CHAPTER IX

PARTIAL DIFFERENTIAL EQUATIONS

9.1. SCOPE OF THE CHAPTER

We can say with a little bit of hyperbolism that to study partial differential equations on smooth manifolds via exterior forms is actually reduced to dealing with a kind of algebraic theory of these equations. The formal treatment of this subject must be based on the theory of jet bundles. However, we prefer here to follow a more direct and concrete path and we attempt to characterise partial differential equations by contact manifolds obtained by extending the main manifold. In Sec. 9.2, we first extend a set of partial differential equations of finite order to a system of first order equations by introducing auxiliary variables. We then show that solutions of this system coincide with solutions of a closed ideal of an exterior algebra defined on an extended manifold. The coordinate cover of this manifold consists of independent and dependent variables, and auxiliary variables corresponding to various order partial derivatives of dependent variables with respect to independent variables. The higher is the order of original system, the huger will be the dimension of the extended manifold. We call 1-forms connecting partial derivatives and auxiliary variables as contact forms and the closure of the ideal generated by them as the contact ideal. The structure of this ideal plays a significant part in the so-called algebraic theory of partial differential equations. The fundamental ideal is constructed through exterior forms describing differential equations together with the contact forms. The first approach that comes to mind to find solutions of the fundamental ideal seems to determine its characteristic vectors in order to be able to apply the Cartan theorem. But, this method proves to be quite unfruitful except for a first order non-linear partial differential equation with one dependent variable. That is the reason why we have chosen to concentrate our efforts to discuss in detail the symmetry transformations that enable us to generate a new family of solutions from a known solution. Since we know that symmetry transformations are generated by isovectors of an ideal, we are first concerned with unravelling the structure of the

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isovector fields of the contact ideal in Sec. 9.3. Sec. 9.4 is devoted to the derivation of determining equations of isovector components of the fundamental ideal, especially the balance ideal associated with balance equations. Sec. 9.5 deals with the similarity solution that remains invariant under a symmetry transformation. In order to benefit substantially from a symmetry transformation, we need first to find a solution, albeit simple, of the system. This of course creates a serious problem. To overcome this obstacle to some extent, we present a method of generalised characteristics in Sec. 9.6 by making use of the isovector fields that helps us to generate a solution from given initial data satisfying certain conditions on an initial manifold. We propose another method in Sec. 9.7 by generalising the contact forms as to include undetermined coefficient functions so that one may be able to explore various possibilities to generate a solution. Some closed horizontal ideals of the exterior algebra introduced that way may prove to be instrumental in obtaining certain solutions. Finally, we investigate in Sec. 9.8 the groups of equivalence transformations that are much more general than the symmetry transformations. When we are given a family of partial differential equations, by means of such a transformation we can transform a member of the family to another member of the same family. The general solutions of the determining equations of isovector fields inducing these kind of transformations are also provided.

9.2. IDEALS FORMED BY DIFFERENTIAL EQUATIONS

We consider an n-dimensional smooth manifold M^n. A set of partial differential equations with A number of members of order m involving the dependent variables $u^\alpha,\ \alpha = 1, \dots, N$ might be locally represented by

$$F^a(x^i, u^\alpha, u^\alpha_{,i}, u^\alpha_{,ij}, \dots, u^\alpha_{,i_1 i_2 \cdots i_m}) = 0,\ a = 1, \dots, A \qquad (9.2.1)$$

where the local coordinates $x^i, i = 1, \dots, n$ in the n-dimensional open set of a chart of the atlas in M denote the independent variables. We assume that all functions F^a are differentiable with respect to their arguments. We define all partial derivatives of order r of u^α with respect to the independent variables x^i as follows

$$\frac{\partial^r u^\alpha}{\partial x^{i_1} \partial x^{i_2} \cdots \partial x^{i_r}} = u^\alpha_{,i_1 i_2 \cdots i_r},\ 1 \le i_1, i_2, \cdots, i_r \le n$$

where $i_1 + i_2 + \cdots + i_r = r$ and $0 \le r \le m$. We adopt the convention that the index i_0 does not exist, hence $u^\alpha_{,i_0} = u^\alpha$ for $r = 0$. In order to identify the global properties of solutions of the system of partial differential equations, we have to solve a rather difficult problem of joining smoothly the results found in local charts, To avoid this problem we shall usually select our manifold as the Euclidean space $M^n = \mathbb{R}^n$ and we shall suppose that the system of differential equations are defined on an open set $\mathcal{D}_n \subseteq \mathbb{R}^n$. In other words, this will mean that all future developments in this chapter will actually be of local character.

In order to study a system of partial differential equations via exterior forms we have to enlarge this system to that of first order partial differential equations by introducing auxiliary variables because of the fact that only the first order exterior derivatives are not identically nil. Introduction of auxiliary variables requires necessarily to enlarge the dimension of the relevant smooth manifold extensively.

The $(n+N)$-dimensional product manifold $G = \mathbb{R}^n \times \mathbb{R}^N = \mathbb{R}^{n+N}$ whose local coordinates are $\{x^i, u^\alpha : 1 \le i \le n, 1 \le \alpha \le N\}$ will be called the ***graph space***. A smooth mapping $\phi : \mathcal{D}_n \to G$ will be propounded as a *regular mapping* if it carries the property

$$\phi^* \mu \neq 0 \tag{9.2.2}$$

where $\mu = dx^1 \wedge \cdots \wedge dx^n$ is the volume form in $\mathbb{R}^n$. This mapping ϕ may be designated by smooth functions $x^i = \Phi^i(\xi^j), u^\alpha = \Phi^\alpha(\xi^j), 1 \le j \le n$ where $(\xi^1, \xi^2, \dots, \xi^n) \in \mathcal{D}_n$. However, if ϕ is a regular mapping we ought to have

$$\phi^* \mu = \det\Big(\frac{\partial \Phi^i}{\partial \xi^j}\Big) d\xi^1 \wedge d\xi^2 \wedge \cdots \wedge d\xi^n \neq 0$$

due to the condition (9.2.2) which leads to $\det\big(\partial \Phi^i / \partial \xi^j\big) \neq 0$. Hence, at least locally the variables ξ^j are expressible in terms of the variables x^i so that the mapping ϕ may be equally represented by

$$u^\alpha = \Phi^\alpha\big(\xi^j(x^i)\big) = \phi^\alpha(x^i) \tag{9.2.3}$$

without loss of generality. *A function in the form* (9.2.3) *constitutes a solution of the system of differential equations* (9.2.1) *when inserted in those expressions the equality*

$$F^a(x^i, \phi^\alpha, \phi^\alpha_{,i}, \phi^\alpha_{,ij}, \dots, \phi^\alpha_{,i_1 i_2 \cdots i_m}) \equiv 0$$

is satisfied identically.

We shall now try to represent a system of partial differential equations via exterior differential forms. In order to achieve this, we have to transform a system of higher order partial differential equations into a much larger system of first order partial differential equations by introducing auxiliary variables as we had mentioned above. To this end, let us define

$$u^\alpha_{,i_1\cdots i_r} = v^\alpha_{i_1\cdots i_r} = v^\alpha_{i_1\cdots i_{r-1},i_r} \tag{9.2.4}$$

where $0 \leq r \leq m, 1 \leq i_1, \ldots, i_r \leq n$. We take of course $v^\alpha_{i_0} = u^\alpha$. Due to their definition, the auxiliary variables $v^\alpha_{i_1\cdots i_r}$ of order r are completely symmetric in indices $i_1, \ldots, i_r$. Thus their number reduces to $N\binom{n+r-1}{r}$ from Nn^r. Hence, when we incorporate the variables u^α $(r = 0)$ into auxiliary variables, their total number reaches to

$$D = N\sum_{r=0}^{m}\binom{n+r-1}{r} = N\binom{n+m}{m} = N\frac{(n+m)!}{n!\,m!}$$

which may be quite a huge number if m is large. The $(n+D)$-dimensional manifold whose coordinate cover is given by $\{x^i, v^\alpha_{i_1 i_2\cdots i_r} : 0 \leq r \leq m\}$ is called the ***jet bundle*** on the base manifold M. The theory of jet bundles that makes it possible to define various order partial derivatives on smooth manifolds has been brought forward first by French mathematician Charles Ehresmann (1905-1979). Since we will be interested in a local approach here, we shall not treat partial differential equations within the formalism of jet bundles. That is the reason why we call this manifold by a more familiar term as the ***mth order contact manifold*** and we denote by $\mathcal{C}_m$. We shall now introduce the following 1-forms on $\mathcal{C}_m$

$$\sigma^\alpha_{i_1 i_2\cdots i_r} = dv^\alpha_{i_1 i_2\cdots i_r} - v^\alpha_{i_1 i_2\cdots i_r i}dx^i \in \Lambda^1(\mathcal{C}_m) \tag{9.2.5}$$

where $0 \leq r \leq m-1$. Their number is obviously given by $N\binom{n+m-1}{m-1}$. We name these forms as ***contact* 1*-forms***. In accordance with our convention, we evidently get $\sigma^\alpha_{i_0} = \sigma^\alpha = du^\alpha - v^\alpha_i dx^i$ for $r = 0$. Since the exterior product of all contact 1-forms may be written as

$$\bigwedge_{1\leq\alpha\leq N;1\leq i_r\leq n,0\leq r\leq m-1} \sigma^\alpha_{i_1 i_2\cdots i_r} = \bigwedge_{1\leq\alpha\leq N;1\leq i_r\leq n,0\leq r\leq m-1} dv^\alpha_{i_1 i_2\cdots i_r} + \cdots \neq 0$$

we see that they are linearly independent on the manifold $\mathcal{C}_m$.

The system of mth order partial differential equations (9.2.1) is now reduced to a system of first order partial differential equations described by the relations (9.2.4) and the algebraic equations

$$F^a(x^i, u^\alpha, v^\alpha_i, v^\alpha_{ij}, \ldots, v^\alpha_{i_1 i_2\cdots i_m}) = 0,\ a = 1, \ldots, A. \tag{9.2.6}$$

(9.2.6) merely represents a functional relation among the coordinates of the contact manifold. Therefore, they only help to define a submanifold of $\mathcal{C}_m$.

We can now lift a regular mapping $\phi : \mathcal{D}_n \to G$ depicted by $u^\alpha = \phi^\alpha(x^i)$ to the regular mapping $\phi : \mathcal{D}_n \to \mathcal{C}_m$ if we choose this mapping in such a way that the pull-back relations

$$\phi^* \sigma^\alpha_{i_1 i_2 \cdots i_r} = (v^\alpha_{i_1 i_2 \cdots i_r, i} - v^\alpha_{i_1 i_2 \cdots i_r i})\, dx^i = 0, \;\; \phi^* F^a = 0$$

are satisfied, in other words, we get

$$v^\alpha_{i_1 i_2 \cdots i_r i} = \frac{\partial v^\alpha_{i_1 i_2 \cdots i_r}}{\partial x^i}, \quad 0 \le r \le m-1.$$

On applying successively the above equality, we immediately observe that the independent coordinates in the manifold $\mathcal{C}_m$ are reduced to the form $v^\alpha_{i_1 i_2 \cdots i_r} = \phi^\alpha_{,i_1 i_2 \cdots i_r}, 0 \le r \le m$ and the mapping ϕ constitutes a solution of the system of partial differential equations. According to Theorem 5.8.2 we find that $\phi^* dF^a = d(\phi^* F^a) = 0$. Thus, this solution is also a solution of the ideal

$$\mathfrak{I}(\sigma^\alpha_{i_1 i_2 \cdots i_r}, 0 \le r \le m-1; dF^a)$$

generated by 1-forms. Since an ideal generated solely by 1-forms is complete (*see* Theorem 5.13.1), then the ideal $\mathfrak{I}$ contains all forms annihilated by the solution of the system (9.2.1). Furthermore, because of the commutation relation $\phi^* d\sigma^\alpha_{i_1 i_2 \cdots i_r} = d(\phi^* \sigma^\alpha_{i_1 i_2 \cdots i_r}) = 0$ the mapping ϕ annihilates also the closure

$$\overline{\mathfrak{I}}(\sigma^\alpha_{i_1 i_2 \cdots i_r}; d\sigma^\alpha_{i_1 i_2 \cdots i_r}; dF^a; 1 \le \alpha \le N, 1 \le i_r \le n, 0 \le r \le m-1, 1 \le a \le A)$$

of $\mathfrak{I}$. Thus, with the purpose of applying the Cartan theorem we can take the closed ideal $\overline{\mathfrak{I}}$ into account instead of the ideal $\mathfrak{I}$. However, due to the symmetries of $v^\alpha_{i_1 i_2 \cdots i_r}$ with respect to their subscripts, we can write

$$\begin{aligned} d\sigma^\alpha_{i_1 i_2 \cdots i_r} &= -\, dv^\alpha_{i_1 i_2 \cdots i_r i} \wedge dx^i = -\, \sigma^\alpha_{i_1 i_2 \cdots i_r i} \wedge dx^i - v^\alpha_{i_1 i_2 \cdots i_r i j} dx^j \wedge dx^i \\ &= -\, \sigma^\alpha_{i_1 i_2 \cdots i_r i} \wedge dx^i \end{aligned}$$

for $0 \le r \le m-2$. This means that the forms $d\sigma^\alpha_{i_1 i_2 \cdots i_r}, 0 \le r \le m-2$ are already in the ideal $\mathfrak{I}$. Therefore, it becomes sufficient to add only the forms $d\sigma^\alpha_{i_1 i_2 \cdots i_{m-1}}$ that cannot be expressed in this way to the ideal to obtain its closure. The closed ideal

$$\mathcal{I}_m = \overline{\mathcal{I}}(\sigma^\alpha, \sigma^\alpha_{i_1}, \sigma^\alpha_{i_1 i_2}, \ldots \sigma^\alpha_{i_1 i_2 \cdots i_{m-1}}; d\sigma^\alpha_{i_1 i_2 \cdots i_{m-1}})$$

will henceforth be called the ***m*th order contact ideal**. On the other hand, we have to consider in essence the closed ideal

$$\mathfrak{I}_m = \overline{\mathfrak{I}}(\sigma^\alpha, \sigma^\alpha_{i_1}, \sigma^\alpha_{i_1 i_2}, \dots, \sigma^\alpha_{i_1 i_2 \cdots i_{m-2}}, \sigma^\alpha_{i_1 i_2 \cdots i_{m-1}}; d\sigma^\alpha_{i_1 i_2 \cdots i_{m-1}}; dF^a)$$

called the ***fundamental ideal***. The most systematic method that we may have recourse to find a solution of this ideal is to determine its characteristic vector fields to utilise Theorem 5.13.5. We first wish to implement this procedure on a rather simple example. Let a first order partial differential equation with a single variable be given by

$$F(x^i, u, u_{,i}) = 0,\ 1 \le i \le n \tag{9.2.7}$$

Since $m = 1$, we write $v_i = u_{,i}$. $(2n+1)$-dimensional contact manifold $\mathcal{C}_1$ has the coordinate cover $\{x^i, u, v_i\}$. On this manifold, we define the forms

$$\sigma = du - v_i\, dx^i, d\sigma = -\, dv_i \wedge dx^i, dF = \frac{\partial F}{\partial x^i}\, dx^i + \frac{\partial F}{\partial u}\, du + \frac{\partial F}{\partial v_i}\, dv_i.$$

A vector field

$$V = X^i \frac{\partial}{\partial x^i} + U \frac{\partial}{\partial u} + V_i \frac{\partial}{\partial v_i} \in T(\mathcal{C}_1)$$

is a characteristic vector field of the closed ideal $\mathfrak{I}_1 = \overline{\mathfrak{I}}(\sigma, d\sigma, dF)$ if one is able to find functions $\lambda, \mu \in \Lambda^0(\mathcal{C}_1)$ so that the relations below are satisfied

$$\mathbf{i}_V(\sigma) = 0, \quad \mathbf{i}_V(dF) = V(F) = 0, \quad \mathbf{i}_V(d\sigma) = \lambda\sigma + \mu dF$$

from which one obtains the following equations that must be satisfied by the components of the characteristic vector field:

$$U - v_i X^i = 0, \qquad \frac{\partial F}{\partial x^i} X^i + \frac{\partial F}{\partial u} U + \frac{\partial F}{\partial v_i} V_i = 0, \tag{9.2.8}$$
$$-\, V_i\, dx^i + X^i\, dv_i = \Big(\lambda + \mu \frac{\partial F}{\partial u}\Big) du - \Big(\lambda v_i - \mu \frac{\partial F}{\partial x^i}\Big) dx^i + \mu \frac{\partial F}{\partial v_i}\, dv_i.$$

$(9.2.8)_3$ and $(9.2.8)_1$ lead to the result

$$\lambda = -\,\mu \frac{\partial F}{\partial u},\ X^i = \mu \frac{\partial F}{\partial v_i},\ U = \mu \frac{\partial F}{\partial v_i} v_i,\ V_i = -\,\mu\Big(\frac{\partial F}{\partial u} v_i + \frac{\partial F}{\partial x^i}\Big).$$

Hence, the characteristic vector field is determined as follows:

$$V = \mu\Big[\frac{\partial F}{\partial v_i}\frac{\partial}{\partial x^i} + v_i \frac{\partial F}{\partial v_i}\frac{\partial}{\partial u} - \Big(\frac{\partial F}{\partial x^i} + \frac{\partial F}{\partial u} v_i\Big)\frac{\partial}{\partial v_i}\Big]. \tag{9.2.9}$$

This vector field is 1-dimensional. We verify at once that $(9.2.8)_2$ becomes identically zero when we insert into it the vector field (9.2.9). As is well known, the solution manifold is produced by the integral curves of the characteristic vector field (9.2.9). If we denote the parameter of the curve by t, then $2n+1$ autonomous ordinary differential equations determining this family of ***characteristic curves*** on the manifold $\mathcal{C}_1$ are given by

$$\begin{aligned}\frac{dx^i}{dt} &= \frac{\partial F}{\partial v_i}, \quad \frac{du}{dt} = v_i \frac{\partial F}{\partial v_i}, \\ \frac{dv_i}{dt} &= -\Big(\frac{\partial F}{\partial x^i} + \frac{\partial F}{\partial u} v_i\Big)\end{aligned} \tag{9.2.10}$$

where we have chosen $\mu = 1$ in (9.2.9) without loss of generality. The variation of the function F along a characteristic curve is found to be

$$\frac{dF}{dt} = \frac{\partial F}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial F}{\partial u}\frac{du}{dt} + \frac{\partial F}{\partial v_i}\frac{dv_i}{dt} = 0$$

when we take (9.2.10) into consideration. Thus, F remains constant along a characteristic curve. This means that if the differential equation is satisfied at a point of the manifold $\mathcal{C}_1$, it is then satisfied along the characteristic curve through that point. A solution in the form $u(x^1, \dots, x^n) - u = 0$ of the equation (9.2.7) represents an n-dimensional submanifold, or a hypersurface, in the graph space. The normal vector to this hypersurface is determined by its components $(v_i = u_{,i}, 1 \le i \le n;\ -1)$. Since we have

$$\frac{du}{dt} = \frac{\partial u}{\partial x^i}\frac{dx^i}{dt} = v_i \frac{dx^i}{dt} = v_i \frac{\partial F}{\partial v_i}$$

on this hypersurface, characteristic curves are also on it. However, to each point of the curve we attach a surface element perpendicular to the normal at that point. Hence, we form a ***characteristic strip*** as was reflected in the classical terminology. In order to find the solution we need to consider characteristic strips emanating from an $(n-1)$-*dimensional initial manifold* S that is not tangent to the characteristic vector field and prescribed by the initial conditions. Let us assume that initial submanifold S is depicted through parameters $\mathbf{s} = (s^1, s^2, \dots, s^{n-1})$ as follows:

$$x^i = x^i_0(\mathbf{s}), \quad i = 1, \dots, n.$$

We suppose that the initial data on this manifold are given by the relations

$$u = u_0(\mathbf{s}), \quad v_i = v_i^0(\mathbf{s}),\ i = 1, \dots, n$$

in terms of parameters $\mathbf{s} = (s^1, s^2, \dots, s^{n-1})$. But, these data cannot be chosen arbitrarily. They have to satisfy the conditions

$$F(x_0^i, u_0, v_i^0) = 0; \quad \frac{\partial u_0}{\partial s^\alpha} = v_i^0 \frac{\partial x_0^i}{\partial s^\alpha}, \ \alpha = 1, \dots, n-1.$$

We thus obtain n equations to determine n initial conditions v_i^0. We shall assume that these equations have at least one solution. Let us now denote the solution of ordinary differential equations (9.2.10) under the initial conditions $x^i(0) = x_0^i(\mathbf{s}), u(0) = u_0(\mathbf{s}), v_i(0) = v_i^0(\mathbf{s})$ by the relations

$$x^i = \mathcal{X}^i(t; \mathbf{s}), \quad u = \mathcal{U}(t; \mathbf{s}), \quad v_i = \mathcal{V}_i(t; \mathbf{s}).$$

Since we have assumed that the characteristic vector field does not belong to the tangent bundle of the initial manifold, we can write

$$\frac{\partial(\mathcal{X}^1, \mathcal{X}^2, \dots, \mathcal{X}^n)}{\partial(s^1, \dots, s^{n-1}, t)} = \begin{vmatrix} \dfrac{\partial \mathcal{X}^1}{\partial s^1} & \cdots & \dfrac{\partial \mathcal{X}^1}{\partial s^{n-1}} & \dfrac{\partial \mathcal{X}^1}{\partial t} \\ \dfrac{\partial \mathcal{X}^2}{\partial s^1} & \cdots & \dfrac{\partial \mathcal{X}^2}{\partial s^{n-1}} & \dfrac{\partial \mathcal{X}^2}{\partial t} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial \mathcal{X}^n}{\partial s^1} & \cdots & \dfrac{\partial \mathcal{X}^n}{\partial s^{n-1}} & \dfrac{\partial \mathcal{X}^n}{\partial t} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial \mathcal{X}^1}{\partial s^1} & \cdots & \dfrac{\partial \mathcal{X}^1}{\partial s^{n-1}} & \dfrac{\partial F}{\partial v_1} \\ \dfrac{\partial \mathcal{X}^2}{\partial s^1} & \cdots & \dfrac{\partial \mathcal{X}^2}{\partial s^{n-1}} & \dfrac{\partial F}{\partial v_2} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial \mathcal{X}^n}{\partial s^1} & \cdots & \dfrac{\partial \mathcal{X}^n}{\partial s^{n-1}} & \dfrac{\partial F}{\partial v_n} \end{vmatrix} \neq 0.$$

Hence, in the neighbourhood of the initial manifold, n variables t, s^α can be expressed in terms of variables x^i by resorting to the inverse mapping theorem whence we arrive at the solution of the partial differential equation (9.2.7) in the following form

$$u = \mathcal{U}(t(\mathbf{x}), \mathbf{s}(\mathbf{x})) = u(x^1, x^2, \dots, x^n).$$

We shall now deal with some applications of the general solution discussed above.

Example 9.2.1. We consider the equation

$$F = \sum_{i=1}^n \Big(\frac{\partial u}{\partial x^i}\Big)^2 - 1 = \sum_{i=1}^n v_i^2 - 1 = 0$$

known as the *eiconal equation* in the geometrical optics. Since

$$\frac{\partial F}{\partial x^i} = 0,\ \frac{\partial F}{\partial u} = 0,\ \frac{\partial F}{\partial v_i} = 2v_i$$

the characteristic equations (9.2.10) take the form

$$\frac{dx^i}{dt} = 2v_i,\ \ \frac{du}{dt} = 2\sum_{i=1}^{n} v_i^2 = 2,\ \ \frac{dv_i}{dt} = 0$$

from which we reach to the conclusion

$$x^i = 2v_i^0(\mathbf{s})t + x_0^i(\mathbf{s});\ \ u = 2t + u_0(\mathbf{s});\ \ v_i = v_i^0(\mathbf{s}),\ \sum_{i=1}^{n}(v_i^0)^2 = 1.$$

Thus, we can express the solution implicitly as

$$x^i = v_i^0(\mathbf{s})\big[u - u_0(\mathbf{s})\big] + x_0^i(\mathbf{s}),\ \sum_{i=1}^{n}(v_i^0)^2 = 1,\ \ v_i^0\frac{\partial x_0^i}{\partial s^\alpha} = 0$$

by eliminating the parameter t. Consequently, the solution manifold $x^i(\mathbf{s})$ corresponding to a chosen value for u is obtained by translating the initial manifold by an amount $u - u_0(\mathbf{s})$ along a unit vector field $\mathbf{v}^0(\mathbf{s})$ which is *orthogonal* to that manifold and the solution $u = u(\mathbf{x})$ is determined by this family of $(n-1)$-dimensional *level manifolds* in $\mathbb{R}^n$. ■

Example 9.2.2. *Quasilinear Equations*. Let us consider the equation

$$a^i(\mathbf{x}, u)\frac{\partial u}{\partial x^i} - b(\mathbf{x}, u) = 0.$$

Since $F = a^i(\mathbf{x}, u)v_i - b(\mathbf{x}, u) = 0$, we find

$$\frac{\partial F}{\partial v_i} = a^i,\ \ \frac{\partial F}{\partial u} = \frac{\partial a^i}{\partial u}v_i - \frac{\partial b}{\partial u},\ \frac{\partial F}{\partial x^i} = \frac{\partial a^j}{\partial x^i}v_j - \frac{\partial b}{\partial x^i}.$$

Hence, the equations $(9.2.10)_{1-2}$ take the form

$$\frac{dx^i}{dt} = a^i(\mathbf{x}, u),\quad \frac{du}{dt} = a^i(\mathbf{x}, u)v_i = b(\mathbf{x}, u).$$

The solution of a first order quasilinear equation then follows from the solution of the above ordinary differential equations. ■

Example 9.2.3. *Hamilton-Jacobi equation.*

The Hamilton-Jacobi partial differential equations governing the motion of a dynamical system of n degrees of freedom [*see* (11.5.18)] [after mathematicians Hamilton and Jacobi] are given by

$$\frac{\partial S}{\partial t} + H\Big(q^1, \dots, q^n, t, \frac{\partial S}{\partial q^1}, \dots, \frac{\partial S}{\partial q^n}\Big) = 0$$

where $S = S(\mathbf{q}, t)$. We denote the generalised coordinates by $(q^1, \dots, q^n)$, time by t and the action function by S. Generalised momenta are defined by $p_i = \partial S/\partial q^i, i = 1, \dots, n$. H is the Hamiltonian function. If we introduce $p = \partial S/\partial t$, we obtain

$$F = p + H(q^1, \dots, q^n, t, p_1, \dots, p_n) = 0.$$

If we denote the parameter of a characteristic curve by s, then it follows from $(9.2.10)_1$ that

$$\frac{dt}{ds} = \frac{\partial F}{\partial p} = 1.$$

Thus, we can choose $s = t$ without loss of generality. Since $\partial F/\partial S = 0$, then equations associated with characteristic strips are found to be

$$\begin{aligned}
&\frac{dq^i}{dt} = \frac{\partial F}{\partial p_i} = \frac{\partial H}{\partial p_i},\\
&\frac{dS}{dt} = p_i \frac{\partial F}{\partial p_i} + p\frac{\partial F}{\partial p} = p_i \frac{\partial H}{\partial p_i} + p = p_i \frac{\partial H}{\partial p_i} - H\\
&\frac{dp}{dt} = -\frac{\partial F}{\partial t} = -\frac{\partial H}{\partial t},\ \frac{dp_i}{dt} = -\frac{\partial F}{\partial q^i} = -\frac{\partial H}{\partial q^i}.
\end{aligned}$$

As a result, we obtain the well known Hamilton equations of analytical mechanics:

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i},\ \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i},\ \frac{dS}{dt} = p_i \frac{\partial H}{\partial p_i} - H,\ \frac{dp}{dt} = -\frac{\partial H}{\partial t}. \quad \blacksquare$$

The method of characteristics that works quite well for the partial differential equation involving a single dependent variable turns out to be rather ineffective when looking for the solution of the general system (9.2.6). Let us denote the *characteristic vector field* V of the ideal $\mathfrak{I}_m$ generated by that system as follows

$$\begin{aligned}
V &= X^i \frac{\partial}{\partial x^i} + U^\alpha \frac{\partial}{\partial u^\alpha} + V^\alpha_{i_1} \frac{\partial}{\partial v^\alpha_{i_1}} + \cdots + V^\alpha_{i_1 \cdots i_m} \frac{\partial}{\partial v^\alpha_{i_1 \cdots i_m}}\\
&= X^i \frac{\partial}{\partial x^i} + \sum_{r=0}^{m} V^\alpha_{i_1 \cdots i_r} \frac{\partial}{\partial v^\alpha_{i_1 \cdots i_r}}.
\end{aligned}$$

If we note that

$$dF^a = \frac{\partial F^a}{\partial x^i}\,dx^i + \sum_{r=0}^{m}\frac{\partial F^a}{\partial v^\alpha_{i_1\cdots i_r}}\,dv^\alpha_{i_1\cdots i_r},$$

then the vector field V must satisfy the relations

$$\begin{aligned}\mathbf{i}_V(\sigma^\alpha_{i_1 i_2\cdots i_r}) &= V^\alpha_{i_1\cdots i_r} - v^\alpha_{i_1\cdots i_r i}X^i = 0,\\ \mathbf{i}_V(dF^a) &= V(F^a) = \frac{\partial F^a}{\partial x^i}X^i + \sum_{r=0}^{m}\frac{\partial F^a}{\partial v^\alpha_{i_1\cdots i_r}}V^\alpha_{i_1\cdots i_r} = 0\\ \mathbf{i}_V(d\sigma^\alpha_{i_1 i_2\cdots i_{m-1}}) &= -V^\alpha_{i_1\cdots i_{m-1}i}\,dx^i + X^i\,dv^\alpha_{i_1\cdots i_{m-1}i}\\ &= \sum_{s=0}^{m-1}\lambda^{\alpha j_1\cdots j_s}_{\beta i_1\cdots i_{m-1}}\sigma^\beta_{j_1\cdots j_s} + \Lambda^\alpha_{i_1\cdots i_{m-1}a}dF^a\end{aligned} \tag{9.2.11}$$

where $\lambda^{\alpha j_1\cdots j_s}_{\beta i_1\cdots i_{m-1}}, \Lambda^\alpha_{i_1\cdots i_{m-1}a} \in \Lambda^0(\mathcal{C}_m)$, $0 \le s \le m-1$. $(9.2.11)_{1-2}$ then yields

$$\begin{aligned}&V^\alpha_{i_1\cdots i_r} = v^\alpha_{i_1\cdots i_r i}X^i, \quad 0 \le r \le m-1,\\ &\Big(\frac{\partial F^a}{\partial x^i} + \sum_{r=0}^{m-1}\frac{\partial F^a}{\partial v^\alpha_{i_1\cdots i_r}}v^\alpha_{i_1\cdots i_r i}\Big)X^i + \frac{\partial F^a}{\partial v^\alpha_{i_1\cdots i_m}}V^\alpha_{i_1\cdots i_m} = 0\end{aligned} \tag{9.2.12}$$

whereas $(9.2.11)_3$ results in

$$\begin{aligned}&-V^\alpha_{i_1\cdots i_{m-1}i}\,dx^i + X^i\,dv^\alpha_{i_1\cdots i_{m-1}i}\\ &\qquad= \sum_{s=0}^{m-1}\Big[\lambda^{\alpha j_1\cdots j_s}_{\beta i_1\cdots i_{m-1}} + \Lambda^\alpha_{i_1\cdots i_{m-1}a}\frac{\partial F^a}{\partial v^\beta_{j_1\cdots j_s}}\Big]dv^\beta_{j_1\cdots j_s}\\ &\qquad\quad + \Big[\Lambda^\alpha_{i_1\cdots i_{m-1}a}\frac{\partial F^a}{\partial x^i} - \sum_{s=0}^{m-1}\lambda^{\alpha j_1\cdots j_s}_{\beta i_1\cdots i_{m-1}}v^\beta_{j_1\cdots j_s i}\Big]dx^i\\ &\qquad\quad + \Lambda^\alpha_{i_1\cdots i_{m-1}a}\frac{\partial F^a}{\partial v^\beta_{j_1\cdots j_m}}\,dv^\beta_{j_1\cdots j_m}.\end{aligned}$$

We thus see that the following relations must be satisfied

$$\begin{aligned}\lambda^{\alpha j_1\cdots j_s}_{\beta i_1\cdots i_{m-1}} &= -\Lambda^\alpha_{i_1\cdots i_{m-1}a}\frac{\partial F^a}{\partial v^\beta_{j_1\cdots j_s}}, \quad 0 \le s \le m-1\\ V^\alpha_{i_1\cdots i_{m-1}i} &= -\Lambda^\alpha_{i_1\cdots i_{m-1}a}\Big[\frac{\partial F^a}{\partial x^i} + \sum_{s=0}^{m-1}\frac{\partial F^a}{\partial v^\beta_{j_1\cdots j_s}}v^\beta_{j_1\cdots j_s i}\Big],\end{aligned} \tag{9.2.13}$$

$$X^i\, dv^\alpha_{i_1\cdots i_{m-1}i} = \Lambda^\alpha_{i_1\cdots i_{m-1}a}\frac{\partial F^a}{\partial v^\beta_{j_1\cdots j_m}}\, dv^\beta_{j_1\cdots j_m}.$$

It then follows from $(9.2.13)_3$ and $(9.2.12)_2$ that

$$X^i\delta^\alpha_\beta\delta^{j_1}_{i_1}\cdots\delta^{j_{m-1}}_{i_{m-1}} = \Lambda^\alpha_{i_1\cdots i_{m-1}a}\frac{\partial F^a}{\partial v^\beta_{j_1\cdots j_{m-1}i}} \tag{9.2.14}$$
$$\Big(X^i\delta^a_b - \Lambda^\alpha_{i_1\cdots i_{m-1}b}\frac{\partial F^a}{\partial v^\alpha_{i_1\cdots i_{m-1}i}}\Big)\Big(\frac{\partial F^b}{\partial x^i} + \sum_{r=0}^{m-1}\frac{\partial F^b}{\partial v^\beta_{j_1\cdots j_r}}v^\beta_{j_1\cdots j_r i}\Big) = 0$$

