$, then $(\det \mathbf{A})^2 = 1$ so that $\det \mathbf{A} = \pm 1$. The Lie group

$$SO(n) = \{\mathbf{A} \in O(n) : \det \mathbf{A} = 1\}$$

whose dimension is also $n(n-1)/2$ is known as the ***special orthogonal group*** because it preserves the length of a vector $\mathbf{x}$ and volumes in $\mathbb{R}^n$. In fact, we obtain for $\mathbf{A} \in O(n)$

$$(\mathbf{Ax})^{\mathrm{T}}\mathbf{Ax} = \mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{Ax} = \mathbf{x}^{\mathrm{T}}\mathbf{x}.$$

The orthogonal group is in fact a disconnected Lie group that is expressible as the union of two disjoint connected groups as

$$O(n) = SO(n) \cup \mathbf{\Omega} SO(n)$$

where $\mathbf{\Omega}$ is the $n \times n$ matrix

$$\mathbf{\Omega} = \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

so that $\det \mathbf{\Omega} = -1$.

Example 3.2.3. The complex plane $\mathbb{C} - \{0\}$ is the 2-dimensional smooth manifold $\mathbb{R}^2 - \{(0,0)\}$. This manifold is also a group with respect to the complex multiplication. On the other hand, if $z_1, z_2 \in \mathbb{C} - \{0\}$, then $z_1 z_2^{-1}$ is a smooth function of real coordinates. Hence, this manifold is a Lie group.

Example 3.2.4. Let us consider the smooth manifold $\mathbb{S}^1$, the unit circle. The points of this manifold can be determined by complex numbers with unit moduli such as $|z| = 1$. If $z_1, z_2 \in \mathbb{S}^1$, then $|z_1 z_2| = |z_1||z_2| = 1$ and this means that $z_1 z_2 \in \mathbb{S}^1$. This is tantamount to say that the manifold $\mathbb{S}^1$ is a Lie group.

Example 3.2.5. The m-torus defined as $\mathbb{T}^m = (\mathbb{S}^1)^m$ is a Lie group because it is the m-fold Cartesian product of a Lie group.

Subgroup. A *submanifold* H of a Lie group G is called a ***subgroup*** if for all elements $h_1, h_2 \in H$ one finds $h_1 * h_2 \in H$ and $h_1^{-1} \in H$. Therefore, a subgroup is a submanifold of a Lie group that is closed with respect to operations of group multiplication and inversion.

If a Lie group is *connected*, then the following theorem states that it can be generated by an open neighbourhood of its identity element.

Theorem 3.2.1. *Let G be a connected Lie group and U be an open neighbourhood of the identity element e. We denote the set of all n-fold products of elements of U by $U^n = \{u_1 * u_2 * \cdots * u_n : u_i \in U\}$. Then one can write*

$$G = \bigcup_{n=1}^{\infty} U^n.$$

In other words, each group element $g \in G$ is expressible as a finite product of some elements in the open set U. Hence, we can say that U generates the group G.

Let us choose a fixed $g \in G$, and define a function $\sigma_g : G \to G$ by the rule $\sigma_g(h) = \sigma(g, h) = g * h$. σ_g is a diffeomorphism [*see* Sec. 3.3]. Hence, if U is an open set, then the set $\sigma_g(U) = \{g * u : u \in U\} \subset G$ will also be open. Consequently, the set U^n is open for all n. Since ι is a diffeomorphism, the set $U^{-1} = \{u^{-1} = \iota(u) : u \in U\}$ is also open. We then conclude that the sets $V = U \cap U^{-1} \subset U$ and V^n are all open. Furthermore, the obvious relationship $V = V^{-1}$ would be valid. Because $e \in U$ and $e = e^{-1}$, we see at once that $e \in V$, i.e., V is not empty. Let us now define the set

$$H = \bigcup_{n=1}^{\infty} V^n \subseteq \bigcup_{n=1}^{\infty} U^n \subseteq G.$$

H is an open set since it is the union of countably many open sets, and it is, consequently, an open submanifold [*see* p. 77]. Due to the property of the set V, H will be a subgroup. We now consider the family of open sets $\sigma_g(H) = \{g{*}h : h \in H\}$ defined for all $g \in G$. One has evidently the relation $H = \sigma_{g \in H}(H)$. Thus we can obviously write

$$G = H \cup \bigcup_{g \in G, g \notin H} \sigma_g(H).$$

But the open set H is the complement of the open set $\bigcup_{g \in G, g \notin H} \sigma_g(H)$ with respect to G so it must also be a closed set. In a connected topological space only the empty set or the space itself can be both open and closed. H cannot be empty since $e \in H$ so it must be equal to G. We therefore reach to the conclusion that $G = \bigcup_{n=1}^{\infty} U^n$. □

The above theorem indicates that if a Lie group is a connected topological space, then an open neighbourhood of the identity element determines the entire group.

A subgroup H of the group G is called a ***normal*** or ***invariant subgroup*** if for all $h \in H$ we get $h_g = g^{-1}{*}h{*}g \in H$ for all $g \in G$ so that H is invariant under ***conjugation***. In other words, if H is a normal subgroup, a ***conjugate*** element $h_g \in H$ corresponds to each element $h \in H$ so that

$$g{*}h_g = h{*}g \quad \text{for each } g \in G.$$

This property is symbolically reflected by the notation $g{*}H = H{*}g$ for all $g \in G$. Let H be a normal subgroup, the ***quotient group*** is defined as the set $G/H = \{g{*}H : g \in G\}$. The ***coset*** $g{*}H$ is the subset of G defined by $\{g{*}h : \forall h \in H\}$. It is easy to verify that G/H is actually a group. Let us consider the direct product which can be written as follows

$$(g_1{*}H){*}(g_2{*}H) = (g_1{*}g_2){*}(H{*}H) = (g_1{*}g_2){*}H \in G/H$$

since one obviously observe the symbolic relation $H{*}H = H$ because H is a subgroup.

3.3. LIE ALGEBRAS

Let G be a Lie group. We choose a fixed element $g \in G$ to define a mapping $L_g : G \to G$ in such a way that

$$L_g(h) = \sigma(g, h) = g{*}h \tag{3.3.1}$$

for all $h \in G$. L_g is evidently a smooth mapping on the manifold G. The mapping L_g is called the ***left translation*** of the Lie group G by the element $g \in G$. We can obviously define a left translation for each element g of the group G. It can easily be seen that the relation $(L_g)^{-1} = L_{g^{-1}}$ is valid. Indeed, for each $h \in G$ we can write

$$L_g\big(L_{g^{-1}}(h)\big) = g{*}g^{-1}{*}h = e{*}h = h$$

so that we obtain $L_g \circ L_{g^{-1}} = i_G$. Similarly, it is found that $L_{g^{-1}} \circ L_g = i_G$. Hence, the inverse mapping $(L_g)^{-1} = L_{g^{-1}}$ is also smooth. Consequently, the left translation L_g *is a diffeomorphism.* The set of mappings

$$G_1 = \{L_g : g \in G\}$$

constitutes a group with respect to the operation of composition of mappings. In fact, if $L_{g_1}, L_{g_2} \in G_1$, then owing to the relation

$$L_{g_1}\big(L_{g_2}(h)\big) = g_1{*}g_2{*}h = L_{g_1*g_2}(h)$$

for all $h \in G$, we obtain $L_{g_1} \circ L_{g_2} = L_{g_1*g_2} \in G_1$ since $g_1{*}g_2 \in G$. Because $L_e = i_G$, it then follows that

$$L_e \circ L_g = L_g \circ L_e = L_g\,.$$

Thus, the identity element of G_1 is L_e and the inverse of L_g in G_1 is clearly $L_{g^{-1}}$. Since the composition is an associative binary operation, we finalise the realisation of the group structure of G_1. Therefore, there exists a mapping $\mathcal{L} : G \to G_1$ such that $\mathcal{L}(g) = L_g$. This mapping $\mathcal{L}$ is evidently surjective. Let us further suppose that $\mathcal{L}(g_1) = \mathcal{L}(g_2)$. If $L_{g_1}(h) = L_{g_2}(h)$ for all $h \in G$, the relation $g_1{*}h = g_2{*}h$ then leads to $g_1 = g_2$ if we multiply both sides by h^{-1} from left which means that $\mathcal{L}$ is injective, and consequently is *bijective*. On the other hand, due to the relation $\mathcal{L}(g_1{*}g_2) = L_{g_1} \circ L_{g_2} = \mathcal{L}(g_1) \circ \mathcal{L}(g_2)$, we infer that that the mapping $\mathcal{L}$ preserves group operations. In other words, it is a *group isomorphism*. Hence, *the groups G and* G_1 *are isomorphic*.

In exactly same fashion, we can define the ***right translation*** of the Lie group G by the element $g \in G$ as the mapping $R_g : G \to G$ such that

$$R_g(h) = h{*}g \tag{3.3.2}$$

for all $h \in G$. We can readily verify that a right translation is also a diffeomorphism and due to the relation $R_{g_1}\big(R_{g_2}(h)\big) = h{*}g_2{*}g_1 = R_{g_2*g_1}(h)$ for all $h \in G$, one obtains $R_{g_1} \circ R_{g_2} = R_{g_2*g_1}$. It is then straightforward to observe that the set of mappings $G_2 = \{R_g : g \in G\}$ constitutes a group with respect to the operation of composition. The identity element of this group is $R_e = i_G$ and the inverse of an element is given by $(R_g)^{-1} = R_{g^{-1}}$. It is clear that this group is also isomorphic to G. Therefore, the groups G_1 and G_2 are isomorphic to one another as well. It is now evident that left and right translations are connected through the following relation

$$R_g(h) = g^{-1}{*}g{*}h{*}g = g^{-1}{*}L_g(h){*}g.$$

Therefore, a right translation of an element of the group G is conjugate to its left translation, and vice versa. Moreover, it follows from $(L_{g_1} \circ R_{g_2})(h) = g_1{*}(h{*}g_2) = (g_1{*}h){*}g_2 = (R_{g_2} \circ L_{g_1})(h)$ for all $h \in G$ that these mappings commute, that is,

$$L_{g_1} \circ R_{g_2} = R_{g_2} \circ L_{g_1}. \tag{3.3.3}$$

In case G is a commutative group, we find that $L_g(h) = g{*}h = h{*}g = R_g(h)$ for all $h \in G$. Hence, we deduce that $L_g = R_g$ for all $g \in G$ in such an *Abelian group*.

Inasmuch as the mapping L_g is a diffeomorphism on G, its differential $dL_g|_h : T_h(G) \to T_{g*h}(G)$ is an isomorphism [*see p.* 124] transforming vector fields onto vector fields. A vector field V on the Lie group G is called a ***left-invariant vector field*** if it satisfies the equality

$$dL_g\big(V(h)\big) = V\big(L_g(h)\big) = V(g{*}h) \tag{3.3.4}$$

for all $g, h \in G$. This means that the image of a vector of such a field at the point h under the linear operator dL_g will be a vector of the same field at the point $g{*}h$. Thus the operator dL_g transforms a left-invariant vector field onto itself. So it is permissible to write symbolically

$$dL_g(V) = V$$

for all $g \in G$. If we take $h = e$ in (3.3.4), we obtain

$$dL_g\big(V(e)\big) = V(g) \tag{3.3.5}$$

for all $g \in G$. This relation implies that a left-invariant vector field on G is completely determined by *a vector in the tangent space* $T_e(G)$ *of the* *identity element e of the Lie group G.* So it becomes quite reasonable to interpret left-invariant vector fields as '*constant vector fields*' on the manifold G [Fig. 3.3.1].

