CHAPTER IV

TENSOR FIELDS ON MANIFOLDS

4.1. SCOPE OF THE CHAPTER

In this chapter, tensors[1] that were defined previously on linear vector spaces and their duals will be restructured as tensor fields in such a way that they would inhabit in a natural fashion on differentiable manifolds. To this end, we first construct in Sec. 4.2 the *cotangent bundle* by conjoining the dual space of the tangent space at each point of the manifold to this point. That fibre bundle is then equipped with a differentiable structure to make it a smooth manifold. Afterwards it is demonstrated in Sec. 4.3 that multilinear functionals on certain Cartesian products of tangent spaces and their duals at a point of the manifold are represented by elements called contravariant and covariant tensors of a vector space defined as some tensor products of these spaces. The basis of a tensor product space is determined as usual as tensor products of natural bases for a tangent space and its dual. A tensor bundle is built by attaching the associated tensor product vector space to each point of the manifold. Tensor fields are obtained as sections of the tensor bundle. A tensor is now being completely determined through its components on natural bases in tangent spaces and their duals. Transformation rules of these components under the change of local coordinates are then derived quite easily. An exterior form field on a manifold will then reasonably be defined as a completely antisymmetric covariant tensor field and, as it should be, the concept of exterior products is linked to the alternation of tensor products. The contraction is defined as an operation that produces an associated tensor to a given tensor whose order is reduced by two compared to the original tensor. After that, the quotient rule that helps us to recognise whether a given indicial quantity are actually components of a particular tensor is discussed. Finally, the Lie derivative of

[1]The term 'tensor' was first used in the present context by the German physicist Woldemar Voigt (1850-1919) in 1898 while he was studying crystal elasticities.

Exterior Analysis, DOI: 10.1016/B978-0-12-415902-0.50004-9

tensor products of finitely many vector fields on the tangent bundle is calculated.

Tensor analysis is today an indispensable tool in many branches of mathematics and physics. It was mainly developed by Italian mathematicians Gregorio Ricci-Curbastro (1853-1925) and Levi-Civita, and it has turned out to be a great impetus in the development of the theory of general relativity. A sentence from a letter of Einstein to Levi-Civita around 1917 reflects his appraisal of the tensor analysis: "I admire the elegance of your method of computation; it must be nice to ride through these fields upon the horse of true mathematics while the like of us have to make our way laboriously on foot."

4.2. COTANGENT BUNDLE

We consider an m-dimensional smooth manifold M and the tangent space $T_p(M)$ at a point $p \in M$. As is well known, the dual of the tangent space is a linear vector space formed by all linear functionals on the tangent space [*see* p. 11]. We denote this m-dimensional dual space by $T_p^*(M)$ and we also call it the ***cotangent space*** at the point p. When we choose the natural basis of the tangent space at the point p as the vectors $\{\partial/\partial x^i : i = 1, \dots, m\}$ generated by the local coordinates in the chart containing the point p, we have seen on p. 125 that reciprocal basis vectors in the dual space are given by linear functionals as differentials $\{dx^i : i = 1, \dots, m\}$ so that the following relations

$$dx^i\Big(\frac{\partial}{\partial x^j}\Big) = \Big\langle dx^i, \frac{\partial}{\partial x^j}\Big\rangle = \delta^i_j \tag{4.2.1}$$

are satisfied. Hence, at a point $p \in M$, a vector $V \in T_p(M)$ and a linear functional $\omega \in T_p^*(M)$ can be expressed as

$$V = v^i \frac{\partial}{\partial x^i}, \quad \omega = \omega_i dx^i, \quad v^i, \omega_i \in \mathbb{R}. \tag{4.2.2}$$

The value of the functional ω on the vector V at p then happens to be

$$\begin{aligned} \omega(V) &= \langle \omega, V\rangle = \Big\langle \omega_i dx^i, v^j \frac{\partial}{\partial x^j}\Big\rangle \\ &= \omega_i v^j \delta^i_j = \omega_i v^i \in \mathbb{R}. \end{aligned} \tag{4.2.3}$$

We shall call elements of the dual space $T_p^*(M)$ as **1-*forms*** at the point p. Next, we define the set

$$T^*(M) = \bigcup_{p\in M} T_p^*(M) = \{(p,\omega) : p \in M, \omega \in T_p^*(M)\}. \quad (4.2.4)$$

By repeating exactly our approach in Sec. 2.8, we see that $T^*(M)$ can be endowed with a differentiable structure making it a $2m$-dimensional smooth manifold which will be called henceforth as the ***cotangent bundle***. The local coordinates of $T^*(M)$ are evidently given by $\{x^1, \ldots, x^m, \omega_1, \ldots, \omega_m\}$. A *section* of the bundle $T^*(M)$ as we have already done in p. 130 characterises this time a 1-form field on the smooth manifold M. In terms of local coordinates in the relevant chart, this field is of course expressible as follows

$$\omega(p) = \omega_i(\mathbf{x})dx^i \in T^*(M), \;\; \mathbf{x} = \varphi(p). \quad (4.2.5)$$