After having performed contraction operations on indices $(\alpha,\beta),(j_1,i_1),\ldots$ (j_{m-1},i_{m-1}) of Kronecker deltas on the left hand side of the expression $(9.2.14)_1$ we find that

$$n^{m-1}NX^i = \Lambda^\alpha_{i_1\cdots i_{m-1}a}\frac{\partial F^a}{\partial v^\alpha_{i_1\cdots i_{m-1}i}}. \tag{9.2.15}$$

If we insert the above expression in $(9.2.14)_{1-2}$, we deduce that the functions $\Lambda^\alpha_{i_1\cdots i_{m-1}a}$ must satisfy the equations

$$\Lambda^\gamma_{k_1\cdots k_{m-1}a}\frac{\partial F^a}{\partial v^\gamma_{k_1\cdots k_{m-1}i}}\delta^\alpha_\beta\delta^{j_1}_{i_1}\cdots\delta^{j_{m-1}}_{i_{m-1}} - n^{m-1}N\Lambda^\alpha_{i_1\cdots i_{m-1}a}\frac{\partial F^a}{\partial v^\beta_{j_1\cdots j_{m-1}i}} = 0,$$
$$\Big(\Lambda^\alpha_{i_1\cdots i_{m-1}c}\frac{\partial F^c}{\partial v^\alpha_{i_1\cdots i_{m-1}i}}\,\delta^a_b - n^{m-1}N\Lambda^\alpha_{i_1\cdots i_{m-1}b}\frac{\partial F^a}{\partial v^\alpha_{i_1\cdots i_{m-1}i}}\Big)\times$$
$$\Big(\frac{\partial F^b}{\partial x^i} + \sum_{r=0}^{m-1}\frac{\partial F^b}{\partial v^\beta_{j_1\cdots j_r}}v^\beta_{j_1\cdots j_r i}\Big) = 0.$$

When $N > 1$, we can always pick out the indices α and β as to be $\alpha \neq \beta$. In this case, if all partial differential equations are of order m, then none of the derivatives $\dfrac{\partial F^a}{\partial v^\beta_{j_1\cdots j_{m-1}i}}$ vanish implying that $\Lambda^\alpha_{i_1\cdots i_{m-1}a} = 0$ and $X^i = 0$. Consequently, we find $V^\alpha_{i_1\cdots i_r} = 0$ for $0 \le r \le m$. Hence, the dimension of the characteristic vector space is zero and we end up only with the trivial solution that consists of constants satisfying the equations (9.2.1). If $N = 1$ and $m > 1$, then we immediately see that we obtain the same result. If some equations in the system have lesser orders than m, some coefficients $\Lambda^\alpha_{i_1\cdots i_{m-1}a}$ may not be necessarily zero. In the case $N = 1,\ m = 1,\ A > 1$, the first set of equations above are satisfied identically. On arranging the second equations, we obtain

$$F^{ab}\Lambda_b = 0$$

where the antisymmetric $A \times A$ matrix $\mathbf{F}$ is given by

$$F^{ab} = -F^{ba} = \frac{\partial F^a}{\partial x^i}\frac{\partial F^b}{\partial v_i} - \frac{\partial F^b}{\partial x^i}\frac{\partial F^a}{\partial v_i} + v_i\Big(\frac{\partial F^a}{\partial u}\frac{\partial F^b}{\partial v_i} - \frac{\partial F^b}{\partial u}\frac{\partial F^a}{\partial v_i}\Big).$$

If only $\det \mathbf{F} = 0$ (when A is an odd number this determinant will always be zero) then all coefficients Λ_a do not have to vanish and we may have the opportunity to write

$$X^i = \Lambda_a \frac{\partial F^a}{\partial v_i},\ U^\alpha = V_0 = \lambda_a \frac{\partial F^a}{\partial v_i} v_i,\ V_i = -\Lambda_a\Big(\frac{\partial F^a}{\partial x^i} + \frac{\partial F^a}{\partial u} v_i\Big),$$
$$V = \Lambda_a\Big[\frac{\partial F^a}{\partial v_i}\Big(\frac{\partial}{\partial x^i} + v_i\frac{\partial}{\partial u}\Big) - \Big(\frac{\partial F^a}{\partial x^i} + \frac{\partial F^a}{\partial u}v_i\Big)\frac{\partial}{\partial v_i}\Big].$$

The dimension of the characteristic subspace is equal to the number of independent functions Λ_a. On the other hand, if $N = 1$, $m = 1, A = 1$ then we arrive at the previously found solution

$$X^i = \Lambda \frac{\partial F}{\partial v_i},\ \ U = V_0 = \Lambda \frac{\partial F}{\partial v_i} v_i,\ \ V_i = -\Lambda\Big(\frac{\partial F}{\partial x^i} + \frac{\partial F}{\partial u} v_i\Big),$$
$$V = \Lambda\Big[\frac{\partial F}{\partial v_i}\Big(\frac{\partial}{\partial x^i} + v_i\frac{\partial}{\partial u}\Big) - \Big(\frac{\partial F}{\partial x^i} + \frac{\partial F}{\partial u}v_i\Big)\frac{\partial}{\partial v_i}\Big].$$

A nontrivial solution is likewise obtained for a system of quasilinear first order partial differential equations with same principal parts

$$a^i(\mathbf{x},\mathbf{u})\frac{\partial u^\alpha}{\partial x^i} - b^\alpha(\mathbf{x},\mathbf{u}) = 0.$$

In this case, the characteristic vector field can be written as follows

$$V = X^i\frac{\partial}{\partial x^i} + U^\alpha\frac{\partial}{\partial u^\alpha} + V_i^\alpha\frac{\partial}{\partial v_i^\alpha}.$$

On the other hand, since we have to take $F^\alpha = a^i(\mathbf{x},\mathbf{u})v_i^\alpha - b^\alpha(\mathbf{x},\mathbf{u}) = 0$, then the equations (9.2.12-13-14) lead to the relations

$$U^\alpha = v_i^\alpha X^i,\ \lambda_\beta^\alpha = -\Lambda_\gamma^\alpha\frac{\partial F^\gamma}{\partial u^\beta},\ X^i\delta_\beta^\alpha = \Lambda_\gamma^\alpha\frac{\partial F^\gamma}{\partial v_i^\beta} = \Lambda_\gamma^\alpha\delta_\beta^\gamma a^i = \Lambda_\beta^\alpha a^i$$

whence we deduce that $\Lambda_\beta^\alpha = \delta_\beta^\alpha$ and $\lambda_\beta^\alpha = -\partial F^\alpha/\partial u^\beta$. Therefore, the components of the characteristic vector field are found as

$$X^i = a^i, \;\; U^\alpha = v_i^\alpha a^i, \;\; V_i^\alpha = -\Big(\frac{\partial F^\alpha}{\partial x^i} + \frac{\partial F^\alpha}{\partial u^\beta} v_i^\beta\Big).$$

These components satisfy identically the relation $(9.2.12)_2$. Since one must write $a^i v_i^\alpha = b^\alpha$, the solution of the system of partial differential equations is constructed by means of the solutions of the following system of ordinary differential equations

$$\frac{dx^i}{dt} = a^i(\mathbf{x}, \mathbf{u}),$$
$$\frac{du^\alpha}{dt} = b^\alpha(\mathbf{x}, \mathbf{u}).$$

To study general solutions of differential equations we usually make use of Lie transformation groups. In such kind of methods, the isovectors of closed ideals generated by differential equations play quite a significant part. Although symmetry groups have emerged at the beginning of 20th Century, their investigation through exterior differential forms started by a seminal paper published on 1971[1].

9.3. ISOVECTOR FIELDS OF THE CONTACT IDEAL

Let $\mathcal{C}_m$ be the contact manifold defined in Sec. 9.2. We first consider the closed ideal

$$\mathcal{I}_m = \overline{\mathcal{I}}(\sigma^\alpha, \sigma^\alpha_{i_1}, \sigma^\alpha_{i_1 i_2}, \dots, \sigma^\alpha_{i_1 i_2 \cdots i_{m-1}}; d\sigma^\alpha_{i_1 i_2 \cdots i_{m-1}}) \qquad (9.3.1)$$

which we have called the *mth order contact ideal*. *The properties of this ideal will remain the same for all system of mth order partial differential equations*. We know that a vector field $V \in T(\mathcal{C}_m)$ is an isovector field of the ideal $\mathcal{I}_m$ if it satisfies the relation $\pounds_V \mathcal{I}_m \subset \mathcal{I}_m$. On the other hand, since the ideal

$$\mathcal{I}(\sigma^\alpha, \sigma^\alpha_{i_1}, \sigma^\alpha_{i_1 i_2}, \dots, \sigma^\alpha_{i_1 i_2 \cdots i_{m-1}}) \qquad (9.3.2)$$

is generated by forms of the same degree (1-forms in the present case), isovector fields of this ideal will be the same as those of its closure $\mathcal{I}_m$ in accordance with Theorem 5.12.4. We may represent a vector field V by the expression

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[1]Harrison, B. K., Estabrook, F. B., Geometric Approach to invariance groups and solution of partial differential systems, *Journal of Mathematical Physics*, **12**, 653-666, 1971.

$$V = X^i \frac{\partial}{\partial x^i} + \sum_{r=0}^{m} V^\alpha_{i_1 \cdots i_r} \frac{\partial}{\partial v^\alpha_{i_1 \cdots i_r}} \in T(\mathcal{C}_m) \tag{9.3.3}$$

where $X^i, V^\alpha_{i_1 \cdots i_r} \in \Lambda^0(\mathcal{C}_m)$ with $0 \le r \le m$. Here, we adopt the conventions

$$V^\alpha_{i_0} = V^\alpha = U^\alpha$$

and

$$\sum_{r=0}^{m} V^\alpha_{i_1 \cdots i_r} \frac{\partial}{\partial v^\alpha_{i_1 \cdots i_r}} = \\ U^\alpha \frac{\partial}{\partial u^\alpha} + V^\alpha_{i_1} \frac{\partial}{\partial v^\alpha_{i_1}} + V^\alpha_{i_1 i_2} \frac{\partial}{\partial v^\alpha_{i_1 i_2}} + \cdots + V^\alpha_{i_1 \cdots i_m} \frac{\partial}{\partial v^\alpha_{i_1 \cdots i_m}}.$$

There are of course summations on all repeated dummy indices. Since the variables $v^\alpha_{i_1 \cdots i_r}$ are completely symmetric with respect to their subscripts for $r \ge 2$, only the completely symmetric parts of corresponding components $V^\alpha_{i_1 \cdots i_r}$ will survive in summations above. Therefore, without loss of generality we may assume that $V_{i_1 \cdots i_r}$ are completely symmetric with respect to their subscripts for $r \ge 2$. As is well known, the necessary and sufficient conditions for a vector field V to be an isovector of the ideal (9.3.2) are the satisfaction of the following relations

$$£_V \sigma^\alpha_{i_1 \cdots i_k} = \sum_{r=0}^{m-1} \lambda^{\alpha j_1 \cdots j_r}_{\beta i_1 \cdots i_k} \sigma^\beta_{j_1 \cdots j_r}, \;\; k = 0, 1, \ldots, m-1 \tag{9.3.4}$$

for certain functions $\lambda^{\alpha j_1 \cdots j_r}_{\beta i_1 \cdots i_k} \in \Lambda^0(\mathcal{C}_m), 0 \le k, r \le m-1$. The discussions presented in this section and the subsequent one are borrowed from the work cited below[2]. On employing the formula (5.11.5) to calculate the Lie derivative, we obtain

$$\begin{aligned} £_V \sigma^\alpha_{i_1 \cdots i_k} &= \mathbf{i}_V(d\sigma^\alpha_{i_1 \cdots i_k}) + d\mathbf{i}_V(\sigma^\alpha_{i_1 \cdots i_k}) \\ &= -V^\alpha_{i_1 \cdots i_k i} dx^i + X^i dv^\alpha_{i_1 \cdots i_k i} + d(V^\alpha_{i_1 \cdots i_k} - X^i v^\alpha_{i_1 \cdots i_k i}) \\ &= -V^\alpha_{i_1 \cdots i_k i} dx^i + dV^\alpha_{i_1 \cdots i_k} - v^\alpha_{i_1 \cdots i_k i} dX^i \end{aligned}$$

by recalling the relation $d\sigma^\alpha_{i_1 \cdots i_k} = -dv^\alpha_{i_1 \cdots i_k i} \wedge dx^i$. Therefore, (9.3.4) yields

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[2]Şuhubi, E. S., Isovector fields and similarity solutions for general balance equations, *International Journal of Engineering Science*, **29**, 133-150, 1991.

$$\left[\frac{\partial V^\alpha_{i_1\cdots i_k}}{\partial x^i} - v^\alpha_{i_1\cdots i_k j}\frac{\partial X^j}{\partial x^i} - V^\alpha_{i_1\cdots i_k i}\right]dx^i + \sum_{r=0}^{m}\left[\frac{\partial V^\alpha_{i_1\cdots i_k}}{\partial v^\beta_{j_1\cdots j_r}} - v^\alpha_{i_1\cdots i_k i}\frac{\partial X^i}{\partial v^\beta_{j_1\cdots j_r}}\right]dv^\beta_{j_1\cdots j_r} = \sum_{r=0}^{m-1}\lambda^{\alpha j_1\cdots j_r}_{\beta i_1\cdots i_k}(dv^\beta_{j_1\cdots j_r} - v^\beta_{j_1\cdots j_r i}dx^i)$$

On equating the coefficients of linearly independent like 1-forms at both sides of the foregoing expression, we arrive at the following relations

$$\begin{aligned} -\sum_{r=0}^{m-1}\lambda^{\alpha j_1\cdots j_r}_{\beta i_1\cdots i_k}v^\beta_{j_1\cdots j_r i} &= \frac{\partial V^\alpha_{i_1\cdots i_k}}{\partial x^i} - v^\alpha_{i_1\cdots i_k j}\frac{\partial X^j}{\partial x^i} - V^\alpha_{i_1\cdots i_k i}, \; 0 \le k \le m-1 \\ \lambda^{\alpha j_1\cdots j_r}_{\beta i_1\cdots i_k} &= \frac{\partial V^\alpha_{i_1\cdots i_k}}{\partial v^\beta_{j_1\cdots j_r}} - v^\alpha_{i_1\cdots i_k i}\frac{\partial X^i}{\partial v^\beta_{j_1\cdots j_r}}, \; 0 \le k, r \le m-1 \qquad (9.3.5) \\ \frac{\partial V^\alpha_{i_1\cdots i_k}}{\partial v^\beta_{j_1\cdots j_m}} - v^\alpha_{i_1\cdots i_k i}\frac{\partial X^i}{\partial v^\beta_{j_1\cdots j_m}} &= 0, \qquad 0 \le k \le m-1 \end{aligned}$$

Equations $(9.3.5)_2$ determine the functions $\lambda^{\alpha j_1\cdots j_r}_{\beta i_1\cdots i_k}$. Inserting these functions into equations $(9.3.5)_1$, we reach to the recurrence relations given below that relate the components $V^\alpha_{i_1\cdots i_k i}$ to the components $V^\alpha_{i_1\cdots i_k}$ and X^i

$$\begin{aligned} V^\alpha_{i_1\cdots i_k i} = {} & \frac{\partial V^\alpha_{i_1\cdots i_k}}{\partial x^i} - v^\alpha_{i_1\cdots i_k j}\frac{\partial X^j}{\partial x^i} \qquad (9.3.6) \\ & + \sum_{r=0}^{m-1}\left[\frac{\partial V^\alpha_{i_1\cdots i_k}}{\partial v^\beta_{j_1\cdots j_r}} - v^\alpha_{i_1\cdots i_k j}\frac{\partial X^j}{\partial v^\beta_{j_1\cdots j_r}}\right]v^\beta_{j_1\cdots j_r i}, 0 \le k \le m-1. \end{aligned}$$

Let us now consider equations $(9.3.5)_3$ and start with equations corresponding to $k = m-1$. If we differentiate these equations with respect to the variables $v^\gamma_{k_1\cdots k_m}$, we then find that

$$\frac{\partial^2 V^\alpha_{i_1\cdots i_{m-1}}}{\partial v^\beta_{j_1\cdots j_m}\partial v^\gamma_{k_1\cdots k_m}} - v^\alpha_{i_1\cdots i_{m-1}i}\frac{\partial^2 X^i}{\partial v^\beta_{j_1\cdots j_m}\partial v^\gamma_{k_1\cdots k_m}} - \delta^\alpha_\gamma\delta^{k_1}_{i_1}\cdots\delta^{k_{m-1}}_{i_{m-1}}\delta^{k_m}_i\frac{\partial X^i}{\partial v^\beta_{j_1\cdots j_m}} = 0.$$

When we take into account the symmetry of the second order derivatives with respect to the variables $v^\alpha_{i_1\cdots i_m}$, the above equations give rise to

$$\delta^\alpha_\gamma \delta^{k_1}_{i_1} \cdots \delta^{k_{m-1}}_{i_{m-1}} \frac{\partial X^{k_m}}{\partial v^\beta_{j_1\cdots j_m}} = \delta^\alpha_\beta \delta^{j_1}_{i_1} \cdots \delta^{j_{m-1}}_{i_{m-1}} \frac{\partial X^{j_m}}{\partial v^\gamma_{k_1\cdots k_m}} \tag{9.3.7}$$

Contraction operations on indices $(\alpha, \gamma), (k_1, i_1), \ldots, (k_{m-1}, i_{m-1})$ lead to the result

$$N n^{m-1} \frac{\partial X^{k_m}}{\partial v^\beta_{j_1\cdots j_{m-1} j_m}} = \frac{\partial X^{j_m}}{\partial v^\beta_{j_1\cdots j_{m-1} k_m}}. \tag{9.3.8}$$

Introducing (9.3.8) into the right hand side of (9.3.7), we get

$$\delta^\alpha_\gamma \delta^{k_1}_{i_1} \cdots \delta^{k_{m-1}}_{i_{m-1}} \frac{\partial X^{k_m}}{\partial v^\beta_{j_1\cdots j_m}} = \delta^\alpha_\beta \delta^{j_1}_{i_1} \cdots \delta^{j_{m-1}}_{i_{m-1}} N n^{m-1} \frac{\partial X^{k_m}}{\partial v^\gamma_{k_1\cdots k_{m-1} j_m}}.$$

Contracting this time the indices $(\alpha, \beta), (j_1, i_1), \ldots, (j_{m-1}, i_{m-1})$ above, we finally reach to the conclusion

$$(N^2 n^{2(m-1)} - 1) \frac{\partial X^{k_m}}{\partial v^\gamma_{k_1\cdots k_{m-1} j_m}} = 0.$$

When we take partial differential equations into consideration, we clearly have $n > 1$. Furthermore, if we assume that $m > 1$, then for $N \geq 1$ we get $N n^{m-1} \neq 1$. *When $m = 1$ we will have to distinguish the case $N = 1$ from the case $N > 1$.* We thus find

$$\frac{\partial X^i}{\partial v^\beta_{j_1\cdots j_m}} = 0 \tag{9.3.9}$$

and it follows from equations $(9.3.5)_3$ that

$$\frac{\partial V^\alpha_{i_1\cdots i_k}}{\partial v^\beta_{j_1\cdots j_m}} = 0, \quad k = 0, 1, \ldots, m-1. \tag{9.3.10}$$

This means that the functions X^i and $V^\alpha_{i_1\cdots i_k}$ cannot depend on the variables $v^\alpha_{i_1\cdots i_m}$. Let us now write the relation (9.3.6) for $k = 0, 1, \ldots, m-2$:

$$\begin{aligned} V^\alpha_{i_1\cdots i_k i} &= \frac{\partial V^\alpha_{i_1\cdots i_k}}{\partial x^i} - v^\alpha_{i_1\cdots i_k j} \frac{\partial X^j}{\partial x^i} + \sum_{r=0}^{m-2} \left[\frac{\partial V^\alpha_{i_1\cdots i_k}}{\partial v^\beta_{j_1\cdots j_r}} - v^\alpha_{i_1\cdots i_k j} \frac{\partial X^j}{\partial v^\beta_{j_1\cdots j_r}} \right] v^\beta_{j_1\cdots j_r i} \\ &+ \left[\frac{\partial V^\alpha_{i_1\cdots i_k}}{\partial v^\beta_{j_1\cdots j_{m-1}}} - v^\alpha_{i_1\cdots i_k j} \frac{\partial X^j}{\partial v^\beta_{j_1\cdots j_{m-1}}} \right] v^\beta_{j_1\cdots j_{m-1} i}, \quad 0 \leq k \leq m-2 \end{aligned}$$

Because of (9.3.10), the functions $V^{\alpha}_{i_1\cdots i_k i}$ must be independent of the variables $v^{\beta}_{j_1\cdots j_{m-1}i}$, hence their coefficients have to vanish:

$$\frac{\partial V^{\alpha}_{i_1\cdots i_k}}{\partial v^{\beta}_{j_1\cdots j_{m-1}}} - v^{\alpha}_{i_1\cdots i_k j}\frac{\partial X^j}{\partial v^{\beta}_{j_1\cdots j_{m-1}}} = 0,\ 0 \le k \le m-2. \qquad (9.3.11)$$

Equations (9.3.11) carry the same structural properties as equations $(9.3.5)_3$. Therefore, they lead similarly to

$$\frac{\partial X^i}{\partial v^{\beta}_{j_1\cdots j_{m-1}}} = 0,\ \frac{\partial V^{\alpha}_{i_1\cdots i_k}}{\partial v^{\beta}_{j_1\cdots j_{m-1}}} = 0,\ k = 0, 1, \ldots, m-2$$

if $Nn^{m-2} \neq 1$. Starting from this result we can verify by mathematical induction that the following relations are to be satisfied if $Nn^{m-s-1} \neq 1$

$$\frac{\partial X^i}{\partial v^{\beta}_{j_1\cdots j_{m-s}}} = 0, \quad \frac{\partial V^{\alpha}_{i_1\cdots i_k}}{\partial v^{\beta}_{j_1\cdots j_{m-s}}} = 0, \qquad (9.3.12)$$

where $k = 0, 1, \ldots, m-s-1$, $s = 0, 1, \ldots, m-2$. We have shown above that these relations are true for $s = 0, 1$. We shall now assume that they are true for s and try to prove that they are also true for $s+1$. On writing the relation (9.3.6) for $k = 0, 1, \ldots,\ m-s-2$, we obtain

$$\begin{aligned} V^{\alpha}_{i_1\cdots i_k i} = \frac{\partial V^{\alpha}_{i_1\cdots i_k}}{\partial x^i} - v^{\alpha}_{i_1\cdots i_k j}\frac{\partial X^j}{\partial x^i} \\ + \sum_{r=0}^{m-2-s}\left[\frac{\partial V^{\alpha}_{i_1\cdots i_k}}{\partial v^{\beta}_{j_1\cdots j_r}} - v^{\alpha}_{i_1\cdots i_k j}\frac{\partial X^j}{\partial v^{\beta}_{j_1\cdots j_r}}\right] v^{\beta}_{j_1\cdots j_r i} \\ + \left[\frac{\partial V^{\alpha}_{i_1\cdots i_k}}{\partial v^{\beta}_{j_1\cdots j_{m-s-1}}} - v^{\alpha}_{i_1\cdots i_k j}\frac{\partial X^j}{\partial v^{\beta}_{j_1\cdots j_{m-s-1}}}\right] v^{\beta}_{j_1\cdots j_{m-s-1} i}, 0 \le k \le m-s-2. \end{aligned}$$

But because of (9.3.12), the functions $V^{\alpha}_{i_1\cdots i_k i}$ cannot depend on variables $v^{\beta}_{j_1\cdots j_{m-s-1}i}$ so that their coefficients must vanish:

$$\frac{\partial V^{\alpha}_{i_1\cdots i_k}}{\partial v^{\beta}_{j_1\cdots j_{m-s-1}}} - v^{\alpha}_{i_1\cdots i_k j}\frac{\partial X^j}{\partial v^{\beta}_{j_1\cdots j_{m-s-1}}} = 0,\ k = 0, 1, \ldots, m-s-2.$$

We thus obtain in the similar fashion

$$\frac{\partial X^i}{\partial v^{\beta}_{j_1\cdots j_{m-s-1}}} = 0,\ \frac{\partial V^{\alpha}_{i_1\cdots i_k}}{\partial v^{\beta}_{j_1\cdots j_{m-s-1}}} = 0,\ \ k = 0, 1, \ldots, m-s-2$$

if $Nn^{m-s-2} \neq 1$. This justifies the proposition (9.3.12). However, we have to be a little bit more careful for the case $s = m - 1$. If we write the relation (9.3.6) for $k = 0$, we then find

$$V_i^\alpha = \frac{\partial U^\alpha}{\partial x^i} - v_j^\alpha \frac{\partial X^j}{\partial x^i} + \Big(\frac{\partial U^\alpha}{\partial u^\beta} - v_j^\alpha \frac{\partial X^j}{\partial u^\beta}\Big) v_i^\beta + \Big(\frac{\partial U^\alpha}{\partial v_j^\beta} - v_k^\alpha \frac{\partial X^k}{\partial v_j^\beta}\Big) v_{ji}^\beta.$$

On the other hand, the functions V_i^α must be independent of the variables v_{ji}^β so that one gets

$$\frac{\partial U^\alpha}{\partial v_i^\beta} - v_j^\alpha \frac{\partial X^j}{\partial v_i^\beta} = 0. \tag{9.3.13}$$

Next, we differentiate (9.3.13) with respect to the variables v_k^γ. The symmetry of the second order derivatives leads eventually to the result

$$\delta_\gamma^\alpha \frac{\partial X^k}{\partial v_i^\beta} = \delta_\beta^\alpha \frac{\partial X^i}{\partial v_k^\gamma}.$$

A contraction on indices (α, γ) gives

$$N \frac{\partial X^k}{\partial v_i^\beta} = \frac{\partial X^i}{\partial v_k^\beta}. \tag{9.3.14}$$

On inserting this result into the above expression and contracting this time on indices (α, β), we finally obtain

$$(N^2 - 1) \frac{\partial X^k}{\partial v_i^\gamma} = 0. \tag{9.3.15}$$

In evaluating this inequality, we have to distinguish two cases:

(i). We assume that $N > 1$. Hence the number of dependent variables is more than one. In this case (9.3.15) and (9.3.13) yield

$$\frac{\partial X^i}{\partial v_j^\alpha} = 0 \quad \text{and} \quad \frac{\partial U^\alpha}{\partial v_i^\beta} = 0.$$

We thus obtain

$$X^i = X^i(\mathbf{x}, \mathbf{u}), \qquad V_{i_0}^\alpha = U^\alpha = U^\alpha(\mathbf{x}, \mathbf{u}). \tag{9.3.16}$$

Thus X^i and U^α components of the isovector field have to depend solely on coordinates x^i and u^α of the graph space. If the components (9.3.16) are inserted into (9.3.6) on taking notice of (9.3.12), we realise that other components of isovector fields of the contact ideal are determined by the following recurrence relations

$$\begin{aligned} V^{\alpha}_{i_1\cdots i_k i} = {} & \frac{\partial V^{\alpha}_{i_1\cdots i_k}}{\partial x^i} - v^{\alpha}_{i_1\cdots i_k j}\frac{\partial X^j}{\partial x^i} \\ & + \left[\frac{\partial V^{\alpha}_{i_1\cdots i_k}}{\partial u^{\beta}} - v^{\alpha}_{i_1\cdots i_k j}\frac{\partial X^j}{\partial u^{\beta}}\right] v^{\beta}_i \\ & + \sum_{r=1}^{k} v^{\beta}_{j_1\cdots j_r i}\frac{\partial V^{\alpha}_{i_1\cdots i_k}}{\partial v^{\beta}_{j_1\cdots j_r}}, \quad k = 0, 1, \dots, m-1. \end{aligned} \tag{9.3.17}$$

Let us now define a set of vector fields, or differential operators, $D_i^{(k)}$ where $i = 1, \dots, n$, $k = 0, 1, \dots, m-1$ as follows

$$\begin{aligned} D_i^{(k)} & = \frac{\partial}{\partial x^i} + v_i^{\alpha}\frac{\partial}{\partial u^{\alpha}} + \sum_{r=1}^{k} v^{\alpha}_{i_1\cdots i_r i}\frac{\partial}{\partial v^{\alpha}_{i_1\cdots i_r}} \\ & = \frac{\partial}{\partial x^i} + v_i^{\alpha}\frac{\partial}{\partial u^{\alpha}} + v^{\alpha}_{ji}\frac{\partial}{\partial v^{\alpha}_j} + \cdots + v^{\alpha}_{i_1\cdots i_k i}\frac{\partial}{\partial v^{\alpha}_{i_1\cdots i_k}}. \end{aligned} \tag{9.3.18}$$

This operator may also be defined by the recurrence relations

$$D_i = D_i^{(0)} = \frac{\partial}{\partial x^i} + v_i^{\alpha}\frac{\partial}{\partial u^{\alpha}}, \quad D_i^{(k+1)} = D_i^{(k)} + v^{\alpha}_{i_1\cdots i_{k+1} i}\frac{\partial}{\partial v^{\alpha}_{i_1\cdots i_{k+1}}}$$

By employing the operator defined in (9.3.18), we can express the recurrence relations (9.3.17) connecting isovector components in the form

$$V^{\alpha}_{i_1\cdots i_k i} = D_i^{(k)}(V^{\alpha}_{i_1\cdots i_k}) - v^{\alpha}_{i_1\cdots i_k j} D_i^{(k)}(X^j) = D_i^{(k)}(V^{\alpha}_{i_1\cdots i_k} - v^{\alpha}_{i_1\cdots i_k j}X^j)$$

where $0 \le k \le m-1$. By introducing the functions

$$F^{\alpha}_{i_1\cdots i_k} = V^{\alpha}_{i_1\cdots i_k} - v^{\alpha}_{i_1\cdots i_k j}X^j = \mathbf{i}_V(\sigma^{\alpha}_{i_1\cdots i_k}) \in \Lambda^0(\mathcal{C}_m)$$

we can also write

$$V^{\alpha}_{i_1\cdots i_k i} = D_i^{(k)}(F^{\alpha}_{i_1\cdots i_k}) = D_i^{(k)}\big(\mathbf{i}_V(\sigma^{\alpha}_{i_1\cdots i_k})\big). \tag{9.3.19}$$

Next, we define vector fields $V_G \in T(G)$ by

$$V_G = X^i(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial x^i} + U^{\alpha}(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial u^{\alpha}}. \tag{9.3.20}$$

Since $T(G)$ is a Lie algebra, these vectors generate a Lie group of diffeomorphisms mapping the manifold G onto itself. On the other hand, we know that isovectors of an ideal of the exterior algebra constitute a Lie subalgebra of the tangent bundle, or the module, of the manifold involved. We denote the Lie algebra of the isovectors of the contact ideal $\mathcal{I}_m$ by $\mathfrak{L}_{\mathcal{I}_m} \subset T(\mathcal{C}_m)$. A mere choice of $n+N$ smooth functions $X^i(\mathbf{x},\mathbf{u})$ and $U^\alpha(\mathbf{x},\mathbf{u})$ determines a *uniquely* defined member

$$V = V_G + \sum_{r=1}^{m} D_{i_r}^{(r-1)}(F^\alpha_{i_1\cdots i_{r-1}})\frac{\partial}{\partial v^\alpha_{i_1\cdots i_r}}$$

of the Lie algebra $\mathfrak{L}_{\mathcal{I}_m}$. Therefore, this expression can be regarded as the *lift* of a vector $V_G \in T(G)$ to a vector $V \in \mathfrak{L}_{\mathcal{I}_m} \subset T(\mathcal{C}_m)$. Since $\mathfrak{L}_{\mathcal{I}_m}$ is a Lie algebra, it generates a Lie group of diffeomorphisms on $\mathcal{C}_m$. It is evident that this group is a subgroup of the Lie group of diffeomorphisms on $\mathcal{C}_m$ generated by the Lie algebra $T(\mathcal{C}_m)$. But it is the only group preserving contact 1-forms. If we regard the manifold $\mathcal{C}_m$ as a fibre bundle on the base G, then the isovector V is called the ***mth order prolongation*** of the vector V_G. We adopt the notation $\mathfrak{L}_{\mathcal{I}_m} = \mathrm{pr}^{(m)}\big(T(G)\big)$. The rather complicated structures of prolongations are clearly illustrated in the two examples below:

$$\begin{aligned}
V_i^\alpha &= \frac{\partial U^\alpha}{\partial x^i} - v_j^\alpha\frac{\partial X^j}{\partial x^i} + v_i^\beta\frac{\partial U^\alpha}{\partial u^\beta} - v_j^\alpha v_i^\beta\frac{\partial X^j}{\partial u^\beta} \qquad (9.3.21)\\
V_{ij}^\alpha &= \frac{\partial V_i^\alpha}{\partial x^j} - v_{ik}^\alpha\frac{\partial X^k}{\partial x^j} + \Big(\frac{\partial V_i^\alpha}{\partial u^\beta} - v_{ik}^\alpha\frac{\partial X^k}{\partial u^\beta}\Big)v_j^\beta + v_{kj}^\beta\frac{\partial V_i^\alpha}{\partial v_k^\beta}\\
&= \frac{\partial^2 U^\alpha}{\partial x^i\partial x^j} - v_k^\alpha\frac{\partial^2 X^k}{\partial x^i\partial x^j} + v_i^\beta\frac{\partial^2 U^\alpha}{\partial x^j\partial u^\beta} + v_j^\beta\frac{\partial^2 U^\alpha}{\partial x^i\partial u^\beta} + v_{ij}^\beta\frac{\partial U^\alpha}{\partial u^\beta} -\\
&\quad v_{ik}^\alpha\frac{\partial X^k}{\partial x^j} - v_{jk}^\alpha\frac{\partial X^k}{\partial x^i} - v_i^\beta v_k^\alpha\frac{\partial^2 X^k}{\partial x^j\partial u^\beta} - v_j^\beta v_k^\alpha\frac{\partial^2 X^k}{\partial x^i\partial u^\beta} +\\
&\quad v_i^\gamma v_j^\beta\frac{\partial^2 U^\alpha}{\partial u^\gamma\partial u^\beta} - (v_j^\beta v_{ik}^\alpha + v_k^\alpha v_{ij}^\beta + v_i^\beta v_{jk}^\alpha)\frac{\partial X^k}{\partial u^\beta} - v_i^\beta v_j^\gamma v_k^\alpha\frac{\partial^2 X^k}{\partial u^\beta\partial u^\gamma}.
\end{aligned}$$

If we recall that the variables v_{ij}^α are symmetric with respect to the indices i,j , we observe in the above relation the components V_{ij}^α become symmetric with respect to the same indices as it should be.