Conversely, let us suppose that the relation $dL_g\big(V(e)\big) = V(g)$ is satisfied for all $g \in G$. We then easily deduce that

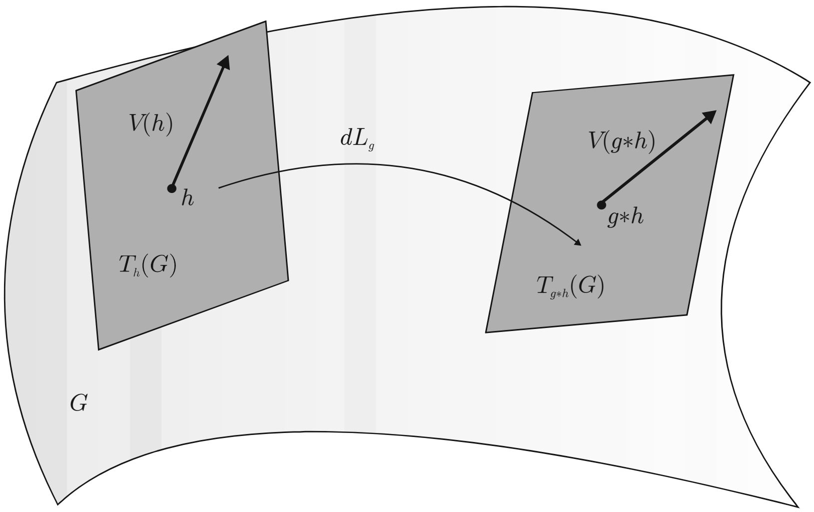

Fig. 3.3.1. A left-invariant vector field.

$$\begin{aligned} V(g{*}h) &= dL_{g*h}\big(V(e)\big) = d(L_g \circ L_h)\big(V(e)\big) \\ &= dL_g\big[dL_h\big(V(e)\big)\big] = dL_g\big(V(h)\big). \end{aligned} \tag{3.3.6}$$

According to (3.3.4), such a vector field V is a left-invariant vector field. We now denote the set of all left-invariant vector fields by $\mathfrak{g}$. It is seen at once that $\mathfrak{g}$ is a linear vector space on real numbers. Indeed, if $V_1, V_2 \in \mathfrak{g}$ and $\alpha_1, \alpha_2 \in \mathbb{R}$, the linearity of the operator dL_g on real numbers leads to the result

$$dL_g(\alpha_1 V_1 + \alpha_2 V_2) = \alpha_1 dL_g(V_1) + \alpha_2 dL_g(V_2) = \alpha_1 V_1 + \alpha_2 V_2$$

from which $\alpha_1 V_1 + \alpha_2 V_2 \in \mathfrak{g}$ follows. If we assume, instead, α_1 are α_2 are smooth functions on G, we realise that the invariance requirement can only be fulfilled if admissible functions are merely constant. The foregoing observations bring to mind the possibility of the existence of a bijective mapping between $\mathfrak{g}$ and $T_e(G)$. To this end, we presently introduce a mapping $\mathcal{G} : \mathfrak{g} \to T_e(G)$ by the rule $\mathcal{G}(V) = V(e)$. Owing to (3.3.5), the operator $\mathcal{G}$

must be linear. Indeed, one can write

$$\begin{aligned}(\alpha_1 V_1 + \alpha_2 V_2)(g) &= dL_g\big(\alpha_1 V_1(e) + \alpha_2 V_2(e)\big)\\ &= \alpha_1 dL_g\big(V_1(e)\big) + \alpha_2 dL_g\big(V_2(e)\big) = \alpha_1 V_1(g) + \alpha_2 V_2(g)\end{aligned}$$

for all $g \in G$. Thus we find that $\mathcal{G}(\alpha_1 V_1 + \alpha_2 V_2) = \alpha_1 \mathcal{G}(V_1) + \alpha_2 \mathcal{G}(V_2)$. The mapping $\mathcal{G}$ is *injective*. Suppose that $\mathcal{G}(V_1) = \mathcal{G}(V_2)$. We then have

$$V_1(g) = dL_g\big(V_1(e)\big) = dL_g\big(V_2(e)\big) = V_2(g)$$

for all $g \in G$ and we conclude that $V_1 = V_2$. $\mathcal{G}$ is *surjective*. Let us consider a vector $V(e) \in T_e(G)$. The vector field defined by $dL_g\big(V(e)\big) = V(g)$ for all $g \in G$ is a left-invariant vector field in view of (3.3.6), hence it is an element of $\mathfrak{g}$. In conclusion, $\mathcal{G}$ is an isomorphism and the vector spaces $\mathfrak{g}$ and $T_e(G)$ are isomorphic. This result dictates that the dimension of $\mathfrak{g}$ will be the same as that of $T_e(G)$. It is, of course, the same as the dimension of the manifold G.

At the identity element e, one writes $dL_g : T_e(G) \to T_g(G)$ so that we have $(dL_g)^{-1} : T_g(G) \to T_e(G)$. Because of the relation $L_{g^{-1}}(g) = e$, we obtain $dL_{g^{-1}} : T_g(G) \to T_e(G)$. On the other hand, the identities $L_g \circ L_{g^{-1}} = L_{g^{-1}} \circ L_g = i_G$ will result in the relations $dL_g \circ dL_{g^{-1}} = I_{T_g(G)}$ and $dL_{g^{-1}} \circ dL_g = I_{T_e(G)}$. It then follow that $(dL_g|_e)^{-1} = dL_{g^{-1}}|_g$.

Since G is a smooth manifold of dimension m, each point of G is contained in an open neighbourhood in G and there is a homeomorphism φ mapping this open set onto an open set of $\mathbb{R}^m$. If local coordinates of a point $h \in G$ are prescribed by $\mathbf{x} = (x^1, \dots, x^m)$ and local coordinates of a point $L_g(h) = g{*}h \in G$ are given by $\mathbf{y} = (y^1, \dots, y^m)$, then we know that there exists a functional relationship in the form $\mathbf{y} = (\varphi \circ L_g \circ \varphi^{-1})(\mathbf{x}) = \mathrm{L}_g(\mathbf{x})$, or $y^i = L_g^i(\mathbf{x})$. Hence the definition (3.3.4) implies that the local components of a left-invariant vector field must satisfy the following expressions

$$v^i(\mathbf{y}) = \frac{\partial L_g^i(\mathbf{x})}{\partial x^j} v^j(\mathbf{x}), \quad \mathbf{y} = \mathrm{L}_g(\mathbf{x}) \tag{3.3.7}$$

for all $\mathbf{x} \in \mathbb{R}^m$ in respective charts.

We now demonstrate that the Lie bracket of vector fields $V_1, V_2 \in \mathfrak{g}$ is also a left-invariant vector field. If we recall (2.10.21) we find that

$$dL_g([V_1, V_2]) = [dL_g(V_1), dL_g(V_2)] = [V_1, V_2],$$

hence, $[V_1, V_2] \in \mathfrak{g}$. As a result of this, we see that *left-invariant vector fields constitute a Lie algebra*. $\mathfrak{g}$ or $T_e(G)$ that is isomorphic to $\mathfrak{g}$ is called the ***Lie algebra*** of the Lie group G. Indeed, since we have $[V_1, V_2] \in \mathfrak{g}$ if

$V_1, V_2 \in \mathfrak{g}$, we understand that the relation

$$\mathcal{G}([V_1, V_2]) = [V_1(e), V_2(e)] = [\mathcal{G}(V_1), \mathcal{G}(V_2)]$$

would also be valid. If the dimension of the manifold G is m, a basis of the vector space $\mathfrak{g}$ are determined by m linearly independent left-invariant vector fields $\{V_i : i = 1, \ldots, m\}$. Properties of a Lie algebra will impose the following restriction on these vectors for all $i, j, k = 1, \ldots, m$

$$\begin{gathered}[V_i, V_j] + [V_j, V_i] = 0, \; . \\ \big[V_i, [V_j, V_k]\big] + \big[V_j, [V_k, V_i]\big] + \big[V_k, [V_i, V_j]\big] = 0\end{gathered} \tag{3.3.8}$$

Since $\mathfrak{g}$ is a Lie algebra, there must exist *constants* c_{ij}^k so that one has

$$[V_i, V_j] = c_{ij}^k V_k. \tag{3.3.9}$$

These constants are called ***structure constants*** of the Lie algebra $\mathfrak{g}$ with respect to the basis $\{V_i\}$. Because of the relations (3.3.9) and (3.3.8), the structure constants should meet the conditions

$$\begin{gathered}c_{ij}^k + c_{ji}^k = 0, \\ c_{jk}^n c_{in}^l + c_{ki}^n c_{jn}^l + c_{ij}^n c_{kn}^l = 0\end{gathered} \tag{3.3.10}$$

for all $i, j, k, l = 1, \ldots, m$ [*see* (2.11.4)]. Structure constants holding the conditions (3.3.10) completely determines the Lie algebra. It is clear that the structure constants depend on the selected basis. Let us choose another basis by the transformation $V_j' = a_j^i V_i$ where $\mathbf{A} = [a_j^i]$ is a regular matrix. If we write $[V_i', V_j'] = c_{ij}'^k V_k'$, we easily find that the following expressions must be satisfied

$$c_{ij}'^k a_k^r V_r = [a_i^p V_p, a_j^q V_q] = a_i^p a_j^q [V_p, V_q] = a_i^p a_j^q c_{pq}^r V_r.$$

Since the vectors V_r are linearly independent, we conclude that

$$c_{ij}'^k = a_i^p \, a_j^q \, b_r^k \, c_{pq}^r \tag{3.3.11}$$

where $\mathbf{B} = \mathbf{A}^{-1} = [b_j^i]$. (3.3.11) clearly indicates that structure constants are components of a third order mixed tensor. This tensor is called the ***structure tensor*** of the Lie algebra. We have seen that the Lie algebra of left-invariant vector fields is isomorphic to the tangent space $T_e(G)$ at the identity element e and the integral manifold of that tangent space locally determines the manifold G. This is tantamount to say that the Lie algebra fully determines the Lie group locally in a neighbourhood of e. However, the correspondence between the Lie groups and the Lie algebras is not unique. Although a given

Lie group determines uniquely its Lie algebra, several Lie groups may generate the same Lie algebra. But, it can be shown that among all the Lie groups with the same Lie algebra, there is only one Lie group that is simply connected. Therefore, a given Lie algebra gives rise to a unique simply connected Lie group locally in a neighbourhood of e. Then in view of Theorem 3.2.1 it determines the Lie group globally if the manifold G is connected. Because features of a Lie algebra are entirely elucidated by its structure constants, to investigate the properties of constants satisfying the algebraic relations (3.3.10) provides quite a significant information about the associated Lie group itself.

If structure constants are all zero, we then have $[V_i, V_j] = 0$ so that $\mathfrak{g}$ becomes a commutative Lie algebra. Such algebras are named as *Abelian Lie algebras*.

In exactly the same fashion as we have introduced the left-invariant vectors, we can define the ***right-invariant*** vector fields through the relation $dR_g(V) = V$. We immediately observe that these vector fields constitute a Lie algebra that is isomorphic to the vector space $T_e(G)$. Let us denote Lie algebras of left- and right-invariant vectors by $\mathfrak{g}_L$ and $\mathfrak{g}_R$, respectively. Since both algebras are isomorphic to the tangent space $T_e(G)$, they are of course isomorphic to one another through the isomorphism $\mathcal{G}_R^{-1} \circ \mathcal{G}_L$.

In view of (2.7.7), the relation (3.3.3) yields

$$dL_g \circ dR_g = dR_g \circ dL_g.$$

If V is a left-invariant vector field, we find

$$dL_g\big(dR_g(V)\big) = dR_g\big(dL_g(V)\big) = dR_g(V)$$

for all $g \in G$. This result means that the vector field $dR_g(V)$ turns out also to be a left- invariant vector field. Conversely, if V is a right-invariant vector field, then the same expression implies that the vector field $dL_g(V)$ is a right-invariant vector field.