Different charts containing the point p gives rise to a coordinate transformation given by invertible functions $y^i = y^i(x^j)$. When we write the 1-form ω in different local coordinates, the relation

$$\omega(p) = \omega_j dx^j = \omega'_i dy^i = \omega'_i \frac{\partial y^i}{\partial x^j} dx^j$$

leads to the following relations between components of ω in two different coordinate systems

$$\omega_j = \omega'_i \frac{\partial y^i}{\partial x^j} \quad \text{or} \quad \omega'_i = \frac{\partial x^j}{\partial y^i}\omega_j. \quad (4.2.6)$$

Because of this transformation rule, the elements of the cotangent bundle are usually called ***covariant vector*** or ***covector fields***. We have already seen that the transformation rule between components of vectors in two different charts in the tangent bundle are given by [*see* (2.6.9)]

$$v'^i = \frac{\partial y^i}{\partial x^j} v^j. \quad (4.2.7)$$

That is the reason why we call vectors in the tangent bundle as ***contravariant vector fields***.

4.3. TENSOR FIELDS

Let us consider an m-dimensional smooth manifold M and vector spaces $T_p(M)$ and $T_p^*(M)$ at a point $p \in M$. We introduce the following Cartesian product set whose two parts are k-times and l-times cartesian products of $T_p^*(M)$ and $T_p(M)$, respectively

$$\mathrm{T}_p(M)^k_l = \underbrace{T^*_p(M) \times \cdots \times T^*_p(M)}_{k} \times \underbrace{T_p(M) \times \cdots \times T_p(M)}_{l}.$$

We now specify a multilinear functional $\mathcal{T} : \mathrm{T}_p(M)^k_l \to \mathbb{R}$ as a ***k-contravariant*** and ***l-covariant mixed tensor***. We next define a vector space $\mathfrak{T}_p(M)^k_l$ as a tensor product of two vector spaces formed by k-times tensor products of $T_p(M)$ and l-times tensor product of $T^*_p(M)$:

$$\mathfrak{T}_p(M)^k_l = \underbrace{T_p(M) \otimes \cdots \otimes T_p(M)}_{k} \otimes \underbrace{T^*_p(M) \otimes \cdots \otimes T^*_p(M)}_{l}. \quad (4.3.1)$$

The tensor $\mathcal{T}$ can then be expressible as an element of $\mathfrak{T}(M)^k_l$ and we say that it is a $\binom{k}{l}$-tensor. With respect to the basis vectors produced by natural local coordinates in $T_p(M)$ and $T^*_p(M)$ we can write $\mathcal{T}$ in the form

$$\mathcal{T} = t^{i_1 i_2 \cdots i_k}_{j_1 j_2 \cdots j_l} \frac{\partial}{\partial x^{i_1}} \otimes \frac{\partial}{\partial x^{i_2}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes dx^{j_2} \otimes \cdots \otimes dx^{j_l}$$

[*see p.* 23]. Here the repeated indices, i.e., dummy indices, $i_1, i_2, \ldots, i_k$ and $j_1, j_2, \ldots, j_l$ are all taking the values from 1 to m. The coefficients $t^{i_1 \cdots i_k}_{j_1 \cdots j_l}$ are called the components of the tensor $\mathcal{T}$. *We frequently identify a tensor $\mathcal{T}$ with its components*. The value of the tensor, or multilinear functional, $\mathcal{T}$ on $\mathrm{T}_p(M)^k_l$, or on k *linear functionals* (*covariant vectors*) $\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(k)}$ and l *contravariant vectors* $V_{(1)}, V_{(2)}, \ldots, V_{(l)}$ are prescribed by

$$\mathcal{T}(\omega^{(1)}, \ldots, \omega^{(k)}, V_{(1)}, , \ldots, V_{(l)}) = t^{i_1 \cdots i_k}_{j_1 \cdots j_l} \omega^{(1)}_{i_1} \cdots \omega^{(k)}_{i_k} v^{j_1}_{(1)} \cdots v^{j_l}_{(l)}.$$

It is straightforward to verify that the m^{k+l} *tensor components* $t^{i_1 \cdots i_k}_{j_1 \cdots j_l}$ are determined by

$$t^{i_1 \cdots i_k}_{j_1 \cdots j_l} = \mathcal{T}(dx^{i_1}, \ldots, dx^{i_k}, \partial_{j_1}, , \ldots, \partial_{j_l}).$$

It is obvious that we end up with different types of k-contravariant and l-covariant tensors if we change the ordering of spaces $T_p(M)$ and $T^*_p(M)$ in the tensor product keeping the numbers of the component spaces constant. When we take into account a coordinate transformation $y^i = y^i(x^j)$ at the point *p*, we can write

$$\frac{\partial}{\partial y^i} = \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}, \quad dy^i = \frac{\partial y^i}{\partial x^j} dx^j$$

to obtain

$$\begin{aligned}\mathcal{T} &= t'^{i_1\cdots i_k}_{j_1\cdots j_l}\frac{\partial}{\partial y^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial y^{i_k}}\otimes dy^{j_1}\otimes\cdots\otimes dy^{j_l}\\ &= t'^{i_1\cdots i_k}_{j_1\cdots j_l}\frac{\partial x^{m_1}}{\partial y^{i_1}}\cdots\frac{\partial x^{m_k}}{\partial y^{i_k}}\frac{\partial y^{j_1}}{\partial x^{n_1}}\cdots\frac{\partial y^{j_l}}{\partial x^{n_l}}\frac{\partial}{\partial x^{m_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{m_k}}\\ &\qquad\otimes dx^{n_1}\otimes\cdots\otimes dx^{n_l}\\ &= t^{m_1\cdots m_k}_{n_1\cdots n_l}\frac{\partial}{\partial x^{m_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{m_k}}\otimes dx^{n_1}\otimes\cdots\otimes dx^{n_l}\end{aligned}$$