(ii). We now assume that $N=1$, that is, there is only one dependent variable. If we write $v^1_{i_1\cdots i_r} = v_{i_1\cdots i_r}, r = 0,1,\dots,m,$ then the isovector field may be represented by

$$V = X^i\frac{\partial}{\partial x^i} + U\frac{\partial}{\partial u} + \sum_{r=1}^{m} V_{i_1\cdots i_r}\frac{\partial}{\partial v_{i_1\cdots i_r}}$$

where we denote $V^1_{i_1\cdots i_r} = V_{i_1\cdots i_r}$. In this case, the relation (9.3.14) becomes

$$\frac{\partial X^k}{\partial v_i} = \frac{\partial X^i}{\partial v_k}.$$

The solution of this set of equations is found to be

$$X^i = X^i(\mathbf{x}, u, \mathbf{v}) = -\frac{\partial F}{\partial v_i} \tag{9.3.22}$$

where $F = F(x^i, u, v_j)$ is an arbitrary smooth function of $2n+1$ variables. Due to this structure of functions X^k, equations (9.3.7) are then satisfied automatically. With $U^1 = U$, equations (9.3.13) lead to

$$\begin{aligned}\frac{\partial U}{\partial v_i} &= v_j \frac{\partial X^j}{\partial v_i} = -v_j \frac{\partial^2 F}{\partial v_j \partial v_i} \\ &= -\frac{\partial}{\partial v_i}\Big(v_j \frac{\partial F}{\partial v_j} - F\Big)\end{aligned}$$

The integration of the above differential equations involves an arbitrary function of variables x^i and u. Absorbing this function into the arbitrary function F, we obtain

$$U = U(\mathbf{x}, u, \mathbf{v}) = F - v_j \frac{\partial F}{\partial v_j}. \tag{9.3.23}$$

Other components of the isovector field are clearly given by the relations

$$V_{i_1\cdots i_k i} = D_i^{(k)}(V_{i_1\cdots i_k}) - v_{i_1\cdots i_k j} D_i^{(k)}(X^j) \tag{9.3.24}$$

where the operator $D_i^{(k)}$ of (9.3.18) should now be expressed as

$$D_i^{(k)} = \frac{\partial}{\partial x^i} + v_i \frac{\partial}{\partial u} + \sum_{r=1}^{k} v_{i_1\cdots i_r i} \frac{\partial}{\partial v_{i_1\cdots i_r}}$$

The recurrence relations (9.3.24) make it possible to determine all components of the isovector field uniquely once one chooses a smooth function $F(\mathbf{x}, u, \mathbf{v})$. The components X^i and U are then determined uniquely through the relations (9.3.22) and (9.3.23). The relation (9.3.24) may be explicitly expressed as

$$V_{i_1\cdots i_k i} = \frac{\partial V_{i_1\cdots i_k}}{\partial x^i} + v_{i_1\cdots i_k j} \frac{\partial^2 F}{\partial x^i \partial v_j} +$$

$$+\Big(\frac{\partial V_{i_1\cdots i_k}}{\partial u}+v_{i_1\cdots i_k j}\frac{\partial^2 F}{\partial u\partial v_j}\Big)v_i+v_{ik}v_{i_1\cdots i_k j}\frac{\partial^2 F}{\partial v_k\partial v_j} \tag{9.3.25}$$
$$+\sum_{r=1}^{k} v_{j_1\cdots j_r i}\frac{\partial V_{i_1\cdots i_k}}{\partial v_{j_1\cdots j_r}}, 0\le k\le m-1.$$

In general, the vector

$$V_G=X^i(\mathbf{x},u,\mathbf{v})\frac{\partial}{\partial x^i}+U(\mathbf{x},u,\mathbf{v})\frac{\partial}{\partial u}$$

is no longer dependent only the coordinates of the graph space. Hence, we cannot interpret the isovector field as a prolongation of a vector field V_G in $T(G)$. In order that an isovector is a prolongation of a member of $T(G)$, the functions X^i and U must be independent of variables $\mathbf{v}$. On the other hand, we easily see that in order to be able to obtain $X^i=\mathcal{X}^i(\mathbf{x},u)$, the equation $\partial F/\partial v_i = -\mathcal{X}^i(\mathbf{x},u)$ requires that the function F must have the form

$$F=-\mathcal{X}^i(\mathbf{x},u)v_i+G(\mathbf{x},u)$$

In such a case, (9.3.23) yields $U=G(\mathbf{x},u)$. Hence, *isovectors are found to be mth order prolongations of vectors $V_G\in T(G)$ if only F is an affine function of variables v_i*. Otherwise, isovectors may be interpreted as prolongations of the tangent bundle $T(\mathcal{C}_1)$ and one may then write $\mathfrak{I}_{\mathcal{I}_m}=\mathrm{pr}^{(m-1)}\big(T(\mathcal{C}_1)\big)$.

The structure of isovectors corresponding to the case $N=1$ might be illustrated to some extent by the following examples:

$$X^i=-\frac{\partial F}{\partial v_i},\quad U=F-v_i\frac{\partial F}{\partial v_i},\quad F=F(\mathbf{x},u,\mathbf{v}) \tag{9.3.26}$$
$$V_i=\frac{\partial F}{\partial x^i}+v_i\frac{\partial F}{\partial u}$$
$$V_{ij}=\frac{\partial^2 F}{\partial x^i\partial x^j}+v_i\frac{\partial^2 F}{\partial u\partial x^j}+v_j\frac{\partial^2 F}{\partial u\partial x^i}+v_{ik}\frac{\partial^2 F}{\partial v_k\partial x^j}+v_{jk}\frac{\partial^2 F}{\partial v_k\partial x^i}$$
$$+v_{ij}\frac{\partial F}{\partial u}+v_iv_j\frac{\partial^2 F}{\partial u^2}+(v_iv_{jk}+v_jv_{ik})\frac{\partial^2 F}{\partial v_k\partial u}+v_{ik}v_{jl}\frac{\partial^2 F}{\partial v_k\partial v_l}$$

We can collect the cases ***(i)*** and ***(ii)*** discussed above in the theorem below:

Theorem 9.3.1. *A vector field $V=X^i\partial/\partial x^i+\sum_{r=0}^{m}V^\alpha_{i_1\cdots i_r}\partial/\partial v^\alpha_{i_1\cdots i_r}$ of $T(\mathcal{C}_m)$ is an isovector field of the contact ideal $\mathcal{I}_m$ if and only if the* *relations* $V^{\alpha}_{i_1\cdots i_k i} = D^{(k)}_i(V^{\alpha}_{i_1\cdots i_k} - v^{\alpha}_{i_1\cdots i_k j}X^j)$ *for* $0 \le k \le m-1$ *are satisfied. The operators* $D^{(k)}_i$ *are given by* (9.3.18). *To determine isovector components completely one has to prescribe* $n+N$ *smooth functions* $X^i = X^i(\mathbf{x},\mathbf{u})$ *and* $U^{\alpha} = U^{\alpha}(\mathbf{x},\mathbf{u})$ *when* $N>1$, *whereas a single function* $F = F(\mathbf{x},u,\mathbf{v})$ *would be sufficient when* $N=1$ *through which the components* X^i *and* U *are found as* $X^i = -\partial F/\partial v_i, U = F - v_i(\partial F/\partial v_i)$. □

Since isovectors forming a Lie algebra produce groups of diffeomorphisms, we can state that this theorem is a somewhat generalised version of the celebrated Bäcklund theorem for $N>1$ [Swedish mathematician Albert Victor Bäcklund (1845-1929]: *The most general diffeomorphisms on* $\mathcal{C}_m$ *preserving the contact structure are prolongations of diffeomorphisms of the graph space.* Since this result restricts substantially admissible diffeomorphisms on $\mathcal{C}_m$, it creates a rather significant obstacle one has to surmount in determining solutions of partial differential equations by resorting to transformations preserving contact structures. We shall be able to overcome this obstacle later by choosing a more convenient ideal of $\Lambda(\mathcal{C}_m)$ instead of the contact ideal $\mathcal{I}_m$ [*see* Sec. 9.7].

The next step after having found isovector fields of the contact ideal would be to determine linearly independent isovector fields of the closed ideal generated by the given system of partial differential equations. Thus, it will become possible to obtain *Lie groups of symmetry transformations* that leave the system of partial differential equations invariant through which one can obtain families of new solutions from a given solution. However, this approach proves to be quite fruitful as far as the analytical procedures are concerned in balance equations derived from conservation laws. Since natural laws are generally of this form, many field equations encountered in physics and engineering fall naturally into this category. Thus, we can say that balance equations are come across most frequently in practical applications. This subject will be discussed in detail in the subsequent section. However, we shall try here to elucidate the approach that we use to employ in determining isovectors associated with a given system of first order partial differential equations through a somewhat complicated example.

Example 9.3.1. We consider the partial differential equations introduced in Example 8.7.3. The functions $u(x,t)$ and $c(x,t)$ satisfy the following first order partial differential equations

$$u_t + uu_x + \alpha cc_x = 0,\ c_t + uc_x + \frac{1}{\alpha}cu_x = 0,\ \ \alpha \neq 1$$

where $(x,t)\in\mathbb{R}^2$. The physical origins of these equations was also explained in that example. In order to simplify a little these equations, let us make the transformations

$$r = u + \alpha c, \quad s = u - \alpha c$$

to readily arrive at

$$r_t + \mathfrak{f}(r,s) r_x = 0, \quad s_t + \mathfrak{g}(r,s) s_x = 0$$

where the functions $\mathfrak{f}$ and $\mathfrak{g}$ are defined by

$$\mathfrak{f}(r,s) = \frac{\alpha+1}{2\alpha} r + \frac{\alpha-1}{2\alpha} s, \;\; \mathfrak{g}(r,s) = \frac{\alpha-1}{2\alpha} r + \frac{\alpha+1}{2\alpha} s.$$

We now introduce the forms $\omega_1, \omega_2 \in \Lambda^2(\mathbb{R}^4)$ as follows

$$\begin{aligned} \omega_1 &= -\,dr \wedge dx + \mathfrak{f}(r,s)\, dr \wedge dt, \\ \omega_2 &= -\,ds \wedge dx + \mathfrak{g}(r,s)\, ds \wedge dt. \end{aligned}$$

The coordinate cover of the manifold $G = \mathbb{R}^4$ is given by (x, t, r, s). If we define a solution mapping $\phi : \mathbb{R}^2 \to \mathbb{R}^4$ by relations $\big(x, t, r(x,t), s(x,t)\big)$, we then obtain

$$\begin{aligned} \phi^*\omega_1 &= \big[r_t + \mathfrak{f}(r,s) r_x\big] dx \wedge dt = 0, \\ \phi^*\omega_2 &= \big[s_t + \mathfrak{g}(r,s) s_x\big] dx \wedge dt = 0. \end{aligned}$$

Thus, the solution mapping ϕ annihilates the ideal generated by the forms ω_1 and ω_2. We can easily check that the exterior derivatives of the forms ω_1 and ω_2 are found to be as

$$\begin{aligned} d\omega_1 = -\,d\omega_2 &= -\,\frac{\alpha-1}{2\alpha} dr \wedge ds \wedge dt \\ &= \frac{\alpha-1}{2(r-s)} (ds \wedge \omega_1 + dr \wedge \omega_2). \end{aligned}$$

Hence, the ideal generated by the forms ω_1 and ω_2 is closed. Since the differential equations are of first order, we can just take the isovector field in the form below

$$V = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + R\frac{\partial}{\partial r} + S\frac{\partial}{\partial s}.$$

The components X, T, R, S are smooth functions of the variables x, t, r, s. In order that V becomes an isovector field we have to find smooth functions $\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}(x,t,r,s)$ so that the relations

$$£_V \omega_1 = \lambda_{11}\omega_1 + \lambda_{12}\omega_2, \;\; £_V \omega_2 = \lambda_{21}\omega_1 + \lambda_{22}\omega_2$$

are satisfied. If we employ the expressions

$$
\begin{aligned}
\mathbf{i}_V(\omega_1) &= -\,R dx + \mathfrak{f}(r,s)\, R dt + \big[X - \mathfrak{f}(r,s)\, T\big] dr \\
\mathbf{i}_V(\omega_2) &= -\,S dx + \mathfrak{g}(r,s)\, S dt + \big[X - \mathfrak{g}(r,s)\, T\big] ds \\
\mathbf{i}_V(d\omega_1) &= -\,\mathbf{i}_V(d\omega_2) = -\,\frac{\alpha-1}{2\alpha}(R ds\wedge dt - S dr\wedge dt + T dr\wedge ds)
\end{aligned}
$$

in the Cartan magic formula, we obtain

$$
\begin{aligned}
&\pounds_V\omega_1 = \big[R_t + \mathfrak{f}(r,s)R_x\big] dx\wedge dt - \big[R_r + X_x - \mathfrak{f}(r,s)T_x\big] dr\wedge dx \\
&-R_s\, ds\wedge dx + \big[\frac{\alpha+1}{2\alpha}R + \frac{\alpha-1}{2\alpha}S + \mathfrak{f}(r,s)(R_r + T_t) - X_t\big] dr\wedge dt \\
&\qquad + \mathfrak{f}(r,s)R_s ds\wedge dt + \big[X_s - \mathfrak{f}(r,s)T_s\big] ds\wedge dr = \\
&-\lambda_{11} dr\wedge dx + \lambda_{11}\mathfrak{f}(r,s)\, dr\wedge dt - \lambda_{12} ds\wedge dx + \lambda_{12}\mathfrak{g}(r,s) ds\wedge dt \\
&\pounds_V\omega_2 = \big[S_t + \mathfrak{g}(r,s)S_x\big] dx\wedge dt - \big[S_s + X_x - \mathfrak{g}(r,s)T_x\big] ds\wedge dx \\
&-S_r\, dr\wedge dx + \big[\frac{\alpha-1}{2\alpha}R + \frac{\alpha+1}{2\alpha}S + \mathfrak{g}(r,s)(S_s + T_t) - X_t\big] ds\wedge dt \\
&\qquad + \big[X_r - \mathfrak{g}(r,s)T_r\big] dr\wedge ds + \mathfrak{g}(r,s)S_r dr\wedge dt = \\
&-\lambda_{21} dr\wedge dx + \lambda_{21}\mathfrak{f}(r,s)\, dr\wedge dt - \lambda_{22} ds\wedge dx + \lambda_{22}\mathfrak{g}(r,s) ds\wedge dt
\end{aligned}
$$

from which we extract the following system of equations

$$
\begin{aligned}
&\lambda_{11} = R_r + X_x - \mathfrak{f}(r,s)T_x, \qquad \lambda_{12} = R_s, \\
&\lambda_{21} = S_r\,, \qquad \lambda_{22} = S_s + X_x - \mathfrak{g}(r,s)T_x \\
&R_t + \mathfrak{f}(r,s)R_x = 0, \quad \frac{1}{\alpha}(r-s)R_s = 0, \quad X_s - \mathfrak{f}(r,s)T_s = 0 \\
&\frac{\alpha+1}{2\alpha}R + \frac{\alpha-1}{2\alpha}S - X_t + \mathfrak{f}(r,s)\big[-X_x + T_t + \mathfrak{f}(r,s)T_x\big] = 0 \\
&S_t + \mathfrak{g}(r,s)S_x = 0, \ -\frac{1}{\alpha}(r-s)S_r = 0, \ X_r - \mathfrak{g}(r,s)T_r = 0 \\
&\frac{\alpha-1}{2\alpha}R + \frac{\alpha+1}{2\alpha}S - X_t + \mathfrak{g}(r,s)\big[-X_x + T_t + \mathfrak{g}(r,s)T_x\big] = 0
\end{aligned}
\tag{9.3.27}
$$

where we have noted that

$$\mathfrak{f}(r,s) - \mathfrak{g}(r,s) = \frac{r-s}{\alpha}.$$

Equations $(9.3.27)_6$ and $(9.3.27)_{10}$ yield obviously

$$R = R(x,t,r), \quad S = S(x,t,s).$$

Next, let us differentiate $(9.3.27)_5$ and $(9.3.27)_9$ with respect to s and r to find, respectively

$$\frac{\alpha-1}{2\alpha}R_x=0,\quad \frac{\alpha-1}{2\alpha}S_x=0$$

and, consequently, $R_t=0$ and $S_t=0$. We thus get

$$R=R(r),\quad S=S(s).$$

On writing the equations $(9.3.27)_7$ and $(9.3.27)_{11}$ in the form

$$\big[X-\mathfrak{f}(r,s)T\big]_s+\frac{\alpha-1}{2\alpha}T=0,\ \ \big[X-\mathfrak{g}(r,s)T\big]_r+\frac{\alpha-1}{2\alpha}T=0,$$

we obtain

$$\big[X-\mathfrak{f}(r,s)T\big]_s=\big[X-\mathfrak{g}(r,s)T\big]_r.$$

This expression implies that the following relations are obtainable

$$\begin{aligned}X-\mathfrak{f}(r,s)T&=\Phi_r,\qquad X-\mathfrak{g}(r,s)T=\Phi_s,\\ T=&-\frac{2\alpha}{\alpha-1}\Phi_{rs}=\frac{\alpha(\Phi_s-\Phi_r)}{r-s}\end{aligned}\tag{9.3.28}$$

where $\Phi=\Phi(x,t,r,s)$. Hence, the function Φ must satisfy the partial differential equation

$$2(r-s)\Phi_{rs}+(\alpha-1)(\Phi_s-\Phi_r)=0.\tag{9.3.29}$$

It follows from $(9.3.27)_8$ and $(9.3.27)_{12}$ that

$$\begin{aligned}R(r)&=X_t+rX_x-rT_t-\frac{1}{4}\Big[(3r-s)(r+s)+\frac{(r-s)^2}{\alpha^2}\Big]T_x,\\ S(s)&=X_t+sX_x-sT_t+\frac{1}{4}\Big[(r-3s)(r+s)-\frac{(r-s)^2}{\alpha^2}\Big]T_x.\end{aligned}$$

By adding the first two expressions in (9.3.28) and using the third one we obtain

$$X=\frac{\big[(\alpha+1)r+(\alpha-1)s\big]\Phi_s-\big[(\alpha-1)r+(\alpha+1)s\big]\Phi_r}{2(r-s)}.$$

Inserting this expression for X together with $(9.3.28)_3$ into $R(r)$ and $S(s)$ given above, we find that

$$\begin{aligned}2R(r)&=-(\alpha-1)\big(\Phi_{ts}+\mathfrak{g}(r,s)\Phi_{xs}\big)+(\alpha+1)\big(\Phi_{tr}+\mathfrak{f}(r,s)\Phi_{xr}\big),\\ 2S(s)&=(\alpha+1)\big(\Phi_{ts}+\mathfrak{g}(r,s)\Phi_{xs}\big)-(\alpha-1)\big(\Phi_{tr}+\mathfrak{f}(r,s)\Phi_{xr}\big).\end{aligned}$$

This result means that the derivatives of the right hand side of the first equation with respect to variables x, t, s, and the derivatives of the right hand side of the second equation with respect to variables x, t, r must vanish. The derivatives with respect to t give

$$\begin{aligned} -(\alpha-1)\big(\Phi_{tts}+\mathfrak{g}(r,s)\Phi_{xts}\big)+(\alpha+1)\big(\Phi_{ttr}+\mathfrak{f}(r,s)\Phi_{xtr}\big)&=0,\\ (\alpha+1)\big(\Phi_{tts}+\mathfrak{g}(r,s)\Phi_{xts}\big)-(\alpha-1)\big(\Phi_{ttr}+\mathfrak{f}(r,s)\Phi_{xtr}\big)&=0 \end{aligned}$$

whence we deduce that

$$\Phi_{trt}+\mathfrak{f}(r,s)\Phi_{xrt}=0,\quad \Phi_{tst}+\mathfrak{g}(r,s)\Phi_{xst}=0$$

since $\alpha \neq 0$. So we can write

$$\Phi_{tr}+\mathfrak{f}(r,s)\Phi_{xr}=\mathfrak{f}A_x(x,r,s),\ \ \Phi_{ts}+\mathfrak{g}(r,s)\Phi_{xs}=\mathfrak{g}B_x(x,r,s)$$

where A and B are arbitrary functions. We then easily obtain

$$\Phi_r=A(x,r,s)+\phi(\xi,r,s),\ \ \Phi_s=B(x,r,s)+\psi(\eta,r,s)$$

where the characteristic variables are

$$\xi=x-\mathfrak{f}(r,s)t,\quad \eta=x-\mathfrak{g}(r,s)t.$$

Similarly, the following equations must hold

$$\Phi_{trx}+\mathfrak{f}(r,s)\Phi_{xrx}=0,\quad \Phi_{tsx}+\mathfrak{g}(r,s)\Phi_{xsx}=0$$

from which we get

$$A_{xx}=0,\ \ B_{xx}=0.$$

We thus conclude that

$$A(x,r,s)=a(r,s)x+b(r,s),\ \ B(x,r,s)=c(r,s)x+d(r,s).$$

Functions A, B, ϕ, ψ must satisfy the compatibility condition $\Phi_{rs}=\Phi_{sr}$, that is, the following equation must hold

$$(a_s-c_r)x-\frac{\alpha-1}{2\alpha}(\phi_\xi-\psi_\eta)t+b_s-d_r+\phi_s-\psi_r=0.$$

If we calculate the variables x and t in terms of ξ and η, and insert them into the above equation, we obtain

$$\begin{aligned} \frac{\alpha(\eta\,\mathfrak{f}-\xi\mathfrak{g})}{r-s}(a_s-c_r)-\frac{(\alpha-1)(\eta-\xi)}{2(r-s)}(\phi_\xi-\psi_\eta)&\\ +b_s-d_r+\phi_s-\psi_r&=0. \end{aligned} \tag{9.3.30}$$

On differentiating this expression successively with respect to variables ξ and η, we find that

$$-\frac{\alpha-1}{2(r-s)}(\phi_{\xi\xi}+\psi_{\eta\eta})=0.$$

We thus have to take

$$\phi_{\xi\xi}(\xi,r,s)=-\psi_{\eta\eta}(\eta,r,s)=2k(r,s)$$

whence we deduce that

$$\begin{aligned}\phi(\xi,r,s)&=k(r,s)\xi^2+m(r,s)\xi+n(r,s),\\ \psi(\eta,r,s)&=-k(r,s)\eta^2+p(r,s)\eta+q(r,s).\end{aligned}$$

If we introduce these functions into (9.3.30) and arrange the resulting terms, we then get the following polynomial in ξ and η

$$\begin{aligned}&\Big[k_s+\frac{\alpha-1}{r-s}k\Big]\xi^2+\Big[k_r-\frac{\alpha-1}{r-s}k\Big]\eta^2+\frac{1}{2(r-s)}\Big[(\alpha-1)(m-p)\\ &\quad-\big[(\alpha-1)r+(\alpha+1)s\big](a_s-c_r)+2(r-s)m_s\Big]\xi\\ &\quad+\frac{1}{2(r-s)}\Big[-(\alpha-1)(m-p)+\big[(\alpha+1)r+(\alpha-1)s\big](a_s-c_r)\\ &\quad-2(r-s)p_r\Big]\eta+(b+n)_s-(d+q)_r=0.\end{aligned}$$

The coefficients above must be zero so that we obtain

$$k(r,s)=\overline{c}_1(r-s)^{\alpha-1},\ (a+m)_s=(c+p)_r,\ (b+n)_s=(d+q)_r.$$

Therefore, we can write

$$m=\omega_r-a,\ \ p=\omega_s-c,\ n=\Omega_r-b,\ q=\Omega_s-d$$

where $\omega=\omega(r,s),\Omega=\Omega(r,s)$. Thus the only equation to be satisfied is

$$\begin{aligned}(\alpha-1)(a-c)+2\alpha\mathfrak{f}a_s-2\alpha\mathfrak{g}c_r+(\alpha-1)(\omega_s-\omega_r)&\\ -2(r-s)\omega_{rs}=0&\end{aligned}\tag{9.3.31}$$

so that we arrive at the result

$$\begin{aligned}\Phi_r&=\overline{c}_1(r-s)^{\alpha-1}(x-\mathfrak{f}t)^2+\omega_r x-\mathfrak{f}(\omega_r-a)t+\Omega_r,\\ \Phi_s&=-\overline{c}_1(r-s)^{\alpha-1}(x-\mathfrak{g}t)^2+\omega_s x-\mathfrak{g}(\omega_s-c)t+\Omega_s.\end{aligned}$$

When we insert these relations into (9.3.29), we see first that we have to take $\overline{c}_1=0$ and the remaining terms give rise to the following equations

$$\begin{aligned}&2(r-s)\omega_{rs}+(\alpha-1)(\omega_s-\omega_r)=0, \\ &2(r-s)\Omega_{rs}+(\alpha-1)(\Omega_s-\Omega_r)=0,\\ &(\alpha-1)\mathfrak{g}(a-c)+(\alpha-1)\mathfrak{g}(\omega_s-\omega_r)-2(r-s)\mathfrak{f}(a_s-\omega_{rs})=0.\end{aligned} \tag{9.3.32}$$

On the other hand, we can now write

$$\begin{aligned}2R(r)&=-(\alpha-1)\mathfrak{g}c+(\alpha+1)\mathfrak{f}a,\\ 2S(s)&=(\alpha+1)\mathfrak{g}c-(\alpha-1)\mathfrak{f}a.\end{aligned}$$

Because of the relations $R_s=S_r=0$, the equations below must be held

$$-(\alpha-1)(\mathfrak{g}c)_s+(\alpha+1)(\mathfrak{f}a)_s=0,\ (\alpha+1)(\mathfrak{g}c)_r-(\alpha-1)(\mathfrak{f}a)_r=0.$$

If we differentiate the first equation with respect to r and the second one with respect to s, we find that

$$(\mathfrak{f}a)_{rs}=0,\ \ (\mathfrak{g}c)_{rs}=0$$

whence we obtain

$$\mathfrak{f}a=\lambda(r)+\mu(s),\mathfrak{g}c=\frac{\alpha-1}{\alpha+1}\lambda(r)+\frac{\alpha+1}{\alpha-1}\mu(s)+2c_1 \tag{9.3.33}$$

and

$$\begin{aligned}R(r)&=\frac{2\alpha}{\alpha+1}\lambda(r)-(\alpha-1)c_1,\\ S(s)&=\frac{2\alpha}{\alpha+1}\mu(s)+(\alpha+1)c_1.\end{aligned}$$

If we insert the expressions (9.3.33) into equations (9.3.31) and $(9.3.32)_3$, solve the resulting expressions for $(\alpha-1)(\omega_s-\omega_r)$ and $2(r-s)\omega_{rs}$ and put them into the equation $(9.3.32)_1$ we reach to the equation

$$\begin{aligned}-2(\alpha-1)\lambda+2(\alpha+1)\mu+(r-s)\big[(\alpha-1)\lambda'+(\alpha+1)\mu'\big]&\\ +2(\alpha^2-1)c_1=0.&\end{aligned} \tag{9.3.34}$$

Differentiating (9.3.34) successively with respect to r and s, we are led to

$$-(\alpha-1)\lambda''(r)+(\alpha+1)\mu''(s)=0$$

from which we find

$$\lambda(r)=c_2r^2+c_3r+c_4,\quad \mu(s)=\frac{\alpha-1}{\alpha+1}c_2s^2+c_5s+c_6$$

On inserting these expressions into (9.3.34) we obtain

$$\lambda(r) = c_2 r^2 + c_3 r + c_4,$$
$$\mu(s) = \frac{\alpha-1}{\alpha+1}c_2 s^2 + \frac{\alpha-1}{\alpha+1}c_3 s + \frac{\alpha-1}{\alpha+1}c_4 - (\alpha-1)c_1.$$

If we employ these relations in (9.3.33) and, (9.3.31) and $(9.3.32)_3$, we come up with the relations

$$\omega_{rs} = -\frac{\alpha(\alpha-1)}{\alpha+1}c_2, \;\; \omega_s - \omega_r = \frac{2\alpha}{\alpha+1}c_2(r-s).$$

Integration of the first equation yields

$$\omega = -\frac{\alpha(\alpha-1)}{\alpha+1}c_2 rs + m(r) + n(s)$$

while the second equation then results in

$$\alpha c_2(r-s) + m'(r) - n'(s) = 0.$$

The solution of this equation is easily found as

$$m(r) = -\frac{1}{2}\alpha c_2 r^2 + c_5 r + c_6,$$
$$n(s) = -\frac{1}{2}\alpha c_2 s^2 + c_5 s + c_7.$$

Hence, by replacing the arbitrary constant $c_6 + c_7$ by c_6, we get

$$\omega(r,s) = -\frac{\alpha(\alpha-1)}{\alpha+1}c_2 rs - \frac{1}{2}\alpha c_2(r^2+s^2) + c_5(r+s) + c_6.$$

On making use of these expressions where they are pertinent and defining new arbitrary constants as appropriate combinations of old constant, we ultimately obtain isovector components depending on constants a_1, a_2, a_3, a_4 and a function $\Omega(r,s)$, being a solution of the partial differential equation $(9.3.32)_2$, as follows

$$X = a_4 x + \{a_3 - a_1(\alpha+1)[\alpha(r+s)^2 - (r-s)^2]\}t + \frac{\alpha[\mathfrak{f}(r,s)\Omega_s - \mathfrak{g}(r,s)\Omega_r]}{r-s}, \tag{9.3.35}$$
$$T = 4\alpha^2 a_1 x - [a_2 - a_4 + 4\alpha(\alpha+1)a_1(r+s)]t + \frac{\alpha(\Omega_s - \Omega_r)}{r-s},$$
$$R = 4\alpha a_1 r^2 + a_2 r + a_3, \qquad S = 4\alpha a_1 s^2 + a_2 s + a_3.$$

Therefore, the linearly independent isovectors are given by

$$
\begin{aligned}
V_1 =& -(\alpha+1)\big[\alpha(r+s)^2-(r-s)^2\big]t\frac{\partial}{\partial x}\\
&+4\alpha\big[\alpha x-(\alpha+1)(r+s)t\big]\frac{\partial}{\partial t}+4\alpha r^2\frac{\partial}{\partial r}+4\alpha s^2\frac{\partial}{\partial s},\\
V_2 =& -t\frac{\partial}{\partial t}+r\frac{\partial}{\partial r}+s\frac{\partial}{\partial s},\ V_3=t\frac{\partial}{\partial x}+\frac{\partial}{\partial r}+\frac{\partial}{\partial s},\ V_4=x\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}\\
V_\Omega =& \frac{\alpha}{r-s}\Big[\big[\mathfrak{f}(r,s)\Omega_s-\mathfrak{g}(r,s)\Omega_r\big]\frac{\partial}{\partial x}+(\Omega_s-\Omega_r)\frac{\partial}{\partial t}\Big]
\end{aligned}
$$

To determine the symmetry groups we have to solve the following autonomous ordinary differential equations

$$\frac{d\overline{x}}{d\epsilon}=X(\overline{x},\overline{t},\overline{r},\overline{s}),\ \frac{d\overline{t}}{d\epsilon}=T(\overline{x},\overline{t},\overline{r},\overline{s}),\ \frac{d\overline{r}}{d\epsilon}=R(\overline{r}),\ \frac{d\overline{s}}{d\epsilon}=S(\overline{s})$$

under the initial conditions $\overline{x}(0)=x,\ \overline{t}(0)=t,\ \overline{r}(0)=r$ and $\overline{s}(0)=s$ where ϵ is taken as the flow parameter. Hence, the one-parameter Lie group generated by the isovector field V_1 becomes

$$
\begin{aligned}
\overline{x}(\epsilon) =& \big[(4\alpha r\epsilon-1)(4\alpha s\epsilon-1)\big]^{\frac{\alpha-1}{2}}\big[\big(2\alpha(\alpha-1)(r+s)\epsilon-16\alpha^3rs\epsilon^2\\
&+1\big)x+\big((r-s)^2-\alpha^2(r+s)^2-4\alpha rs+8\alpha^2(\alpha+1)rs(r+s)\epsilon\big)\epsilon t\big]\\
\overline{t}(\epsilon) =& \big[(4\alpha r\epsilon-1)(4\alpha s\epsilon-1)\big]^{\frac{\alpha-1}{2}}\big[4\alpha^2x\epsilon+\big(1-2\alpha(\alpha+1)(r+s)\epsilon\big)t\big]\\
\overline{r}(\epsilon) =& -r/(4\alpha r\epsilon-1),\qquad \overline{s}(\epsilon)=-s/(4\alpha s\epsilon-1)
\end{aligned}
$$

Similarly, the isovector field V_2 leads up to the Lie group

$$\overline{x}(\epsilon)=x,\ \overline{t}(\epsilon)=t\,e^{-\epsilon},\ \overline{r}(\epsilon)=r\,e^{\epsilon},\ \overline{s}(\epsilon)=s\,e^{\epsilon},$$

the isovector field V_3 to the Lie group

$$\overline{x}(\epsilon)=x+\epsilon t,\ \overline{t}(\epsilon)=t\,,\ \overline{r}(\epsilon)=r+\epsilon,\ \overline{s}(\epsilon)=s+\epsilon,$$

and the isovector field V_4 to the Lie group

$$\overline{x}(\epsilon)=x\,e^{\epsilon},\ \overline{t}(\epsilon)=t\,e^{\epsilon},\ \overline{r}(\epsilon)=r,\ \overline{s}(\epsilon)=s.$$

On the other hand, the function $\Omega(r,s)$ satisfying $(9.3.32)_2$ generates the Lie group

$$
\begin{aligned}
\overline{x}(\epsilon) &= x+\frac{\alpha}{r-s}\big[\mathfrak{f}(r,s)\Omega_s-\mathfrak{g}(r,s)\Omega_r\big]\epsilon,\\
\overline{t}(\epsilon) &= t+\frac{\alpha}{r-s}(\Omega_s-\Omega_r)\epsilon,\ \overline{r}(\epsilon)=r,\ \overline{s}(\epsilon)=s.
\end{aligned}
$$

If we wish to pass to the physically meaningful dependent variables (u, c), then the isovector field should be depicted by

$$V = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + C\frac{\partial}{\partial c}.$$