Example 3.3.1. Consider the affine space $\mathbb{R}^n$ [*see* Example 2.2.1]. This smooth manifold is obviously a commutative Lie group with respect to the following addition operation

$$x + y = (x^1 + y^1, \dots, x^n + y^n)$$

for all $x, y \in \mathbb{R}$. In this case left and right translations are not different and they are given by

$$L_x(y) = R_x(y) = x + y.$$

Let us denote a left-invariant vector field by $V(x) = v^i(x)\,\partial_i$. Then (3.3.7) leads to the relation

$$v^i(x+y) = \frac{\partial(y^i + x^i)}{\partial x^j} v^j(x) = \delta^i_j v^j(x) = v^i(x).$$

Hence the left-invariant vector fields are constant vector fields whose components merely $v^i \in \mathbb{R}$. Of course, they generate a commutative Lie algebra with vanishing structure constant. ■

Example 3.3.2. We wish to compute the Lie algebra of the Lie group $GL(n,\mathbb{R})$. Inasmuch as $GL(n,\mathbb{R})$ is an open submanifold of the manifold $gl(n,\mathbb{R})$, its dimension is n^2. Hence, the tangent space at the identity element $e = \mathbf{I}$ is an n^2-dimensional vector space. We can thus identify the associated Lie algebra with the space $gl(n,\mathbb{R})$ that consists of all $n \times n$ matrices. We can choose as basis vectors the set of following linearly independent $n \times n$ matrices whose only one entry is 1 and all the other entries are 0:

$$V^i_j(e) = \delta^i_k \delta^l_j \frac{\partial}{\partial x^l_k}, \quad i,j,k,l = 1,\dots,n$$

where n^2 matrix entries x^l_k represent the local coordinates of $GL(n,\mathbb{R})$. Left translation is naturally defined as the matrix product $L_g(h) = \mathbf{GH}$ or $\big(L_g(h)\big)^k_l = g^k_m h^m_l$ in terms of components of $\mathbf{G} = [g^i_j]$ and $\mathbf{H} = [h^i_j]$. Hence, according to (3.3.7), the components of a left-invariant vector field must obey the equality

$$\big(V^i_j(g)\big)^k_l = \frac{\partial(g^k_m x^m_l)}{\partial x^p_q} \big(V^i_j(e)\big)^p_q = g^k_m \delta^m_p \delta^q_l \delta^i_q \delta^p_j = g^k_j \delta^i_l.$$

Consequently we can construct left-invariant vector fields by making use of the basis vectors

$$V^i_j(g) = g^k_j \delta^i_l \frac{\partial}{\partial g^k_l} = g^k_j \frac{\partial}{\partial g^k_i}$$

for all $g \in GL(n,\mathbb{R})$. An element of the Lie algebra $\mathfrak{gl}(n)$ will now be expressible as

$$V_{\mathbf{A}} = a^j_i V^i_j(e) = a^j_i \delta^i_k \delta^l_j \frac{\partial}{\partial x^l_k} = a^j_i \frac{\partial}{\partial x^j_i}$$

where the numbers a^j_i are entries of a matrix $\mathbf{A}$. Next, we determine the structure constants of the Lie algebra by evaluating

$$\begin{aligned}[V_j^i, V_l^k] &= \left[g_j^p \frac{\partial}{\partial g_i^p}, g_l^q \frac{\partial}{\partial g_k^q}\right] = g_j^p \frac{\partial}{\partial g_i^p}\left(g_l^q \frac{\partial}{\partial g_k^q}\right) - g_l^q \frac{\partial}{\partial g_k^q}\left(g_j^p \frac{\partial}{\partial g_i^p}\right) \\ &= g_j^p \delta_p^q \delta_l^i \frac{\partial}{\partial g_k^q} + g_j^p g_l^q \frac{\partial^2}{\partial g_i^p \partial g_k^q} - g_l^q \delta_q^p \delta_j^k \frac{\partial}{\partial g_i^p} - g_l^q g_j^p \frac{\partial^2}{\partial g_k^q \partial g_i^p} \\ &= \delta_l^i g_j^p \frac{\partial}{\partial g_k^p} - \delta_j^k g_l^p \frac{\partial}{\partial g_i^p} = \delta_l^i V_j^k - \delta_j^k V_l^i.\end{aligned}$$

It then follows that

$$[V_j^i, V_l^k] = \left(\delta_l^i \delta_p^k \delta_j^q - \delta_j^k \delta_p^i \delta_l^q\right) V_q^p = c_{jlp}^{ikq} V_q^p.$$

Since $\left(V_{\mathbf{A}}(e)\right)_j^i = a_j^i$, the left-invariant vector field generated by a vector $V_{\mathbf{A}}$ becomes

$$V_{\mathbf{A}}(g) = \left(V_{\mathbf{A}}(g)\right)_j^i \frac{\partial}{\partial g_j^i} = \frac{\partial (g_m^i x_j^m)}{\partial x_l^k} a_l^k \frac{\partial}{\partial g_j^i} = g_k^i a_j^k \frac{\partial}{\partial g_j^i}.$$

Therefore the Lie product (bracket) of left-invariant matrices corresponding to matrices **A** and **B** is found to be

$$\begin{aligned}[V_{\mathbf{A}}, V_{\mathbf{B}}](g) &= \left[g_m^i a_j^m \frac{\partial}{\partial g_j^i}, g_p^k b_l^p \frac{\partial}{\partial g_l^k}\right] = g_m^i a_j^m b_l^j \frac{\partial}{\partial g_l^i} - g_p^i b_l^p a_j^l \frac{\partial}{\partial g_j^i} \\ &= (a_j^m b_l^j - b_j^m a_l^j) g_m^i \frac{\partial}{\partial g_l^i} = g_m^i [\mathbf{A}, \mathbf{B}]_l^m \frac{\partial}{\partial g_l^i} = V_{[\mathbf{A},\mathbf{B}]}(g).\end{aligned}$$

where $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$ is the *matrix commutator*. These results clearly indicate that the Lie algebra $\mathfrak{gl}(n)$ is actually generated by the elements of the vector space $gl(n, \mathbb{R})$ on which the Lie product of matrices $\mathbf{A}, \mathbf{B}$ is defined as the matrix commutator $[\mathbf{A}, \mathbf{B}]$. ■

3.4. LIE GROUP HOMOMORPHISMS

Let $(G, *)$ and $(H, \diamond)$ be Lie groups, and $\phi : G \to H$ be a *smooth* function. If, for all $g_1, g_2 \in G$, the relation $\phi(g_1 * g_2) = \phi(g_1) \diamond \phi(g_2)$ is valid, then the function ϕ is called a ***Lie group homomorphism***. Moreover, if the homomorphism ϕ is also a diffeomorphism, ϕ is then a ***Lie group isomorphism***. For the identity element $e \in G$, we simply obtain

$$\phi(g) = \phi(e * g) = \phi(e) \diamond \phi(g).$$

Hence, the unique identity element e' of the Lie group H will necessarily be

$e' = \phi(e)$. Moreover, we can write $e' = \phi(g{*}g^{-1}) = \phi(g) \diamond \phi(g^{-1})$ so that we deduce the relation $\big(\phi(g)\big)^{-1} = \phi(g^{-1})$. Thus, $\phi(G) \subseteq H$ is a subgroup.

If a left translation on G is L_g, then we obtain

$$\phi\big(L_g(g_1)\big) = \phi(g{*}g_1) = \phi(g) \diamond \phi(g_1) = L_{\phi(g)}\big(\phi(g_1)\big)$$

for all $g, g_1 \in G$ from which it follows that $\phi \circ L_g = L_{\phi(g)} \circ \phi : G \to H$ for all $g \in G$. The expression (2.7.7) now leads to the rule

$$d\phi \circ dL_g = dL_{\phi(g)} \circ d\phi. \tag{3.4.1}$$

Let us consider a left-invariant vector field V on G. Since V satisfies the relation $dL_g(V) = V$, (3.4.1) now yields the result

$$d\phi\big(dL_g(V)\big) = d\phi(V) = dL_{\phi(g)}\big(d\phi(V)\big)$$

valid for all $g \in G$. This means that the vector field $d\phi(V)$ is a left-invariant vector field of H on the subgroup $\phi(G) \subseteq H$. Let V_1, V_2 be left-invariant vector fields on G. On taking into account the relation $dL_g([V_1, V_2]) = [V_1, V_2]$, (3.4.1) leads to the conclusion

$$d\phi\big(dL_g([V_1, V_2])\big) = d\phi([V_1, V_2]) = dL_{\phi(g)}\big(d\phi([V_1, V_2])\big)$$

which expresses the fact that $d\phi([V_1, V_2]) = [d\phi(V_1), d\phi(V_2)]$ is a left-invariant vector field on H. In other words, images of left-invariant vector fields under the differential mapping $d\phi$ where ϕ is a homomorphism are elements of a Lie algebra on H. Since the homomorphism ϕ transports the identity element e in G to the identity element $\phi(e)$ in H, we find that $d\phi : T_e(G) \to T_{\phi(e)}(H)$. We denote the Lie algebras on G and H by $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Via isomorphisms $\mathcal{G} : \mathfrak{g} \to T_e(G)$ and $\mathcal{H} : \mathfrak{h} \to T_{\phi(e)}(H)$, which we have discussed on $p.$ 182, we can introduce a *linear operator* $\psi = \mathcal{H}^{-1} \circ d\phi \circ \mathcal{G} : \mathfrak{g} \to \mathfrak{h}$. It is straightforward to see that this operator fulfil the relation

$$\psi([V_1, V_2]) = [\psi(V_1), \psi(V_2)] \tag{3.4.2}$$

for all $V_1, V_2 \in \mathfrak{g}$, that is, ψ preserves the Lie product. We thus conclude that ψ so defined is a Lie algebra homomorphism. The image $\psi(\mathfrak{g})$ of $\mathfrak{g}$ is clearly a *subalgebra* of $\mathfrak{h}$.

When ϕ is an isomorphism, ψ turns out to be likewise an isomorphism and we find that $\mathfrak{h} = \psi(\mathfrak{g})$. In that situation, if the set of vector fields $\{V_i\}$ is a basis for the Lie algebra $\mathfrak{g}$, then the set of vector fields $\{\psi(V_i)\}$ becomes a basis for the Lie algebra $\mathfrak{h}$. Because of relations $[V_i, V_j] = c_{ij}^k V_k$ it follows from (3.4.2) that

$$[\psi(V_i), \psi(V_j)] = \psi([V_i, V_j]) = \psi(c_{ij}^k V_k) = c_{ij}^k \psi(V_k). \qquad (3.4.3)$$

Hence, *such an isomorphism preserves structure constants*.

3.5. ONE-PARAMETER SUBGROUPS

We consider a Lie group G. As is well known, the set $\mathbb{R}$ is an Abelian Lie group with respect to the operation of addition. *Let $\phi : \mathbb{R} \to G$ be a Lie group homomorphism. The subset $\{\phi(t) : t \in \mathbb{R}\} = \phi(\mathbb{R}) \subseteq G$ is called a one-parameter subgroup of G*. By definition, the function ϕ must satisfy the condition

$$\phi(t + s) = \phi(t)*\phi(s) = \phi(s)*\phi(t) \qquad (3.5.1)$$

for all $s, t \in \mathbb{R}$ because $t + s = s + t$. Therefore, one-parameter subgroups would necessarily be commutative. Inasmuch as ϕ is a homomorphism, we observe that $e = \phi(0)$ and $\big(\phi(t)\big)^{-1} = \phi(-\,t)$. The smooth function ϕ will evidently describe a smooth curve on the manifold G through the point e.