whence we deduce the following relations between components of the same tensor in different coordinate systems

$$t'^{i_1\cdots i_k}_{j_1\cdots j_l}\frac{\partial x^{m_1}}{\partial y^{i_1}}\cdots\frac{\partial x^{m_k}}{\partial y^{i_k}}\frac{\partial y^{j_1}}{\partial x^{n_1}}\cdots\frac{\partial y^{j_l}}{\partial x^{n_l}} = t^{m_1\cdots m_k}_{n_1\cdots n_l}.$$

If we recall the chain rule $(\partial x^{j_r}/\partial y^{i_r})(\partial y^{i_r}/\partial x^{j_s}) = \partial x^{j_r}/\partial x^{j_s} = \delta^{j_r}_{j_s}$, we finally find out that the transformation rule for the components of the tensor $\mathcal{T}$ under the change of coordinates $y^i = y^i(x^j)$ is given by

$$t'^{i_1 i_2\cdots i_k}_{j_1 j_2\cdots j_l} = t^{m_1 m_2\cdots m_k}_{n_1 n_2\cdots n_l}\frac{\partial y^{i_1}}{\partial x^{m_1}}\frac{\partial y^{i_2}}{\partial x^{m_2}}\cdots\frac{\partial y^{i_k}}{\partial x^{m_k}}\frac{\partial x^{n_1}}{\partial y^{j_1}}\frac{\partial x^{n_2}}{\partial y^{j_2}}\cdots\frac{\partial x^{n_l}}{\partial y^{j_l}}$$

We can immediately realise that the set

$$\mathfrak{T}(M)^k_l = \bigcup_{p\in M}\mathfrak{T}_p(M)^k_l = \{(p,\mathcal{T}) : p\in M, \mathcal{T}\in\mathfrak{T}_p(M)^k_l\} \quad (4.3.2)$$

can be endowed with a differentiable structure as was done in Sec. 2.8 so as it becomes an $m + m^{k+l}$-dimensional smooth manifold. This manifold will be called the ***tensor bundle of order*** $\boldsymbol{k+l}$ whose local coordinates are given by $\{x^1,\dots,x^m, t^{i_1\cdots i_k}_{j_1\cdots j_l} : i_1,\dots,i_k,j_1,\dots,j_l = 1,\dots,m\}$. A *section* of the bundle $\mathfrak{T}(M)^k_l$ characterises a tensor field on the manifold M. In terms of standard local coordinates this tensor field is expressible as

$$\mathcal{T}(p) = t^{i_1\cdots i_k}_{j_1\cdots j_l}(\mathbf{x})\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_k}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_l}. \quad (4.3.3)$$

The sum of two tensor fields $\mathcal{T}_1,\mathcal{T}_2\in\mathfrak{T}(M)^k_l$ of the same type is the tensor field $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2 \in \mathfrak{T}(M)^k_l$ whose components are given by

$$t^{i_1\cdots i_k}_{j_1\cdots j_l}(\mathbf{x}) = t^{i_1\cdots i_k}_{(1)j_1\cdots j_l}(\mathbf{x}) + t^{i_1\cdots i_k}_{(2)j_1\cdots j_l}(\mathbf{x}).$$

Similarly, if we choose $f\in\Lambda^0(M)$ and $\mathcal{T}\in\mathfrak{T}(M)^k_l$, then the tensor field $f\mathcal{T}\in\mathfrak{T}(M)^k_l$ is determined by its components $f(\mathbf{x})t^{i_1\cdots i_k}_{j_1\cdots j_l}(\mathbf{x})$. Hence, all tensor fields of the same order and of the same type constitute a module on the commutative ring $\Lambda^0(M)$. It is obvious that one can use the representations $\mathfrak{T}(M)^1_0 = T(M)$ and $\mathfrak{T}(M)^0_1 = T^*(M)$.

The ***operation of contraction*** on a tensor field is defined as in Sec. 1.3. If we remove in (4.3.3) the tensor product between dx^{j_s} and ∂_{i_r}, and notice that $dx^{j_s}(\partial_{i_r}) = \delta^{j_s}_{i_r}$, we obtain the ***contracted tensor***

$$\mathcal{T}_c = {}_c t^{i_1\cdots i_{k-1}}_{j_1\cdots j_{l-1}} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_{k-1}}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_{l-1}}$$

whose order is now $k + l - 2$. The components of a once contracted tensor are given, for instance, by

$${}_c t^{i_1\cdots i_{k-1}}_{j_1\cdots j_{l-1}} = t^{i_1\cdots i_{r-1} i i_{r+1}\cdots i_{k-1}}_{j_1\cdots j_{s-1} i j_{s+1}\cdots j_{l-1}}.$$

The contraction operation makes it possible for us to propose a rather simple test to recognise whether a given array of coefficients as an indicial quantity in a particular coordinate system are components of a tensor.