If we take into account the relations

$$\frac{\partial}{\partial r} = \frac{1}{2}\Big(\frac{\partial}{\partial u} + \frac{1}{\alpha}\frac{\partial}{\partial c}\Big), \quad \frac{\partial}{\partial s} = \frac{1}{2}\Big(\frac{\partial}{\partial u} - \frac{1}{\alpha}\frac{\partial}{\partial c}\Big)$$

we readily obtain

$$U(u,c) = \frac{R(u+\alpha c) + S(u-\alpha c)}{2},$$
$$C(u,c) = \frac{R(u+\alpha c) - S(u-\alpha c)}{2\alpha}.$$

Thus, it follows from (9.3.35) that

$$\begin{aligned} X &= a_4 x + \big[a_3 - 4\alpha(\alpha+1)a_1(u^2 - \alpha c^2)\big]t + 2\Omega_u - \frac{2u}{\alpha c}\Omega_c, \\ T &= 4\alpha^2 a_1 x - \big[a_2 - a_4 + 8\alpha(\alpha+1)a_1 u\big]t - \frac{1}{2\alpha c}\Omega_c, \\ U &= 4\alpha a_1(u^2 + \alpha^2 c^2) + a_2 u + a_3, \\ C &= 8\alpha a_1 uc + a_2 c \end{aligned}$$

where the function $\Omega(u, c)$ has now to be taken as a solution of the partial differential equation

$$\alpha^2\Omega_{uu} - \Omega_{cc} - \frac{\alpha - 1}{c}\Omega_c = 0.$$

Hence, the linearly independent isovectors become

$$\begin{aligned} V_1 = &-4\alpha(\alpha+1)(u^2 - \alpha c^2)t\frac{\partial}{\partial x} + 4\alpha\big[\alpha x - 2(\alpha+1)ut\big]\frac{\partial}{\partial t} \\ &+ 4\alpha(u^2 + \alpha^2 c^2)\frac{\partial}{\partial u} + 8\alpha uc\frac{\partial}{\partial c}, \\ V_2 = &-t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + c\frac{\partial}{\partial c}, \ V_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \ V_4 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}, \\ V_\Omega = 2&\Big(\Omega_u - \frac{u}{\alpha c}\Omega_c\Big)\frac{\partial}{\partial x} - \frac{1}{2\alpha c}\Omega_c\frac{\partial}{\partial t}\Big]. \end{aligned}$$

It is easily seen that the Lie transformation group generated by the isovector V_1 is now given by

$$\overline{x}(\epsilon) = \big[1 - 8\alpha u\epsilon + 16\alpha^2(u^2 - \alpha^2 c^2)\epsilon^2\big]^{\frac{\alpha-1}{2}} \big[\{1 + 4\alpha[(\alpha-1)u - \alpha^2(u^2 - \alpha^2 c^2)\epsilon 1\big)x - 4\alpha(\alpha+1)\{(u^2 - \alpha c^2) - 4\alpha u(u+\alpha c)^2\epsilon\}\epsilon t\big],$$
$$\overline{t}(\epsilon) = \big[1 - 8\alpha u\epsilon + 16\alpha^2(u^2 - \alpha^2 c^2)\epsilon^2\big]^{\frac{\alpha-1}{2}} \big[4\alpha^2 x\epsilon + \{1 - 4\alpha(\alpha-1)u\epsilon\}t\big],$$
$$\overline{u}(\epsilon) = \frac{u - 4\alpha(u^2 - \alpha^2 c^2)\epsilon}{1 - 8\alpha u\epsilon + 16\alpha^2(u^2 - \alpha^2 c^2)\epsilon^2},$$
$$\overline{c}(\epsilon) = \frac{c}{1 - 8\alpha u\epsilon + 16\alpha^2(u^2 - \alpha^2 c^2)\epsilon^2}.$$

Similarly, the isovector V_2 gives rise to the Lie group

$$\overline{x}(\epsilon) = x,\ \overline{t}(\epsilon) = t\, e^{-\epsilon},\ \overline{u}(\epsilon) = u\, e^{\epsilon},\ \overline{c}(\epsilon) = c\, e^{\epsilon},$$

the isovector V_3 to the Lie group

$$\overline{x}(\epsilon) = x + \epsilon t,\ \overline{t}(\epsilon) = t\,,\ \overline{u}(\epsilon) = u + \epsilon,\ \overline{c}(\epsilon) = c\,,$$

and the isovector V_4 to the Lie group

$$\overline{x}(\epsilon) = x\, e^{\epsilon},\ \overline{t}(\epsilon) = t\, e^{\epsilon},\ \overline{u}(\epsilon) = u,\ \overline{c}(\epsilon) = c.$$

The function $\Omega(u, c)$ generates the Lie group

$$\overline{x}(\epsilon) = x + 2\Big(\Omega_u - \frac{u}{\alpha c}\Omega_c\Big)\epsilon,$$
$$\overline{t}(\epsilon) = t - \frac{1}{2\alpha c}\Omega_c\epsilon, \quad \overline{u}(\epsilon) = u, \quad \overline{c}(\epsilon) = c.$$ ■

9.4. ISOVECTOR FIELDS OF BALANCE IDEALS

Before dealing with partial differential equations in the form of general balance equations, we would like first to consider the system of non-linear partial differential equations given by (9.2.1). This time we shall represent this system via n-forms

$$\omega^a = F^a \mu \in \Lambda^n(\mathcal{C}_m),\ a = 1, \dots, A$$

defined on the mth order contact manifold $\mathcal{C}_m$. The volume form of the manifold M is the n-form $\mu = dx^1 \wedge \cdots \wedge dx^n$. We shall also need the forms $\mu_i = \mathbf{i}_{\partial_i}(\mu) \in \Lambda^{n-1}(M)$. The reason why we use n-forms in association with the field equations instead of 0-forms as before is that they happen to be more beneficial in determining isovector fields. The regular solution mapping $\phi : \mathcal{D}_n \to \mathcal{C}_m$ introduced on p. 489 gives rise to the relation

$$\phi^*\omega^a = (\phi^* F^a)(\phi^*\mu) = 0$$

since $\phi^*\mu \neq 0$ and $\phi^* F^a = 0$. Thus, it annihilates the forms ω^a. Let us now consider the *fundamental ideal*

$$\mathfrak{I}_m = \mathcal{I}(\sigma^\alpha, \sigma^\alpha_{i_1}, \sigma^\alpha_{i_1 i_2}, \ldots, \sigma^\alpha_{i_1 i_2 \cdots i_{m-1}}; d\sigma^\alpha_{i_1 i_2 \cdots i_{m-1}}; \omega^a)$$

This ideal is closed. Indeed, if we make use of the definitions (9.2.5), we obtain

$$\begin{aligned} d\omega^a = dF^a \wedge \mu &= \Big(\frac{\partial F^a}{\partial x^i}\, dx^i + \sum_{r=0}^{m} \frac{\partial F^a}{\partial v^\alpha_{i_1 \cdots i_r}}\, dv^\alpha_{i_1 \cdots i_r}\Big) \wedge \mu \\ &= \Big[\Big(\frac{\partial F^a}{\partial x^i} + \sum_{r=0}^{m-1} \frac{\partial F^a}{\partial v^\alpha_{i_1 \cdots i_r}} v^\alpha_{i_1 \cdots i_r i}\Big)\, dx^i + \sum_{r=0}^{m-1} \frac{\partial F^a}{\partial v^\alpha_{i_1 \cdots i_r}}\, \sigma^\alpha_{i_1 \cdots i_r}\Big] \wedge \mu \\ &\qquad\qquad - \frac{\partial F^a}{\partial v^\alpha_{i_1 \cdots i_m}}\, d\sigma^\alpha_{i_1 \cdots i_{m-1}} \wedge \mu_{i_m} \\ &= (-1)^n \sum_{r=0}^{m-1} \frac{\partial F^a}{\partial v^\alpha_{i_1 \cdots i_r}} \mu \wedge \sigma^\alpha_{i_1 \cdots i_r} - \frac{\partial F^a}{\partial v^\alpha_{i_1 \cdots i_m}} \mu_{i_m} \wedge d\sigma^\alpha_{i_1 \cdots i_{m-1}} \in \mathfrak{I}_m \end{aligned}$$

where we have utilised the known relations $dx^i \wedge \mu = 0$, $dx^j \wedge \mu_i = \delta^j_i \mu$ and $dv^\alpha_{i_1 \cdots i_{m-1} i_m} \wedge \mu = dv^\alpha_{i_1 \cdots i_{m-1} i} \wedge dx^i \wedge \mu_{i_m} = -\, d\sigma^\alpha_{i_1 \cdots i_{m-1}} \wedge \mu_{i_m}$. In order to determine the isovector fields of this ideal, we first consider the isovector field V of the contact ideal whose general structure has been fully revealed in Sec. 9.3. According to Theorem 5.12.5, we have to impose further the condition $\pounds_V \omega^a \in \mathfrak{I}_m$ on this vector. If we evaluate the Lie derivative of ω^a by noting the relations

$$\mathbf{i}_V(\omega^a) = F^a X^i \mu_i, \;\; \mathbf{i}_V(d\omega^a) = V(F^a)\mu - X^i dF^a \wedge \mu_i$$

we thus conclude that the following conditions should be satisfied

$$\begin{aligned} \pounds_V \omega^a &= V(F^a)\mu + F^a dX^i \wedge \mu_i \\ &= \lambda^a_b F^b \mu + \sum_{r=0}^{m-1} \lambda^{a i_1 \cdots i_r}_\alpha \wedge \sigma^\alpha_{i_1 \cdots i_r} + \Lambda^{a i_1 \cdots i_{m-1}}_\alpha \wedge d\sigma^\alpha_{i_1 \cdots i_{m-1}}. \end{aligned}$$

But, we must of course show that it is possible to find forms $\lambda^a_b \in \Lambda^0(\mathcal{C}_m)$, $\lambda^{a i_1 \cdots i_r}_\alpha \in \Lambda^{n-1}(\mathcal{C}_m), r = 0, 1, \ldots, m, \Lambda^{a i_1 \cdots i_{m-1}}_\alpha \in \Lambda^{n-2}(\mathcal{C}_m)$ satisfying the above relations. If we recall that $X^i = X^i(\mathbf{x}, \mathbf{u})$ if $N > 1$, the foregoing expressions require that we have to take

$$\lambda^a_\alpha = (-1)^{n-1} F^a \frac{\partial X^i}{\partial u^\alpha}\mu_i;\ \lambda^{ai_1\cdots i_r}_\alpha = 0, 1 \leq r \leq m-1;\ \Lambda^{ai_1\cdots i_{m\text{-}1}}_\alpha = 0$$

and the coefficient of μ there yields

$$V(F^a) + \Big[\Big(\frac{\partial X^i}{\partial x^i} + \frac{\partial X^i}{\partial u^\alpha} v^\alpha_i\Big)\delta^a_b - \lambda^a_b\Big] F^b = 0.$$

However, we know that the system of differential equations has to comply with the conditions $\phi^* F^a = 0$. Hence, the isovector components must satisfy the relations

$$\phi^* V(F^a) = \phi^* \Big[\frac{\partial F^a}{\partial x^i} X^i + \sum_{r=0}^{m} \frac{\partial F^a}{\partial v^\alpha_{i_1\cdots i_r}} V^\alpha_{i_1\cdots i_r}\Big] = 0,\ \phi^* F^a = 0$$

for $1 \leq a \leq A$. The functions $X^i(\mathbf{x}, \mathbf{u})$ and $U^\alpha(\mathbf{x}, \mathbf{u})$ determining completely the isovector components can be found in principle from the above equations. These equations are exactly the same as the determining equations for infinitesimal generators of Lie symmetry groups obtained by the classical approach [*see* Olver (1986), Ch. 2]. Consequently, it is not possible to get useful information about isovector components without knowing explicitly the structure of functions F^a. The case $N = 1$ can likewise be discussed in a similar manner.

On the other hand, when partial differential equations are of balance type we can attain to much more feasible results than those obtained above. An $(m+1)$th order balance equations with n independent variables x^i and N dependent variables u^α are specified by

$$\frac{\partial \Sigma^{\alpha i}}{\partial x^i} + \Sigma^\alpha = 0,\ i = 1, 2, \dots, n;\ \alpha = 1, 2, \dots, N \qquad (9.4.1)$$

where $\Sigma^{\alpha i}$ and Σ^α are smooth functions of variables x^i, u^α and partial derivatives $u^\alpha_{,i}, u^\alpha_{,ij}, \dots, u^\alpha_{,i_1 i_2\cdots i_m}$ of functions $u^\alpha = u^\alpha(x^i)$ up to and including mth order. Because of the physical significance, we shall assume that the number of equations are equal to the number of unknowns. However, methods that we shall explore fully in this section and some of subsequent sections will be equally applicable to a case in which the number of equations differs from the number of unknowns, that is, to balance equations in the form

$$\frac{\partial \Sigma^{ai}}{\partial x^i} + \Sigma^a = 0,\ a = 1, 2, \dots, A.$$

As we have mentioned earlier, we suppose that the differential equations are defined on an open set $\mathcal{D}_n \subseteq \mathbb{R}^n$. If we integrate equations (9.4.1) on the region $\mathcal{D}_n$ whose exterior unit normal is **n** and make use of the divergence theorem, we obtain the following integral relation

$$\int_{\partial\mathcal{D}_n} \Sigma^{ai} n_i dS = -\int_{\mathcal{D}_n} \Sigma^a dV.$$

We call $\Sigma^{ai} n_i$ as the ***flux*** along the boundary of the region and $-\Sigma^a$ as the ***source*** inside the region. Thus the total flux is balanced by the total source. In order to say that the set (9.4.1) is of $(m+1)$th order, at least one of the functions $\Sigma^{\alpha i}$ must contain an mth order derivative $u^\alpha_{,i_1 i_2 \cdots i_m}$. The explicit form of equations (9.4.1) is found by resorting to the chain rule as follows

$$\sum_{r=0}^{m} \frac{\partial \Sigma^{\alpha i}}{\partial u^\beta_{,i_1 i_2 \cdots i_r}} u^\beta_{,i_1 i_2 \cdots i_r i} + \frac{\partial \Sigma^{\alpha i}}{\partial x^i} + \Sigma^\alpha = 0 \qquad (9.4.2)$$

where we have again adopted the convention

$$\sum_{r=0}^{m} \frac{\partial \Sigma^{\alpha i}}{\partial u^\beta_{,i_1 i_2 \cdots i_r}} u^\beta_{,i_1 i_2 \cdots i_r i} = \frac{\partial \Sigma^{\alpha i}}{\partial u^\beta} u^\beta_{,i} + \frac{\partial \Sigma^{\alpha i}}{\partial u^\beta_{,i_1}} u^\beta_{,i_1 i} + \cdots + \frac{\partial \Sigma^{\alpha i}}{\partial u^\beta_{,i_1 \cdots i_m}} u^\beta_{,i_1 \cdots i_m i}.$$

In understanding the real extent of above expressions we should recall that all repeated dummy indices indicate summations over their ranges. As we said if the order of this set of partial differential equations is $m+1$, then at least one of the coefficients $\partial \Sigma^{\alpha i} / \partial u^\beta_{,i_1 \cdots i_m}$ must be different from zero. Since equations (9.4.2) are linear with respect to $(m+1)$th order derivatives, they constitute a set of *quasilinear* partial differential equations. In order to utilise exterior forms the set (9.4.2) has to be transformed to a system of first order partial differential equations by introducing again auxiliary variables. Through the auxiliary variables $v^\alpha_{i_1 i_2 \cdots i_r} = u^\alpha_{,i_1 i_2 \cdots i_r}$, $0 \le r \le m$ that are *completely symmetric in its subscripts* defined as in (9.2.4), we can readily enlarge our system to the following first order system

$$v^\alpha_{i_1 i_2 \cdots i_r} = v^\alpha_{i_1 i_2 \cdots i_{r-1}, i_r}, \;\; 0 \le r \le m; \;\; v^\alpha_{i_0} = u^\alpha$$
$$\sum_{r=0}^{m} \frac{\partial \Sigma^{\alpha i}}{\partial v^\beta_{i_1 i_2 \cdots i_r}} v^\beta_{i_1 i_2 \cdots i_r, i} + \frac{\partial \Sigma^{\alpha i}}{\partial x^i} + \Sigma^\alpha = 0$$

Let us now consider the contact 1-forms (9.2.5)

$$\sigma^\alpha_{i_1 i_2 \cdots i_r} = dv^\alpha_{i_1 i_2 \cdots i_r} - v^\alpha_{i_1 i_2 \cdots i_r i} dx^i \in \Lambda^1(\mathcal{C}_m), \;\; 0 \le r \le m-1$$

and N ***balance n-forms***

$$\omega^\alpha = d\Sigma^{\alpha i} \wedge \mu_i + \Sigma^\alpha \mu \in \Lambda^n(\mathcal{C}_m). \tag{9.4.3}$$

In this section, we shall frequently find the opportunity of using the relations (5.5.10-13-14). (9.4.3) balance forms may be explicitly written as

$$\omega^\alpha = \Big(\frac{\partial \Sigma^{\alpha i}}{\partial x^i} + \Sigma^\alpha\Big)\mu + \sum_{r=0}^{m} \frac{\partial \Sigma^{\alpha i}}{\partial v^\beta_{i_1\cdots i_r}}\, dv^\beta_{i_1\cdots i_r} \wedge \mu_i. \tag{9.4.4}$$

A regular mapping $\phi : \mathcal{D}_n \to \mathcal{C}_m$ becomes a solution of balance equations if it satisfies the relations

$$\phi^*\sigma^\alpha_{i_1 i_2\cdots i_r} = 0,\ \ 0 \le r \le m-1;\ \ \phi^*\omega^\alpha = 0.$$

In fact, the equations

$$\begin{aligned}\phi^*\sigma^\alpha_{i_1 i_2\cdots i_r} &= (v^\alpha_{i_1 i_2\cdots i_r,i} - v^\alpha_{i_1 i_2\cdots i_r i})\, dx^i = 0,\ \ 0 \le r \le m-1\\ \phi^*\omega^\alpha &= \Big[\frac{\partial \Sigma^{\alpha i}}{\partial x^i} + \Sigma^\alpha + \sum_{r=0}^{m} \frac{\partial \Sigma^{\alpha i}}{\partial v^\beta_{i_1\cdots i_r}}\, v^\beta_{i_1\cdots i_r,i}\Big]\mu = 0\end{aligned}$$

yield $v^\alpha_{i_1 i_2\cdots i_r i} = v^\alpha_{i_1 i_2\cdots i_r,i}$ and $v^\alpha_{i_0} = u^\alpha$, $v^\alpha_{i_1 i_2\cdots i_r} = u^\alpha_{,i_1 i_2\cdots i_r}$ from which we recover the differential equations (9.4.2). We shall now consider the ideal below of the exterior algebra $\Lambda(\mathcal{C}_m)$

$$\mathfrak{I}_m = \mathcal{I}(\sigma^\alpha_{i_1 i_2\cdots i_r}, 0 \le r \le m-1; d\sigma^\alpha_{i_1 i_2\cdots i_{m-1}}; \omega^\alpha)$$

where $1 \le \alpha \le N$, $1 \le i_r \le n$, $0 \le r \le m-1$. $\mathfrak{I}_m$ will be called henceforth as the ***balance ideal*** or the ***fundamental ideal***. We can immediately verify that the balance ideal $\mathfrak{I}_m$ is closed. On using the definition of contact 1-forms, we obtain

$$\begin{aligned}d\omega^\alpha &= d\Sigma^\alpha \wedge \mu = \frac{\partial \Sigma^\alpha}{\partial x^i}\, dx^i \wedge \mu + \sum_{r=0}^{m} \frac{\partial \Sigma^\alpha}{\partial v^\beta_{i_1\cdots i_r}}\, dv^\beta_{i_1\cdots i_r} \wedge \mu\\ &= \Big[(-1)^n \sum_{r=0}^{m-1} \frac{\partial \Sigma^\alpha}{\partial v^\beta_{i_1\cdots i_r}}\, \mu \wedge \sigma^\beta_{i_1\cdots i_r} - \frac{\partial \Sigma^\alpha}{\partial v^\beta_{i_1\cdots i_{m-1} i}}\, \mu_i \wedge d\sigma^\beta_{i_1\cdots i_{m-1}}\Big]\end{aligned}$$

that amounts to say that $d\omega^\alpha \in \mathfrak{I}_m$. Solutions of balance equations in question annihilate the ideal $\mathfrak{I}_m$. We shall now attempt to determine isovector fields of the closed balance ideal $\mathfrak{I}_m$. To this end, we resort to Theorem 5.12.5. Let $V \in T(\mathcal{C}_m)$ be an isovector field of the contact ideal obtained in the previous section. We shall now try to specify the particular structure of this vector that permits us to determine appropriate forms $\lambda^\alpha_\beta \in \Lambda^0(\mathcal{C}_m)$;

$\gamma_\beta^{\alpha i_1\cdots i_r} \in \Lambda^{n-1}(\mathcal{C}_m), r = 0, 1, \ldots, m-1; \Gamma_\beta^{\alpha i_1\cdots i_{m-1}} \in \Lambda^{n-2}(\mathcal{C}_m)$ such that the following relations are satisfied

$$\pounds_V \omega^\alpha = \lambda_\beta^\alpha \omega^\beta + \sum_{r=0}^{m-1} \gamma_\beta^{\alpha i_1\cdots i_r} \wedge \sigma_{i_1\cdots i_r}^\beta + \Gamma_\beta^{\alpha i_1\cdots i_{m-1}} \wedge d\sigma_{i_1\cdots i_{m-1}}^\beta. \tag{9.4.5}$$

Since we can write

$$\begin{aligned}
\mathbf{i}_V(d\omega^\alpha) &= \mathbf{i}_V(d\Sigma^\alpha)\mu - d\Sigma^\alpha \wedge \mathbf{i}_V(\mu) = V(\Sigma^\alpha)\mu - X^i d\Sigma^\alpha \wedge \mu_i \\
\mathbf{i}_V(\omega^\alpha) &= \mathbf{i}_V(d\Sigma^{\alpha i})\mu_i - d\Sigma^{\alpha i} \wedge \mathbf{i}_V(\mu_i) + \Sigma^\alpha \mathbf{i}_V(\mu) \\
&= V(\Sigma^{\alpha i})\mu_i - X^j d\Sigma^{\alpha i} \wedge \mu_{ji} + X^i \Sigma^\alpha \mu_i,
\end{aligned}$$

the Cartan formula $\pounds_V \omega^\alpha = \mathbf{i}_V(d\omega^\alpha) + d\mathbf{i}_V(\omega^\alpha)$ then leads to

$$\begin{aligned}
\pounds_V \omega^\alpha = V(\Sigma^\alpha)\mu + \big[dV(\Sigma^{\alpha i}) + \Sigma^\alpha dX^i\big] \wedge \mu_i \\
- dX^j \wedge d\Sigma^{\alpha i} \wedge \mu_{ji}.
\end{aligned} \tag{9.4.6}$$

Let us recall that

$$V(f) = X^i \frac{\partial f}{\partial x^i} + \sum_{r=0}^{m} V_{i_1\cdots i_r}^\alpha \frac{\partial f}{\partial v_{i_1\cdots i_r}^\alpha} \in \Lambda^0(\mathcal{C}_m)$$

for a smooth function $f \in \Lambda^0(\mathcal{C}_m)$. We now have to take into consideration two different cases concerning isovectors of the contact ideal.

(i). Let $N > 1$. Therefore we have to choose $X^i = X^i(\mathbf{x}, \mathbf{u})$, $U^\alpha = U^\alpha(\mathbf{x}, \mathbf{u})$ and the components $V_{i_1\cdots i_r}^\alpha, 1 \le r \le m$ are found from (9.3.19). If we evaluate (9.4.6) under this constraint we obtain

$$\begin{aligned}
\pounds_V \omega^\alpha = V(\Sigma^\alpha)\mu + \Big[\frac{\partial V(\Sigma^{\alpha i})}{\partial x^j} dx^j + \sum_{r=0}^{m} \frac{\partial V(\Sigma^{\alpha i})}{\partial v_{i_1\cdots i_r}^\beta} dv_{i_1\cdots i_r}^\beta \\
+ \Sigma^\alpha \Big(\frac{\partial X^i}{\partial x^j} dx^j + \frac{\partial X^i}{\partial u^\beta} du^\beta\Big)\Big] \wedge \mu_i \\
- \Big(\frac{\partial X^j}{\partial x^k} dx^k + \frac{\partial X^j}{\partial u^\gamma} du^\gamma\Big) \wedge \Big[\frac{\partial \Sigma^{\alpha i}}{\partial x^l} dx^l + \sum_{r=0}^{m} \frac{\partial \Sigma^{\alpha i}}{\partial v_{i_1\cdots i_r}^\beta} dv_{i_1\cdots i_r}^\beta\Big] \wedge \mu_{ji}.
\end{aligned} \tag{9.4.7}$$

By making use of the relation $dx^j \wedge \mu_i = \delta_i^j \mu$ we cast (9.4.7) into

$$\begin{aligned}
\pounds_V \omega^\alpha &= A^\alpha \mu + A_\beta^{\alpha i} du^\beta \wedge \mu_i + A_{\beta\gamma}^{\alpha ij} du^\beta \wedge du^\gamma \wedge \mu_{ji} \\
&\quad + \sum_{r=1}^{m} A_\beta^{\alpha i i_1\cdots i_r} dv_{i_1\cdots i_r}^\beta \wedge \mu_i + \sum_{r=1}^{m} A_{\beta\gamma}^{\alpha ij i_1\cdots i_r} dv_{i_1\cdots i_r}^\beta \wedge du^\gamma \wedge \mu_{ji}
\end{aligned}$$

where the smooth functions

$$
\begin{aligned}
A^{\alpha} &= V(\Sigma^{\alpha}) + \frac{\partial V(\Sigma^{\alpha i})}{\partial x^{i}} + \Sigma^{\alpha}\frac{\partial X^{i}}{\partial x^{i}} + \frac{\partial \Sigma^{\alpha i}}{\partial x^{i}}\frac{\partial X^{j}}{\partial x^{j}} - \frac{\partial \Sigma^{\alpha i}}{\partial x^{j}}\frac{\partial X^{j}}{\partial x^{i}} \\
A^{\alpha i}_{\beta} &= \frac{\partial V(\Sigma^{\alpha i})}{\partial u^{\beta}} + \Sigma^{\alpha}\frac{\partial X^{i}}{\partial u^{\beta}} + \frac{\partial \Sigma^{\alpha j}}{\partial x^{j}}\frac{\partial X^{i}}{\partial u^{\beta}} - \frac{\partial \Sigma^{\alpha i}}{\partial x^{j}}\frac{\partial X^{j}}{\partial u^{\beta}} \\
&\qquad + \frac{\partial \Sigma^{\alpha i}}{\partial u^{\beta}}\frac{\partial X^{j}}{\partial x^{j}} - \frac{\partial \Sigma^{\alpha j}}{\partial u^{\beta}}\frac{\partial X^{i}}{\partial x^{j}} \\
A^{\alpha ij}_{\beta\gamma} &= -A^{\alpha ji}_{\beta\gamma} = -A^{\alpha ij}_{\gamma\beta} = \frac{\partial \Sigma^{\alpha[i}}{\partial u^{[\beta}}\frac{\partial X^{j]}}{\partial u^{\gamma]}} \qquad (9.4.8) \\
&= \frac{1}{4}\Big[\frac{\partial \Sigma^{\alpha i}}{\partial u^{\beta}}\frac{\partial X^{j}}{\partial u^{\gamma}} - \frac{\partial \Sigma^{\alpha j}}{\partial u^{\beta}}\frac{\partial X^{i}}{\partial u^{\gamma}} + \frac{\partial \Sigma^{\alpha j}}{\partial u^{\gamma}}\frac{\partial X^{i}}{\partial u^{\beta}} - \frac{\partial \Sigma^{\alpha i}}{\partial u^{\gamma}}\frac{\partial X^{j}}{\partial u^{\beta}}\Big] \\
A^{\alpha i i_1\cdots i_r}_{\beta} &= \frac{\partial V(\Sigma^{\alpha i})}{\partial v^{\beta}_{i_1\cdots i_r}} + \frac{\partial \Sigma^{\alpha i}}{\partial v^{\beta}_{i_1\cdots i_r}}\frac{\partial X^{j}}{\partial x^{j}} - \frac{\partial \Sigma^{\alpha j}}{\partial v^{\beta}_{i_1\cdots i_r}}\frac{\partial X^{i}}{\partial x^{j}}, \qquad 1 \le r \le m \\
A^{\alpha ij i_1\cdots i_r}_{\beta\gamma} &= -A^{\alpha ji i_1\cdots i_r}_{\beta\gamma} = \frac{\partial \Sigma^{\alpha[i}}{\partial v^{\beta}_{i_1\cdots i_r}}\frac{\partial X^{j]}}{\partial u^{\gamma}} \\
&= \frac{1}{2}\Big[\frac{\partial \Sigma^{\alpha i}}{\partial v^{\beta}_{i_1\cdots i_r}}\frac{\partial X^{j}}{\partial u^{\gamma}} - \frac{\partial \Sigma^{\alpha j}}{\partial v^{\beta}_{i_1\cdots i_r}}\frac{\partial X^{i}}{\partial u^{\gamma}}\Big], \qquad 1 \le r \le m
\end{aligned}
$$

are all elements of $\Lambda^0(\mathcal{C}_m)$. It is obvious that the functions $A^{\alpha i i_1\cdots i_r}_{\beta}$ and $A^{\alpha ij i_1\cdots i_r}_{\beta\gamma}$ are completely symmetric in indices $i_1,\dots,i_r$. The antisymmetry in indices i,j arise from the antisymmetry of forms $\mu_{ji} \in \Lambda^{n-2}(M)$ and antisymmetry with respect to indices β,γ in $A^{\alpha ij}_{\beta\gamma}$ from the exterior product $du^{\beta}\wedge du^{\gamma}$. If we make the transformations

$$
dv^{\alpha}_{i_1 i_2\cdots i_r} = \sigma^{\alpha}_{i_1 i_2\cdots i_r} + v^{\alpha}_{i_1 i_2\cdots i_r j}\,dx^{j}, \qquad 0 \le r \le m-1
$$

in the expression above for $\pounds_V\omega^{\alpha}$ and use the relations

$$
\begin{aligned}
dx^{k}\wedge \mu_{ji} &= \delta^{k}_{j}\,\mu_i - \delta^{k}_{i}\,\mu_j, \\
dx^{k}\wedge dx^{l}\wedge \mu_{ji} &= (\delta^{k}_{i}\delta^{l}_{j} - \delta^{k}_{j}\delta^{l}_{i})\mu
\end{aligned}
$$

we arrive at

$$
\begin{aligned}
\pounds_V\omega^{\alpha} = \Big[A^{\alpha} + A^{\alpha i}_{\beta}v^{\beta}_{i} + 2A^{\alpha ij}_{\beta\gamma}v^{\beta}_{i}v^{\gamma}_{j} + \sum_{r=1}^{m-1}(A^{\alpha i i_1\cdots i_r}_{\beta} \qquad (9.4.9) \\
+ 2A^{\alpha ij i_1\cdots i_r}_{\beta\gamma}v^{\gamma}_{j})v^{\beta}_{i_1\cdots i_r i}\Big]\mu + (A^{\alpha i i_1\cdots i_m}_{\beta} + 2A^{\alpha ij i_1\cdots i_m}_{\beta\gamma}v^{\gamma}_{j})\,dv^{\beta}_{i_1\cdots i_m}\wedge\mu_i
\end{aligned}
$$

$$+\Big\{(-1)^{n-1}\Big[A^{\alpha i}_{\beta}+4A^{\alpha ij}_{\beta\gamma}v_j-2\sum_{r=1}^{m-1}A^{\alpha iji_1\cdots i_r}_{\gamma\beta}v^{\gamma}_{i_1\cdots i_r j}\Big]\mu_i$$
$$-A^{\alpha ij}_{\beta\gamma}\mu_{ji}\wedge\sigma^{\gamma}+A^{\alpha iji_1\cdots i_m}_{\gamma\beta}\mu_{ji}\wedge dv^{\gamma}_{i_1\cdots i_m}\Big\}\wedge\sigma^{\beta}$$
$$+\sum_{r=1}^{m-1}\Big[(-1)^{n-1}(A^{\alpha ii_1\cdots i_r}_{\beta}+2A^{\alpha iji_1\cdots i_r}_{\beta\gamma}v^{\gamma}_j)\,\mu_i-A^{\alpha iji_1\cdots i_r}_{\beta\gamma}\mu_{ji}\wedge\sigma^{\gamma}\Big]\wedge\sigma^{\beta}_{i_1\cdots i_r}.$$

The functions in (9.4.8) comprise now solely presently arbitrary functions $X^i(\mathbf{x},\mathbf{u})$ and $U^{\alpha}(\mathbf{x},\mathbf{u})$ as unknowns. On the other hand (9.4.4) can thereby be written as

$$\omega^{\alpha}=\Big(\Sigma^{\alpha}+\frac{\partial\Sigma^{\alpha i}}{\partial x^i}+\sum_{r=0}^{m-1}\frac{\partial\Sigma^{\alpha i}}{\partial v^{\beta}_{i_1\cdots i_r}}v^{\beta}_{i_1\cdots i_r i}\Big)\mu+\frac{\partial\Sigma^{\alpha i}}{\partial u^{\beta}}\,\sigma^{\beta}\wedge\mu_i$$
$$+\sum_{r=1}^{m-1}\frac{\partial\Sigma^{\alpha i}}{\partial v^{\beta}_{i_1\cdots i_r}}\,\sigma^{\beta}_{i_1\cdots i_r}\wedge\mu_i+\frac{\partial\Sigma^{\alpha i}}{\partial v^{\beta}_{i_1\cdots i_m}}\,dv^{\beta}_{i_1\cdots i_m}\wedge\mu_i.$$