Theorem 3.5.1. *A curve on a Lie group G is a one-parameter subgroup if and only if it is an integral curve of a left-invariant or a right-invariant vector field through the identity element e.*

Let $\phi : \mathbb{R} \to G$ give rise to a one-parameter subgroup. As in (2.9.1), we represent symbolically a tangent vector at an element $g = \phi(t)$ in the following manner

$$V\big(\phi(t)\big) = \frac{d\phi(t)}{dt}. \qquad (3.5.2)$$

Owing to the formula $L_{\phi(t)}\big(\phi(s)\big) = \phi(t)*\phi(s) = \phi(t + s)$, the vector field $V\big(\phi(t)\big)$ under the differential operator $dL_{\phi(t)}$ must satisfy the relation

$$dL_{\phi(t)}\Big(\frac{d\phi(s)}{ds}\Big) = \frac{d\phi(t + s)}{ds} = \frac{d\phi(t + s)}{dt}$$

If we insert $s = 0$ into this expression, we obtain

$$dL_{\phi(t)}\big(V(e)\big) = V\big(\phi(t)\big).$$

which indicates that (3.5.2) is a left-invariant vector field. It is evident that $\phi(t)$ is an integral curve of this vector field through the point $e \in G$. If we associate each point $g \in G$ with a curve defined by

$$\phi_g(t) = g*\phi(t)$$

we produce a congruence on G that is tangent to the left-invariant vector field V. However, it is evident that only the curve of this congruence through the point e corresponds to a one-parameter subgroup.

Conversely, let us now consider a left-invariant vector field $V \in \mathfrak{g}$. This vector field associated with a vector in $T_e(G)$ generates a flow on G whose member through the identity element $e \in G$ will be given just like in (2.9.11) by

$$g_t(e) = e^{tV}(e) \in G. \tag{3.5.3}$$

If we make use of the relation (2.9.17) it follows from (3.5.3) that

$$\begin{aligned} g_t {*} g_s &= L_{g_t}(g_s) = L_{g_t}\big(e^{sV}(e)\big) = e^{s\,dL_{g_t}(V)} L_{g_t}(e) = e^{sV}(g_t) \\ &= e^{sV} e^{tV}(e) = e^{tV}(e) {*} e^{sV}(e) = e^{(t+s)V}(e) = g_{t+s}. \end{aligned}$$

This clearly shows that the subset (3.5.3) is a one-parameter subgroup, and we have $e = g_0$ and $(g_t)^{-1} = g_{-t}$.

The case of right-invariant vector fields can be treated in exactly the same manner. □

Let $\phi : \mathbb{R} \to G$ be a one-parameter subgroup. If we write $g(t) = \phi(t)$, this subgroup gives rise to a one-parameter group of transformations of left translations $\{L_{g(t)} : G \to G : t \in \mathbb{R}\}$. At $t = 0$ or equivalently at $g = e$, the tangent vector is determined by $V(e) = dg/dt|_{t=0}$. Hence, the vector field generating this group is found to be

$$\frac{dL_{g(t)}(h)}{dt}\bigg|_{t=0} = \frac{dR_h\big(g(t)\big)}{dt}\bigg|_{t=0} = dR_h \frac{dg}{dt}\bigg|_{t=0} = dR_h\big(V(e)\big) = V^R(h).$$

Thus, *it is a right-invariant vector field*. Similarly, one demonstrates that the generator of a one-parameter group of transformations of right translations $\{R_{g(t)} : G \to G : t \in \mathbb{R}\}$ is a left-invariant vector field:

$$\frac{dR_{g(t)}(h)}{dt}\bigg|_{t=0} = \frac{dL_h(g(t))}{dt}\bigg|_{t=0} = dL_h\big(V(e)\big) = V^L(h). \tag{3.5.4}$$

Exponential mapping $\exp : \mathfrak{g} \to G$ is defined by taking $t = 1$ in the one-parameter group (3.5.3) generated by a vector field $V \in \mathfrak{g}$ as follows:

$$\exp(V) = g_1(V) = e^V(e) \in G.$$

This definition leads automatically to $\exp(\mathbf{0}) = e$. If we regard the vector space $\mathfrak{g}$ as a manifold, its tangent spaces $T_g(\mathfrak{g})$ will be the same everywhere and they will be isomorphic to $\mathfrak{g}$. On the other hand, the tangent space

$T_e(G)$ at the point e is isomorphic to $\mathfrak{g}$. Since the tangent vector field of the curve defined by (3.5.3) is V, the differential of the exponential mapping at the vicinity of the vector $V = 0$ becomes

$$d\text{exp} : \mathfrak{g} \to T_e(G) \simeq \mathfrak{g}$$

yielding $d\text{exp}\,(V) = V$. The symbol $\simeq$ denotes isomorphism. We thus obtain the identity mapping $d\text{exp}|_{V=\mathbf{0}} = i_{\mathfrak{g}}$. We then conclude that at $V = 0$, $d$exp is a regular linear operator. This, of course, indicates that the function exp is a *local* diffeomorphism from the Lie algebra $\mathfrak{g}$ to an open neighbourhood of the identity element e of the Lie group G. Therefore, in a neighbourhood U_e of e, a group element g may be expressible in the form

$$g = \text{exp}\,(V) = \text{exp}\,(t^i V_i) \in U_e \subseteq G$$

where the set $\{V_1, , \dots, V_n\}$ is a basis for the Lie algebra. The ordered n-tuple of real numbers $(t^1, \dots, t^n) \in \mathbb{R}^n$ are called the *canonical coordinates* of g and they must be sufficiently small in order that $g \in U_e$. Owing to some properties of the exponential mapping illustrated in p. 139, g can also be written in the following way for sufficiently small canonical coordinates t_i, $i = 1, \dots, n$

$$g = \text{exp}\,(t^1 V_1) {*} \text{exp}\,(t^2 V_2) {*} \cdots {*} \text{exp}\,(t^n V_n).$$

because we can always choose commuting basis vectors for the Lie algebra. This amounts to say that the Lie algebra determines locally the Lie group at a neighbourhood of the group's identity element. That is the reason why a basis of a Lie algebra is called as ***infinitesimal generators*** of a Lie group. As we have mentioned before, it cannot be claimed that a given Lie algebra generates a uniquely determined global Lie group. However, if a Lie group is a connected manifold in which e has a simply connected neighbourhood, then the Lie algebra determines globally this group [*see* Theorem 3.2.1].

We now try to get the isomorphism between Lie algebras $\mathfrak{g}_L$ and $\mathfrak{g}_R$ whose existence was established on p. 185 to acquire a more concrete structure and we shall show that this isomorphism is provided by the differential $d\iota : T_g(G) \to T_{g^{-1}}(G)$ of the *inversion diffeomorphism* $\iota : G \to G$ that was defined by $\iota(g) = g^{-1}$. Since we can write

$$(\iota \circ L_g)(h) = (g{*}h)^{-1} = h^{-1}{*}g^{-1} = R_{g^{-1}}(h^{-1})$$

for all $h \in G$, we obtain $(\iota \circ L_g)(e) = R_{g^{-1}}(e)$ for $h = h^{-1} = e$ from which it follows that $d\iota \circ dL_g\big|_e = dR_{g^{-1}}\big|_e$ for all $g \in G$. For a vector $V \in T_e(G)$, this equality naturally implies that $d\iota \circ dL_g(V\big|_e) = dR_{g^{-1}}(V\big|_e)$ resulting in the relation

$$d\iota\big(V^L(g)\big) = V^R(g^{-1})$$

for all $g \in G$ where V^L and V^R are left and right invariant vectors. Hence, despite an apparent problem in the arguments, we may expect that the operator $d\iota : \mathfrak{g}_L \to \mathfrak{g}_R$ can be a possible candidate for the isomorphism that we are hoping to find. On the other hand, if a vector field V generates the one-parameter subgroup by $g(t) = \exp(tV)$, we have $g(t)^{-1} = \exp(-tV)$. Thus the tangent vectors to curves $g(t)$ and $g(t)^{-1}$ at the identity element e are prescribed by

$$\frac{dg(t)}{dt}\bigg|_{t=0} = V\big|_e, \quad \frac{dg(t)^{-1}}{dt}\bigg|_{t=0} = -V\big|_e.$$

Hence, at the identity element the operator $d\iota\big|_e : T_e(G) \to T_e(G)$ acts in the manner $d\iota(V\big|_e) = -V\big|_e$. It is then straightforward to realise that a right-invariant vector field produced by a vector $V \in T_e(G)$ has to satisfy the relation $V^R(g^{-1}) = -V^R(g)$. Therefore, the isomorphism between $\mathfrak{g}_L$ and $\mathfrak{g}_R$ is now provided by

$$d\iota\big(V^L(g)\big) = -V^R(g). \tag{3.5.5}$$

Whenever $V_1, V_2 \in T_e(G)$, we can define a vector $W = [V_1, V_2] \in T_e(G)$. We know that the left-invariant vector fields associated with these vectors will satisfy the relation $W^L = [V_1^L, V_2^L]$. We thus find

$$-W^R = d\iota(W^L) = d\iota([V_1^L, V_2^L]) = [d\iota(V_1^L), d\iota(V_2^L)] = [V_1^R, V_2^R]$$

that leads easily to the result $[V_1, V_2]_R = -[V_1, V_2]_L$ from which we deduce that if the structure constants of the left Lie algebra are c_{ij}^k, then the structure constants of the right Lie algebra has to be $-c_{ij}^k$.

Let $\mathfrak{g}$ be an n-dimensional Lie algebra of a Lie group G. A subalgebra $\mathfrak{h}$ of this algebra with dimension $m < n$ is again a Lie algebra. In other words, it is an involutive distribution. Therefore, according to the Frobenius theorem it generates an m-dimensional smooth submanifold through the point e. This submanifold is locally an m-parameter Lie group that is a subgroup of G.

Example 3.5.1. We know that the Lie algebra $\mathfrak{gl}(n)$ of the general linear group $GL(n, \mathbb{R})$ consists of $n \times n$ matrices. Hence, we can express a matrix $\mathbf{X} \in GL(n, \mathbb{R})$ in a neighbourhood of the identity element $\mathbf{I}$ by

$$\mathbf{X} = \exp(\mathbf{A}) = e^{\mathbf{A}}(\mathbf{I})$$

where $\mathbf{A} \in \mathfrak{gl}(n)$. Let us now consider the function $\det : GL(n, \mathbb{R}) \to \mathbb{R}$. In view of the relation (2.9.17), we have

$$\det\big(e^{\mathbf{A}}(\mathbf{I})\big) = e^{d\det(\mathbf{A})}\det(\mathbf{I}) = e^{d\det(\mathbf{A})}.$$

Let us write $\mathbf{A} = a^i_j\,\partial/\partial a^i_j$. Then the relation (2.7.9) yields

$$d\det(\mathbf{A}) = a^i_j\,\frac{\partial\det(\mathbf{X})}{\partial a^i_j}\bigg|_{\mathbf{X}=\mathbf{I}}.$$

Due to the equality $\partial\det(\mathbf{X})/\partial x^k_l = \text{cofactor}\,(x^k_l) = X^l_k = \det(\mathbf{X})\,(\mathbf{X}^{-1})^l_k$, we easily arrive at the expression

$$a^i_j\,\frac{\partial\det(\mathbf{X})}{\partial a^i_j} = \det(\mathbf{X})\,(\mathbf{X}^{-1})^l_k\,\frac{\partial x^k_l}{\partial a^i_j}\,a^i_j.$$

Inasmuch as we define $\mathbf{X}$ as the following series

$$\mathbf{X} = \exp(\mathbf{A}) = \mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \cdots + \frac{1}{n!}\mathbf{A}^n + \cdots,$$

then its entries are prescribed by

$$x^k_l = \delta^k_l + a^k_l + \frac{1}{2!}a^k_m a^m_l + \cdots + \frac{1}{n!}a^k_{m_1}a^{m_1}_{m_2}\cdots a^{m_{n-2}}_{m_{n-1}}\,a^{m_{n-1}}_l + \cdots.$$