Quotient Rule. *Let the coefficients* $t^{i_1\cdots i_k}_{j_1\cdots j_l}$ *be the components of an arbitrary* $\binom{k}{l}$*-tensor* $\mathcal{T}$ *of order* $k + l$ *in a given coordinate system and let* $s^{m_1\cdots m_r}_{n_1\cdots n_s}$ *be an array of numbers considered to be the components of a quantity* $\mathcal{S}$ *of order* $r + s$. *We introduce the quantity* $\mathcal{R} = \mathcal{T} \times \mathcal{S}$ *with the components* $r^{i_1\cdots i_k m_1\cdots m_r}_{j_1\cdots j_l n_1\cdots n_s} = t^{i_1\cdots i_k}_{j_1\cdots j_l} s^{m_1\cdots m_r}_{n_1\cdots n_s}$. *If any contraction of this indicial quantity such as, for instance,* $r^{i_1\cdots i\cdots i_k m_1\cdots m_r}_{j_1\cdots j_l n_1\cdots i\cdots n_s}$ *is found to be components of a tensor* ${}_c\mathcal{R}$, *then the coefficients* $s^{m_1\cdots m_r}_{n_1\cdots n_s}$ *are also components of a* $\binom{r}{s}$*-tensor* $\mathcal{S}$ *of order* $r + s$.

Since ${}_c\mathcal{R}$ and $\mathcal{T}$ are tensors, their components transform according to the well known rule under any coordinate transformations. Therefore, we can write

$$\begin{aligned}
&t'^{a_1\cdots j\cdots a_k}_{b_1\cdots b_l} s'^{p_1\cdots p_r}_{q_1\cdots j\cdots q_s}(\mathbf{y}) = \\
&\frac{\partial y^{a_1}}{\partial x^{i_1}}\cdots\frac{\partial y^{j}}{\partial x^{i}}\cdots\frac{\partial y^{a_k}}{\partial x^{i_k}}\frac{\partial x^{j_1}}{\partial y^{b_1}}\cdots\frac{\partial x^{j_l}}{\partial y^{b_l}} t^{i_1\cdots i\cdots i_k}_{j_1\cdots j_l} s'^{p_1\cdots p_r}_{q_1\cdots j\cdots q_s} = \\
&\frac{\partial y^{a_1}}{\partial x^{i_1}}\cdots\frac{\partial y^{a_k}}{\partial x^{i_k}}\frac{\partial y^{p_1}}{\partial x^{m_1}}\cdots\frac{\partial y^{p_r}}{\partial x^{m_r}}\frac{\partial x^{j_1}}{\partial y^{b_1}}\cdots\frac{\partial x^{j_l}}{\partial y^{b_l}}\frac{\partial x^{n_1}}{\partial y^{q_1}}\cdots\frac{\partial x^{n_s}}{\partial y^{q_s}} t^{i_1\cdots i\cdots i_k}_{j_1\cdots j_l} s^{m_1\cdots m_r}_{n_1\cdots i\cdots n_s}(\mathbf{x})
\end{aligned}$$

whence we deduce that

$$t^{i_1\cdots i\cdots i_k}_{j_1\cdots j_l}\Big(\frac{\partial y^j}{\partial x^i} s'^{p_1\cdots p_r}_{q_1\cdots j\cdots q_s} - \frac{\partial y^{p_1}}{\partial x^{m_1}}\cdots\frac{\partial y^{p_r}}{\partial x^{m_r}}\frac{\partial x^{n_1}}{\partial y^{q_1}}\cdots\frac{\partial x^{n_s}}{\partial y^{q_s}} s^{m_1\cdots m_r}_{n_1\cdots i\cdots n_s}\Big) = 0$$

since we can obviously cancel regular matrices $[\partial y^k/\partial x^l]$ in both sides because their determinants do not vanish. Inasmuch as these relations should be satisfied for every tensor $\mathcal{T}$, the expressions within parentheses must be equal to zero from which it follows that

$$s'^{p_1\cdots p_r}_{q_1\cdots j\cdots q_s}(\mathbf{y}) = \frac{\partial y^{p_1}}{\partial x^{m_1}}\cdots\frac{\partial y^{p_r}}{\partial x^{m_r}}\frac{\partial x^{n_1}}{\partial y^{q_1}}\cdots\frac{\partial x^i}{\partial y^j}\cdots\frac{\partial x^{n_s}}{\partial y^{q_s}}s^{m_1\cdots m_r}_{n_1\cdots i\cdots n_s}(\mathbf{x})$$

This relation shows clearly that $\mathcal{S}$ is a $\binom{r}{s}$-tensor of order $r+s$. □

Let us now consider *a completely antisymmetric* k-covariant tensor field $\omega \in \mathfrak{T}(M)^0_k$. In local natural coordinates, we may represent this tensor in the following form:

$$\omega(p) = \omega_{j_1 j_2\cdots j_k}(\mathbf{x})\, dx^{j_1}\otimes dx^{j_2}\otimes\cdots\otimes dx^{j_k}.$$