On inserting this expression together with (9.4.9) into the relation (9.4.5) and equating the coefficients of linearly independent like forms in both sides we end up with the following result

$$\lambda^{\alpha}_{\beta}\Big(\Sigma^{\beta}+\frac{\partial\Sigma^{\beta i}}{\partial x^i}+\sum_{r=0}^{m-1}\frac{\partial\Sigma^{\beta i}}{\partial v^{\gamma}_{i_1\cdots i_r}}v^{\gamma}_{i_1\cdots i_r i}\Big)= \tag{9.4.10}$$
$$A^{\alpha}+A^{\alpha i}_{\beta}v^{\beta}_i+2A^{\alpha ij}_{\beta\gamma}v^{\beta}_i v^{\gamma}_j+\sum_{r=1}^{m-1}(A^{\alpha ii_1\cdots i_r}_{\beta}+2A^{\alpha iji_1\cdots i_r}_{\beta\gamma}v^{\gamma}_j\Big)v^{\beta}_{i_1\cdots i_r i}$$
$$\gamma^{\alpha}_{\beta}=(-1)^n\lambda^{\alpha}_{\gamma}\frac{\partial\Sigma^{\gamma i}}{\partial u^{\beta}}\mu_i$$
$$+(-1)^{n-1}\Big[A^{\alpha i}_{\beta}+A^{\alpha ij}_{\beta\gamma}v^{\gamma}_j-\sum_{r=1}^{m-1}A^{\alpha iji_1\cdots i_r}_{\gamma\beta}v^{\gamma}_{i_1\cdots i_r j}\Big]\mu_i$$
$$-A^{\alpha ij}_{\beta\gamma}\mu_{ji}\wedge\sigma^{\gamma}+A^{\alpha iji_1\cdots i_m}_{\gamma\beta}\mu_{ji}\wedge dv^{\gamma}_{i_1\cdots i_m}$$
$$\gamma^{\alpha i_1\cdots i_r}_{\beta}=(-1)^n\lambda^{\alpha}_{\gamma}\frac{\partial\Sigma^{\gamma i}}{\partial v^{\beta}_{i_1\cdots i_r}}\,\mu_i$$
$$+(-1)^{n-1}(A^{\alpha ii_1\cdots i_r}_{\beta}+2A^{\alpha iji_1\cdots i_r}_{\beta\gamma}v^{\gamma}_j)\,\mu_i$$
$$-A^{\alpha iji_1\cdots i_r}_{\beta\gamma}\mu_{ji}\wedge\sigma^{\gamma},\quad 1\le r\le m-1$$

whereas the last N equations to be satisfied take the form

$$\Big(A_\beta^{\alpha i i_1\cdots i_m}+2A_{\beta\gamma}^{\alpha i j i_1\cdots i_m}v_j^\gamma-\lambda_\gamma^\alpha\frac{\partial\Sigma^{\gamma i}}{\partial v_{i_1\cdots i_m}^\beta}\Big)\,dv_{i_1\cdots i_m}^\beta\wedge\mu_i$$
$$=-\,\Gamma_\beta^{\alpha i_1\cdots i_{m-1}}\wedge dv_{i_1\cdots i_{m-1}i_m}^\beta\wedge dx^{i_m}$$

when we write $d\sigma_{i_1\cdots i_{m-1}}^\beta=-\,dv_{i_1\cdots i_m i}^\beta\wedge dx^i$. The composition of this expressions suggests that it would be rather adequate to choose the forms $\Gamma_\beta^{\alpha i_1\cdots i_{m-1}}\in\Lambda^{n-2}(\mathcal{C}_m)$ as follows

$$\Gamma_\beta^{\alpha i_1\cdots i_{m-1}}=\gamma_\beta^{\alpha i_1\cdots i_{m-1}ij}\mu_{ji},\ \ \gamma_\beta^{\alpha i_1\cdots i_{m-1}ij}\in\Lambda^0(\mathcal{C}_m).$$

Since the forms μ_{ji} are antisymmetric in indices i,j, we can take without loss of generality $\gamma_\beta^{\alpha i_1\cdots i_{m-1}ij}=-\,\gamma_\beta^{\alpha i_1\cdots i_{m-1}ji}$. We thus obtain

$$\begin{aligned}
-\,\Gamma_\beta^{\alpha i_1\cdots i_{m-1}}&\wedge dv_{i_1\cdots i_{m-1}i_m}^\beta\wedge dx^{i_m}\\
&=-\,\gamma_\beta^{\alpha i_1\cdots i_{m-1}ij}\mu_{ji}\wedge dv_{i_1\cdots i_{m-1}i_m}^\beta\wedge dx^{i_m}\\
&=-\,\gamma_\beta^{\alpha i_1\cdots i_{m-1}ij}dv_{i_1\cdots i_{m-1}i_m}^\beta\wedge dx^{i_m}\wedge\mu_{ji}\\
&=-\,\gamma_\beta^{\alpha i_1\cdots i_{m-1}ij}dv_{i_1\cdots i_{m-1}i_m}^\beta\wedge(\delta_j^{i_m}\mu_i-\delta_i^{i_m}\mu_j)\\
&=-\,\gamma_\beta^{\alpha i_1\cdots i_{m-1}ii_m}dv_{i_1\cdots i_{m-1}i_m}^\beta\wedge\mu_i+\gamma_\beta^{\alpha i_1\cdots i_{m-1}i_m j}dv_{i_1\cdots i_{m-1}i_m}\wedge\mu_j\\
&=2\gamma_\beta^{\alpha i_1\cdots i_{m-1}i_m i}dv_{i_1\cdots i_{m-1}i_m}\wedge\mu_i
\end{aligned}$$

whence we deduce that

$$2\gamma_\beta^{\alpha i_1\cdots i_{m-1}i_m i}=A_\beta^{\alpha i i_1\cdots i_m}+2A_{\beta\gamma}^{\alpha i j i_1\cdots i_m}v_j^\gamma-\lambda_\gamma^\alpha\frac{\partial\Sigma^{\gamma i}}{\partial v_{i_1\cdots i_m}^\beta}.$$

But, because of the antisymmetry of functions $\gamma_\beta^{\alpha i_1\cdots i_{m-1}i_m i}$ with respect to indices i,i_m, we see that the following relations must be satisfied

$$\begin{aligned}
4\gamma_\beta^{\alpha i_1\cdots i_{m-1}i_m i}&=4\gamma_\beta^{\alpha i_1\cdots i_{m-1}[i_m i]}=A_\beta^{\alpha i i_1\cdots i_{m-1}i_m}+2A_{\beta\gamma}^{\alpha i j i_1\cdots i_{m-1}i_m}v_j^\gamma\\
&-A_\beta^{\alpha i_m i_1\cdots i_{m-1}i}-2A_{\beta\gamma}^{\alpha i_m j i_1\cdots i_{m-1}i}v_j^\gamma-\lambda_\gamma^\alpha\Big(\frac{\partial\Sigma^{\gamma i}}{\partial v_{i_1\cdots i_{m-1}i_m}^\beta}-\frac{\partial\Sigma^{\gamma i_m}}{\partial v_{i_1\cdots i_{m-1}i}^\beta}\Big)\\
0&=A_\beta^{\alpha i i_1\cdots i_{m-1}i_m}+2A_{\beta\gamma}^{\alpha i j i_1\cdots i_{m-1}i_m}v_j^\gamma+A_\beta^{\alpha i_m i_1\cdots i_{m-1}i}+2A_{\beta\gamma}^{\alpha i_m j i_1\cdots i_{m-1}i}v_j^\gamma\\
&\qquad-\lambda_\gamma^\alpha\Big(\frac{\partial\Sigma^{\gamma i}}{\partial v_{i_1\cdots i_{m-1}i_m}^\beta}+\frac{\partial\Sigma^{\gamma i_m}}{\partial v_{i_1\cdots i_{m-1}i}^\beta}\Big)
\end{aligned}$$

Consequently, we can state the theorem below:

Theorem 9.4.1. *In order that an isovector field V of the contact ideal* *becomes also an isovector of the balance ideal,* $n+N+N^2$ *number of functions* $X^i(\boldsymbol{x},\boldsymbol{u}), U^\alpha(\boldsymbol{x},\boldsymbol{u}),\ \lambda^\alpha_\beta \in \Lambda^0(\mathcal{C}_m)$ *ought to satisfy the determining equations*

$$\lambda^\alpha_\beta\Big(\Sigma^\beta + \frac{\partial \Sigma^{\beta i}}{\partial x^i} + \sum_{r=0}^{m-1}\frac{\partial \Sigma^{\beta i}}{\partial v^\gamma_{i_1\cdots i_r}} v^\gamma_{i_1\cdots i_r i}\Big) = \tag{9.4.11}$$

$$A^\alpha + A^{\alpha i}_\beta v^\beta_i + 2A^{\alpha ij}_{\beta\gamma} v^\beta_i v^\gamma_j \ + \sum_{r=1}^{m-1}(A^{\alpha i i_1\cdots i_r}_\beta + 2A^{\alpha i j i_1\cdots i_r}_{\beta\gamma} v^\gamma_j\Big) v^\beta_{i_1\cdots i_r i}$$

$$\lambda^\alpha_\gamma\Big(\frac{\partial \Sigma^{\gamma i}}{\partial v^\beta_{i_1\cdots i_{m-1} i_m}} + \frac{\partial \Sigma^{\gamma i_m}}{\partial v^\beta_{i_1\cdots i_{m-1} i}}\Big) =$$

$$A^{\alpha i i_1\cdots i_{m-1} i_m}_\beta + A^{\alpha i_m i_1\cdots i_{m-1} i}_\beta + 2(A^{\alpha i j i_1\cdots i_{m-1} i_m}_{\beta\gamma} + A^{\alpha i_m j i_1\cdots i_{m-1} i}_{\beta\gamma}) v^\gamma_j$$

whenever $N > 1$. □

The number of the equations (9.4.11) that help determine isovector components, or *infinitesimal generators* in the nomenclature of the classical theory of Lie symmetry groups, are considerably less than those in the classical theory because exterior products are quite effective in eliminating some of the redundant equations. However, it is still a large number. It can easily be checked that there can be at most $N + \frac{1}{2}N^2\binom{n+m-2}{m-1}n(n+1)$ number of determining equations. Therefore, we must expect that the number of the determining equations would be much larger than that of unknowns. This property amounts to say that the shape of the solutions would perhaps be restricted to a great extent even if they exist.

If $m = 0$, that is, if functions entering the balance equations are in the form $\Sigma^{\alpha i}(\mathbf{x},\mathbf{u})$ and $\Sigma^\alpha(\mathbf{x},\mathbf{u})$, we get a system of first order equations. In that case we do not need the contact ideal and only the components X^i and U^α of the isovector field survive. One can readily verify that the determining equations are then reduced to

$$\lambda^\alpha_\beta\Big(\Sigma^\beta + \frac{\partial \Sigma^{\beta i}}{\partial x^i}\Big) = A^\alpha,\ \ \lambda^\alpha_\gamma \frac{\partial \Sigma^{\gamma i}}{\partial u^\beta} = A^{\alpha i}_\beta,\ \ A^{\alpha ij}_{\beta\gamma} = 0.$$

***(ii)*.** Let $N = 1$. Hence, the components of an isovector field V of the contact ideal become $X^i = X^i(\mathbf{x},u,\mathbf{v})$, $U = U(\mathbf{x},u,\mathbf{v})$ generated from a function $F = F(\mathbf{x},u,\mathbf{v})$ via the relations (9.3.22-23) and $V_{i_1\cdots i_r} \in \Lambda^0(\mathcal{C}_m)$, $1 \le r \le m$ determined by (9.3.25). In this case, the single balance equation takes the form

$$\frac{\partial \Sigma^i}{\partial x^i} + \Sigma = 0$$

and the *balance n-form* producing this equation is given by

$$\omega = d\Sigma^i \wedge \mu_i + \Sigma\mu = \Big(\frac{\partial\Sigma^i}{\partial x^i} + \Sigma\Big)\mu + \sum_{r=0}^{m}\frac{\partial\Sigma^i}{\partial v_{i_1\cdots i_r}}\,dv_{i_1\cdots i_r}\wedge\mu_i.$$

The vector V will be an isovector field of the balance ideal if one is able to find suitable forms $\lambda \in \Lambda^0(\mathcal{C}_m); \gamma^{i_1\cdots i_r} \in \Lambda^{n-1}(\mathcal{C}_m), r = 0, 1, \ldots, m-1$; $\Gamma^{i_1\cdots i_{m-1}} \in \Lambda^{n-2}(\mathcal{C}_m)$ so that the relation

$$\pounds_V\omega = \lambda\,\omega + \sum_{r=0}^{m-1}\gamma^{i_1\cdots i_r}\wedge\sigma_{i_1\cdots i_r} + \Gamma^{i_1\cdots i_{m-1}}\wedge d\sigma_{i_1\cdots i_{m-1}} \tag{9.4.12}$$

is satisfied. The Lie derivative of the balance form ω with respect to the vector V follows from (9.4.6) as

$$\pounds_V\omega = V(\Sigma)\mu + \big[dV(\Sigma^i) + \Sigma\,dX^i\big]\wedge\mu_i - dX^j\wedge d\Sigma^i\wedge\mu_{ji}.$$

However, because of the possible dependence of the components X^i on the variables v_i the above expression may now lead to a different result from (9.4.7):

$$\begin{aligned}\pounds_V\omega = V(\Sigma)\mu &+ \bigg[\frac{\partial V(\Sigma^i)}{\partial x^j}\,dx^j + \sum_{r=0}^{m}\frac{\partial V(\Sigma^i)}{\partial v_{i_1\cdots i_r}}\,dv_{i_1\cdots i_r}\\ &+ \Sigma\Big(\frac{\partial X^i}{\partial x^j}\,dx^j + \frac{\partial X^i}{\partial u}\,du + \frac{\partial X^i}{\partial v_j}\,dv_j\Big)\bigg]\wedge\mu_i - \Big(\frac{\partial X^j}{\partial x^k}\,dx^k\\ &+ \frac{\partial X^j}{\partial u}\,du + \frac{\partial X^j}{\partial v_k}\,dv_k\Big)\wedge\bigg[\frac{\partial\Sigma^i}{\partial x^l}\,dx^l + \sum_{r=0}^{m}\frac{\partial\Sigma^i}{\partial v_{i_1\cdots i_r}}\,dv_{i_1\cdots i_r}\bigg]\wedge\mu_{ji}.\end{aligned}$$

We can arrange this expression into the following form

$$\begin{aligned}\pounds_V\omega = A\mu &+ A^i\,du\wedge\mu_i + A^{ij}\,dv_j\wedge\mu_i + B^{ijk}\,dv_k\wedge du\wedge\mu_{ji}\\ &+ C^{ijkl}dv_l\wedge dv_k\wedge\mu_{ji} + \sum_{r=2}^{m}A^{ii_1\cdots i_r}dv_{i_1\cdots i_r}\wedge\mu_i\\ &+ \sum_{r=2}^{m}B^{iji_1\cdots i_r}dv_{i_1\cdots i_r}\wedge du\wedge\mu_{ji} + + \sum_{r=2}^{m}C^{ijki_1\cdots i_r}dv_{i_1\cdots i_r}\wedge dv_k\wedge\mu_{ji}\end{aligned} \tag{9.4.13}$$

where the smooth functions

$$A = V(\Sigma) + \frac{\partial V(\Sigma^i)}{\partial x^i} + \Sigma\frac{\partial X^i}{\partial x^i} + \frac{\partial\Sigma^i}{\partial x^i}\frac{\partial X^j}{\partial x^j} - \frac{\partial\Sigma^i}{\partial x^j}\frac{\partial X^j}{\partial x^i} \tag{9.4.14}$$

$$\begin{aligned}
A^i &= \frac{\partial V(\Sigma^i)}{\partial u} + \Sigma\frac{\partial X^i}{\partial u} + \frac{\partial \Sigma^j}{\partial x^j}\frac{\partial X^i}{\partial u} - \frac{\partial \Sigma^i}{\partial x^j}\frac{\partial X^j}{\partial u} \\
&\qquad + \frac{\partial \Sigma^i}{\partial u}\frac{\partial X^j}{\partial x^j} - \frac{\partial \Sigma^j}{\partial u}\frac{\partial X^i}{\partial x^j} \\
A^{ij} &= \frac{\partial V(\Sigma^i)}{\partial v_j} + \Sigma\frac{\partial X^i}{\partial v_j} + \frac{\partial \Sigma^k}{\partial x^k}\frac{\partial X^i}{\partial v_j} - \frac{\partial \Sigma^i}{\partial x^k}\frac{\partial X^k}{\partial v_j} \\
&\qquad + \frac{\partial \Sigma^i}{\partial v_j}\frac{\partial X^k}{\partial x^k} - \frac{\partial \Sigma^k}{\partial v_j}\frac{\partial X^i}{\partial x^k} \\
B^{ijk} &= -B^{jik} = \frac{\partial \Sigma^{[i}}{\partial v_k}\frac{\partial X^{j]}}{\partial u} - \frac{\partial \Sigma^{[i}}{\partial u}\frac{\partial X^{j]}}{\partial v_k} \\
&= \frac{1}{2}\Big[\frac{\partial \Sigma^i}{\partial v_k}\frac{\partial X^j}{\partial u} - \frac{\partial \Sigma^j}{\partial v_k}\frac{\partial X^i}{\partial u} + \frac{\partial \Sigma^j}{\partial u}\frac{\partial X^i}{\partial v_k} - \frac{\partial \Sigma^i}{\partial u}\frac{\partial X^j}{\partial v_k}\Big] \\
C^{ijkl} &= -C^{jikl} = -C^{ijlk} = \frac{\partial \Sigma^{[i}}{\partial v_{[l}}\frac{\partial X^{j]}}{\partial v_{k]}} \\
&= \frac{1}{4}\Big[\frac{\partial \Sigma^i}{\partial v_l}\frac{\partial X^j}{\partial v_k} - \frac{\partial \Sigma^j}{\partial v_l}\frac{\partial X^i}{\partial v_k} + \frac{\partial \Sigma^j}{\partial v_k}\frac{\partial X^i}{\partial v_l} - \frac{\partial \Sigma^i}{\partial v_k}\frac{\partial X^j}{\partial v_l}\Big] \\
A^{ii_1\cdots i_r} &= \frac{\partial V(\Sigma^i)}{\partial v_{i_1\cdots i_r}} + \frac{\partial \Sigma^i}{\partial v_{i_1\cdots i_r}}\frac{\partial X^j}{\partial x^j} - \frac{\partial \Sigma^j}{\partial v_{i_1\cdots i_r}}\frac{\partial X^i}{\partial x^j}, \\
B^{iji_1\cdots i_r} &= -B^{jii_1\cdots i_r} = \frac{\partial \Sigma^{[i}}{\partial v_{i_1\cdots i_r}}\frac{\partial X^{j]}}{\partial u} \\
&= \frac{1}{2}\Big[\frac{\partial \Sigma^i}{\partial v_{i_1\cdots i_r}}\frac{\partial X^j}{\partial u} - \frac{\partial \Sigma^j}{\partial v_{i_1\cdots i_r}}\frac{\partial X^i}{\partial u}\Big], \\
C^{ijki_1\cdots i_r} &= -C^{jiki_1\cdots i_r} = \frac{\partial \Sigma^{[i}}{\partial v_{i_1\cdots i_r}}\frac{\partial X^{j]}}{\partial v_k} \\
&= \frac{1}{2}\Big[\frac{\partial \Sigma^i}{\partial v_{i_1\cdots i_r}}\frac{\partial X^j}{\partial v_k} - \frac{\partial \Sigma^j}{\partial v_{i_1\cdots i_r}}\frac{\partial X^i}{\partial v_k}\Big]
\end{aligned}$$

are elements of $\Lambda^0(\mathcal{C}_m)$. By writing again $dv_{i_1\cdots i_r} = \sigma_{i_1\cdots i_r} + v_{i_1\cdots i_r j}\,dx^j$, $0 \le r \le m-1$ and arranging suitably the resulting expression, we can express the Lie derivative $\pounds_V\omega$ in the following manner:

$$\begin{aligned}
\pounds_V\omega = \Big[&A + A^i v_i + A^{ij}v_{ji} + 2B^{ijk}v_j v_{ki} + 2C^{ijkl}v_{kj}v_{li} \\
&+ \sum_{r=2}^{m-1}(A^{ii_1\cdots i_r} + 2B^{iji_1\cdots i_r}v_j + 2C^{ijki_1\cdots i_r}v_{kj})v_{i_1\cdots i_r i}\Big]\mu \\
&+ (A^{ii_1\cdots i_m} + 2B^{iji_1\cdots i_m}v_j + 2C^{ijki_1\cdots i_m}v_{kj})\,dv_{i_1\cdots i_m}\wedge\mu_i
\end{aligned} \tag{9.4.15}$$

$$
\begin{aligned}
+ \Big[A^i - 2B^{ijk}v_{kj} - 2\sum_{r=2}^{m-1} B^{iji_1\cdots i_r} v_{i_1\cdots i_r j}\Big]\sigma \wedge \mu_i \\
+ B^{iji_1\cdots i_m} dv_{i_1\cdots i_m} \wedge \sigma \wedge \mu_{ji} \\
+ \Big[A^{ij} + 2B^{ikj}v_k + 4C^{iklj}v_{lk} - 2\sum_{r=2}^{m-1} C^{ikji_1\cdots i_r} v_{i_1\cdots i_r k}\Big]\sigma_j \wedge \mu_i \\
- B^{ikj}\sigma \wedge \sigma_j \wedge \mu_{ki} + C^{ikji_1\cdots i_m} dv_{i_1\cdots i_m} \wedge \sigma_j \wedge \mu_{ki} + C^{ikjl}\sigma_l \wedge \sigma_j \wedge \mu_{ki} \\
+ \sum_{r=2}^{m-1}(A^{ii_1\cdots i_r} + 2B^{iji_1\cdots i_r}v_j + 2C^{ijki_1\cdots i_r}v_{kj})\,\sigma_{i_1\cdots i_r} \wedge \mu_i \\
- \sum_{r=2}^{m-1} B^{iji_1\cdots i_r}\sigma \wedge \sigma_{i_1\cdots i_r} \wedge \mu_{ji} - \sum_{r=2}^{m-1} C^{ijki_1\cdots i_r}\sigma_k \wedge \sigma_{i_1\cdots i_r} \wedge \mu_{ji}
\end{aligned}
$$

Next, we transform (9.4.12) into

$$
\begin{aligned}
\pounds_V\omega = \lambda\bigg[\Big(\Sigma + \frac{\partial \Sigma^i}{\partial x^i} + \sum_{r=0}^{m-1} \frac{\partial \Sigma^i}{\partial v_{i_1\cdots i_r}}\, v_{i_1\cdots i_r i}\Big)\mu + \frac{\partial \Sigma^i}{\partial u}\,\sigma \wedge \mu_i + \frac{\partial \Sigma^i}{\partial v_j}\,\sigma_j \wedge \mu_i \\
+ \sum_{r=2}^{m-1} \frac{\partial \Sigma^i}{\partial v_{i_1\cdots i_r}}\,\sigma_{i_1\cdots i_r} \wedge \mu_i + \frac{\partial \Sigma^i}{\partial v_{i_1\cdots i_m}}\, dv_{i_1\cdots i_m} \wedge \mu_i\bigg] + \gamma \wedge \sigma + \gamma^j \wedge \sigma_j \\
+ \sum_{r=2}^{m-1} \gamma^{i_1\cdots i_r} \wedge \sigma_{i_1\cdots i_r} - \Gamma^{i_1\cdots i_{m-1}} \wedge dv_{i_1\cdots i_{m-1}i_m} \wedge dx^{i_m}
\end{aligned}
$$

and compare it with (9.4.14) to obtain

$$
\begin{aligned}
\lambda\Big(\Sigma + \frac{\partial \Sigma^i}{\partial x^i} + \sum_{r=0}^{m-1} \frac{\partial \Sigma^i}{\partial v_{i_1\cdots i_r}}\, v_{i_1\cdots i_r i}\Big) \qquad (9.4.16) \\
= A + A^i v_i + A^{ij}v_{ji} + 2B^{ijk}v_j v_{ki} \\
+ 2C^{ijkl}v_{kj}v_{li} + \sum_{r=2}^{m-1}(A^{ii_1\cdots i_r} + 2B^{iji_1\cdots i_r}v_j + 2C^{ijki_1\cdots i_r}v_{kj})v_{i_1\cdots i_r i} \\
\gamma = (-1)^{n-1}\Big[A^i - 2B^{ijk}v_{kj} - 2\sum_{r=2}^{m-1} B^{iji_1\cdots i_r} v_{i_1\cdots i_r j} - \lambda\frac{\partial \Sigma^i}{\partial u}\Big]\mu_i \\
+ B^{iji_1\cdots i_m}\mu_{ji} \wedge dv_{i_1\cdots i_m} \\
\gamma^j = -B^{ikj}\mu_{ki} \wedge \sigma + C^{ikji_1\cdots i_m}\mu_{ki} \wedge dv_{i_1\cdots i_m} + C^{ikjl}\mu_{ki} \wedge \sigma_l \\
+ \Big[(-1)^{n-1}(A^{ij} + 2B^{ikj}v_k + 4C^{iklj}v_{lk}
\end{aligned}
$$

$$- 2\sum_{r=2}^{m-1} C^{ikji_1\cdots i_r} v_{i_1\cdots i_r k} - \lambda \frac{\partial \Sigma^i}{\partial v_j}\Big]\mu_i$$

$$\gamma^{i_1\cdots i_r} = (-1)^{n-1}\Big(A^{ii_1\cdots i_r} + 2B^{iji_1\cdots i_r} v_j + 2C^{ijki_1\cdots i_r} v_{kj} - \lambda \frac{\partial \Sigma^i}{\partial v_{i_1\cdots i_r}}\Big)\mu_i - B^{iji_1\cdots i_r}\mu_{ji}\wedge\sigma - C^{ijki_1\cdots i_r}\mu_{ji}\wedge\sigma_k, \quad 2 \le r \le m-1$$

while the remaining expression is given by

$$(A^{ii_1\cdots i_m} + 2B^{iji_1\cdots i_m} v_j + 2C^{ijki_1\cdots i_m} v_{kj} - \lambda \frac{\partial \Sigma^i}{\partial v_{i_1\cdots i_m}})\, dv_{i_1\cdots i_m}\wedge \mu_i = -\Gamma^{i_1\cdots i_{m-1}}\wedge dv_{i_1\cdots i_{m-1}i_m}\wedge dx^{i_m}$$

whose structure suggests that it would really be appropriate to choose the forms $\Gamma^{i_1\cdots i_{m-1}} \in \Lambda^{n-2}(\mathcal{C}_m)$ as follows

$$\Gamma^{i_1\cdots i_{m-1}} = \gamma^{i_1\cdots i_{m-1}ij}\mu_{ji}$$

where $\gamma^{i_1\cdots i_{m-1}ij} \in \Lambda^0(\mathcal{C}_m)$. Due to the antisymmetry of the forms μ_{ji} with respect to the indices i,j, we can take without loss of generality $\gamma_\beta^{\alpha i_1\cdots i_{m-1}ij} = -\gamma_\beta^{\alpha i_1\cdots i_{m-1}ji}$. We thus get

$$\begin{aligned} -\Gamma^{i_1\cdots i_{m-1}} \wedge dv_{i_1\cdots i_{m-1}i_m}\wedge dx^{i_m} &= -\gamma^{i_1\cdots i_{m-1}ij}\mu_{ji}\wedge dv_{i_1\cdots i_{m-1}i_m}\wedge dx^{i_m} \\ &= 2\gamma^{i_1\cdots i_{m-1}i_m i} dv_{i_1\cdots i_{m-1}i_m}\wedge\mu_i \end{aligned}$$

whence we conclude that

$$\begin{aligned} 4\gamma^{i_1\cdots i_{m-1}i_m i} &= 4\gamma^{i_1\cdots i_{m-1}[i_m i]} = A^{ii_1\cdots i_{m-1}i_m} + 2B^{iji_1\cdots i_{m-1}i_m} v_j \\ &\quad + 2C^{ijki_1\cdots i_{m-1}i_m} v_{kj} - A^{i_m i_1\cdots i_{m-1}i} - 2B^{i_m j i_1\cdots i_{m-1}i} v_j - 2C^{i_m jki_1\cdots i_{m-1}i} v_{kj} \\ &\quad - \lambda\Big(\frac{\partial \Sigma^i}{\partial v_{i_1\cdots i_{m-1}i_m}} - \frac{\partial \Sigma^{i_m}}{\partial v_{i_1\cdots i_{m-1}i}}\Big), \\ 0 &= A^{ii_1\cdots i_{m-1}i_m} + 2B^{iji_1\cdots i_{m-1}i_m} v_j + 2C^{ijki_1\cdots i_{m-1}i_m} v_{kj} + A^{i_m i_1\cdots i_{m-1}i} \\ &\quad + 2B^{i_m j i_1\cdots i_{m-1}i} v_j + 2C^{i_m jki_1\cdots i_{m-1}i} v_{kj} - \lambda\Big(\frac{\partial \Sigma^i}{\partial v_{i_1\cdots i_{m-1}i_m}} + \frac{\partial \Sigma^{i_m}}{\partial v_{i_1\cdots i_{m-1}i}}\Big). \end{aligned}$$

Therefore, we can state the theorem below:

Theorem 9.4.2. *An isovector field V of a contact ideal generated by a single dependent variable u can also be an isovector field of the balance* *ideal created by a single balance equation* $(N = 1)$ *if and only if the smooth functions* $F = F(\mathbf{x}, u, \mathbf{v})$ *producing the isovector components and* $\lambda \in \Lambda^0(\mathcal{C}_m)$ *must satisfy the determining equations*

$$
\begin{aligned}
\lambda\Big(\Sigma + \frac{\partial \Sigma^i}{\partial x^i} + \sum_{r=0}^{m-1} \frac{\partial \Sigma^i}{\partial v_{i_1\cdots i_r}}\, v_{i_1\cdots i_r i}\Big) \\
&= A + A^i v_i + A^{ij} v_{ji} + 2B^{ijk} v_j v_{ki} + 2C^{ijkl} v_{li} v_{kj} \\
&\quad + \sum_{r=2}^{m-1} (A^{ii_1\cdots i_r} + 2B^{iji_1\cdots i_r} v_j + 2C^{ijki_1\cdots i_r} v_{kj}) v_{i_1\cdots i_r i}, \\
\lambda\Big(\frac{\partial \Sigma^i}{\partial v_{i_1\cdots i_{m-1} i_m}} + \frac{\partial \Sigma^{i_m}}{\partial v_{i_1\cdots i_{m-1} i}}\Big) &= A^{ii_1\cdots i_{m-1} i_m} + A^{i_m i_1\cdots i_{m-1} i} \qquad (9.4.17) \\
&\quad + 2(B^{iji_1\cdots i_{m-1} i_m} + B^{i_m j i_1\cdots i_{m-1} i}) v_j \\
&\quad + 2(C^{ijki_1\cdots i_{m-1} i_m} + C^{i_m jki_1\cdots i_{m-1} i}) v_{kj} \qquad \square
\end{aligned}
$$

The above theorem looses its validity for $m = 1$. In this case, the isovector field is represented by

$$V = X^i \frac{\partial}{\partial x^i} + U \frac{\partial}{\partial u} + V_i \frac{\partial}{\partial v_i}$$

and its components are given by (9.3.26)$_{1-3}$. The expression (9.4.6) takes of course now the form

$$
\begin{aligned}
\pounds_V \omega = V(\Sigma)\mu &+ \Big[\frac{\partial V(\Sigma^i)}{\partial x^j}\, dx^j + \frac{\partial V(\Sigma^i)}{\partial u}\, du + \frac{\partial V(\Sigma^i)}{\partial v_j}\, dv_j \\
&+ \Sigma\Big(\frac{\partial X^i}{\partial x^j}\, dx^j + \frac{\partial X^i}{\partial u}\, du + \frac{\partial X^i}{\partial v_j}\, dv_j\Big)\Big] \wedge \mu_i - \Big(\frac{\partial X^j}{\partial x^k}\, dx^k \\
&+ \frac{\partial X^j}{\partial u}\, du + \frac{\partial X^j}{\partial v_k}\, dv_k\Big) \wedge \Big[\frac{\partial \Sigma^i}{\partial x^l}\, dx^l + \frac{\partial \Sigma^i}{\partial u}\, du + \frac{\partial \Sigma^i}{\partial v_l}\, dv_l\Big] \wedge \mu_{ji}.
\end{aligned}
$$

By employing the transformation $du = \sigma + v_i\, dx^i$ on the above relation, we can finally obtain

$$
\begin{aligned}
\pounds_V \omega = (A + A^i v_i)\mu + A^i \sigma \wedge \mu_i + (A^{ij} + 2B^{ikj} v_k)\, dv_j \wedge \mu_i \\
+ B^{ijk} dv_k \wedge \sigma \wedge \mu_{ji} + C^{ijkl} dv_l \wedge dv_k \wedge \mu_{ji}.
\end{aligned}
$$

On the other hand, (9.4.12) can now be cast into

$$\pounds_V \omega = \lambda\, \omega + \gamma \wedge \sigma + \Gamma \wedge d\sigma = \lambda\Big(\Sigma + \frac{\partial \Sigma^i}{\partial x^i} + \frac{\partial \Sigma^i}{\partial u} v_i\Big)\mu +$$

$$+\lambda\frac{\partial\Sigma^i}{\partial v_j}\,dv_j\wedge\mu_i+\gamma\wedge\sigma-\Gamma\wedge dv_m\wedge dx^m$$

where $\lambda\in\Lambda^0(\mathcal{C}_m),\ \gamma\in\Lambda^{n-1}(\mathcal{C}_m),\ \ \Gamma\in\Lambda^{n-2}(\mathcal{C}_m)$. In order to be able to compare the two expressions above for $£_V\omega$, let us choose this time

$$\Gamma=\gamma^{ij}\mu_{ij}+\gamma^{ijkl}dv_l\wedge\mu_{ijk},\qquad\gamma^{ij},\gamma^{ijkl}\in\Lambda^0(\mathcal{C}_m).$$

Because of the antisymmetry of the forms $\mu_{ijk}\in\Lambda^3(M)$, we can take the functions γ^{ijkl} as completely antisymmetric with respect to indices i,j,k without loss of generality. Since $(5.5.16)_2$ allows us to write

$$dx^m\wedge\mu_{ijk}=\delta_i^m\mu_{jk}+\delta_j^m\mu_{ki}+\delta_k^m\mu_{ij}$$

we easily obtain

$$\begin{aligned}-\Gamma\wedge dv_m\wedge dx^m&=-dv_m\wedge dx^m\wedge\Gamma=-\gamma^{ij}dv_m\wedge dx^m\wedge\mu_{ij}\\&\quad+\gamma^{ijkl}dv_m\wedge dv_l\wedge dx^m\wedge\mu_{ijk}=\gamma^{ij}\left(-dv_i\wedge\mu_j+dv_j\wedge\mu_i\right)\\&\quad+\gamma^{ijkl}(dv_i\wedge dv_l\wedge\mu_{jk}+dv_j\wedge dv_l\wedge\mu_{ki}+dv_k\wedge dv_l\wedge\mu_{ij})\\&=2\gamma^{ij}dv_j\wedge\mu_i+3\gamma^{ijkl}dv_l\wedge dv_k\wedge\mu_{ji}\end{aligned}$$