Taking into account the relation

$$\begin{aligned}\frac{\partial a^k_{m_1}a^{m_1}_{m_2}\cdots a^{m_{n-1}}_l}{\partial a^i_j}a^i_j &= \delta^k_i\delta^j_{m_1}a^{m_1}_{m_2}\cdots a^{m_{n-1}}_l a^i_j + \delta^{m_1}_i\delta^j_{m_2}a^k_{m_1}\cdots a^{m_{n-1}}_l a^i_j\\ \cdots + \delta^{m_{n-1}}_i\delta^j_l a^k_{m_1}a^{m_1}_{m_2}\cdots a^i_j &= a^k_{m_1}a^{m_1}_{m_2}\cdots a^{m_{n-1}}_l + a^k_{m_1}a^{m_1}_{m_2}\cdots a^{m_{n-1}}_l\\ &+ a^k_{m_1}a^{m_1}_{m_2}\cdots a^{m_{n-1}}_l + \cdots + a^k_{m_1}a^{m_1}_{m_2}\cdots a^{m_{n-1}}_l = n(\mathbf{A}^n)^k_l,\end{aligned}$$

we finally find

$$\frac{\partial x^k_l}{\partial a^i_j}a^i_j = \big[\mathbf{A} + \mathbf{A}^2 + \cdots\frac{1}{(n-1)!}\mathbf{A}^n + \cdots\big]^k_l = (\mathbf{X}\mathbf{A})^k_l = (\mathbf{X})^k_m(\mathbf{A})^m_l$$

and reach to the conclusion

$$d\det(\mathbf{A}) = \det(\mathbf{X})\,(\mathbf{X}^{-1})^l_k(\mathbf{X})^k_m(\mathbf{A})^m_l\big|_{\mathbf{X}=\mathbf{I}} = \delta^l_m\boldsymbol{a}^m_l = a^m_m = \text{tr}\,(\mathbf{A}).$$

We thus obtain the rather elegant result

$$\det(e^{\mathbf{A}}) = e^{\text{tr}\,(\mathbf{A})}.$$

If the matrix $\mathbf{X}$ belongs to the subgroup $SL(n,\mathbb{R})$, then we must have $\det(\mathbf{X}) = 1$. Hence, if the matrix $\mathbf{A}$ is an element of the Lie subalgebra $\mathfrak{sl}(n)$, the condition

$$\det(e^{\mathbf{A}}) = e^{\operatorname{tr}(\mathbf{A})} = 1$$

must hold. This requires that $\operatorname{tr}(\mathbf{A}) = 0$. Consequently, the Lie algebra $\mathfrak{sl}(n)$ consists $n \times n$ traceless matrices.

Next, we consider the orthogonal group $O(n)$. If the matrix $\mathbf{X}$ belongs to that subgroup, the relation $\mathbf{X}\mathbf{X}^{\mathrm{T}} = \mathbf{X}^{\mathrm{T}}\mathbf{X} = \mathbf{I}$ must be satisfied. Let us take again $\mathbf{X} = e^{\mathbf{A}}$. It can easily be verified that $\mathbf{X}^{\mathrm{T}} = \left(e^{\mathbf{A}}\right)^{\mathrm{T}} = e^{\mathbf{A}^{\mathrm{T}}}$. We thus obtain the condition

$$e^{\mathbf{A}}e^{\mathbf{A}^{\mathrm{T}}} = e^{\mathbf{A}^{\mathrm{T}}}e^{\mathbf{A}} \quad \text{or} \quad [e^{\mathbf{A}}, e^{\mathbf{A}^{\mathrm{T}}}] = \mathbf{0}.$$

But this leads to the conclusion $[\mathbf{A}, \mathbf{A}^{\mathrm{T}}] = \mathbf{0}$ [*see p.* 148]. Hence the relation

$$\mathbf{X}\mathbf{X}^{\mathrm{T}} = e^{\mathbf{A}}e^{\mathbf{A}^{\mathrm{T}}} = e^{\mathbf{A}+\mathbf{A}^{\mathrm{T}}} = \mathbf{I}$$

requires that $\mathbf{A} + \mathbf{A}^{\mathrm{T}} = \mathbf{0}$, or $\mathbf{A}^{\mathrm{T}} = -\mathbf{A}$. Therefore, the Lie algebra $\mathfrak{o}(n)$ of the orthogonal group consists of antisymmetric $n \times n$ matrices. ■

3.6. ADJOINT REPRESENTATION

Let G be a Lie group. We choose an element $g \in G$ and define a mapping $\mathcal{I}_g : G \to G$ by the operation of *conjugation* prescribed by

$$\mathcal{I}_g(h) = g{*}h{*}g^{-1} \in G \tag{3.6.1}$$

for all $h \in G$. It is clear that this mapping is a diffeomorphism, Moreover, because it satisfies the relation

$$\mathcal{I}_g(h_1{*}h_2) = g{*}h_1{*}g^{-1}{*}g{*}h_2{*}g^{-1} = \mathcal{I}_g(h_1){*}\mathcal{I}_g(h_2)$$

for all $h_1, h_2 \in G$, it preserves the group operation. Hence, $\mathcal{I}_g$ is an automorphism on G called the ***inner automorphism***. All other automorphisms of G are named as *outer automorphisms.* The composition of two inner automorphisms yield

$$\mathcal{I}_{g_1} \circ \mathcal{I}_{g_2}(h) = g_1{*}g_2{*}h{*}g_2^{-1}{*}g_1^{-1} = (g_1{*}g_2){*}h{*}(g_1{*}g_2)^{-1} = \mathcal{I}_{g_1*g_2}(h)$$

for all $h \in G$ from which we deduce that $\mathcal{I}_{g_1} \circ \mathcal{I}_{g_2} = \mathcal{I}_{g_1*g_2}$. We immediately see that $\mathcal{I}_e = i_G$ is the identity mapping. Since $\mathcal{I}_g \circ \mathcal{I}_{g^{-1}} = \mathcal{I}_{g^{-1}} \circ \mathcal{I}_g = \mathcal{I}_e = i_G$ we realise that $(\mathcal{I}_g)^{-1} = \mathcal{I}_{g^{-1}}$. Furthermore, we obtain $\mathcal{I}_g(e) = e$ for all $g \in G$. Therefore all inner automorphisms $\{\mathcal{I}_g : g \in G\}$ transform any curve on the manifold G through the identity element e to another curve passing again through e. The definition (3.6.1) leads to

$$\mathcal{I}_g(h) = L_g\big(R_{g^{-1}}(h)\big) = R_{g^{-1}}\big(L_g(h)\big)$$

for all $h \in G$. We then conclude that

$$\mathcal{I}_g = L_g \circ R_{g^{-1}} = R_{g^{-1}} \circ L_g. \tag{3.6.2}$$

It is now obvious that the set $\mathcal{G} = \{\mathcal{I}_g : g \in G\}$ constitutes a group with respect to the composition of mappings. On taking into account properties of the mappings L_g and R_g, it is easily understood that the expressions (3.6.2) indicate the existence of an isomorphism between this group and the Lie group G.

If G is an Abelian group, then we obtain $\mathcal{I}_g(h) = h$ for each $g \in G$ so that we get $\mathcal{I}_g = i_G$. Hence, in commutative groups the mapping $\mathcal{I}_g$ acquires quite a trivial structure.

Let us now consider the differential $d\mathcal{I}_g$. (3.6.2) yields naturally

$$d\mathcal{I}_g = dL_g \circ dR_{g^{-1}} = dR_{g^{-1}} \circ dL_g. \tag{3.6.3}$$

If $V \in \mathfrak{g}$ is a left-invariant vector field, then it follows from (3.6.3) that

$$d\mathcal{I}_g(V) = dR_{g^{-1}} \circ dL_g(V) = dR_{g^{-1}}(V) = dL_g \circ dR_{g^{-1}}(V)$$

that may be expressed in the way

$$dL_g\big(d\mathcal{I}_g(V)\big) = d\mathcal{I}_g(V).$$

Thus $d\mathcal{I}_g(V)$ becomes also a left-invariant vector field so that we can write $d\mathcal{I}_g(V) \in \mathfrak{g}$ and conclude that $d\mathcal{I}_g : \mathfrak{g} \to \mathfrak{g}$. Since $\mathcal{I}_g$ is a diffeomorphism, its differential $d\mathcal{I}_g$ is a regular linear operator, i.e., an isomorphism. For all vectors $V_1, V_2 \in \mathfrak{g}$, we have $d\mathcal{I}_g([V_1, V_2]) = [d\mathcal{I}_g(V_1), d\mathcal{I}_g(V_2)]$. Therefore, the isomorphism $d\mathcal{I}_g$ preserves the Lie product. In other words, it is an automorphism on the Lie algebra $\mathfrak{g}$. Thus, to each element $g \in G$, there corresponds an automorphism on the Lie algebra $\mathfrak{g}$. Let us denote the linear vector space formed by these automorphism, or to be more concrete, by regular matrices representing these automorphisms, as $Aut(\mathfrak{g})$. We now rename the operator $d\mathcal{I}_g$ as $d\mathcal{I}_g = Ad_g : \mathfrak{g} \to \mathfrak{g}$ for convenience. Let us next introduce the mapping

$$Ad : G \to Aut(\mathfrak{g})$$

in the following manner: $Ad(g) = Ad_g \in Aut(\mathfrak{g})$ for each $g \in G$. On the other hand, one can easily verify that the equality $d\mathcal{I}_{g_1} \circ d\mathcal{I}_{g_2} = d\mathcal{I}_{g_1 * g_2}$ entails the relation

$$Ad(g_1 * g_2) = Ad_{g_1} \circ Ad_{g_2}.$$

Hence Ad is a group homomorphism assigning to each element of the group G a matrix representing an automorphism. That is the reason why it is called the ***adjoint representation*** of the Lie group G over the Lie algebra $\mathfrak{g}$. One the most outstanding successes of the group theory was to predict that every abstract group is homomorphic to a general linear group $GL(n, \mathbb{R})$ which is called a ***representation*** or more precisely an ***unfaithful representation*** of the group. Whenever this homomorphism is an isomorphism, we obtain a ***faithful representation***. *The theory of group representation* deals with the quite difficult, but practically very important problem of determining the number n and the specific form of matrices involved in such a representation.

It is straightforward to verify that what we have discussed above would be equally valid when we replace an element g by its inverse g^{-1} in case Lie algebra is derived from right-invariant vector fields.

Let V be a *left-invariant* vector field. We consider the one-parameter subgroup exp (tV) produced by V. If we recall (2.9.17), we observe that we can write

$$\begin{aligned} \mathcal{I}_g\big(\exp(tV)\big) &= g * \exp(tV) * g^{-1} = \mathcal{I}_g\big(e^{tV}(e)\big) \\ &= e^{t\, d\mathcal{I}_g(V)}\big(\mathcal{I}_g(e)\big) = \exp\big(tAd_g(V)\big) \end{aligned} \tag{3.6.4}$$

for all $g \in G$. This result simply means that under the mapping $\mathcal{I}_g$, the one-parameter subgroup generated by the vector field V is transformed into the one-parameter subgroup generated by the vector field $Ad_g(V)$.