Here, the components of this tensor are completely antisymmetric, namely, they must obey the rule $\omega_{i_1\cdots i_p\cdots i_q\cdots i_k}(\mathbf{x}) = -\,\omega_{i_1\cdots i_q\cdots i_p\cdots i_k}(\mathbf{x})$ for every pair of indices. Therefore, by making use of the generalised Kronecker deltas we can write as in (1.4.8)

$$\delta^{i_1 i_2\cdots i_k}_{j_1 j_2\cdots j_k}\omega_{i_1 i_2\cdots i_k}(\mathbf{x}) = k!\,\omega_{[j_1 j_2\cdots j_k]}(\mathbf{x}) = k!\,\omega_{j_1 j_2\cdots j_k}(\mathbf{x})$$

to obtain

$$\omega = \frac{1}{k!}\,\omega_{i_1 i_2\cdots i_k}(\mathbf{x})\,\delta^{i_1 i_2\cdots i_k}_{j_1 j_2\cdots j_k} dx^{j_1}\otimes dx^{j_2}\otimes\cdots\otimes dx^{j_k}.$$

Just like we did in Sec. 1.4, we can again define the ***exterior product*** of basis vectors in the tensor bundle $\mathfrak{T}(M)^0_k$ as follows

$$\begin{aligned} dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_k} &= \delta^{i_1 i_2\cdots i_k}_{j_1 j_2\cdots j_k} dx^{j_1}\otimes dx^{j_2}\otimes\cdots\otimes dx^{j_k} \\ &= k!\, dx^{[i_1}\otimes dx^{i_2}\otimes\cdots\otimes dx^{i_k]}. \end{aligned}$$

Hence, we reach to the conclusion

$$\omega(p) = \frac{1}{k!}\,\omega_{i_1 i_2\cdots i_k}(\mathbf{x})\, dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_k}. \qquad (4.3.4)$$

At the point $p\in M$, the tensor $\omega(p)$ is an *alternating k-linear functional* assigning a scalar number to k vectors $V_1, V_2, \ldots, V_k$ in the tangent space $T_p(M)$, and consequently, a smooth function to vector fields $V_1(p), V_2(p), \ldots, V_k(p)$. If we write

$$V_\alpha(p) = v^i_\alpha(\mathbf{x})\,\partial/\partial x^i, \alpha = 1, 2, \ldots, k,$$

this function is determined by the expression

$$\begin{aligned}\omega(V_1, V_2, \ldots, V_k)(p) &= \omega_{i_1 i_2\cdots i_k}(\mathbf{x})\, dx^{[i_1}(V_1)\, dx^{i_2}(V_2)\cdots dx^{i_k]}(V_k)\\ &= \omega_{i_1 i_2\cdots i_k}(\mathbf{x})\, v_1^{[i_1}(\mathbf{x}) v_2^{i_2}(\mathbf{x})\cdots v_k^{i_k]}(\mathbf{x})\\ &= \omega_{i_1 i_2\cdots i_k}(\mathbf{x})\, v_1^{i_1}(\mathbf{x}) v_2^{i_2}(\mathbf{x})\cdots v_k^{i_k}(\mathbf{x})\\ &= \frac{1}{k!}\,\omega_{i_1 i_2\cdots i_k}(\mathbf{x})\begin{vmatrix} v_1^{i_1}(\mathbf{x}) & v_2^{i_1}(\mathbf{x}) & \cdots & v_k^{i_1}(\mathbf{x})\\ v_1^{i_2}(\mathbf{x}) & v_2^{i_2}(\mathbf{x}) & \cdots & v_k^{i_2}(\mathbf{x})\\ \vdots & \vdots & & \vdots\\ v_1^{i_k}(\mathbf{x}) & v_2^{i_k}(\mathbf{x}) & \cdots & v_k^{i_k}(\mathbf{x})\end{vmatrix}\end{aligned}$$

Since coefficient functions $\omega_{i_1 i_2\cdots i_k}$ are completely antisymmetric, the second and the third lines above will of course yield the same numerical value. The completely antisymmetric k-covariant tensor given by (4.3.4) will be called henceforth a ***k-exterior differential form*** or a ***k-exterior form*** or simply a ***k-form*** on the manifold M.

It is evident that the exterior product will not be confined solely on cotangent spaces. A *completely antisymmetric* k-contravariant tensor field $\mathcal{V} \in \mathfrak{T}(M)_0^k$ is expressible as follows in local natural coordinates

$$\mathcal{V}(p) = v^{i_1 i_2\cdots i_k}(\mathbf{x})\frac{\partial}{\partial x^{i_1}} \otimes \frac{\partial}{\partial x^{i_2}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}}$$

where the coefficient functions $v^{i_1 i_2\cdots i_k}(\mathbf{x})$ are completely antisymmetric. Thus, making use of generalised Kronecker deltas, we can again write

$$\delta^{i_1 i_2\cdots i_k}_{j_1 j_2\cdots j_k} v^{j_1 j_2\cdots j_k}(\mathbf{x}) = k!\, v^{[i_1 i_2\cdots i_k]}(\mathbf{x}) = k!\, v^{i_1 i_2\cdots i_k}(\mathbf{x}).$$

This of course leads to the representation

$$\mathcal{V}(p) = \frac{1}{k!}\, v^{i_1 i_2\cdots i_k}(\mathbf{x})\frac{\partial}{\partial x^{i_1}} \wedge \frac{\partial}{\partial x^{i_2}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}}.$$