But, exterior products appearing in the second form on the right hand side in the last line above are antisymmetric in indices k and l. This entails that the functions γ^{ijkl} should be completely antisymmetric with respect to all superscripts. We thus arrive at the following relations

$$\begin{aligned}\lambda\Big(\Sigma+\frac{\partial\Sigma^i}{\partial x^i}+\frac{\partial\Sigma^i}{\partial u}v_i\Big)&=A+A^iv_i,\\\lambda\frac{\partial\Sigma^i}{\partial v_j}+2\gamma^{ij}&=A^{ij}+2B^{ikj}v_k,\\\gamma&=(-1)^{n-1}A^i\mu_i+B^{ijk}\mu_{ji}\wedge dv_k,\\3\gamma^{ijkl}&=C^{ijkl}.\end{aligned}$$

Since the functions $\gamma^{ij},\gamma^{ijkl}$ are completely antisymmetric, we then obtain

$$\begin{aligned}4\gamma^{ij}&=A^{ij}-A^{ji}-(B^{kij}-B^{kji})v_k-\lambda\Big(\frac{\partial\Sigma^i}{\partial v_j}-\frac{\partial\Sigma^j}{\partial v_i}\Big),\\3\gamma^{ijkl}&=C^{[ijkl]}=C^{i[jk]l}\end{aligned}$$

where in the last line, we have made use of antisymmetries of the functions C^{ijkl} with respect to pairs of indices (i,j) and (k,l). Hence, *the determining equations for the isovector components corresponding to the case* $m=1$

take the following special forms

$$\lambda\Big(\Sigma+\frac{\partial\Sigma^i}{\partial x^i}+\frac{\partial\Sigma^i}{\partial u}v_i\Big)=A+A^iv_i, \tag{9.4.18}$$
$$\lambda\Big(\frac{\partial\Sigma^i}{\partial v_j}+\frac{\partial\Sigma^j}{\partial v_i}\Big)=A^{ij}+A^{ji}-2(B^{kij}+B^{kji})v_k,$$
$$C^{ijkl}+C^{ikjl}=0.$$

We know that isovector fields of the balance ideal constitute a Lie algebra, and this algebra in turn induces a Lie group of transformations. This group is called the ***symmetry group*** of the system of differential equations. If we obtain r linearly independent isovectors V_a from the determining equations, then any isovector field may be represented by $V=c^aV_a$ where $c^a\in\mathbb{R}, a=1,\dots,r$ are arbitrary constants. In this case, the symmetry group becomes an r-dimensional Lie group. If arbitrary functions are involved in isovector components, then the Lie group turns out to be infinite dimensional. Let a regular mapping $\phi:\mathcal{D}_n\to\mathcal{C}_m$ be a solution of the balance ideal $\mathfrak{I}_m$ satisfying the conditions $\phi^*\sigma^\alpha_{i_1i_2\cdots i_r}=0, r=0,1,\dots,m-1;$ $\phi^*\omega^\alpha=0$. If V is an isovector field of this ideal, we had already shown in Theorem 5.13.7 that the mappings

$$\phi_V(t)=e^{tV}\circ\phi:\mathcal{D}_n\to\mathcal{C}_m,$$

where t is a real parameter, constitute a 1-parameter family of solutions of the ideal $\mathfrak{I}_m$. Let us recall that the mapping $\phi:\mathcal{D}_n\to\mathcal{C}_m$ is obtained by *lifting* the solution mapping $\phi:\mathcal{D}_n\to G$ specified by $u^\alpha=\phi^\alpha(x^i)$. Hence, to determine the family of solutions $\phi_V(t):\mathcal{D}_n\to\mathcal{C}_m$ when $N>1$, we have to solve the following set of autonomous ordinary differential equations

$$\frac{d\overline{x}^i}{dt}=X^i(\overline{\mathbf{x}},\overline{\mathbf{u}}),\qquad\qquad \frac{d\overline{u}^\alpha}{dt}=U^\alpha(\overline{\mathbf{x}},\overline{\mathbf{u}}) \tag{9.4.19}$$
$$\frac{d\overline{v}^\alpha_{i_1\cdots i_r}}{dt}=V_{i_1\cdots i_r}(\overline{\mathbf{x}},\overline{\mathbf{u}},\overline{v}^\alpha_{i_1},\dots,\overline{v}^\alpha_{i_1\cdots i_r}), r=1,\dots,m$$

under the initial conditions $\overline{\mathbf{x}}(0)=\mathbf{x}$, $\overline{\mathbf{u}}(0)=\mathbf{u}$, $\overline{v}^\alpha_{i_1\cdots i_r}(0)=v^\alpha_{i_1\cdots i_r}$ where $r=1,\dots,m$. Let us now consider the vector field

$$V_G=X^i(\mathbf{x},\mathbf{u})\frac{\partial}{\partial x^i}+U^\alpha(\mathbf{x},\mathbf{u})\frac{\partial}{\partial u^\alpha}$$

which is the projection of the isovector onto the tangent bundle of the graph space G. We know that this vector induces a Lie group of diffeomorphisms mapping the manifold G onto itself. Let us suppose that a solution $\phi : \mathcal{D}_n \to G$ of the system of differential equations is depicted by the given expressions $u^\alpha = \phi^\alpha(x^i)$. If we represent the solution of $n + N$ ordinary differential equations $(9.4.19)_{1-2}$ under the initial conditions $\overline{\mathbf{x}}(0) = \mathbf{x}$ and $\overline{\mathbf{u}}(0) = \mathbf{u}$ by

$$\overline{x}^i = \psi^i(t; \mathbf{x}, \mathbf{u}), \quad \overline{u}^\alpha = \psi^\alpha(t; \mathbf{x}, \mathbf{u})$$

we find

$$\overline{x}^i = \psi^i(t; \mathbf{x}, \boldsymbol{\phi}(\mathbf{x})) = \Psi^i(t; \mathbf{x}), \quad \overline{u}^\alpha = \psi^\alpha(t; \mathbf{x}, \boldsymbol{\phi}(\mathbf{x})) = \Psi^i(t; \mathbf{x})$$

when we insert the original solution $\mathbf{u} = \boldsymbol{\phi}(\mathbf{x})$ into these relations. On solving the variables x^i in terms of $\overline{x}^i$ from the first set of equations and introduce the result into the second set, we ultimately obtain the family of solutions $\phi_V(t) : \mathcal{D}_n \to G$ in the form $\overline{u}^\alpha = \Phi_t^\alpha(\overline{x}^i)$. Hence, this procedure based on isovectors of the graph space enables us to produce a family of new, probably more complicated, solutions if we have at hand a solution, however simple, of the set of partial differential equations. But, it is clear that if we do not know a particular solution, this approach cannot help us at all to generate any new solution.

In the case of $N = 1$, the components X^i and U of the isovector fields will depend on the variables x^i, u, v_i. Hence, we can project isovectors only on the tangent bundle of the manifold $\mathcal{C}_1$. Consequently, to determine the group of transformations, we have to solve the following set of ordinary differential equations

$$\frac{d\overline{x}^i}{dt} = -\frac{\partial F}{\partial \overline{v}_i}, \ \frac{d\overline{u}}{dt} = F - \overline{v}_i \frac{\partial F}{\partial \overline{v}_i}, \ \frac{d\overline{v}_i}{dt} = \frac{\partial F}{\partial \overline{x}^i} + \overline{v}_i \frac{\partial F}{\partial \overline{u}},$$

under the initial conditions $\overline{x}^i(0) = x^i$, $\overline{u}(0) = u$, $\overline{v}_i(0) = v_i$. Here, the function $F = F(\mathbf{x}, u, \mathbf{v})$ determines the isovector components. When we accomplish to integrate these differential equations, we arrive at the result

$$\overline{x}^i = \psi^i(t; \mathbf{x}, u, \mathbf{v}), \ \ \overline{u} = \psi(t; \mathbf{x}, u, \mathbf{v}), \ \ \overline{v}_i = \psi_i(t; \mathbf{x}, u, \mathbf{v}).$$

Since, the transformations between (x^i, u) and $(\overline{x}^i, \overline{u})$ now involve derivatives $u_{,i}$, they become now ***Bäcklund transformations*** forming a group. A Bäcklund transformation reduces to a Lie transformation if and only if the function F is an affine function of variables v_i.

The approach we have developed so far to determine isovector fields of balance equations may also be used to find isovector fields associated with mth order non-linear partial differential equations given by (9.2.1) and taken into account at the beginning of this section. In this case, we have to take $\Sigma^{ai} \equiv 0$ and we write $\omega^a = \Sigma^a(x^i, u^\alpha, v_i^\alpha, \dots, v_{i_1 i_2 \cdots i_m}^\alpha)\mu$. Evidently, the *fundamental ideal* induced by n-forms ω^a is closed. Thus, if V is an isovector field of the contact ideal, then the functions X^i and U^α should be found from equations (9.4.11) such that $V(\Sigma^a) = 0$ when $\Sigma^a = 0$ or the function F from equations (9.4.17) or (9.4.18) such that $V(\Sigma) = 0$ when $\Sigma = 0$. This is quite a difficult procedure to accomplish. Nonetheless, if we consider a first order equation in the form $\Sigma(x^i, u, u_{,i}) = 0$, the isovector field can be found easily. In this case, we may choose $F(x^i, u, v_i) = -\Sigma$ to find the isovector components as follows

$$X^i = \frac{\partial \Sigma}{\partial v_i},\ U = -\Sigma + v_i \frac{\partial \Sigma}{\partial v_i},\ V_i = -\Big(\frac{\partial \Sigma}{\partial x^i} + v_i \frac{\partial \Sigma}{\partial u}\Big). \quad (9.4.20)$$

We thus obtain

$$V(\Sigma) = \frac{\partial \Sigma}{\partial v_i}\frac{\partial \Sigma}{\partial x^i} + \Big(v_i \frac{\partial \Sigma}{\partial v_i} - \Sigma\Big)\frac{\partial \Sigma}{\partial u} - \Big(\frac{\partial \Sigma}{\partial x^i} + v_i \frac{\partial \Sigma}{\partial u}\Big)\frac{\partial \Sigma}{\partial v_i} = -\frac{\partial \Sigma}{\partial u}\Sigma$$

implying that $V(\Sigma) = 0$ whenever $\Sigma = 0$. In this situation, we can obviously write

$$\mathbf{i}_V(\sigma) = U - v_i X^i = \Sigma = 0,\ \mathbf{i}_V(\omega) = \Sigma X^i \mu_i = 0,$$
$$\mathbf{i}_V(d\sigma) = -V_i dx^i + X^i dv_i = d\Sigma + \frac{\partial \Sigma}{\partial u}\sigma = \frac{\partial \Sigma}{\partial u}\sigma.$$

Hence, this isovector field is likewise a characteristic vector field of the ideal. Consequently, we again find the previously given solution (9.2.10) by employing this isovector field.

Example 9.4.1. The time-dependent, one-dimensional heat equation in a homogeneous medium, or more generally the one-dimensional diffusion equation modelling various physical phenomena is given by

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(\kappa(u)\frac{\partial u}{\partial x}\Big) + h(x, t, u) \quad (9.4.21)$$

where u is the temperature, t is the time, x is the spatial variable and $\kappa(u)$ is a constitutive quantity called the *coefficient of thermal diffusivity* that may be dependent on temperature and h is the heat source. Let us denote $x = x^1,\ t = x^2,\ v_1 = u_x,\ v_2 = u_t,\ \mu = dx \wedge dt,\ \mu_1 = dt,\ \mu_2 = -dx$. Then we arrive at the balance equation

$$\frac{\partial \Sigma^1}{\partial x^1} + \frac{\partial \Sigma^2}{\partial x^2} + \Sigma = 0$$

where

$$\Sigma^1 = \kappa(u)\,v_1,\ \Sigma^2 = -u,\ \Sigma = h(x^1, x^2, u).$$

In this case, the isovector field must be prescribed by

$$V = X^1 \frac{\partial}{\partial x^1} + X^2 \frac{\partial}{\partial x^2} + U \frac{\partial}{\partial u} + V_1 \frac{\partial}{\partial v_1} + V_2 \frac{\partial}{\partial v_2}$$

and its components are specified by relations $(9.3.26)_{1-2}$ through an arbitrary function $F = F(x^1, x^2, u, v_1, v_2) = F(x, t, u, v_1, v_2)$. With these data, non-zero coefficient functions in (9.4.13) become

$$\begin{aligned}
V(\Sigma^1) &= \kappa' v_1 U + \kappa V_1,\\
V(\Sigma^2) &= -U,\\
V(\Sigma) &= X^1 \frac{\partial h}{\partial x} + X^2 \frac{\partial h}{\partial t} + U \frac{\partial h}{\partial u}\\
A &= V(\Sigma) + \kappa' v_1 \frac{\partial U}{\partial x} + \kappa \frac{\partial V_1}{\partial x} - \frac{\partial U}{\partial t} + h\Big(\frac{\partial X^1}{\partial x} + \frac{\partial X^2}{\partial t}\Big)\\
A^1 &= \kappa'' v_1 U + \kappa' v_1 \frac{\partial U}{\partial u} + \kappa' V_1 + \kappa \frac{\partial V_1}{\partial u} + h \frac{\partial X^1}{\partial u} + \kappa' v_1 \frac{\partial X^2}{\partial t} + \frac{\partial X^1}{\partial t}\\
A^2 &= -\frac{\partial U}{\partial u} + h \frac{\partial X^2}{\partial u} - \frac{\partial X^1}{\partial x} - \kappa' v_1 \frac{\partial X^2}{\partial x}\\
A^{11} &= \kappa' U + \kappa' v_1 \frac{\partial U}{\partial v_1} + \kappa \frac{\partial V_1}{\partial v_1} + h \frac{\partial X^1}{\partial v_1} + \kappa \frac{\partial X^2}{\partial t}\\
A^{22} &= -\frac{\partial U}{\partial v_2} + h \frac{\partial X^2}{\partial v_2},\\
A^{12} &= \kappa' v_1 \frac{\partial U}{\partial v_2} + \kappa \frac{\partial V_1}{\partial v_2} + h \frac{\partial X^1}{\partial v_2},\\
A^{21} &= -\frac{\partial U}{\partial v_1} + h \frac{\partial X^2}{\partial v_1} - \kappa \frac{\partial X^2}{\partial x},\\
2B^{121} &= \kappa \frac{\partial X^2}{\partial u} - \frac{\partial X^1}{\partial v_1} - \kappa' v_1 \frac{\partial X^2}{\partial v_1},\\
2B^{122} &= -\frac{\partial X^1}{\partial v_2} - \kappa' v_1 \frac{\partial X^2}{\partial v_2}.
\end{aligned}$$

Hence, the determining equations (9.4.18) are found to be

$$\begin{aligned}
\lambda(\kappa' v_1^2 - v_2 + h) &= A + A^1 v_1 + A^2 v_2,\\
\lambda\kappa &= A^{11} + 2B^{121} v_2,\\
A^{22} - 2B^{122} v_1 &= 0,
\end{aligned}$$

$$A^{12} + A^{21} - 2B^{121}v_1 + 2B^{122}v_2 = 0$$

whence we extract, respectively, the following four equations to be satisfied by the single function F

$$\begin{aligned}
&\lambda(v_2 - h - \kappa' v_1^2) + \Big(\kappa'' v_1^2 + \frac{\partial h}{\partial u}\Big)F + 2\kappa' v_1 \frac{\partial F}{\partial x} + (2\kappa' v_1^2 - v_2)\frac{\partial F}{\partial u} \\
&\quad - \frac{\partial F}{\partial t} - \Big(\kappa'' v_1^3 + \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}v_1\Big)\frac{\partial F}{\partial v_1} - (\kappa'' v_1^2 v_2 + \frac{\partial h}{\partial t} + \frac{\partial h}{\partial u}v_2)\frac{\partial F}{\partial v_2} \\
&\quad + (v_2 - h - \kappa' v_1^2)\Big(\frac{\partial^2 F}{\partial x \partial v_1} + \frac{\partial^2 F}{\partial t \partial v_2} + v_1 \frac{\partial^2 F}{\partial u \partial v_1} + v_2 \frac{\partial^2 F}{\partial u \partial v_2}\Big) \\
&\quad + \kappa \frac{\partial^2 F}{\partial x^2} + 2\kappa v_1 \frac{\partial^2 F}{\partial x \partial u} + \kappa v_1^2 \frac{\partial^2 F}{\partial u^2} = 0, \\
&\kappa'\Big(F - v_1 \frac{\partial F}{\partial v_1} - v_2 \frac{\partial F}{\partial v_2}\Big) + (v_2 - h - \kappa' v_1^2)\frac{\partial^2 F}{\partial v_1^2} + \kappa\Big(-\lambda + \frac{\partial F}{\partial u} \\
&\quad + \frac{\partial^2 F}{\partial x \partial v_1} - \frac{\partial^2 F}{\partial t \partial v_2} + v_1 \frac{\partial^2 F}{\partial u \partial v_1} - v_2 \frac{\partial^2 F}{\partial u \partial v_2}\Big) = 0, \qquad (9.4.22) \\
&(v_2 - h - \kappa' v_1^2)\frac{\partial^2 F}{\partial v_2^2} = 0, \\
&(v_2 - h - \kappa' v_1^2)\frac{\partial^2 F}{\partial v_1 \partial v_2} + \kappa\Big(\frac{\partial^2 F}{\partial x \partial v_2} + v_1 \frac{\partial^2 F}{\partial u \partial v_2}\Big) = 0.
\end{aligned}$$

$(9.4.22)_3$ yields

$$F = f(x,t,u,v_1)\, v_2 + g(x,t,u,v_1).$$

On inserting this expression into the equation $(9.4.22)_4$, we find that

$$\kappa\Big(\frac{\partial f}{\partial x} + v_1 \frac{\partial f}{\partial u}\Big) - (\kappa' v_1^2 + h)\frac{\partial f}{\partial v_1} + v_2 \frac{\partial f}{\partial v_1} = 0$$

from which we obviously obtain

$$\frac{\partial f}{\partial v_1} = 0, \;\; \frac{\partial f}{\partial u} = 0, \;\; \frac{\partial f}{\partial x} = 0.$$

Hence, we see that F must take the form

$$F = f(t)\, v_2 + g(x,t,u,v_1).$$

Let us now introduce this function F into equations $(9.4.22)_{1-2}$ and eliminate the function λ between these two equations so obtained. If we then equate the coefficient of the variable v_2^2 to zero in this expression, we are led to the simple partial differential equation $\partial^2 g/\partial v_1^2 = 0$ whose solution is immediately obtained as

$$g = \alpha(x,t,u)v_1 + \beta(x,t,u)$$

The coefficient of v_2 in the same expression gives

$$-\kappa f' + \kappa'\beta + 2\kappa\frac{\partial\alpha}{\partial x} + 2\kappa\frac{\partial\alpha}{\partial u}v_1 = 0$$

whence we deduce $\partial\alpha/\partial u = 0$ and $g = \alpha(x,t)\,v_1 + \beta(x,t,u)$. Let us insert this function g into that expression. Then the vanishing of the coefficients of variables v_1, v_1^2 and v_2 together with the remaining expression lead, respectively, to the equations

$$\begin{aligned}
&-\kappa\frac{\partial\alpha}{\partial t} + \kappa^2\frac{\partial^2\alpha}{\partial x^2} + 2\kappa\kappa'\frac{\partial\beta}{\partial x} + 2\kappa^2\frac{\partial^2\beta}{\partial x\partial u} = 0, \qquad (9.4.23)\\
&\big[\kappa\kappa'' - (\kappa')^2\big]\beta + \kappa\kappa'\frac{\partial\beta}{\partial u} + \kappa^2\frac{\partial^2\beta}{\partial u^2} = 0,\\
&-\kappa f' + \kappa'\beta + 2\kappa\frac{\partial\alpha}{\partial x} = 0,\\
&\Big(h\frac{\kappa'}{\kappa} - \frac{\partial h}{\partial u}\Big)\beta + h\Big(2\frac{\partial\alpha}{\partial x} + \frac{\partial\beta}{\partial u}\Big) + \frac{\partial\beta}{\partial t} + f\frac{\partial h}{\partial t} + \alpha\frac{\partial h}{\partial x} - \kappa\frac{\partial^2\beta}{\partial x^2} = 0,
\end{aligned}$$

The second order differential equation in $(9.4.23)_2$ can be written as

$$\frac{\partial^2\beta}{\partial u^2} + \frac{\kappa'}{\kappa}\frac{\partial\beta}{\partial u} + \Big(\frac{\kappa'}{\kappa}\Big)'\beta = \frac{\partial^2\beta}{\partial u^2} + \frac{\partial}{\partial u}\Big(\frac{\kappa'}{\kappa}\beta\Big) = 0$$

from which we get

$$\frac{\partial\beta}{\partial u} + \frac{\kappa'}{\kappa}\beta = n(x,t)$$

yielding

$$\frac{\partial}{\partial u}(\kappa\beta) = n(x,t)\kappa(u).$$

We thus obtain

$$\beta(x,t,u) = \frac{1}{\kappa(u)}\big[m(x,t) + n(x,t)K(u)\big]$$

where m and n are arbitrary functions of their arguments and we introduce the indefinite integral $K(u) = \int\kappa(u)\,du$ so that we have $\kappa(u) = K'(u)$.

When we insert this expression into the first equation in (9.4.23), we get

$$-\frac{\partial\alpha}{\partial t}+\kappa(u)\Big(2\frac{\partial n}{\partial x}+\frac{\partial^2\alpha}{\partial x^2}\Big)=0.$$

Whenever the function $\kappa(u)$ is not a constant, the foregoing equation can only be satisfied if

$$\alpha=\alpha(x),\qquad n=-\tfrac{1}{2}\alpha'(x)+n_0(t)$$

Therefore, the function F must be in the form

$$F=f(t)\,v_2+\alpha(x)\,v_1+\frac{1}{K'(u)}\big[m(x,t)+[-\tfrac{1}{2}\alpha'(x)+n_0(t)]K(u)\big]$$

Furthermore, the third and the fourth equations in (9.4.23) should also be satisfied:

$$K'(u)^2\big(2\alpha'(x)+f'(t)\big)+ K''(u)\Big[m(x,t)+\big(n_0(t)-\tfrac{1}{2}\alpha'(x)\big)K(u)\Big]=0$$

$$K'(u)h(x,t,u)\big(2n_0(t)+3\alpha'(x)\big)+K(u)\Big[2n_0'(t)+K'(u)\frac{\partial^3\alpha}{\partial x^3}+\big(\alpha'(x)-2n_0(t)\big)\frac{\partial h}{\partial u}\Big]+2\Big(\frac{\partial m}{\partial t}-m(x,t)\frac{\partial h}{\partial u}\Big)+2K'(u)\Big(f(t)\frac{\partial h}{\partial t}+\alpha(x)\frac{\partial h}{\partial x}-\frac{\partial^2 m}{\partial x^2}\Big)=0.$$

These equations restrict the admissible forms of functions $f(t)$, $\alpha(x)$, $n_0(t)$, $m(x,t)$ and structures of physical data $\kappa(u)$ and $h(x,t,u)$ so that isovector fields are realisable. Interested readers can determine admissible choices without experiencing too much difficulties by scrutinising these equations. It is clearly seen that the function F is an affine function of the variables v_1 and v_2. Therefore, in this case isovectors will be prolongations of the vectors V_G.

As a special case, we take $\kappa=1, h=0$. Hence, the field equation becomes

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}.$$

This equation is obtained by non-dimensionalising the heat conduction equation in the absence of the heat source and by assuming that the coefficient of thermal diffusivity is constant. Equations (9.4.23) are reduced in this case to the form

$$\begin{aligned}-\frac{\partial\alpha}{\partial t}+\frac{\partial^2\alpha}{\partial x^2}+2\frac{\partial^2\beta}{\partial x\partial u}&=0,\\ \frac{\partial^2\beta}{\partial u^2}=0,\ \frac{\partial\beta}{\partial t}-\frac{\partial^2\beta}{\partial x^2}&=0,\\ -f'+2\frac{\partial\alpha}{\partial x}&=0.\end{aligned}$$

The second and the fourth equations yield

$$\beta=\lambda(x,t)u+\mu(x,t),\quad \alpha=\frac{1}{2}f'(t)x+\gamma(t).$$

If we insert these expressions into the first and the third equations, we obtain

$$-\frac{1}{2}f''(t)x-\gamma'(t)+2\frac{\partial\lambda}{\partial x}=0,\ \frac{\partial\lambda}{\partial t}-\frac{\partial^2\lambda}{\partial x^2}=0,\ \frac{\partial\mu}{\partial t}-\frac{\partial^2\mu}{\partial x^2}=0$$

We then introduce the function

$$\lambda=\frac{1}{8}f''(t)x^2+\frac{1}{2}\gamma'(t)x+\delta(t),$$

found from integrating the first equation above, into the second equation, we arrive at

$$\frac{1}{8}f'''(t)x^2+\frac{1}{2}\gamma''(t)x+\delta'(t)-\frac{1}{4}f''(t)=0$$

whence we obviously deduce the relations

$$f'''(t)=0,\ \gamma''(t)=0,\ \delta'(t)=\frac{1}{4}f''(t).$$

We therefore find

$$f(t)=4c_1t^2+2c_2t+c_3,\quad \gamma(t)=2c_4t+c_5,\quad \delta(t)=2c_1t+c_6$$

where $c_1,\dots,c_6$ are arbitrary constants. Hence, the function F is expressible as follows

$$\begin{aligned}F&=\big[(4c_1t+c_2)x+2c_4t+c_5\big]v_1+(4c_1t^2+2c_2t+c_3)v_2\\&\qquad+\big[c_1(x^2+2t)+c_4x+c_6\big]u+\mu(x,t)\\&=c_1\big[4txv_1+4t^2v_2+(x^2+2t)u\big]\\&\quad+c_2(xv_1+2tv_2)+c_3v_2+c_4(2tv_1+xu)+c_5v_1+c_6u+\mu(x,t).\end{aligned}$$

The function $\mu(x,t)$ is any solution of the linear equation $\dfrac{\partial\mu}{\partial t}-\dfrac{\partial^2\mu}{\partial x^2}=0$. Thus, the linearly independent isovectors are found to be

$$\begin{aligned}
V^1 =& -4xt\frac{\partial}{\partial x}-4t^2\frac{\partial}{\partial t}+(x^2+2t)u\frac{\partial}{\partial u}+\big[(x^2+6t)v_1+2xu\big]\frac{\partial}{\partial v_1}\\
&+\big[4xv_1+(x^2+10t)v_2+2u\big]\frac{\partial}{\partial v_2},\\
V^2 =& -x\frac{\partial}{\partial x}-2t\frac{\partial}{\partial t}+v_1\frac{\partial}{\partial v_1}+2v_2\frac{\partial}{\partial v_2},\ V^3=-\frac{\partial}{\partial t},\ V^5=-\frac{\partial}{\partial x},\\
V^4 =& -2t\frac{\partial}{\partial x}+xu\frac{\partial}{\partial u}+(u+xv_1)\frac{\partial}{\partial v_1}+(2v_1+xv_2)\frac{\partial}{\partial v_2},\\
V^6 =& u\frac{\partial}{\partial u}+v_1\frac{\partial}{\partial v_1}+v_2\frac{\partial}{\partial v_2},\qquad V_\mu=\mu\frac{\partial}{\partial u}+\frac{\partial\mu}{\partial x}\frac{\partial}{\partial v_1}+\frac{\partial\mu}{\partial t}\frac{\partial}{\partial v_2}.
\end{aligned}$$

That these vectors constitute a Lie algebra as it should be can be observed at once from the relations

$$\begin{aligned}
&[V^1,V^2]=2V^1,[V^1,V^3]=4V^2+2V^6,[V^1,V^4]=0,[V^1,V^5]=2V^4,\\
&[V^1,V^6]=0,\ [V^2,V^3]=-2V^3,\ [V^2,V^4]=-V^4,\ [V^2,V^5]=V^5,\\
&[V^2,V^6]=0,\quad [V^3,V^4]=-2V^5,\quad [V^3,V^5]=0,\quad [V^3,V^6]=0,\\
&[V^4,V^5]=V^6,\quad [V^4,V^6]=0,\quad [V^5,V^6]=0,\quad [V^3,V_\mu]=-V_{\mu_t},\\
&[V^1,V_\mu]=-2V_{x^2\mu+2\mu t+4xt\mu_x+4t^2\mu_t},\qquad [V^2,V_\mu]=-V_{x\mu_x+2t\mu_t},\\
&[V^4,V_\mu]=-V_{x\mu+2t\mu_x},[V^5,V_\mu]=-V_{\mu_x},[V^6,V_\mu]=-V_\mu,[V_\mu,V_\nu]=0.
\end{aligned}$$

Here, we have defined $\mu_x=\partial\mu/\partial x$ and $\mu_t=\partial\mu/\partial t$. Since isovectors are prolongations of vectors of the form V_G, it would suffice to integrate the differential equations $(9.4.19)_{1,2}$ in order to determine the associated symmetry groups. To simplify the operations, let us take isovectors into consideration one by one:

The isovector V^1 gives rise to the ordinary differential equations

$$\frac{d\overline{x}}{ds}=-4\overline{x}\,\overline{t},\ \ \frac{d\overline{t}}{ds}=-4\overline{t}^2,\ \ \frac{d\overline{u}}{ds}=(\overline{x}^2+2\,\overline{t})\overline{u}$$

whose solutions under the initial conditions $\overline{x}(0)=x,\overline{t}(0)=t,\overline{u}(0)=u$ are readily found to be

$$\overline{x}(s)=\frac{x}{1+4st},\ \ \overline{t}(s)=\frac{t}{1+4st},\ \ \overline{u}(s)=u(x,t)\sqrt{1+4st}\,e^{\frac{sx^2}{1+4st}}.$$

It then follows from these relations that

$$x = \frac{\overline{x}}{1-4s\overline{t}}, \qquad t = \frac{\overline{t}}{1-4s\overline{t}}.$$

We thus manufacture a 1-parameter family of new solutions of the partial differential equation

$$\frac{\partial \overline{u}}{\partial \overline{t}} = \frac{\partial^2 \overline{u}}{\partial \overline{x}^2}$$

from a given solution $u(x,t)$ by the following manner

$$\overline{u}(\overline{x},\overline{t};s) = u\Big(\frac{\overline{x}}{1-4s\overline{t}}, \frac{\overline{t}}{1-4s\overline{t}}\Big)\frac{1}{\sqrt{1-4s\overline{t}}}\, e^{\frac{s\overline{x}^2}{1-4s\overline{t}}}.$$

In other words, if a function $u(x,t)$ is a solution of the heat conduction equation under consideration, then the family of functions

$$u\Big(\frac{x}{1-4st}, \frac{t}{1-4st}\Big)\frac{1}{\sqrt{1-4st}}\, e^{\frac{sx^2}{1-4st}}$$

become also solutions of the same equation. For instance, the trivially obtained simple solution $u(x,t)=1$ gives rise to the family of new solutions $u(x,t) = e^{\frac{sx^2}{1-4st}}/\sqrt{1-4st}$.

If we consider the equations

$$\frac{d\overline{x}}{ds} = -\,\overline{x}\,, \quad \frac{d\overline{t}}{ds} = -\,2\overline{t}, \quad \frac{d\overline{u}}{ds} = 0$$

corresponding to the isovector V^2, we obtain

$$\begin{aligned} &\overline{x}(s) = x\,e^{-s}, \ \ \overline{t}(s) = t\,e^{-2s}, \\ &\overline{u}(s) = u. \end{aligned}$$

This result implies that a solution is invariant under a *scaling transformation*: $u(x,t) = u(\lambda x, \lambda^2 t)$ where λ is a constant.

The isovector V^4 generates the differential equations

$$\frac{d\overline{x}}{ds} = -\,2\overline{t}, \quad \frac{d\overline{t}}{ds} = 0, \quad \frac{d\overline{u}}{ds} = \overline{x}\,\overline{u}$$

whose solution is

$$\overline{x}(s) = x\,-2st, \ \ \overline{t}(s) = t, \ \ \overline{u}(s) = u\,e^{sx-s^2t}.$$

Hence, if $u(x,t)$ is a solution, then the function

$$u(x+2st,t)\,e^{sx+s^2t}$$

provides a family of solutions. For example, the trivial solution $u = x + a$ leads to

$$u = (x+2st+a)\,e^{sx+s^2t}.$$

We can easily check that the isovectors V^3, V^5 and V^6, respectively, give rise to transformations

$$\begin{aligned}
&\overline{x} = x, \quad \overline{t} = t - s, \quad \overline{u} = u;\\
&\overline{x} = x\ - s, \quad \overline{t} = t, \quad \overline{u} = u;\\
&\overline{x} = x, \quad \overline{t} = t, \quad \overline{u} = u\,e^s.
\end{aligned}$$

These transformations mean that solutions are invariant under translations in the temporal and spatial variables, and by multiplications with constants.