Let us now consider another one-parameter subgroup generated by a left-invariant vector field U whose elements are, of course, given by $g(s) = \exp(sU)$. If we resort to the relation (2.10.16), we arrive at the following expression

$$Ad_{g(s)}(V) = e^{s\pounds_U} V, \;\; U, V \in \mathfrak{g} \tag{3.6.5}$$

which measures the change in the vector field $Ad_g(V)$ over the subgroup exp (sU). By employing (3.6.5), we can evaluate the following expression at the point e:

$$\frac{d}{ds} Ad_{g(s)}(V)\bigg|_{s=0} = ad_U(V) = \pounds_U V = [U, V],\; V \in \mathfrak{g}. \tag{3.6.6}$$

We have already seen that $\mathcal{I}_g = i_G$ if G is an Abelian group. Hence, in this case, we obtain $Ad_g = I_{\mathfrak{g}}$ for all $g \in G$ and (3.6.6) leads to $[U, V] = 0$. Thus the Lie algebra of such a Lie group becomes also Abelian. Conversely, it can be shown that if G is a *connected* Lie group whose Lie algebra is Abelian then G, too, will be an Abelian group.

3.7. LIE TRANSFORMATION GROUPS

We assume that we are given a Lie group G of r-parameters and an m-dimensional smooth manifold M. Let us consider a *differentiable* mapping $\Psi : G \times M \to M$ on the product manifold that manifests the *action* of the group G on the manifold M. We thus obtain $\Psi(g, p) = g \blacklozenge p \in M$ for all $g \in G$ and $p \in M$. We can now form a function $\Psi_g : M \to M$ mapping the manifold M onto itself by the relation $\Psi_g(p) = \Psi(g, p)$ where $g \in G$ is a fixed element of the group G. The set $\{\Psi_g : g \in G\}$ will be called a ***Lie transformation group*** if it possesses group properties with respect to composition, that is, when the conditions

(i) $\Psi_{g_1} \circ \Psi_{g_2} = \Psi_{g_1 * g_2}$ *or* $g_1 \blacklozenge (g_2 \blacklozenge p) = (g_1 * g_2) \blacklozenge p$ *for* $g_1, g_2 \in G$

(ii) *if* $e \in G$ *is the identity element, then* $\Psi_e = i_M$ *is the identity mapping on* M *so that one can write* $\Psi_e(p) = p$ *or* $e \blacklozenge p = p$

are satisfied. Hence, the following properties are valid

$$\Psi_{g_1}\big(\Psi_{g_2}(p)\big) = \Psi\big(g_1, \Psi(g_2, p)\big) = \Psi(g_1 * g_2, p),\ \Psi_e(p) = \Psi(e, p) = p.$$

It is easy to observe that the foregoing expressions lead to relations

$$\Psi_g^{-1} = \Psi_{g^{-1}}, \quad \Psi\big(g^{-1}, \Psi(g, p)\big) = p \quad \text{or} \quad (g^{-1} * g) \blacklozenge p = p.$$

We say that the group G acts ***effectively*** on the manifold M if the relation $\Psi_g(p) = p$ for all $p \in M$ implies $g = e$. If the stronger condition $\Psi_g(p) \neq p$ unless $g = e$ holds, then the group G acts ***freely*** (*without a fixed point*) on the manifold M. If for all $p, q \in M$, there exists an element $g \in G$ such that $\Psi_g(p) = g \blacklozenge p = q$, then the group G acts ***transitively*** on the manifold M.

We now define the set

$$G_p = \{g \in G : \Psi_g(p) = g \blacklozenge p = p\} \subset G$$

for a *fixed point* $p \in M$. It can be demonstrated that G_p is a subgroup of G. If $g \in G_p$, then one has $g \blacklozenge p = p$ from which $g^{-1} \blacklozenge (g \blacklozenge p) = g^{-1} \blacklozenge p$ follows at once. On the other hand, we can write $g^{-1} \blacklozenge (g \blacklozenge p) = (g^{-1} * g) \blacklozenge p = p$ so that we find $g^{-1} \blacklozenge p = p$. Therefore, we find that $g^{-1} \in G_p$. Next, let us consider $g, h \in G_p$, then we obtain $(g*h) \blacklozenge p = g \blacklozenge (h \blacklozenge p) = g \blacklozenge p = p$ implying that $g*h \in G_p$. Moreover, we observe that $e \in G_p$ since $\Psi_e(p) = p$. Thus G_p is a subgroup of G. The subgroup G_p so defined is known as the ***isotropy group*** of a point $p \in M$. The isotropy groups at the points $p \in M$ and $g \blacklozenge p \in M$ are connected by the conjugation relation

$$G_{g \blacklozenge p} = g*G_p*g^{-1}, \text{ for any } g \in G.$$

Indeed, for any $h \in G_p$, one deduce $(g*h*g^{-1}) \blacklozenge (g \blacklozenge p) = g \blacklozenge (h \blacklozenge p) = g \blacklozenge p$ so that $g*h*g^{-1} \in G_{g \blacklozenge p}$. This, of course, means that $g*G_p*g^{-1} \subseteq G_{g \blacklozenge p}$. Now, consider an element $h \in G_{g \blacklozenge p}$ so that $h \blacklozenge (g \blacklozenge p) = (h*g) \blacklozenge p = g \blacklozenge p$ or $(g^{-1}*h*g) \blacklozenge p = p$ implying that $g^{-1}*h*g \in G_p$ and $g^{-1}*G_{g \blacklozenge p}*g \subseteq G_p$ from which we immediately obtain $G_{g \blacklozenge p} \subseteq g*G_p*g^{-1}$. Thus we arrive at the desired equality given above. However, the statement $g^{-1}*h*g \in G_p$ for all $h \in G_p$ and for all $g \in G$ implies that G_p is a *normal subgroup*. If G is a freely acting group, then it is clear that $G_p = \{e\}$ at each point $p \in M$.

The ***orbit*** of the group G at a point $p_0 \in M$ is defined as the set

$$\{g \blacklozenge p_0 : g \in G\} = \mathcal{O}_{p_0} \subseteq M.$$

When $p, q \in \mathcal{O}_{p_0}$, then there are $g_1, g_2 \in G$ such that one has $p = g_1 \blacklozenge p_0$ and $q = g_2 \blacklozenge p_0$. Consequently, we can write $q = (g_2*g_1^{-1}) \blacklozenge p$. Thus, the group G acts transitively on any orbit $\mathcal{O}_{p_0}$.

Example 3.7.1. Let us consider the smooth manifold $M = \mathbb{R}^n$ and the Lie group $G = GL(n, \mathbb{R})$. If $\mathbf{x} = (x^1, x^2, \dots, x^n) \in \mathbb{R}^n$ and $\mathbf{A} \in GL(n, \mathbb{R})$, we define the group action on the manifold by $\Psi(\mathbf{A}, \mathbf{x}) = \mathbf{A}\mathbf{x} \in \mathbb{R}^n$. Hence, the isotropy group of a point $\mathbf{x}_0 \in \mathbb{R}^n$ is determined by the following set

$$G_{\mathbf{x}_0} = \{\mathbf{A} \in GL(n, \mathbb{R}) : \mathbf{A}\mathbf{x}_0 = \mathbf{x}_0\}.$$

Thus, elements of the isotropy group can only be $n \times n$ matrices with an eigenvalue 1 and admitting the vector $\mathbf{x}_0$ as an eigenvector associated with that eigenvalue. Therefore, the necessary condition imposed on matrices $\mathbf{A}$ should be $\det(\mathbf{A} - \mathbf{I}) = 0$. For instance, the isotropy group of the point $\mathbf{x}_0 = (1, 0, \dots, 0)$ consists of matrices of the form

$$\mathbf{A} = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \ \det \mathbf{A} = \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0.$$

Obviously, the condition $\det(\mathbf{A} - \mathbf{I}) = 0$ are satisfied. We can easily verify that the orbit of a point $\mathbf{x}_0$ is $\mathcal{O}_{\mathbf{x}_0} = M - \{\mathbf{0}\}$. Indeed, if $\mathbf{x} \neq \mathbf{0} \in M$ is an arbitrary point, we can always construct a matrix $\mathbf{A} \in GL(n, \mathbb{R})$ so as the relation $\mathbf{A}\mathbf{x}_0 = \mathbf{x}$ is satisfied. If we choose $n^2 - n$ entries of that matrix arbitrarily, then the foregoing n equation will help determine the remaining n entries. For example, if $x^1 x^2 \cdots x^n \neq 0$ and $x_0^1 x_0^2 \cdots x_0^n \neq 0$, then we may choose a diagonal matrix such that $a_{11} = x^1/x_0^1, \ldots, a_{nn} = x^n/x_0^n$. ■

By employing the smooth function $\Psi : G \times M \to M$ represented by $\Psi(g, p) = g \blacklozenge p$, we can now introduce two functions for a fixed $g \in G$ and a fixed $p \in M$, respectively, and their differentials in the following manner

$$\Psi_g : M \to M,\ \Psi_g(p) = g \blacklozenge p,\ \ d\Psi_g : T_p(M) \to T_{g\blacklozenge p}(M) \quad (3.7.1)$$
$$\Psi_p : G \to M,\ \Psi_p(g) = g \blacklozenge p,\ \ d\Psi_p : T_g(G) \to T_{g\blacklozenge p}(M).$$

Consider a member $V \in T_e(G)$ of the Lie algebra on G. By means of the linear operator $d\Psi_p : T_e(G) \to T_p(M)$, we can construct a vector field $V^K(p) \in T_p(M)$ on M through the relation

$$V^K(p) = d\Psi_p\big(V\big). \tag{3.7.2}$$

Such vector fields V^K are called ***Killing vector fields*** after German mathematician Wilhelm Karl Joseph Killing (1847-1923). To attribute a more concrete meaning to Killing vectors, let us consider a right-invariant vector field $V^R \in \mathfrak{g}_R$ produced by the vector $V \in T_e(G)$ through the usual relation $V^R(g) = dR_g(V)$. For each $h \in G$, we can evidently write

$$\Psi_p \circ R_g(h) = \Psi_p(h{*}g) = (h{*}g) \blacklozenge p = h \blacklozenge (g \blacklozenge p) = \Psi_{g\blacklozenge p}(h).$$

Thus, it follows from (3.7.2) that

$$V^K(g \cdot p) = d\Psi_{g\cdot p}(V) = d\Psi_p \circ dR_g(V) = d\Psi_p\big(V^R(g)\big). \tag{3.7.3}$$

Hence, the Lie product of two Killing vector fields becomes

$$[V_i^K, V_j^K]\big|_{g\blacklozenge p} = [d\Psi_p(V_i^R), d\Psi_p(V_j^R)]\big|_g = d\Psi_p[V_i^R, V_j^R]\big|_g.$$

Inasmuch as $[V_i^R, V_j^R] = -c_{ij}^k V_k^R$ [*see p.* 192], then we find that

$$[V_i^K, V_j^K] = -c_{ij}^k d\Psi_p(V_k^R) = -c_{ij}^k V_k^K, \quad i, j, k = 1, \ldots, r.$$

This relation states that Killing vectors, too, constitute a Lie algebra. If an r-dimensional Lie algebra with structure constants c_{ij}^k is given on a smooth manifold M, then we conclude, conversely, that there exists a Lie group G whose Lie algebra has those structure constants with respect to a basis V_1, $\ldots, V_r$ and the local action of G on M is described by the vector fields $V_i^K = d\Psi_p(V_i), i = 1, \ldots, r$.