Once more, we define the ***exterior product*** of basis vectors in the tensor bundle $\mathfrak{T}(M)_0^k$ in the following way:

$$\frac{\partial}{\partial x^{i_1}} \wedge \frac{\partial}{\partial x^{i_2}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}} = \delta^{i_1 i_2\cdots i_k}_{j_1 j_2\cdots j_k}\frac{\partial}{\partial x^{i_1}} \otimes \frac{\partial}{\partial x^{i_2}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}}.$$

At the point $p \in M$, the tensor $\mathcal{V}(p)$ is an *alternating k-linear functional* assigning a scalar number to k covectors, or linear functionals, $\omega_1, \omega_2, \ldots,$ ω_k in the cotangent space $T_p^*(M)$, and consequently a smooth function to fields $\omega_1(p), \omega_2(p), \ldots, \omega_k(p)$. It is clear that tensor fields of this form constitute a module $\mathfrak{X}^k(M)$. If we take $\mathfrak{X}^0(M) = \Lambda^0(M)$, we immediately observe that the direct sum $\mathfrak{X}(M) = \overset{m}{\underset{k=0}{\oplus}} \mathfrak{X}^k(M)$ is a 2^m-dimensional Grassmann algebra with respect to the exterior product operation $\wedge$. It is evident that one has $\mathfrak{X}^1(M) = T(M)$.

It is obvious that a mixed completely antisymmetric tensor is now expressible in the form

$$\mathcal{T}(p) = t^{i_1\cdots i_k}_{j_1\cdots j_l}(\mathbf{x})\frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l}.$$

The components $t^{i_1\cdots i_k}_{j_1\cdots j_l}(\mathbf{x})$ must be completely antisymmetric functions with respect to its subscripts and superscripts.

Although we would mainly be interested in exterior forms and their exterior products in this work, it becomes now clear that the use of exterior products are not restricted to such types of entities only.

Finally, we shall try to calculate the Lie derivative of elements in an arbitrary tensor bundle with respect to a vector field V in the tangent bundle of a manifold M. To this end, we illustrate a fundamental property of Lie derivatives. Let us consider the tensor product

$$U_1(p) \otimes U_2(p) \otimes \cdots \otimes U_k(p)$$

of vectors $U_1, U_2, \cdots, U_k$. Its Lie derivative at a point $p \in M$ can now be evaluated as in (2.10.2):

$$\pounds_V(U_1 \otimes \cdots \otimes U_k) = \lim_{t\to 0}\frac{U_1^*(p;t) \otimes \cdots \otimes U_k^*(p;t) - U_1(p) \otimes \cdots \otimes U_k(p)}{t}.$$

In view of (2.10.4), we can write

$$U_i^*(p;t) = U_i(p) + t\pounds_V U_i + o(t), \quad i = 1, \ldots, k.$$

Hence, employing the rules of the tensor product, we obtain

$$\begin{aligned} U_1^*(p;t) \otimes U_2^*(p;t) \otimes \cdots \otimes U_k^*(p;t) = \\ \big(U_1(p) + t\pounds_V U_1(p) + o(t)\big) \otimes \\ \big(U_2(p) + t\pounds_V U_2(p) + o(t)\big) \otimes \cdots \otimes \big(U_k(p) + t\pounds_V U_k(p) + o(t)\big) = \\ U_1(p) \otimes U_2(p) \otimes \cdots \otimes U_k(p) \\ + t\big[\pounds_V U_1(p) \otimes U_2(p) \otimes \cdots \otimes U_k(p) \\ + U_1(p) \otimes \pounds_V U_2(p) \otimes \cdots \otimes U_k(p) + \\ \cdots + U_1(p) \otimes U_2(p) \otimes \cdots \otimes \pounds_V U_k(p)\big] + o(t). \end{aligned}$$

We thus conclude that

$$\begin{aligned}\pounds_V(U_1 \otimes U_2 \otimes \cdots \otimes U_k) = \pounds_V U_1 \otimes U_2 \otimes \cdots \otimes U_k \\ + U_1 \otimes \pounds_V U_2 \otimes \cdots \otimes U_k \\ + \cdots + U_1 \otimes U_2 \otimes \cdots \otimes \pounds_V U_k.\end{aligned} \tag{4.3.5}$$

This clearly means that the Lie derivative obeys the classical Leibniz' rule. We utilise this property in Sec. 5.11 to evaluate quite easily the Lie derivative of any tensor.

IV. EXERCISES

4.1. The tensor field $\mathcal{T} \in \mathfrak{T}(\mathbb{R}^2)^0_2$ is given by $\mathcal{T} = dx \otimes dx + dy \otimes dy$. (a) Find the value of this covariant tensor field of order 2 on the vector fields given below

$$U = u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y}, \quad V = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y}.$$

(b) Show that under the coordinate transformation $x = r\cos\theta, y = r\sin\theta$, the same tensor can be written as $\mathcal{T} = dr \otimes dr + r^2\, d\theta \otimes d\theta$.