The isovector V_μ gives

$$\overline{x} = x, \quad \overline{t} = t, \quad \overline{u} = u + s\mu(x,t).$$

This is an expected result associated with linear equations reflecting the fact that solutions may be superimposed. ■

Example 9.4.2. As a more complicated example, let us consider the non-homogeneous ***Korteweg-de Vries equation*** [after Dutch mathematicians Diederik Johannes Korteweg (1848-1941) and Gustav de Vries (1866-1934)]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = f(x,t,u). \tag{9.4.24}$$

where t and x denote the time and the space variables. This equation models the propagation of solitons in a medium. We denote the independent variables by $x^1 = x$, $x^2 = t$. We introduce the auxiliary variables $v_1 = u_{,1}$, $v_2 = u_{,2}$, $v_{11} = u_{,11}$, $v_{22} = u_{,22}$ and $v_{12} = v_{21} = u_{,12}$. Hence, (9.4.24) is transformed into the first order equation

$$\frac{\partial v_{11}}{\partial x} + uv_1 + v_2 - f = 0.$$

We thus have

$$\begin{aligned}
\Sigma^1 &= v_{11}, \quad \Sigma^2 = 0,\\
\Sigma &= uv_1 + v_2 - f.
\end{aligned}$$

The isovector field is prescribed by

$$V = X^1\frac{\partial}{\partial x^1} + X^2\frac{\partial}{\partial x^2} + U\frac{\partial}{\partial u} + V_1\frac{\partial}{\partial v_1} + V_2\frac{\partial}{\partial v_2} + V_{11}\frac{\partial}{\partial v_{11}} + V_{12}\frac{\partial}{\partial v_{12}} + V_{22}\frac{\partial}{\partial v_{22}}.$$

The components of this vector field are given by the expressions (9.3.26). Because of the relations

$$V(\Sigma^1) = V_{11}, \qquad V(\Sigma^2) = 0,$$
$$V(\Sigma) = -X^1\frac{\partial f}{\partial x} - X^2\frac{\partial f}{\partial t} + U\Big(v_1 - \frac{\partial f}{\partial u}\Big) + V_1 u + V_2$$

we realise that we need only to know explicit forms of the following relevant components

$$X^1 = -\frac{\partial F}{\partial v_1}, \quad X^2 = -\frac{\partial F}{\partial v_2},$$
$$U = F - v_1\frac{\partial F}{\partial v_1} - v_2\frac{\partial F}{\partial v_2},$$
$$V_1 = \frac{\partial F}{\partial x} + v_1\frac{\partial F}{\partial u},$$
$$V_2 = \frac{\partial F}{\partial t} + v_2\frac{\partial F}{\partial u},$$
$$V_{11} = \frac{\partial^2 F}{\partial x^2} + 2v_1\frac{\partial^2 F}{\partial x\partial u} + v_1^2\frac{\partial^2 F}{\partial u^2} + v_{11}\Big(2\frac{\partial^2 F}{\partial x\partial v_1} + 2v_1\frac{\partial^2 F}{\partial u\partial v_1} + \frac{\partial F}{\partial u}\Big) + 2v_{12}\Big(\frac{\partial^2 F}{\partial x\partial v_2} + v_1\frac{\partial^2 F}{\partial u\partial v_2}\Big) + v_{11}^2\frac{\partial^2 F}{\partial v_1^2} + v_{12}^2\frac{\partial^2 F}{\partial v_2^2} + 2v_{11}v_{12}\frac{\partial^2 F}{\partial v_1\partial v_2}$$

where $F = F(x,t,u,v_1,v_2)$ is presently an arbitrary function. The coefficients given in (9.4.13) that are not identically zero can now be evaluated as follows

$$A = -X^1\frac{\partial f}{\partial x} - X^2\frac{\partial f}{\partial t} + U\Big(v_1 - \frac{\partial f}{\partial u}\Big) + V_1 u + V_2 + \frac{\partial V_{11}}{\partial x} - (uv_1 + v_2 - f)\Big(\frac{\partial^2 F}{\partial x\partial v_1} + \frac{\partial^2 F}{\partial t\partial v_2}\Big);$$
$$A^1 = \frac{\partial V_{11}}{\partial u} - (uv_1 + v_2 - f)\frac{\partial^2 F}{\partial u\partial v_1}, \quad A^2 = -(uv_1 + v_2 - f)\frac{\partial^2 F}{\partial u\partial v_2};$$

$$
\begin{aligned}
&A^{11} = \frac{\partial V_{11}}{\partial v_1} - (uv_1 + v_2 - f)\frac{\partial^2 F}{\partial v_1^2}, \qquad A^{22} = -(uv_1 + v_2 - f)\frac{\partial^2 F}{\partial v_2^2},\\
&A^{12} = \frac{\partial V_{11}}{\partial v_2} - (uv_1 + v_2 - f)\frac{\partial^2 F}{\partial v_1 \partial v_2}, A^{21} = -(uv_1 + v_2 - f)\frac{\partial^2 F}{\partial v_1 \partial v_2};\\
&A^{111} = \frac{\partial V_{11}}{\partial v_{11}} - \frac{\partial^2 F}{\partial t \partial v_2}, \qquad A^{112} = A^{121} = \frac{\partial V_{11}}{\partial v_{12}}, \qquad A^{122} = \frac{\partial V_{11}}{\partial v_{22}},\\
&A^{211} = \frac{\partial^2 F}{\partial x \partial v_2}, \qquad A^{212} = A^{221} = A^{222} = 0;\\
&B^{ijk} = 0; \quad C^{ijkl} = 0;\\
&B^{1211} = -B^{2111} = -\frac{\partial^2 F}{\partial u \partial v_2},\\
&C^{12111} = -C^{21111} = -\frac{1}{2}\frac{\partial^2 F}{\partial v_1 \partial v_2}, \; C^{12211} = -C^{21211} = -\frac{1}{2}\frac{\partial^2 F}{\partial v_2^2}.
\end{aligned}
$$

Thus $(9.4.17)_2$ takes the form

$$
\begin{aligned}
\lambda\Big(\frac{\partial \Sigma^i}{\partial v_{mn}} + \frac{\partial \Sigma^n}{\partial v_{mi}}\Big) &= A^{imn} + A^{nmi} + 2(B^{ijmn} + B^{njmi})v_j\\
&\quad + 2(C^{ijkmn} + C^{njkmi})v_{kj}.
\end{aligned}
$$

If we introduce the coefficient calculated above into these expressions, we end up with the independent equations given below

$$
\begin{aligned}
\frac{\partial V_{11}}{\partial v_{11}} - \frac{\partial^2 F}{\partial t \partial v_2} - \frac{\partial^2 F}{\partial u \partial v_2}v_2 - \frac{\partial^2 F}{\partial v_1 \partial v_2}v_{12} - \frac{\partial^2 F}{\partial v_2^2}v_{22} &= \lambda \quad (9.4.25)\\
\frac{\partial^2 F}{\partial x \partial v_2} + \frac{\partial^2 F}{\partial u \partial v_2}v_1 + \frac{\partial^2 F}{\partial v_2^2}v_{12} + \frac{\partial^2 F}{\partial v_1 \partial v_2}v_{11} &= 0.
\end{aligned}
$$

$(9.4.25)_2$ yields first

$$
\frac{\partial^2 F}{\partial v_2^2} = \frac{\partial^2 F}{\partial v_1 \partial v_2} = 0
$$

and consequently $F = \alpha(x,t,u)v_2 + \beta(x,t,u,v_1)$. We then get

$$
\frac{\partial \alpha}{\partial x} = \frac{\partial \alpha}{\partial u} = 0
$$

so we find that

$$
F = \alpha(t)v_2 + \beta(x,t,u,v_1).
$$

Hence, it follows from $(9.4.25)_1$ that

$$\lambda = 2\frac{\partial^2\beta}{\partial x\partial v_1} + 2\frac{\partial^2\beta}{\partial u\partial v_1}v_1 + \frac{\partial\beta}{\partial u} + 2\frac{\partial^2\beta}{\partial v_1^2}v_{11} - \alpha'(t).$$

Therefore, the expression

$$\lambda(uv_1 + v_2 - f) = \\ A + A^1 v_1 + A^2 v_2 + A^{11} v_{11} + A^{22} v_{22} + (A^{12} + A^{21})v_{12}$$

given by $(9.4.17)_1$ can be written as

$$\frac{\partial^3\beta}{\partial v_1^3}v_{11}^3 + 3\Big(\frac{\partial^3\beta}{\partial u\partial v_1^2}v_1 + \frac{\partial^2\beta}{\partial u\partial v_1} + \frac{\partial^3\beta}{\partial x\partial v_1^2}\Big)v_{11}^2 + 3\Big(-\frac{\partial^2\beta}{\partial v_1^2}(uv_1 - f) \\ + \frac{\partial^3\beta}{\partial u^2\partial v_1}v_1^2 + \frac{\partial^2\beta}{\partial u^2}v_1 + 2\frac{\partial^3\beta}{\partial x\partial u\partial v_1}v_1 + \frac{\partial^2\beta}{\partial x\partial u} + \frac{\partial^3\beta}{\partial x^2\partial v_1}\Big)v_{11} + \frac{\partial^3\beta}{\partial u^3}v_1^3 \\ + \Big(3\frac{\partial^3\beta}{\partial x\partial u^2} - 3\frac{\partial^2\beta}{\partial u\partial v_1}u - \frac{\partial\beta}{\partial v_1}\Big)v_1^2 + \Big(\beta + \frac{\partial f}{\partial u}\frac{\partial\beta}{\partial v_1} + 3\frac{\partial^2\beta}{\partial u\partial v_1}f - \\ 3\frac{\partial^2\beta}{\partial x\partial v_1}u + 3\frac{\partial^3\beta}{\partial x^2\partial u}\Big)v_1 - \Big(3\frac{\partial^2\beta}{\partial v_1^2}v_{11} + 3\frac{\partial^2\beta}{\partial u\partial v_1}v_1 + 3\frac{\partial^2\beta}{\partial x\partial v_1} - \alpha'\Big)v_2 \\ - \frac{\partial f}{\partial u}\beta + \frac{\partial f}{\partial t}\alpha + \frac{\partial f}{\partial x}\frac{\partial\beta}{\partial v_1} + f\frac{\partial\beta}{\partial u} + \frac{\partial\beta}{\partial t} + \frac{\partial\beta}{\partial x}u + 3f\frac{\partial^2\beta}{\partial x\partial v_1} + \frac{\partial^3\beta}{\partial x^3} = 0.$$

From the coefficient of v_2, we first obtain

$$\frac{\partial^2\beta}{\partial v_1^2} = 0 \quad \text{and} \quad \beta = \phi(x,t,u)v_1 + \psi(x,t,u),$$

then the relation

$$3\frac{\partial\phi}{\partial u}v_1 + \Big(3\frac{\partial\phi}{\partial x} - \alpha'(t)\Big) = 0$$

leads to

$$\frac{\partial\phi}{\partial u} = 0, \quad 3\frac{\partial\phi}{\partial x} - \alpha'(t) = 0$$

from which we obtain $\phi = \phi(x,t)$, and

$$\phi(x,t) = \frac{1}{3}\alpha'(t)x + \gamma(t).$$

If we insert these relations into the above equation, we see that coefficients of v_{11}^3 and v_{11}^2 vanish automatically while the coefficient of v_{11} gives

$$\frac{\partial^2\psi}{\partial u^2}v_1 + \frac{\partial^2\psi}{\partial x\partial u} = 0$$

whence we get

$$\psi(x,t,u) = \mu(t)u + \nu(x,t).$$

If we introduce this function into the remaining expression above and set the coefficient of v_1 to zero, we get

$$\Big(\mu(t) - \frac{2}{3}\alpha'(t)\Big)u + \nu(x,t) + \gamma'(t) + \frac{1}{3}\alpha''(t)x = 0$$

from which we obtain

$$\mu(t) = \frac{2}{3}\alpha'(t),\;\; \nu(x,t) = -\frac{1}{3}\alpha''(t)x - \gamma'(t).$$

The remaining term imposes the following restriction on the admissible forms of the functions f, α and γ

$$\Big(\frac{1}{3}\alpha'(t)x + \gamma(t)\Big)\frac{\partial f}{\partial x} + \alpha(t)\,\frac{\partial f}{\partial t} + \Big(\frac{1}{3}\alpha''(t)x + \gamma'(t) - \frac{2}{3}\alpha'(t)u\Big)\frac{\partial f}{\partial u} + \frac{5}{3}\alpha'(t)f + \frac{1}{3}\alpha''(t)u - \frac{1}{3}\alpha'''(t)x - \gamma''(t) = 0$$

in order that a nontrivial symmetry group exists. Together with this side condition, the function F is expressible as

$$F = \Big(\frac{1}{3}\alpha'(t)x + \gamma(t)\Big)v_1 + \alpha(t)v_2 + \frac{2}{3}\alpha'(t)u - \frac{1}{3}\alpha''(t)x - \gamma'(t)$$

depending on somewhat arbitrary functions $\alpha(t)$ and $\gamma(t)$. Therefore, isovectors are prolongation's of vectors V_G in tangent bundle of the graph space. Their components are given by

$$X^1 = -\frac{1}{3}\alpha'(t)x - \gamma(t),\qquad X^2 = -\alpha(t),$$
$$U = \frac{2}{3}\alpha'(t)u - \frac{1}{3}\alpha''(t)x - \gamma'(t).$$

In homogeneous Korteweg-de Vries equation we have $f = 0$ so that the functions $\alpha(t)$ and $\gamma(t)$ ought to satisfy the additional constraint

$$\frac{1}{3}\alpha''(t)u - \frac{1}{3}\alpha'''(t)x - \gamma''(t) = 0$$

whence we immediately obtain

$$\alpha(t) = 3c_1 t + c_2, \quad \gamma(t) = c_3 t + c_4.$$

Hence, the relevant isovector components become

$$X^1 = -c_1 x - c_3 t - c_4,\ X^2 = -3c_1 t - c_2,\ U = 2c_1 u - c_3$$

Consequently, parts of linearly independent isovectors in the tangent bundle of the graph space are designated as follows

$$\begin{aligned} V^1 &= -x\frac{\partial}{\partial x} - 3t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u}, \quad & V^2 &= -\frac{\partial}{\partial t}, \\ V^3 &= -t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}, & V^4 &= -\frac{\partial}{\partial x}. \end{aligned}$$

As an example, let us determine the admissible form of the function f leading to these isovectors. Assuming $c_1 \neq 0$, we define new constants by $c_2/c_1 = a_2$, $c_3/c_1 = a_3,\ c_4/c_1 = a_4$. Then the general solution of the differential equation

$$(x + a_3 t + a_4)\frac{\partial f}{\partial x} + (3t + a_2)\frac{\partial f}{\partial t} + (a_3 - 2u)\frac{\partial f}{\partial u} + 5f = 0$$

is found by resorting the method of characteristics as

$$f(x, t, u) = (a_3 - 2u)^{5/2} g(\xi, \eta)$$

where the characteristic variables are defined by

$$\begin{aligned} \xi &= \frac{x - \frac{1}{2}(a_2 a_3 - 2a_4 + a_3 t)}{(3t + a_2)^{1/3}}, \\ \eta &= (3t + a_2)^{2/3}(u - \frac{1}{2}a_3) \end{aligned}$$

It is immediately observed that the isovector V^1 generates the *scaling transformation*

$$\overline{x} = \lambda x,\ \ \overline{t} = \lambda^3 t\,,\ \ \overline{u} = u/\lambda^2$$

with $\lambda = e^{-s}$ whereas the isovector V^3 produces the group

$$\overline{x} = x - st,\ \ \overline{t} = t\,,\ \ \overline{u} = u - s.$$

The isovectors V^2 and V^4 induce, respectively, translations in the temporal and spatial variables.

We have to determine the integral curves of the isovector field in order to derive a family of solutions from a known solution $u(x,t)$. The differential equations to be integrated are

$$\begin{aligned}\frac{d\overline{x}}{ds} &= -c_2(a\overline{x}+b\overline{t}+c),\\ \frac{d\overline{t}}{ds} &= -c_2(3a\overline{t}+1),\\ \frac{d\overline{u}}{ds} &= c_2(2a\overline{u}-b),\ c_2\neq 0\end{aligned}$$

where we have defined $c_1/c_2 = a, c_3/c_2 = b, c_4/c_2 = c$. These equations are to be solved under the initial conditions $\overline{x}(0) = x, \overline{t}(0) = t, \overline{u}(0) = u$. This solution is easily found as

$$\begin{aligned}\overline{x}(s) &= \frac{2(b-3ac)+3\big[2a^2x+2ac-b(1+at)\big]e^{-c_2 s}+b(1+3at)e^{-3c_2 s}}{6a^2}\\ \overline{t}(s) &= \frac{(1+3at)e^{-3c_2 s}-1}{3a},\\ \overline{u}(s) &= \frac{\big[2au(x,t)-b\big]e^{2c_2 s}+b}{2a}\end{aligned}$$

■

Example 9.4.3. As an example to the case $N>1$, we shall treat the ***boundary layer*** equations associated with a semi-infinite flat plate along x-axis placed in a unidirectional flow of an incompressible viscous fluid. The field equations governing this flow are given by

$$\nu\frac{\partial^2 u}{\partial y^2} - u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} + \mathcal{U}(x)\mathcal{U}'(x) = 0,\ \ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

where u and v are velocity components along x- and y-axes, respectively, and the constant ν is the kinematic viscosity. $\mathcal{U}(x)$ is the velocity field in the direction of x-axis before the plate is installed into the flow. The second equation above represents the incompressibility condition. Boundary layer equations are highly useful approximations to exact equations of viscous flow known as Navier-Stokes equations [*see* Exercise **9.15**] Let us denote $x^1 = x$, $x^2 = y$, $u^1 = u$, $u^2 = v$, $v^1_1 = u_{,1}$, $v^1_2 = u_{,2}$, $v^2_1 = v_{,1}$ and $v^2_2 = v_{,2}$. Field equations then become

$$\nu\frac{\partial v^1_2}{\partial x^2} - u^1 v^1_1 - u^2 v^1_2 + f(x^1) = 0,\quad v^1_1 + v^2_2 = 0 \qquad (9.4.26)$$

where $f = \mathcal{U}\mathcal{U}'$. Since the variable v^2_2 is eliminated by $v^2_2 = -v^1_1$, the isovector field can be taken as

$$V = X^1\frac{\partial}{\partial x^1} + X^2\frac{\partial}{\partial x^2} + U^1\frac{\partial}{\partial u^1} + U^2\frac{\partial}{\partial u^2} + V_1^1\frac{\partial}{\partial v_1^1} + V_2^1\frac{\partial}{\partial v_1^1} + V_1^2\frac{\partial}{\partial v_1^2}.$$

Indeed, the relation $dv_1^1 + dv_2^2 = 0$ yields $V_2^2 = -V_1^1$. The functions X^1, X^2, U^1 and U^2 depend only on the variables x^1, x^2, u^1, u^2. The other components of the isovector field of the contact ideal follow from $(9.3.21)_1$:

$$\begin{aligned}
V_1^1 &= \frac{\partial U^1}{\partial x^1} - \Big(\frac{\partial X^1}{\partial x^1} - \frac{\partial U^1}{\partial u^1}\Big)v_1^1 + \frac{\partial U^1}{\partial u^2}v_1^2 - \frac{\partial X^2}{\partial x^1}v_2^1 - \frac{\partial X^1}{\partial u^1}(v_1^1)^2 \\
&\quad - \frac{\partial X^2}{\partial u^1}v_1^1 v_2^1 - \frac{\partial X^1}{\partial u^2}v_1^1 v_1^2 - \frac{\partial X^2}{\partial u^2}v_2^1 v_1^2 \qquad (9.4.27)\\
V_2^2 &= \frac{\partial U^2}{\partial x^2} - \Big(\frac{\partial X^2}{\partial x^2} - \frac{\partial U^2}{\partial u^2}\Big)v_2^2 + \frac{\partial U^2}{\partial u^1}v_2^1 - \frac{\partial X^1}{\partial x^2}v_1^2 - \frac{\partial X^2}{\partial u^2}(v_2^2)^2 \\
&\quad - \frac{\partial X^2}{\partial u^1}v_2^2 v_2^1 - \frac{\partial X^1}{\partial u^2}v_2^2 v_1^2 - \frac{\partial X^1}{\partial u^1}v_2^1 v_1^2 \\
V_2^1 &= \frac{\partial U^1}{\partial x^2} - \frac{\partial X^1}{\partial x^2}v_1^1 - \Big(\frac{\partial X^2}{\partial x^2} - \frac{\partial U^1}{\partial u^1}\Big)v_2^1 + \frac{\partial U^1}{\partial u^2}v_2^2 - \frac{\partial X^1}{\partial u^2}v_1^1 v_2^2 \\
&\quad - \frac{\partial X^2}{\partial u^1}(v_2^1)^2 - \frac{\partial X^1}{\partial u^1}v_1^1 v_2^1 - \frac{\partial X^2}{\partial u^2}v_2^2 v_2^1 \\
V_1^2 &= \frac{\partial U^2}{\partial x^1} - \frac{\partial X^2}{\partial x^1}v_2^2 + \frac{\partial U^2}{\partial u^1}v_1^1 - \Big(\frac{\partial X^1}{\partial x^1} - \frac{\partial U^2}{\partial u^2}\Big)v_1^2 - \frac{\partial X^2}{\partial u^1}v_1^1 v_2^2 \\
&\quad - \frac{\partial X^1}{\partial u^2}(v_1^2)^2 - \frac{\partial X^1}{\partial u^1}v_1^1 v_1^2 - \frac{\partial X^2}{\partial u^2}v_2^2 v_1^2
\end{aligned}$$

If we replace v_2^2 in these expressions by $-v_1^1$, we see that the condition the $V_2^2 + V_1^1 = 0$ is fulfilled provided that equations below are satisfied

$$\begin{aligned}
\frac{\partial U^1}{\partial x^1} + \frac{\partial U^2}{\partial x^2} &= 0, \qquad (9.4.28)\\
\frac{\partial X^1}{\partial x^1} - \frac{\partial X^2}{\partial x^2} - \frac{\partial U^1}{\partial u^1} + \frac{\partial U^2}{\partial u^2} &= 0,\\
\frac{\partial X^1}{\partial x^2} - \frac{\partial U^1}{\partial u^2} &= 0,\\
\frac{\partial X^2}{\partial x^1} - \frac{\partial U^2}{\partial u^1} &= 0,\\
\frac{\partial X^1}{\partial u^1} + \frac{\partial X^2}{\partial u^2} &= 0.
\end{aligned}$$

In view of the balance equation $(9.4.26)_1$, we have to take

$$\Sigma^{1i} = \nu v_2^1 \delta_2^i, \quad \Sigma^1 = -u^1 v_1^1 - u^2 v_2^1 + f.$$

We thus find $V(\Sigma^{11}) = 0$ and

$$V(\Sigma^{12}) = \nu V_2^1,$$
$$V(\Sigma^1) = -v_1^1 U^1 - v_2^1 U^2 - u^1 V_1^1 - u^2 V_2^1 + X^1 f'.$$

We then obtain from (9.4.8) that

$$\begin{aligned}
A^1 &= X^1 f' - v_1^1 U^1 - v_2^1 U^2 - u^1 V_1^1 - u^2 V_2^1 + \nu \frac{\partial V_2^1}{\partial x^2} \\
&\quad + (f - u^1 v_1^1 - u^2 v_2^1)\Big(\frac{\partial X^1}{\partial x^1} + \frac{\partial X^2}{\partial x^2}\Big), \\
A_\beta^{1i} &= \nu \frac{\partial V_2^1}{\partial u^\beta}\, \delta_2^i + (f - u^1 v_1^1 - u^2 v_2^1)\frac{\partial X^i}{\partial u^\beta}, \\
A_\beta^{1ij} &= \nu \Big[\frac{\partial V_2^1}{\partial v_j^\beta}\, \delta_2^i + \Big(\frac{\partial X^1}{\partial x^1} + \frac{\partial X^2}{\partial x^2}\Big)\, \delta_2^i \delta_2^j \delta_\beta^1 - \frac{\partial X^i}{\partial x^2}\, \delta_2^j \delta_\beta^1\Big], \; A_{\beta\gamma}^{1ij} = 0, \\
2A_{\beta\gamma}^{1ijk} &= -2A_{\beta\gamma}^{1jik} = \nu \Big(\frac{\partial X^j}{\partial u^\gamma}\, \delta_2^i - \frac{\partial X^i}{\partial u^\gamma}\, \delta_2^j\Big)\, \delta_2^k \delta_\beta^1.
\end{aligned}$$

Hence, the equations (9.4.11) can now be written as

$$\begin{aligned}
\lambda\Big(\frac{\partial \Sigma^{1i}}{\partial x^i} + \frac{\partial \Sigma^{1i}}{\partial u^\alpha} v_i^\alpha + \Sigma^1\Big) &= A^1 + A_\beta^{1i} v_i^\beta, \\
\lambda\Big(\frac{\partial \Sigma^{1i}}{\partial v_j^\beta} + \frac{\partial \Sigma^{1j}}{\partial v_i^\beta}\Big) &= A_\beta^{1ij} + A_\beta^{1ji} + 2(A_{\beta\gamma}^{1ikj} + A_{\beta\gamma}^{1jki}) v_k^\gamma
\end{aligned}$$

whose explicit forms become

$$\begin{aligned}
\lambda(f - u^1 v_1^1 - u^2 v_2^1) &= X^1 f' - v_1^1 U^1 - v_2^1 U^2 - u^1 V_1^1 - u^2 V_2^1 + \nu \frac{\partial V_2^1}{\partial x^2} \\
&\quad + \nu \frac{\partial V_2^1}{\partial u^\beta} v_2^\beta + (f - u^1 v_1^1 - u^2 v_2^1)\Big(\frac{\partial X^1}{\partial x^1} + \frac{\partial X^2}{\partial x^2} + \frac{\partial X^i}{\partial u^\beta} v_i^\beta\Big) \qquad (9.4.29) \\
2\lambda\, \delta_2^i \delta_2^j \delta_\beta^1 &= \frac{\partial V_2^1}{\partial v_j^\beta}\, \delta_2^i + \frac{\partial V_2^1}{\partial v_i^\beta}\, \delta_2^j \\
&\quad + 2\Big(\frac{\partial X^1}{\partial x^1} + \frac{\partial X^2}{\partial x^2} + \frac{\partial X^k}{\partial u^\gamma} v_k^\gamma\Big)\delta_2^i \delta_2^j \delta_\beta^1 \\
&\quad - \Big(\frac{\partial X^i}{\partial x^2} + \frac{\partial X^i}{\partial u^\gamma} v_2^\gamma\Big)\delta_2^j \delta_\beta^1 - \Big(\frac{\partial X^j}{\partial x^2} + \frac{\partial X^j}{\partial u^\gamma} v_2^\gamma\Big)\delta_2^i \delta_\beta^1
\end{aligned}$$

The second set of equations above leads immediately to

$$\begin{aligned}\lambda &= \frac{\partial V_2^1}{\partial v_2^1} + \frac{\partial X^1}{\partial x^1} + \frac{\partial X^1}{\partial u^1} v_1^1 + \frac{\partial X^1}{\partial u^2} v_1^2 \\ &= \frac{\partial U^1}{\partial u^1} + \frac{\partial X^1}{\partial x^1} - \frac{\partial X^2}{\partial x^2} + \frac{\partial X^2}{\partial u^2} v_1^1 - 2\frac{\partial X^2}{\partial u^1} v_2^1 + \frac{\partial X^1}{\partial u^2} v_1^2 \\ 0 &= \frac{\partial V_2^1}{\partial v_1^1} - \frac{\partial X^1}{\partial x^2} - \frac{\partial X^1}{\partial u^1} v_2^1 + \frac{\partial X^1}{\partial u^2} v_1^1, \\ 0 &= \frac{\partial V_2^1}{\partial v_2^2} = \frac{\partial V_2^1}{\partial v_1^2}.\end{aligned} \tag{9.4.30}$$

Because we had replaced v_2^2 by $-v_1^1$ the equations $(9.4.30)_3$ are satisfied identically whereas $(9.4.30)_2$ gives

$$2\frac{\partial X^1}{\partial x^2} + \frac{\partial U^1}{\partial u^2} - 3\frac{\partial X^1}{\partial u^2} v_1^1 + \Big(2\frac{\partial X^1}{\partial u^1} - \frac{\partial X^2}{\partial u^2}\Big) v_2^1 = 0$$

whence we extract the relations

$$\frac{\partial X^1}{\partial u^2} = 0, \quad \frac{\partial U^1}{\partial u^2} = -2\frac{\partial X^1}{\partial x^2}, \quad \frac{\partial X^2}{\partial u^2} = 2\frac{\partial X^1}{\partial u^1}.$$

If we insert these results into equations $(9.4.28)_{3,5}$, we find that

$$\frac{\partial X^1}{\partial x^2} = \frac{\partial X^1}{\partial u^1} = \frac{\partial X^2}{\partial u^2} = \frac{\partial U^1}{\partial u^2} = 0.$$

Consequently, at this stage we reach to the following components

$$\begin{aligned}&X^1 = \xi(x^1), \;\; X^2 = X^2(x^1, x^2, u^1), \\ &U^1 = U^1(x^1, x^2, u^1).\end{aligned}$$

Introducing these together with the relation $(9.4.30)_1$ into $(9.4.29)_1$, taking $v_2^2 = -v_1^1$ and arranging the resulting expression, we conclude that

$$\begin{aligned}&\nu\frac{\partial^2 X^2}{\partial (u^1)^2}(v_2^1)^3 - \Big(\nu\frac{\partial^2 U^1}{\partial (u^1)^2} - 2u^2\frac{\partial X^2}{\partial u^1} - 2\nu\frac{\partial^2 X^2}{\partial x^2 \partial u^1}\Big)(v_2^1)^2 + 2u^1\frac{\partial X^2}{\partial u^1} v_1^1 v_2^1 \\ &\quad - \Big(\nu\frac{\partial^2 X^2}{\partial (x^2)^2} + u^1\frac{\partial X^2}{\partial x^1} - u^2\frac{\partial X^2}{\partial x^2} + 3f\frac{\partial X^2}{\partial u^1} + 2\nu\frac{\partial^2 U^1}{\partial x^2 \partial u^1} - U^2\Big) v_2^1 + \\ &\Big[u^1\Big(2\frac{\partial X^2}{\partial x^2} - \frac{d\xi}{dx^1}\Big) + U^1\Big] v_1^1 - \nu\frac{\partial^2 U^1}{\partial (x^2)^2} + u^1\frac{\partial U^1}{\partial x^1} + u^2\frac{\partial U^1}{\partial x^2} \\ &\qquad\qquad + f\frac{\partial U^1}{\partial u^1} - 2f\frac{\partial X^2}{\partial x^2} - \xi f' = 0.\end{aligned}$$

Equating the coefficients of powers of v_1^1 and v_2^1 to zero, we readily see that the following relations are to be satisfied

$$\frac{\partial X^2}{\partial u^1} = 0, \qquad \frac{\partial^2 U^1}{\partial (u^1)^2} = 0, \tag{9.4.31}$$
$$U^1 = \Big(\frac{dX^1}{dx^1} - 2\frac{\partial X^2}{\partial x^2}\Big)u^1,\ U^2 = u^1\frac{\partial X^2}{\partial x^1} - u^2\frac{\partial X^2}{\partial x^2},$$
$$\frac{\partial U^1}{\partial x^2} = 0, \qquad \frac{\partial^2 X^2}{\partial (x^2)^2} = 0,$$
$$0 = u^1\frac{\partial U^1}{\partial x^1} + u^2\frac{\partial U^1}{\partial x^2} + f\frac{\partial U^1}{\partial u^1} - 2f\frac{\partial X^2}{\partial x^2} - X^1 f'.$$

These relations imply that

$$X^2 = \alpha(x^1)x^2 + \beta(x^1),$$
$$U^1 = \big[\xi'(x^1) - 2\alpha(x^1)\big]u^1,$$
$$U^2 = \big[\alpha'(x^1)x^2 + \beta'(x^1)\big]u^1 - \alpha(x^1)u^2.$$

If we insert these expressions into equations (9.4.28), then the first equation yields

$$\xi''(x^1) - \alpha'(x^1) = 0 \ \text{ and } \ \alpha(x^1) = \xi'(x^1) + c_0.$$

The other equations are satisfied identically. The last equation (9.4.31) takes the form

$$\xi''(u^1)^2 + \xi f' + (3\alpha + c_0)f = 0$$

so that we obtain $\xi'' = 0$ and

$$\xi(x^1) = c_1x^1 + c_2,\ \alpha(x^1) = c_1 + c_0 = c_3.$$

Therefore, the relevant components of the isovector field are found as

$$X^1 = c_1x^1 + c_2, \qquad X^2 = c_3x^2 + \beta(x^1), \tag{9.4.32}$$
$$U^1 = (c_1 - 2c_3)u^1, \quad U^2 = \beta'(x^1)u^1 - c_3u^2.$$

We see that the function f must satisfy $(c_1x^1 + c_2)f' + (4c_3 - c_1)f = 0$. On assuming $c_1 \neq 0$ and writing $c_2/c_1 = a, c_3/c_1 = b$ we realise that f has to be chosen in the form

$$\frac{1}{2}(\mathcal{U}^2)' = f(x^1) = A(x^1 + a)^{1-4b}$$

to be admitted by the symmetry group. Thus, the admissible velocity field is

$$\mathcal{U}(x^1) = \sqrt{B + \frac{A(x^1+a)^{2(1-2b)}}{1-2b}}.$$

If we take $c_1 = 0$, then we get $f(x^1) = Ae^{-cx^1}$ where $c = 4c_3/c_2$. Linearly independent isovectors are then given by

$$\begin{aligned}
V^1 &= x^1\frac{\partial}{\partial x^1} + u^1\frac{\partial}{\partial u^1},\\
V^2 &= \frac{\partial}{\partial x^1},\\
V^3 &= x^2\frac{\partial}{\partial x^2} - 2u^1\frac{\partial}{\partial u^1} - u^2\frac{\partial}{\partial u^2},\\
V_\beta &= \beta(x^1)\frac{\partial}{\partial x^2} + \beta'(x^1)u^1\frac{\partial}{\partial u^2}.
\end{aligned}$$

■

Example 9.4.4. As an example to a problem to determine symmetries of which proves to be quite difficult by using classical methods, we consider the equations governing the motion of a hyperelastic body[1]. In order to simplify the discussion, we shall employ Cartesian coordinates. We had denoted the location of a particle in the undeformed body by *material coordinates* X_K with $K = 1,2,3$ and the location of the same point at time t by *spatial coordinates* x_k with $k = 1,2,3$ [*see p.* 453]. The motion of the body was specified by a diffeomorphism represented by relations $x_k = x_k(\mathbf{X}, t)$. We know that a homogeneous hyperelastic medium is characterised by a given *stress potential* $\Sigma(\mathbf{C})$ where $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ is the deformation tensor. $\mathbf{F} = [x_{k,K}]$ is the tensor of deformation gradients. Here, we use the notation

$$F_{kK} = x_{k,K} = \frac{\partial x_k}{\partial X_K}$$

The equations of motion of an hyperelastic material are designated by

$$\frac{\partial}{\partial X_K}\Big(\frac{\partial \Sigma}{\partial x_{k,K}}\Big) - \rho_0\frac{\partial v_k}{\partial t} = 0,\ \ v_k = \frac{\partial x_k}{\partial t} \tag{9.4.33}$$

in the absence of body forces [*see* (8.7.4-5)]. These equations will turn out to be an example to the case $m = 1, n = 4, N = 3$. In order to utilise directly the determining equations (9.4.11), let us introduce the notations