We know that a particular vector $V \in T_e(G)$ determines a one-parameter subgroup of the group G via the curve $g(t) = \exp(tV) \in G$ and the relation $V = dg(t)/dt|_{t=0}$ is satisfied. The curve $g(t)$ generates a group of mappings on M through $\{\Psi\big(g(t), p\big) = \Psi_{g(t)}(p) = \Psi_p\big(g(t)\big) : t \in \mathbb{R}\}$. On the other hand, the integral curve passing through the point $p \in M$ has to satisfy the relation

$$\frac{d\Psi_{g(t)}(p)}{dt} = \frac{d\Psi_p\big(g(t)\big)}{dt} = W[\Psi_p\big(g(t)\big)], \quad \Psi_p(e) = p.$$

where W is the vector field tangent to the integral curve. Consequently, the tangent vector to that curve at $t = 0$ should be given by

$$W(p) = \frac{d\Psi_p\big(g(t)\big)}{dt}\bigg|_{t=0} = d\Psi_p\Big(\frac{dg(t)}{dt}\bigg|_{t=0}\Big) = d\Psi_p(V) = V^K(p).$$

Therefore the vector field $W(p)$ is a Killing vector field. The dimension s of the Lie algebra of Killing vectors depends on the rank of the linear operator $d\Psi_p$. If the dimension of the vector space $T_e(G)$ is r, then it is known that one can write $r = n(d\Psi_p) + s$ where $n(d\Psi_p)$ is the dimension of the null space of $d\Psi_p$. Thus this relation implies that $s \leq r$. *If only $d\Psi_p$ is injective, namely, if $n(d\Psi_p) = 0$, we obtain $s = r$.* In this case, we have $V^K = 0$ if and only if $V = 0$. This becomes possible if only the group G acts effectively on the manifold M. To demonstrate this statement, let us consider an effectively acting group G and assume that $V^K = 0$ for a vector $V \neq 0$. The vector V now determines a subgroup $g(t) = \exp(tV)$ in G. This subgroup then generates a curve $p(t) = \Psi_{g(t)}(p_0)$ on the manifold M going through the point $p_0 \in M$. Then we find of course

$$V^K\big(p(t)\big) = \frac{d\Psi_{g(t)}(p_0)}{dt} = 0$$

implying that $\Psi_{g(t)}(p_0)$ is *constant*. Hence, we obtain $\Psi_{g(t)}(p_0) = \Psi_e(p_0) = p_0$. However, this contradicts the effectiveness of the group G. As a result of this, we find $s = r$. In other words, *if the group G is acting effectively on the manifold M, then $d\Psi_p$ is an isomorphism if $d\Psi_p$ is injective.*

III. EXERCISES

3.1. The circle $\mathbb{S}^1$ is given by $|z| = 1, z \in \mathbb{C}$. Let us consider the smooth manifold $G = \mathbb{R} \times \mathbb{R} \times \mathbb{S}^1$ and define an operation $* : G \times G \to G$ by

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, e^{iy_1 x_2} z_1 z_2).$$

Show that $(G, *)$ is a Lie group.

3.2. The mapping $\phi : (\mathbb{R}, +) \to SO(2)$ is defined by

$$\phi(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Show that ϕ is a Lie group homomorphism.

3.3. G is a Lie group. Show that $T(G)$ is also a Lie group.

3.4. We define the set of *symplectic matrices* by the following relation

$$Sp(n, \mathbb{R}) = \{\mathbf{A} \in GL(2n, \mathbb{R}) : \mathbf{A}^{\mathrm{T}} \boldsymbol{J} \mathbf{A} = \boldsymbol{J}\},\ \boldsymbol{J} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_n \\ -\mathbf{I}_n & \mathbf{0} \end{bmatrix}$$

where $\mathbf{I}_n$ is the $n \times n$ unit matrix. Let $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ and $\mathbf{A}_4$ be $n \times n$ matrices. We introduce a matrix $\mathbf{A}$ by

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix}.$$

In order that $\mathbf{A} \in Sp(n, \mathbb{R})$, show that the matrices $\mathbf{A}_1^{\mathrm{T}} \mathbf{A}_3$ and $\mathbf{A}_2^{\mathrm{T}} \mathbf{A}_4$ must be symmetric, and the relation $\mathbf{A}_1^{\mathrm{T}} \mathbf{A}_4 - \mathbf{A}_3^{\mathrm{T}} \mathbf{A}_2 = \mathbf{I}_n$ must be satisfied. Prove that $Sp(n, \mathbb{R})$ is a Lie group. Find the dimension of this group. Show that if $\mathbf{A} \in Sp(n, \mathbb{R})$, then one also finds $\mathbf{A}^{\mathrm{T}} \in Sp(n, \mathbb{R})$.

3.5. Show that the Le algebra of the Lie group $Sp(n, \mathbb{R})$ is determined as follows

$$\mathfrak{sp}(n, \mathbb{R}) = \{\mathbf{B} \in gl(2n, \mathbb{R}) : \mathbf{B}^{\mathrm{T}} \boldsymbol{J} + \boldsymbol{J} \mathbf{B} = \mathbf{0}\}.$$

If we choose a matrix $\mathbf{B} \in \mathfrak{sp}(n, \mathbb{R})$ in the following manner

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{bmatrix},$$

show that the relations $\mathbf{B}_1^{\mathrm{T}} = -\mathbf{B}_4, \mathbf{B}_2^{\mathrm{T}} = \mathbf{B}_2$ and $\mathbf{B}_3^{\mathrm{T}} = \mathbf{B}_3$ must hold.

3.6. Show that all upper triangular $n \times n$ matrices whose entries on the principal diagonal are all 1 constitute a Lie group $T^u(n, \mathbb{R})$. Evaluate the Lie algebra of $T^u(3, \mathbb{R})$.

3.7. Let $\mathbf{S} \in GL(n, \mathbb{R})$ be a given symmetric matrix and consider the set $R_S = \{\mathbf{A} \in GL(n, \mathbb{R}) : \mathbf{A}^{\mathrm{T}} \mathbf{S} \mathbf{A} = \mathbf{S}\}$. (a) Show that R_S is a Lie group with respect to the matrix product. (b) Show that the mapping $\phi : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$ prescribed by the relation $\phi(\mathbf{A}) = \mathbf{A}^{\mathrm{T}} \mathbf{S} \mathbf{A} - \mathbf{S}$ is a submersion and the set $R_S = \phi^{-1}(\mathbf{0})$ is a submanifold. Find the dimension of R_S. (c) Show that the Lie algebra of R_S is $\mathfrak{r}_{\mathfrak{s}} = \{\mathbf{B} \in gl(n, \mathbb{R}) : \mathbf{B}^{\mathrm{T}} \mathbf{S} + \mathbf{S} \mathbf{B} = \mathbf{0}\}$. Show further that if $\mathbf{B}_1, \mathbf{B}_2 \in \mathfrak{r}_{\mathfrak{s}}$, then the commutator $[\mathbf{B}_1, \mathbf{B}_2] = \mathbf{B}_1 \mathbf{B}_2 - \mathbf{B}_2 \mathbf{B}_1$ is also an element of the algebra. (d) Which is the group that will be obtained if one chooses $\mathbf{S} = \mathbf{I}$?

3.8. Find bases and structure constants of Lie algebras $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{so}(n)$ and $\mathfrak{o}(n)$ of the Lie groups $SL(n,\mathbb{R})$, $SO(n)$ and $O(n)$ for $n=2$ and $n=3$.

3.9. Show that the vector space $\mathbb{R}^3$ acquires a Lie algebra structure with the usual vectorial product $\times$. We define a mapping $\phi:\mathbb{R}^3\to\mathfrak{o}(3)$ by the relation

$$\phi(\mathbf{v})=\begin{bmatrix}0 & -v_3 & v_2\\ v_3 & 0 & -v_1\\ -v_2 & v_1 & 0\end{bmatrix}$$

where $\mathbf{v}=(v_1,v_2,v_3)\in\mathbb{R}^3$. Show that ϕ is a Lie algebra isomorphism and the equalities

$$\phi(\mathbf{u}\times\mathbf{v})=[\phi(\mathbf{u}),\phi(\mathbf{v})],\quad \phi(\mathbf{u})\,\mathbf{v}=\mathbf{u}\times\mathbf{v},\quad \mathbf{u}\cdot\mathbf{v}=-\frac{1}{2}\mathrm{tr}\,[\phi(\mathbf{u})\,\phi(\mathbf{v})]$$

are satisfied where $\cdot$ denotes the standard scalar product. Show that the *length* of a vector is given by $\|\mathbf{u}\|=\sqrt{\mathbf{u}\cdot\mathbf{u}}=-\,\mathrm{tr}\,(\phi(\mathbf{u})^2)/2.$

3.10. Show that the mapping $\phi:\mathbb{R}^3\to\mathfrak{o}(3)$ in Ex. **3.9** satisfies the relation

$$e^{\phi(\mathbf{u})}=\mathbf{I}+\frac{\sin\|\mathbf{u}\|}{\|\mathbf{u}\|}\,\phi(\mathbf{u})+\frac{1-\cos\|\mathbf{u}\|}{\|\mathbf{u}\|^2}\,\phi(\mathbf{u})^2$$

which is known as *Rodrigues' equality* [after French mathematician Benjamin Olinde Rodrigues (1795-1851)].

3.11. Let $\mathbf{A},\mathbf{B}\in O(3)$. We define the inner automorphism on $O(3)$ by the usual relation $\mathcal{I}_{\mathbf{A}}(\mathbf{B})=\mathbf{A}\mathbf{B}\mathbf{A}^{-1}=\mathbf{A}\mathbf{B}\mathbf{A}^{\mathrm{T}}$. By employing the foregoing mapping ϕ, show that one can write $Ad_{\mathbf{A}}\big(\phi(\mathbf{u})\big)=\phi(\mathbf{A}\mathbf{u})$ and the relation [*see* (3.6.6)] $ad_{\phi(\mathbf{u})}\big(\phi(\mathbf{v})\big)=[\phi(\mathbf{u}),\phi(\mathbf{v})]$ leads to $\phi(\mathbf{u}\times\mathbf{v})=[\phi(\mathbf{u}),\phi(\mathbf{v})]$.

3.12. Show that an eigenvalue of a matrix $\mathbf{A}\in SO(3)$ must be 1. Exploiting this fact, prove that every matrix $\mathbf{A}\in SO(3)$ corresponds to a rotation by an amount θ about a vector $\mathbf{u}$ in $\mathbb{R}^3$ and by choosing an appropriate basis in $\mathbb{R}^3$ this matrix can be reduced to the form

$$\mathbf{A}=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}.$$

3.13. The matrices $\mathbf{A},\mathbf{B}\in gl(n,\mathbb{R})$ are satisfying the condition $\mathbf{A}\mathbf{B}+\mathbf{B}\mathbf{A}^{\mathrm{T}}=\mathbf{0}$. Show that

$$e^{t\mathbf{A}}\,\mathbf{B}\,e^{t\mathbf{A}^{\mathrm{T}}}=\mathbf{B}.$$

Discuss the special cases $\mathbf{B}=\mathbf{I}$ and $\mathbf{B}=\boldsymbol{J}$.

3.14. Show that the set of all ***unitary matrices*** that are defined by the relation $U(n)=\{\mathbf{A}\in GL(n,\mathbb{C}):\mathbf{A}\overline{\mathbf{A}}^{\mathrm{T}}=\mathbf{A}\mathbf{A}^{\dagger}=\mathbf{I}_n\}$ constitutes a Lie group whose Lie algebra is given by $\mathfrak{u}(n,\mathbb{C})=\{\mathbf{A}\in gl(n,\mathbb{C}):\mathbf{A}+\mathbf{A}^{\dagger}=\mathbf{0}\}$. Show also that members of this group preserve the standard inner product $z_i\overline{w}_i$ in $\mathbb{C}^n$.

3.15. Show that the set $SU(n) = U(n) \cap SL(n, \mathbb{C})$ is also a Lie group and its Lie algebra is given by $\mathfrak{su}(n) = \{\mathbf{A} \in gl(n, \mathbb{C}) : \mathbf{A} + \mathbf{A}^\dagger = \mathbf{0}, \operatorname{tr} \mathbf{A} = \mathbf{0}\}$.

3.16. Show that the group $U(n)$ is diffeomorphic to $\mathbb{S}^1 \times SU(n)$.

3.17. Show that the $SU(2)$ is a connected manifold and the mapping $\phi : \mathbb{S}^3 \to SU(2)$ between the sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ and $SU(2)$ defined by

$$\phi(\mathbf{x}) = \begin{bmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{bmatrix}, \quad x_1^2 + x_2^2 + x_3^2 + x_4^4 = 1$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ is a diffeomorphism.