4.2. Let the tensor $\mathcal{T} \in \mathfrak{T}(\mathbb{R}^3)^0_2$ be given by $\mathcal{T} = dx \otimes dx + dy \otimes dy + dz \otimes dz$. (a) Find the value of $\mathcal{T}$ on vector fields given below

$$U = u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}, \quad V = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z},$$
$$W = w_x \frac{\partial}{\partial x} + w_y \frac{\partial}{\partial y} + w_z \frac{\partial}{\partial z}.$$

(b) Show that under the coordinate transformation $x = r\cos\theta,\ y = r\sin\theta,\ z = z$ this tensor takes the form $\mathcal{T} = dr \otimes dr + r^2\, d\theta \otimes d\theta + dz \otimes dz$, and (c) under the transformation $x = r\cos\varphi\sin\theta,\ y = r\sin\varphi\sin\theta,\ z = r\cos\theta$ it becomes $\mathcal{T} = dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin\theta\, d\varphi \otimes d\varphi$.

4.3. Components of a tensor $\mathcal{T} \in \mathfrak{T}(M)^3_2$ are given by $t^{ijk}_{lm}(p)$. How many $\binom{2}{1}$-tensors are obtainable through contraction operations? Further contraction operations result in how many $\binom{1}{0}$-tensors?

4.4. Show that the components of the tensor $\mathcal{T} = V \otimes \omega \in \mathfrak{T}(M)^1_1$ are given by $t^i_j(p) = v^i(p)\,\omega_j(p)$.

4.5. Let us consider $\mathfrak{T}(M)^k = \underbrace{T(M) \otimes \cdots \otimes T(M)}_{k}$. What type of a tensor can be regarded as representing a multilinear mapping $\mathfrak{T}(M)^k \to T(M)$?

4.6. If the components of a $\binom{1}{1}$-tensor are the same with respect to every basis, then show that they should be written in the form $t^i_j(p) = t(p)\,\delta^i_j$.

4.7. If the components of a $\binom{k}{l}$-tensor are the same with respect to every basis, then show that either $\mathcal{T} = 0$ or $k = l$.

4.8. If the components of a $\binom{1}{1}$-tensor is symmetric with respect to their indices, that is, if the equalities $t^i_j = t^j_i$ are numerically valid, then show that $t^i_j(p) = t(p)\,\delta^i_j$.

4.9. Show that the generalised Kronecker deltas $\delta^{i_1 i_2 \cdots i_k}_{j_1 j_2 \cdots j_k}$ are components of a $\binom{k}{k}$-tensor and verify that these components remain unchanged in every set of coordinates.

4.10. Let the structure constants of a Lie algebra $\mathfrak{g}$ be c^k_{ij}. Show that $\mathfrak{c}_{ij} = c^k_{il} c^l_{kj}$ are components of a symmetric tensor, whereas $\mathfrak{c}_{ijk} = c^l_{ij}\mathfrak{c}_{lk}$ are components of a completely antisymmetric tensor.

4.11. Assume that $\mathcal{T} \in \mathfrak{T}(M)^0_2$ is a symmetric tensor. We define the components of a tensor $\mathcal{S} \in \mathfrak{T}(M)^0_4$ by the relations

$$s_{ijkl} = t_{ik}t_{jl} - t_{il}t_{jk}.$$

Verify that the following equalities

$$s_{ijkl} = -\,s_{jikl} = -\,s_{ijlk},\; s_{ijkl} + s_{ijkl} + s_{ijkl} = 0$$

are satisfied. Let $U, V \in T(M)$. Show that

$$\mathcal{S}(U,V,U,V) = \mathcal{T}(U,U)\,\mathcal{T}(V,V) - \mathcal{T}(U,V)^2$$

and, if vectors U and V are linearly independent, then one finds for $U \neq \mathbf{0}$ $\mathcal{S}(U,V,U,V) > 0$ whenever $\mathcal{T}(U,U) > 0$.

4.12. A mapping $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ is prescribed by $\phi(x,y,z) = (x+y, 2y-x, z^3)$. Evaluate the action of this mapping on the tensor

$$\mathcal{T} = 3x\frac{\partial}{\partial x} \otimes dy \otimes dz + y\,\frac{\partial}{\partial y} \otimes dx \otimes dz + \sin x\,\frac{\partial}{\partial x} \otimes dx \otimes dy.$$

4.13. A tensor field $\mathcal{T} \in \mathfrak{T}(\mathbb{R}^2)^2_0$ and a vector field $V \in T(\mathbb{R}^2)$ are given, respectively, by

$$\mathcal{T} = x\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} - y^2\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y},\quad V = y\frac{\partial}{\partial x} - x^2\frac{\partial}{\partial y}.$$

Evaluate the Lie derivative $\pounds_V \mathcal{T}$.

4.14. Prove that elements $(v_1, v_2, \dots, v_k)$ of a vector space are linearly independent if and only if $v_1 \wedge v_2 \wedge \cdots \wedge v_k \neq \mathbf{0}$.

4.15. Prove that the linearly independent sets $(u_1, u_2, \dots, u_k)$ and $(v_1, v_2, \dots, v_k)$ are bases of the same k-dimensional subspace of a vector space if and only if

$$u_1 \wedge u_2 \wedge \cdots \wedge u_k = A v_1 \wedge v_2 \wedge \cdots \wedge v_k$$

where $A \neq 0$. Show further that there exist a regular $k \times k$ matrix $\mathbf{A} = [a^i_j]$ such that $u_i = a^j_i v_j$ and $A = \det \mathbf{A}$.

4.16. If $\mathcal{U}, \mathcal{V} \in \mathfrak{X}(M)$ and $V \in \mathfrak{X}^1(M)$, then show that one can write

$$£_V(\mathcal{U} \wedge \mathcal{V}) = £_V\mathcal{U} \wedge \mathcal{V} + \mathcal{U} \wedge £_V\mathcal{V}.$$

4.17. Let us consider vector fields $U_i \in \mathfrak{X}^1(M), i = 1, \ldots, k$, and let us denote the exterior product of these vectors by $\boldsymbol{\mathcal{U}} = U_1 \wedge \cdots \wedge U_k \in \mathfrak{X}^k(M)$. We define the ***Schouten-Nijenhuis bracket*** [Dutch mathematician Albert Nijenhuis] $\langle\,,\,\rangle : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ through the following expression:

$$\begin{aligned}\langle \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{V}} \rangle &= \sum_{i=1}^{k} (-1)^{i+1} U_1 \wedge \cdots \wedge U_{i-1} \wedge U_{i+1} \wedge \cdots \wedge U_k \wedge £_{U_i}\mathcal{V} \\ &= \sum_{i=1}^{k}\sum_{j=1}^{l} (-1)^{i+j} [U_i, V_j] \wedge U_1 \wedge \cdots \wedge U_{i-1} \wedge U_{i+1} \wedge \cdots \wedge U_k \\ &\qquad \wedge V_1 \wedge \cdots \wedge V_{j-1} \wedge V_{j+1} \wedge \cdots \wedge V_l.\end{aligned}$$

where $\boldsymbol{\mathcal{U}} \in \mathfrak{X}^k(M)$ and $\boldsymbol{\mathcal{V}} \in \mathfrak{X}^l(M)$. Assume that $\boldsymbol{\mathcal{U}} \in \mathfrak{X}^k(M), \boldsymbol{\mathcal{V}} \in \mathfrak{X}^l(M)$, $\boldsymbol{\mathcal{W}} \in \mathfrak{X}^m(M), U, V \in \mathfrak{X}^1(M)$ and $f, g \in C^\infty(M)$. Then show that Schouten -Nijenhuis bracket satisfies the following relations:
(a) $\langle \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{V}} \rangle \in \mathfrak{X}^{k+l-1}(M)$, (b) $\langle f, g \rangle = 0$, (c) $\langle U, f \rangle = U(f)$,
(d) $\langle U, V \rangle = [U, V]$,
(e) $\langle \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{V}} \wedge \boldsymbol{\mathcal{W}} \rangle = \langle \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{V}} \rangle \wedge \boldsymbol{\mathcal{W}} + (-1)^{(k+1)l} \boldsymbol{\mathcal{V}} \wedge \langle \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{W}} \rangle$,
(f) $\langle \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{V}} \rangle = (-1)^{kl} \langle \boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{U}} \rangle$,
(g) *the generalised Jacobi identity*

$$(-1)^{km}\langle\langle \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{V}} \rangle, \boldsymbol{\mathcal{W}} \rangle + (-1)^{kl}\langle\langle \boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{W}} \rangle, \boldsymbol{\mathcal{U}} \rangle + (-1)^{lm}\langle\langle \boldsymbol{\mathcal{W}}, \boldsymbol{\mathcal{U}} \rangle, \boldsymbol{\mathcal{V}} \rangle = \mathbf{0}$$

4.18. The fields $\mathcal{U} \in \mathfrak{X}^k(M)$ and $\mathcal{V} \in \mathfrak{X}^l(M)$ are given by

$$\mathcal{U}(p) = \frac{1}{k!}\, u^{i_1\cdots i_k}(\mathbf{x}) \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}},\ \ \mathcal{V}(p) = \frac{1}{l!}\, v^{i_1\cdots i_l}(\mathbf{x}) \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_l}}.$$

Show that

$$\langle \mathcal{U}, \mathcal{V} \rangle = \frac{1}{(k+l-1)!} \langle \mathcal{U}, \mathcal{V} \rangle^{i_1\cdots i_{k+l-1}}(\mathbf{x}) \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_{k+l-1}}}$$

and the coefficient functions $\langle \mathcal{U}, \mathcal{V} \rangle^{i_1\cdots i_{k+l-1}}(\mathbf{x})$ are determined by the expressions

$$\begin{aligned}\langle \mathcal{U}, \mathcal{V} \rangle^{i_1\cdots i_{k+l-1}}(\mathbf{x}) = {} & \frac{(-1)^k}{k!(l-1)!} \delta^{i_1\cdots\cdots i_{k+l-1}}_{j_1\cdots j_k k_1\cdots k_{l-1}} v^{ik_1\cdots k_{l-1}} \frac{\partial\, u^{j_1\cdots j_k}}{\partial x^i} \\ & + \frac{1}{l!(k-1)!} \delta^{i_1\cdots\cdots i_{k+l-1}}_{j_1\cdots j_{k-1} k_1\cdots k_l} u^{ij_1\cdots j_{k-1}} \frac{\partial\, v^{k_1\cdots k_l}}{\partial x^i}.\end{aligned}$$