3.18. Show that every matrix $\mathbf{A} \in GL(n, \mathbb{R})$ can be represented in the form $\mathbf{A} = \mathbf{Q}_1\mathbf{S}_1 = \mathbf{S}_2\mathbf{Q}_2$ where $\mathbf{S}_1$ and $\mathbf{S}_2$ are positive definite symmetric matrices, and $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are orthogonal matrices. Prove that this operation called ***polar decomposition*** is uniquely determined and $\mathbf{Q}_1 = \mathbf{Q}_2 = \mathbf{Q}$ so that

$$\mathbf{S}_2 = \mathbf{Q}\mathbf{S}_1\mathbf{Q}^{\mathrm{T}}.$$

3.19. Show that every matrix $\mathbf{A} \in GL(n, \mathbb{C})$ can be represented in the form $\mathbf{A} = \mathbf{U}_1\mathbf{S}_1 = \mathbf{S}_2\mathbf{U}_2$ where $\mathbf{S}_1$ and $\mathbf{S}_2$ are Hermitean matrices [A matrix satisfying the condition $\mathbf{A} = \overline{\mathbf{A}}^{\mathrm{T}} = \mathbf{A}^\dagger$ is called a *Hermitean matrix* after French mathematician Charles Hermite (1822-1901)] and $\mathbf{U}_1$, $\mathbf{U}_2$ are unitary matrices. Prove that this operation is uniquely determined and $\mathbf{U}_1 = \mathbf{U}_2 = \mathbf{U}$ so that one gets

$$\mathbf{S}_2 = \mathbf{U}\mathbf{S}_1\mathbf{U}^\dagger.$$

3.20. Show that if a matrix $\mathbf{A}$ satisfies the equality $\mathbf{A} = \mathbf{A}^\dagger$, then the matrix $\mathbf{B} = i\mathbf{A}$ holds the relation $\mathbf{B} = -\mathbf{B}^\dagger$. Utilising this property show that a basis for the Lie algebra $\mathfrak{su}(2)$ can be chosen as $(i\boldsymbol{\sigma}_1, i\boldsymbol{\sigma}_2, i\boldsymbol{\sigma}_3)$ where *Pauli spin matrices* [Austrian physicist Wolfgang Ernst Pauli (1900-1958)] are given as follows

$$\boldsymbol{\sigma}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Find the structure constants of the Lie algebra (in Quantum mechanics the conventional basis is taken as $i\boldsymbol{\sigma}_k/2$).

3.21. $(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3)$ are the foregoing Pauli spin matrices. We define a mapping $\phi : \mathbb{R}^3 \to \mathfrak{su}(2)$ in the following manner

$$\phi(\mathbf{u}) = \frac{1}{2i} u^k \boldsymbol{\sigma}_k = \frac{1}{2} \begin{bmatrix} -iu^3 & -iu^1 - u^2 \\ -iu^1 + u^2 & iu^3 \end{bmatrix}, \mathbf{u} = (u^1, u^2, u^3) \in \mathbb{R}^3.$$

Show that (a) the inverse mapping $\phi^{-1} : \mathfrak{su}(2) \to \mathbb{R}^3$ is provided by

$$u^1 = i\mathrm{tr}\,\big(\phi(\mathbf{u})\,\boldsymbol{\sigma}_1\big),\ u^2 = i\mathrm{tr}\,\big(\phi(\mathbf{u})\,\boldsymbol{\sigma}_2\big),\ u^3 = i\mathrm{tr}\,\big(\phi(\mathbf{u})\,\boldsymbol{\sigma}_3\big),$$

(b) the mapping ϕ is a Lie algebra isomorphism so that the relation $\phi(\mathbf{u} \times \mathbf{v})$

$= [\phi(\mathbf{u}), \phi(\mathbf{v})]$ and (c) the equalities $\|\mathbf{u}\|^2 = -\det(u^k \sigma_k)$ and $\mathbf{u}\cdot\mathbf{v} = 2\mathrm{tr}\,\big(\phi(\mathbf{u})\,\phi(\mathbf{v})\big)$ are satisfied.

3.22. We define a mapping $\phi : SU(2) \to GL(3,\mathbb{R})$ in such a way that for each vector $\mathbf{u} \in \mathbb{R}^3$, the relation

$$[\phi(\mathbf{A})\mathbf{u}]^i \boldsymbol{\sigma}_i = \mathbf{A}(u^i \boldsymbol{\sigma}_i)\mathbf{A}^{-1}, \ \ \mathbf{A} \in SU(2)$$

will be satisfied. Show that (a) ϕ has the properties $\phi(\mathbf{A}) = \phi(-\mathbf{A})$, $\phi(\mathbf{I}_2) = \mathbf{I}_3$ and $\phi(\mathbf{A}) \in SO(3)$ and (b) the mapping $\phi : SU(2) \to SO(3)$ is a submersion.

3.23. According to the celebrated *Hamilton-Cayley theorem* [Irish mathematician Sir William Rowan Hamilton (1805-1865) and English mathematician Arthur Cayley (1821-1895)] every 2×2 matrix $\mathbf{A}$ satisfies its characteristic equation

$$\mathbf{A}^2 - \mathrm{tr}\,(\mathbf{A})\,\mathbf{A} + (\det \mathbf{A})\,\mathbf{I} = \mathbf{0}.$$

Utilising this equation, show that if $\mathbf{A} \in \mathfrak{sl}(2,\mathbb{R})$ and $\det \mathbf{A} = \delta$ the relations

$$e^{\mathbf{A}} = \begin{cases} \cos\sqrt{\delta}\,\mathbf{I} + \frac{1}{\sqrt{\delta}}\sin\sqrt{\delta}\,\mathbf{A}, & \delta > 0 \\ \mathbf{I} + \mathbf{A} & \delta = 0 \\ \cosh\sqrt{-\delta}\,\mathbf{I} + \frac{1}{\sqrt{-\delta}}\sinh\sqrt{-\delta}\,\mathbf{A} & \delta < 0 \end{cases}$$

are valid. Show further that $\mathrm{tr}\,e^{\mathbf{A}} \geq -2$. Verify whether the mapping

$$\exp : \mathfrak{sl}(2,\mathbb{R}) \to SL(2,\mathbb{R})$$

is surjective (*Hint*: consider the matrix $\begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix} \in SL(2,\mathbb{R})$).

3.24. $\phi : G \to H$ is a Lie group isomorphism, and $\psi : \mathfrak{g} \to \mathfrak{h}$ is the Lie algebra isomorphism produced by the mapping ϕ [*see p.* 188]. Show that the exponential mappings $\exp_G : \mathfrak{g} \to G$ and $\exp_H : \mathfrak{h} \to H$ satisfy the equality $\phi \circ \exp_G = \exp_H \circ \psi$. [*Hint*: Define two one-parameter groups and evaluate tangents of the curves $\gamma_1 : t \to \phi\big(\exp_G(tU)\big)$ and $\gamma_2 : t \to \exp_H\big(t\psi(U)$ at $t = 0$]. This means that the following diagram

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ \exp_G \uparrow & & \uparrow \exp_H \\ \mathfrak{g} & \xrightarrow{\psi} & \mathfrak{h} \end{array}$$

commutes.

3.25. We define the length of a vector $V \in T(\mathbb{R}^n)$ by $\|V\|^2 = \sum_{i=1}^{n} (v^i)^2$ where v^i are the components of V. A mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is called an ***isometry*** if the condition $\|\phi_* V\| = \|V\|$ is met for all vectors $V \in T(\mathbb{R}^n)$. Show that a vector field $V = v^i(\mathbf{x})\,\partial_i$ generates a one-parameter group of isometries if and only if its components satisfy the following partial differential equations

$$\frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i} = 0.$$

Show further that this group is the ***Euclidian group*** $E(n)$ of dimension $n(n+1)/2$ which consists of rotations and translations.

3.26. Show that the set of matrices $G = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \in GL(2,\mathbb{R}) : x, y \in \mathbb{R}, x \neq 0 \right\}$ forms a Lie group with respect to the matrix product. Utilise this group to define an appropriate multiplication on $\mathbb{R}^2$ so that a homomorphism between them can be constructed. Consider two curves with tangent vectors $\partial/\partial x$ and $\partial/\partial y$ at $(1,0) \in \mathbb{R}^2$ and verify that these vectors become $x\partial/\partial x$ and $x\partial/\partial y$ under a left translation so that they generate left-invariant vector fields. Hence, they constitute the Lie algebra $\mathfrak{g}$. Employing the fact that the one-parameter subgroup associated with the vector $c_1\partial/\partial x + c_2\partial/\partial y \in \mathfrak{g}$ must be tangent to the vector field $c_1 x\,\partial/\partial x + c_2 x\,\partial/\partial y$, we deduce that it is expressible by the relation

$$x = e^{c_1 t}, \;\; y = \frac{c_2}{c_1}(e^{c_1 t} - 1)$$

which is also obtainable by evaluating the matrix $\exp t \begin{bmatrix} c_1 & c_2 \\ 0 & 0 \end{bmatrix}$.

3.27. We define the action of the Lie group $SO(3)$ on the manifold $\mathbb{R}^3$ with the mapping $\Psi : SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3$ where Ψ is prescribed by

$$\Psi(\mathbf{A}, \mathbf{u}) = \mathbf{A}\mathbf{u}.$$

Here, $\mathbf{A} \in SO(3)$, $\mathbf{u} \in \mathbb{R}^3$. Discuss the properties of this mapping (freely, effectively or transitively acting) and show that orbits are submanifolds $\mathbb{S}^2$. Determine the Killing vector fields.

3.28. We consider the product manifold $Aff(n,\mathbb{R}) = GL(n,\mathbb{R}) \times \mathbb{R}^n$. We define an operation of multiplication $*$ on $Aff(n,\mathbb{R})$ as follows

$$(\mathbf{A}, \mathbf{u}) * (\mathbf{B}, \mathbf{v}) = (\mathbf{AB}, \mathbf{A}\mathbf{v} + \mathbf{u})$$

where $(\mathbf{A}, \mathbf{u}), (\mathbf{B}, \mathbf{v}) \in Aff(n,\mathbb{R})$. Show that $(Aff(n,\mathbb{R}), *)$ is a Lie group called the ***group of affine motions***. Let us further introduce the mapping $\Psi : Aff(n,\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n$ representing the action of this group on the manifold $\mathbb{R}^n$ by the relation $\Psi\big((\mathbf{A}, \mathbf{u}), \mathbf{v}\big) = \mathbf{A}\mathbf{v} + \mathbf{u}$. Ψ is called an ***affine mapping***. Discuss its properties (freely, effectively or transitively acting).

3.29. $\mathbb{Z}$ is the set of integers. Let us consider the group $G = \mathbb{Z}^2$ and the manifold $M = \mathbb{R}^2$, and define the mapping $\Psi : G \times M \to M$ by

$$\Psi\big((a,b),(x,y)\big) = (ax - by, bx).$$

Discuss the properties of this mapping (freely, effectively or transitively acting)

3.30. The action of the Lie group G on the manifold M is given by the mapping $\Psi : G \times M \to M$. Show that each orbit $\mathcal{O}_p$ is a submanifold of M and is diffeomorphic to the quotient manifold G/G_p. G_p is the isotropy group of a point $p \in M$.

3.31. We define the action of the Lie group G on its Lie algebra manifold $M = \mathfrak{g}$ by the mapping $\Psi : G \times \mathfrak{g} \to \mathfrak{g}$ by means of the relation $\Psi(g, V) = Ad_g(V)$. Show that the Killing vector field is then given by $V^K(U) = [V, U]$ where $U \in M$.