[1]Adetailed discussion of this problem involving heterogenous materials can be found in the following work: Şuhubi, E. S. and A. Bakkaloğlu, Symmetry groups for arbitrary motions of hyperelastic solids, *International Journal of Engineering Science*, **35**, 637-657, 1997.

$$X_4 = t,\ \ \Sigma_{kK}(\mathbf{F}) = \frac{\partial \Sigma}{\partial F_{kK}},\ \ \Sigma_{k4}(\mathbf{v}) = -\rho_0 v_k.$$

Hence, the equations of motion are reduced to the form

$$\frac{\partial \Sigma_{kK}}{\partial X_K} + \frac{\partial \Sigma_{k4}}{\partial X_4} = 0.$$

We take the isovector field as follows

$$V = -\Phi_K \frac{\partial}{\partial X_K} - \Psi \frac{\partial}{\partial t} + \Omega_k \frac{\partial}{\partial x_k} + V_{kK} \frac{\partial}{\partial F_{kK}} + V_k \frac{\partial}{\partial v_k}$$

where the functions Φ_K, Ψ, Ω_k depend only on the variables X_K, t, x_k and for the components V_{kK} and V_k, we have the expressions below

$$V_{kK} = \frac{\partial \Omega_k}{\partial X_K} + F_{kL}\frac{\partial \Phi_L}{\partial X_K} + v_k \frac{\partial \Psi}{\partial X_K} + F_{lK}\frac{\partial \Omega_k}{\partial x_l} + F_{kL}F_{lK}\frac{\partial \Phi_L}{\partial x_l} + F_{lK}v_k\frac{\partial \Psi}{\partial x_l}$$
$$V_k = \frac{\partial \Omega_k}{\partial t} + F_{kL}\frac{\partial \Phi_L}{\partial t} + v_k\frac{\partial \Psi}{\partial t} + v_l\frac{\partial \Omega_k}{\partial x_l} + F_{kL}v_l\frac{\partial \Phi_L}{\partial x_l} + v_k v_l\frac{\partial \Psi}{\partial x_l}.$$

We thus obtain

$$B_{kK} = V(\Sigma_{kK}) = C_{kKlL}V_{lL},\ V(\Sigma_{k4}) = -\rho_0 V_k$$

where we had defined

$$C_{kKlL} = \frac{\partial \Sigma_{kK}}{\partial F_{lL}} = \frac{\partial^2 \Sigma}{\partial F_{kK}\partial F_{lL}} = C_{lLkK}.$$

We had already called the tensor $C_{kKlL}(\mathbf{F})$ enjoying the block symmetry shown above as the elasticities of the material in Example 8.7.4. Hence, the coefficients appearing in the determining equations (9.4.11) become

$$A_k = B_{kK,K} - \rho_0 \dot{V}_k,\ \ A_{Kkl} = B_{kK,l},\ \ A_{4kl} = -\rho_0 V_{k,l},$$
$$A_{KLkl} = \frac{\partial B_{kK}}{\partial F_{lL}} - C_{kKlL}(\Phi_{M,M} + \dot{\Psi}) + C_{kMlL}\Phi_{K,M},$$
$$A_{4Lkl} = -\rho_0\frac{\partial V_k}{\partial F_{lL}} + C_{kKlL}\Psi_{,K},\ \ A_{K4kl} = \frac{\partial B_{kK}}{\partial v_l} - \rho_0 \dot{\Phi}_K\,\delta_{kl},$$
$$A_{44kl} = -\rho_0\frac{\partial V_k}{\partial v_l} + \rho_0\Phi_{K,K}\,\delta_{kl}, A_{4LMklm} = -A_{L4Mklm} = \frac{1}{2}C_{kLlM}\Psi_{,m},$$
$$A_{KLMklm} = -A_{LKMklm} = \frac{1}{2}(C_{kLlM}\Phi_{K,m} - C_{kKlM}\Phi_{L,m}),$$

$$A_{4L4klm} = -A_{L44klm} = \frac{1}{2}\rho_0 \Phi_{L,m}\,\delta_{kl}$$

where an overdot $(\cdot)$ denotes the derivative with respect to the time variable t. All other coefficients turn out be zero. Therefore, the equations (9.4.11) take the form

$$\begin{aligned}
\lambda_{kl}\frac{\partial \Sigma_{lK}}{\partial X_k} &= A_k + A_{Kkl}F_{lK} + A_{4kl}v_l \\
\lambda_{km}\Big(\frac{\partial \Sigma_{mK}}{\partial F_{lL}} + \frac{\partial \Sigma_{mL}}{\partial F_{lK}}\Big) &= A_{KLkl} + A_{LKkl} + 2(A_{KMLklm}+A_{LMKklm})F_{mM} \\
&\quad + 2(A_{K4Lklm} + A_{L4Kklm})v_m \\
-\rho_0\lambda_{kl} &= A_{44kl} + 2A_{4M4klm}F_{mM} \\
0 &= A_{K4kl} + A_{4Kkl} + 2A_{4LKklm}F_{mL} + 2A_{K44klm}v_m
\end{aligned}$$

where λ_{km} are arbitrary functions of the coordinates $(X_K, t, x_k, F_{kK}, v_k)$ of the contact manifold. The above equations thus yield the following result

$$\begin{aligned}
& B_{kK,K} - \rho_0\dot{V}_k + B_{kK,l}F_{lK} - \rho_0 V_{k,l}v_l = 0, \qquad (9.4.34)\\
& \lambda_{km}(C_{mKlL} + C_{mLlK}) = \frac{\partial B_{kK}}{\partial F_{lL}} + \frac{\partial B_{kL}}{\partial F_{lK}} - (C_{kKlL} + C_{kLlK})(\Phi_{M,M} + \dot{\Psi} \\
&\qquad + \Phi_{M,m}F_{mM} + \Psi_{,m}v_m\,) + C_{kMlL}\Phi_{K,M} + C_{kMlK}\Phi_{L,M} \\
&\qquad\qquad + (C_{kMlL}\Phi_{K,m} + C_{kMlK}\Phi_{L,m}\big]F_{mM}, \\
& \lambda_{kl} = \frac{\partial V_k}{\partial v_l} - (\Phi_{K,K} + \Phi_{M,m}F_{mM})\,\delta_{kl} \\
& \frac{\partial B_{kK}}{\partial v_l} - \rho_0\dot{\Phi}_K\,\delta_{kl} - \rho_0\frac{\partial V_k}{\partial F_{lK}} + C_{kLlK}(\Psi_{,L} + \Psi_{,m}F_{mL}) - \rho_0\Phi_{K,m}v_m\delta_{kl} = 0
\end{aligned}$$

Because of the relations

$$\frac{\partial B_{kK}}{\partial v_l} = C_{kLlK}(\Psi_{,L} + \Psi_{,m}F_{mL}), \quad \frac{\partial V_k}{\partial F_{lK}} = (\dot{\Phi}_K + \Phi_{K,m}v_m)\delta_{kl}$$

the equation $(9.4.34)_4$ takes the form

$$(C_{kKlL} + C_{kLlK})(\Psi_{,L} + \Psi_{,m}F_{mL}) - 2\rho_0(\dot{\Phi}_K + \Phi_{K,m}v_m)\delta_{kl} = 0$$

Since the components $C_{kKlL} + C_{kLlK}$ are coefficients of the terms like $x_{k,KL}$ in the field equations (9.4.33), they cannot be all zero. It then follows from the above equation that

$$\Phi_{K,m} = 0,\ \dot{\Phi}_K = 0,\ \Psi_{,L} = 0,\ \Psi_{,m} = 0.$$

Thus, we must have $\Phi_K = \Phi_K(\mathbf{X})$, $\Psi = \Psi(t)$. In this case, we get

$$B_{kK} = C_{kKlL}(\Omega_{l,L} + F_{lM}\Phi_{M,L} + F_{mL}\Omega_{l,m}),$$
$$\lambda_{kl} = \Omega_{k,l} + (\dot{\Psi} - \Phi_{K,K})\delta_{kl}$$

so that $(9.4.34)_1$ can be written as

$$C_{kKlL}\big[\Omega_{l,LK} + F_{lM}\Phi_{M,LK} + F_{mL}\Omega_{l,mK} + (\Omega_{l,Lm} + F_{nL}\Omega_{l,mn})F_{mK}\big] \\ -\rho_0\big[\Omega_{k,lm}v_l v_m + (\ddot{\Psi}\delta_{kl} + 2\dot{\Omega}_{k,l})v_l + \ddot{\Omega}_k\big] = 0. \tag{9.4.35}$$

This requires that we have to take

$$\frac{\partial^2\Omega_k}{\partial x_l \partial x_m} = 0, \quad \frac{\partial^2\Omega_k}{\partial x_l \partial t} = -\frac{1}{2}\ddot{\Psi}(t)\,\delta_{kl}.$$

These equations yield easily

$$\Omega_k(\mathbf{X}, t, \mathbf{x}) = -\frac{1}{2}\dot{\Psi}(t)x_k + \gamma_{kl}(\mathbf{X})x_l + \Gamma_k(\mathbf{X}, t) = \Lambda_{kl}(\mathbf{X}, t)x_l + \Gamma_k(\mathbf{X}, t)$$

where we have defined

$$\Lambda_{kl}(\mathbf{X}, t) = -\frac{1}{2}\dot{\Psi}(t)\delta_{kl} + \gamma_{kl}(\mathbf{X}).$$

Then the equations $(9.4.34)_2$ reduce to the form

$$(C_{mKlL} + C_{mLlK})\Omega_{k,m} - (C_{kKmL} + C_{kLmK})\Omega_{m,l} + \{2(C_{kKlL} + C_{kLlK})\dot{\Psi} \\ - (C_{kKlM} + C_{kMlK})\Phi_{L,M} - (C_{kMlL} + C_{kLlM})\Phi_{K,M} \\ - (C_{kKlLmM} + C_{kLlKmM})(\Omega_{m,M} + F_{mN}\Phi_{N,M} + F_{nM}\Omega_{m,n})\} = 0 \tag{9.4.36}$$

where we have introduced the tensor

$$C_{kKlLmM} = \frac{\partial^3\Sigma}{\partial F_{kK}\partial F_{lL}\partial F_{mM}}.$$

The block symmetries manifested by this tensor are obvious. It is plainly observed that the term within braces in the expression (9.4.36) is symmetric in indices k and l due to the block symmetry of the components C_{kKlL}. Hence, the antisymmetric part of that expression must satisfy the relations

$$(C_{mKlL} + C_{mLlK})(\Omega_{k,m} + \Omega_{m,k}) = (C_{mKkL} + C_{mLkK})(\Omega_{l,m} + \Omega_{m,l}).$$

Let us define matrices $\mathbf{A}^{(KL)}$ via $A^{(KL)}_{kl} = C_{kKlL} + C_{kLlK}$. These matrices are symmetric. Consequently, we obtain

$$\Omega_{(k,m)}A^{(KL)}_{ml} = A^{(KL)}_{km}\Omega_{(m,l)} \quad \text{or} \quad \Lambda_{(km)}A^{(KL)}_{ml} = A^{(KL)}_{km}\Lambda_{(ml)}$$

Thus, the symmetric part of the matrix $\boldsymbol{\Lambda}$ commutes with every symmetric matrix $\mathbf{A}^{(KL)}$. According to the well known *Schur lemma* of the group theory $\Lambda_{(kl)}$ can only be a multiple of the unit matrix. Therefore, on noting that $\Lambda_{[kl]}$ may be represented by an axial vector a_m, we can write

$$\Lambda_{kl} = \Lambda_{(kl)} + \Lambda_{[kl]} = \Lambda_0(\mathbf{X}, t)\delta_{kl} + e_{klm}a_m(\mathbf{X}, t)$$

whence we get

$$\gamma_{(kl)}(\mathbf{X}) = \Big[\Lambda_0(\mathbf{X}, t) + \frac{1}{2}\dot{\Psi}(t)\Big]\delta_{kl} = \lambda_0(\mathbf{X})\delta_{kl},$$
$$\gamma_{[kl]}(\mathbf{X}) = e_{klm}a_m(\mathbf{X}).$$

We thus conclude that

$$\Omega_k(\mathbf{X}, t, \mathbf{x}) = \Big[\lambda_0(\mathbf{X}) - \frac{1}{2}\dot{\Psi}(t)\Big]x_k + e_{klm}a_m(\mathbf{X})x_l + \Gamma_k(\mathbf{X}, t).$$

If we insert this expression into (9.4.35-36) and equate the coefficients of x_n to zero, we get $\dddot{\Psi} = 0$ together with

$$\lambda_{0,M}\,\delta_{mn} + e_{mnr}a_{r,M} = 0$$

and consequently $\lambda_{0,M} = 0, a_{r,M} = 0$ leading to

$$\lambda_0(\mathbf{X}) = a_0, \quad a_k(\mathbf{X}) = a_k,$$
$$\Psi(t) = b_1t^2 + 2b_2t + b_3.$$

The equation (9.4.35) now takes the form

$$C_{kKlL}(\Gamma_{l,LK} + F_{lN}\Phi_{N,LK}) - \rho_0\ddot{\Gamma}_k = 0.$$

We differentiate this expression with respect to F_{mM} to obtain

$$C_{kKlLmM}(\Gamma_{l,LK} + F_{lN}\Phi_{N,LK}) + C_{kKmL}\Phi_{M,LK} = 0$$

that can be satisfied for any non-linear elastic material if only $\Phi_{M,LK} = 0$ and $\Gamma_{l,LK} = 0$. This of course implies that $\ddot{\Gamma}_k = 0$. We thus easily obtain

$$\Phi_K(\mathbf{X}) = A_{KL}X_L + B_K,$$
$$\Gamma_k(\mathbf{X}, t) = (\alpha_{kK}t + \beta_{kK})X_K + \mu_k t + \nu_k.$$

Replacing the functions in (9.4.36) by the above expressions, we see at once that the coefficient of the variable t vanishes if only we take $b_1 = 0$ and

$\alpha_{kK} = 0$. If we take into consideration the identity

$$F_{nN}\, C_{kKlLmM} = F_{nN} \frac{\partial^3 \Sigma}{\partial F_{kK} \partial F_{lL} \partial F_{mM}} = \frac{\partial^2}{\partial F_{kK} \partial F_{lL}} \Big(F_{nN} \frac{\partial \Sigma}{\partial F_{mM}} \Big) \\ - C_{kKmM}\, \delta_{nl} \delta_{NL} - C_{lLmM}\, \delta_{nk} \delta_{NK}$$

the remaining terms in the expressions (9.4.36) can be arranged in the following manner

$$\frac{\partial^2 \mathcal{F}}{\partial F_{kK} \partial F_{lL}} + \frac{\partial^2 \mathcal{F}}{\partial F_{kL} \partial F_{lK}} = 0 \tag{9.4.37}$$

where the function $\mathcal{F}$ is defined as

$$\mathcal{F} = \big[A_{NM} F_{mN} + (a_0 - b_2) F_{mM} + e_{klm} a_l F_{kM} + \beta_{mM} \big] \frac{\partial \Sigma}{\partial F_{mM}} \\ - 2(a_0 + b_2)\Sigma.$$

A rather straightforward but somewhat tedious calculation for details of which we may refer to the work cited above shows that the solution of the equations (9.4.37) is expressible as

$$\mathcal{F} = \frac{1}{3!} \gamma e_{klm} e_{KLM} F_{kK} F_{lL} F_{mM} + \frac{1}{2} \gamma_{mM} e_{klm} e_{KLM} F_{kK} F_{lL} \\ + \delta_{kK} F_{kK} + \delta.$$

If we recall identities

$$J = \det \mathbf{F} = \frac{1}{3!} e_{klm} e_{KLM} F_{kK} F_{lL} F_{mM}$$
$$\frac{\partial J}{\partial F_{kK}} = J F^{-1}_{Kk} = \frac{1}{2} e_{klm} e_{KLM} F_{kK} F_{lL}$$

where F^{-1}_{Kk} are entries of the inverse matrix $\mathbf{F}^{-1}$, $\mathcal{F}$ can also be written in the form

$$\mathcal{F} = \gamma J + \gamma_{kK} J F^{-1}_{Kk} + \delta_{kK} F_{kK} + \delta.$$

We had emphasised the fact that the stress potential Σ is actually dependent on the components $C_{KL} = F_{kK} F_{kL}$ of the deformation tensor. Once this transformation is fulfilled, we see that we have to take $\beta_{mM} = \gamma_{kK} = \delta_{kK} = 0$ in order to remove the dependence on $\mathbf{F}$. The relevant components of the isovector field are then given by

$$\Phi_K = A_{KL} X_L + B_K,$$

$$\Psi = 2b_2 t + b_3,$$
$$\Omega_k = (a_0 - b_2)x_k + e_{klm}a_m x_l + \mu_k t + \nu_k$$

and the admissible functions Σ must be solutions of the equation

$$2\big[A_{ML} + (a_0 - b_2)\delta_{ML}\big]C_{KM}\frac{\partial\Sigma}{\partial C_{KL}} - 2(a_0 + b_2)\Sigma = \gamma\sqrt{\det\mathbf{C}} + \delta.$$

In the components Ω_k, the terms ν_k indicate that the space is homogeneous whereas the terms $e_{klm}a_m x_l = -(\mathbf{a}\times\mathbf{x})_k$ imply that the space is isotropic. The terms $(a_0 - b_2)x_k + \mu_k t$ mean that the field equations are invariant under a Galilean transformation [Italian physicist and astronomer Galileo Galilei (1564-1642)]. Of course, these symmetry groups must be present in all classical mechanical system modelled correctly. ■

9.5. SIMILARITY SOLUTIONS

As we have mentioned several times we can produce a new family of solutions from a known solution of a system of partial differential equations if we possess an isovector field of the fundamental ideal generating a symmetry group of transformations. In this section, however, we shall try to determine structural properties of certain solutions that remain *invariant* under a particular symmetry group. If a mapping $\phi : \mathcal{D}_n \to \mathcal{C}_m$ corresponds to a solution to a system of partial differential equations which remains invariant with respect to an isovector field V, then it has to satisfy the requirement $\phi_V(t)\circ\phi = e^{tV}\circ\phi = \phi$. Let us suppose that such a solution is given in the form $f^\alpha = \phi^\alpha(\mathbf{x}) - u^\alpha = 0$. *If these functions are to be invariant under the flow generated by an isovector V, then it must satisfy the condition*

$$\pounds_V f^\alpha = V(f^\alpha) = 0$$

[*see* (2.9.14)]. This means that a ***group-invariant solution***, in other words, a ***similarity solution*** must satisfy the system of quasilinear partial differential equations

$$X^i\big(x^j, \phi^\beta(\mathbf{x})\big)\frac{\partial\phi^\alpha}{\partial x^i} - U^\alpha\big(x^j, \phi^\beta(\mathbf{x})\big) = 0 \tag{9.5.1}$$

when $N > 1$, or the non-linear partial differential equation

$$X^i\big(x^j, \phi(\mathbf{x}), \phi(\mathbf{x})_{,j}\big)\frac{\partial\phi}{\partial x^i} - U\big(x^j, \phi(\mathbf{x}), \phi(\mathbf{x})_{,j}\big) = 0 \tag{9.5.2}$$

when $N = 1$. These partial differential equations usually specify the structure of a similarity solution in the following manner

$$u^\alpha = \phi^\alpha(\xi^a), \quad a = 1, 2, \ldots, p < n$$

where ξ^a are known functions of independent variables x^i. After having installed these functions in the original system of differential equations we are led to a new set of partial differential equations with a smaller number of novel independent variables since x^i are replaced by ξ^a. That is why we expect that to solve them may be somewhat easier compared to original equations. If we manage to find a solution of these equations we then reach to the functional form of a similarity solution. Inserting this form into original field equation we can find an explicit solution. If we denote a solution of a given system of partial differential equations by a regular mapping $\phi : \mathcal{D}_n \subseteq \mathbb{R}^n \to \mathcal{C}_m$, then the relations (9.5.1) or (9.5.2) require that the mapping ϕ must satisfy the condition

$$\phi^*\big(\mathbf{i}_V(\sigma^\alpha)\big) = 0 \tag{9.5.3}$$

to be a similarity solution associated with an isovector V. In fact, this result follows immediately from $\mathbf{i}_V(\sigma^\alpha) = U^\alpha - v^\alpha_i X^i$ and $\phi^* v^\alpha_i = u^\alpha_{,i}$. Actually, we readily observe that a similarity solution satisfies as well the condition $\phi^*\mathbf{i}_V(\mathcal{I}_m) = 0$ where $\mathcal{I}_m$ is the contact ideal defined in (9.3.1). If we take into account the relations $V^\alpha_{i_1\cdots i_k i} = D^{(k)}_i(V^\alpha_{i_1\cdots i_k}) - v^\alpha_{i_1\cdots i_k j} D^{(k)}_i(X^j)$ for isovector components of the contact ideal given by (9.3.19) in the expression $\mathbf{i}_V(\sigma^\alpha_{i_1 i_2\cdots i_r}) = V^\alpha_{i_1\cdots i_r} - v^\alpha_{i_1\cdots i_r i}X^i$, we get

$$\begin{aligned}\phi^*\big(\mathbf{i}_V(\sigma^\alpha_{i_1\cdots i_r})\big) &= (\phi^* V^\alpha_{i_1\cdots i_{r-1}})_{,i_r} - u^\alpha_{,i_1\cdots i_{r-1}i}(\phi^* X^i)_{,i_r} - u^\alpha_{,i_1\cdots i_r i}\phi^* X^i \\ &= (\phi^* V^\alpha_{i_1\cdots i_{r-1}} - u^\alpha_{,i_1\cdots i_{r-1}i}\phi^* X^i)_{,i_r} = \big[\phi^*\big(\mathbf{i}_V(\sigma^\alpha_{i_1\cdots i_{r-1}})\big)\big]_{,i_r.}\end{aligned}$$

If we continue to utilise this recurrence relation successively, we finally obtain

$$\phi^*\big(\mathbf{i}_V(\sigma^\alpha_{i_1\cdots i_r})\big) = \big[\phi^*\big(\mathbf{i}_V(\sigma^\alpha)\big)\big]_{,i_1\cdots i_r} = 0$$

where $r = 0, 1, \ldots, m-1$. We thus conclude that

$$\phi^* V^\alpha_{i_1\cdots i_r} = (\phi^* X^i)\, u^\alpha_{,i_1\cdots i_r i},\ r = 0, 1, \ldots, m-1.$$

On the other hand, we have

$$\begin{aligned}\phi^*\big(\mathbf{i}_V(d\sigma^\alpha_{i_1\cdots i_{m-1}})\big) &= -\,\phi^*(V^\alpha_{i_1\cdots i_{m-1}i}\, dx^i - X^i dv^\alpha_{i_1\cdots i_{m-1}i}) \\ &= -\,\big(\phi^* V^\alpha_{i_1\cdots i_{m-1}i} - (\phi^* X^j)u^\alpha_{,i_1\cdots i_{m-1}ij}\big)dx^i.\end{aligned}$$

But inserting the relation

$$\phi^* V^\alpha_{i_1\cdots i_{m-1}i} = (\phi^* V_{i_1\cdots i_{m-1}})_{,i} - (\phi^* X^j)_{,i} u^\alpha_{,i_1\cdots i_{m-1}j}$$

into the foregoing expression, we find

$$\phi^*\big(\mathbf{i}_V(d\sigma^\alpha_{i_1\cdots i_{m-1}})\big) = (\phi^* V^\alpha_{i_1\cdots i_{m-1}} - (\phi^* X^j)\, u^\alpha_{,i_1\cdots i_{m-1}j})_{,i} dx^i = 0$$

whence we draw the conclusion $\phi^*\mathbf{i}_V(\mathcal{I}_m) = 0$. Next, we consider the balance ideal $\mathfrak{I}_m$ and write

$$\begin{aligned}\phi^*\big(\mathbf{i}_V(\omega^\alpha)\big) &= \phi^*\big(\mathbf{i}_V(d\Sigma^{\alpha i}\wedge\mu_i + \Sigma^\alpha\mu)\big)\\ &= \phi^*\big(V(\Sigma^{\alpha i})\mu_i - X^j d\Sigma^{\alpha i}\wedge\mu_{ji} + X^j\Sigma^\alpha\mu_j)\big).\end{aligned}$$

However, because of the relations

$$\begin{aligned}\phi^*\big(V(\Sigma^{\alpha i})\big) &= \phi^*\Big[X^j\frac{\partial\Sigma^{\alpha i}}{\partial x^j} + \sum_{r=0}^m V^\beta_{i_1\cdots i_r}\frac{\partial\Sigma^{\alpha i}}{\partial v^\beta_{i_1\cdots i_r}}\Big]\\ &= (\phi^* X^j)\Big[\frac{\partial\Sigma^{\alpha i}}{\partial x^j} + \sum_{r=0}^m u^\beta_{,i_1\cdots i_r j}\frac{\partial\Sigma^{\alpha i}}{\partial u^\beta_{,i_1\cdots i_r}}\Big] = (\phi^* X^j)\frac{\partial(\phi^*\Sigma^{\alpha i})}{\partial x^j}\\ \phi^* d\Sigma^{\alpha i} &= \frac{\partial(\phi^*\Sigma^{\alpha i})}{\partial x^j}\,dx^j\end{aligned}$$

we obtain

$$\begin{aligned}\phi^*\big(\mathbf{i}_V(\omega^\alpha)\big) &= (\phi^* X^j)\Big[\frac{\partial(\phi^*\Sigma^{\alpha i})}{\partial x^j}\mu_i - \frac{\partial(\phi^*\Sigma^{\alpha i})}{\partial x^j}\mu_i + \frac{\partial(\phi^*\Sigma^{\alpha i})}{\partial x^i}\mu_j\\ &\quad + \phi^*(\Sigma^\alpha)\mu_j\Big] = (\phi^* X^j)\Big[\frac{\partial(\phi^*\Sigma^{\alpha i})}{\partial x^i} + \phi^*(\Sigma^\alpha)\Big]\mu_j = 0\end{aligned}$$

so we arrive at the result $\phi^*\mathbf{i}_V(\mathfrak{I}_m) = 0$. It is clear that this property will be equally valid for a fundamental ideal generated by forms $\omega^a = \Sigma^a\mu$.

Example 9.5.1. Let us consider the isovector field obtained previously in Example 9.4.1 for the heat conduction equation

$$-V^1_G = 4xt\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (x^2+2t)u\frac{\partial}{\partial u}$$

except for a sign difference. The similarity solution associated with this vector field must satisfy the partial differential equation

$$4xt\frac{\partial u}{\partial x} + 4t^2\frac{\partial u}{\partial t} + (x^2+2t)u = 0$$

whose characteristics are described by the ordinary differential equations

$$\frac{dx}{4xt} = \frac{dt}{4t^2} = -\frac{du}{(x^2+2t)u}.$$

Hence the solution becomes

$$u(x,t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{t}} T(\xi), \quad \xi = \frac{x}{t}.$$

where $T(\xi)$ is an arbitrary function. On inserting this expression into the field equation $u_t = u_{xx}$, we simply obtain $T'' = 0$. Thus, this similarity solution takes the form

$$u(x,t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{t}} \Big(c_1 \frac{x}{t} + c_2\Big) \qquad \blacksquare$$

Example 9.5.2. We now consider an isovector field associated with the Korteweg-de Vries equation given by

$$-V^1 - cV^3 = (x+ct)\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - (2u-c)\frac{\partial}{\partial u}$$

where c is a constant. The similarity solution associated with this isovector must satisfy the partial differential equation

$$(x+ct)\frac{\partial u}{\partial x} + 3t\frac{\partial u}{\partial t} + 2u - c = 0$$

whose characteristics are determined via the ordinary differential equations

$$\frac{dx}{x+ct} = \frac{dt}{3t} = -\frac{du}{2u-c}.$$

Hence, the solution is found as

$$u(x,t) = \frac{c}{2} + \phi(\xi)t^{-2/3}, \;\; \xi = t^{-1/3}\Big(x - \frac{1}{2}ct\Big).$$

where $\phi(\xi)$ is an arbitrary function. If we insert this expression into the equation $u_t + uu_x + u_{xxx} = 0$, we deduce the following non-linear ordinary differential equation

$$3\phi''' + (3\phi - \xi)\phi' - 2\phi = 0. \tag{9.5.4}$$

In order to get an idea about the structure of solutions of this equation, a numerically obtained solution under the initial conditions $\phi(0) = 0$,

$\phi'(0) = 1,\ \phi''(0) = 1$ is depicted in Fig. 9.5.1.

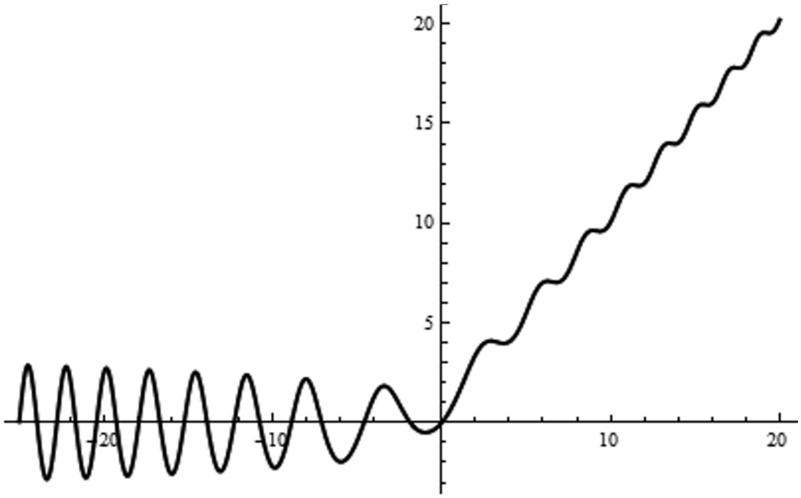

Fig. 9.5.1. A numerical solution of the equation (9.5.4).

As another isovector, we choose

$$-CV^4 - V^2 = C\frac{\partial}{\partial x} + \frac{\partial}{\partial t}.$$

where C is a constant. The similarity solution now satisfies the simple partial differential equation

$$C\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$$

whose solution is in the form $u = u(\xi)$ where $\xi = x - Ct$. Hence, we have to solve the non-linear equation

$$u''' + uu' - Cu' = 0$$

or its first integral

$$u'' + \frac{1}{2}u^2 - Cu = c_1.$$

The general solution of this equation can be found in terms of elliptic functions. However, if we impose the condition $u \to 0$ for $\xi \to \pm\infty$, we have to take $c_1 = 0$. In this case, the solution is expressible in elementary functions and the advancing wave type of a solution of Korteweg-de Vries equation yields the well known ***soliton*** solution

$$u(x,t) = 3C\,\mathrm{sech}^2\Big[\frac{C^{1/2}}{2}(x - Ct) + \delta\Big].$$

Starting from this particular solution, we can construct a new family of solutions parametrically by making use of the relations obtained at the end of Example 9.4.2 as follows

$$\begin{aligned}
\overline{x}(s) &= \frac{2(b-3ac) + 3\big[2a^2x + 2ac - b(1+at)\big]e^{-c_1 s} + b(1+3at)e^{-3c_1 s}}{6a^2}\\
\overline{t}(s) &= \frac{(1+3at)e^{-3c_1 s} - 1}{3a}\\
\overline{u}(s) &= \frac{\big[6aC\mathrm{sech}^2\big[\frac{C^{1/2}}{2}(x-Ct)+\delta\big] - b\big]e^{2c_1 s} + b}{2a}
\end{aligned}$$

where s is the parameter of the family. ■

Example 9.5.3. We consider the isovector field

$$V_G = (x+a)\frac{\partial}{\partial x} + by\frac{\partial}{\partial x} + (1-2b)u\frac{\partial}{\partial u} - bv\frac{\partial}{\partial v}$$

associated with partial differential equations governing the boundary layer flow past a flat plate discussed in Example 9.4.3. Equations (9.5.1) now take the form

$$(x+a)\frac{\partial u}{\partial x} + by\frac{\partial u}{\partial x} - (1-2b)u = 0,\ (x+a)\frac{\partial v}{\partial x} + by\frac{\partial v}{\partial x} + bv = 0$$

the solution of which is easily obtained as

$$u(x,y) = (x+a)^{1-2b}\phi(\xi),\ v(x,y) = (x+a)^{-b}\psi(\xi);\ \xi = y(x+a)^{-b}$$

where $\phi(\xi)$ and $\psi(\xi)$ are arbitrary functions. Introduction of these expressions together with the admissible function $f(x^1) = A(x^1+a)^{1-4b}$ into the field equations (9.4.26) gives rise to the following set of ordinary differential equations

$$\begin{aligned}
&\nu\phi''' + b\xi\phi\phi' - (1-2b)\phi^2 - \phi'\psi + A = 0,\\
&b\xi\phi' - \psi' - (1-2b)\phi = 0
\end{aligned} \tag{9.5.5}$$

A numerical solution of the above equations corresponding to $b = 1/2$, $\nu = 1, A = 1$ under the initial conditions $\phi(0) = 0, \phi'(0) = 0, \phi''(0) = 1.5$ and $\psi(0) = 0$ that may not reflect an actual physical situation is depicted in Fig. 9.5.2.

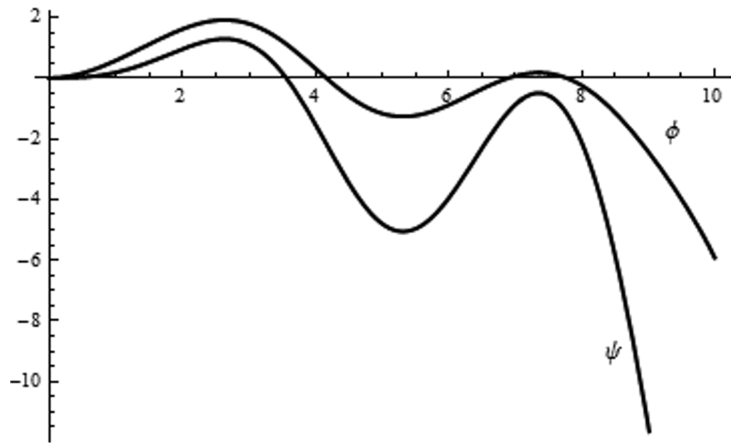

Fig. 9.5.2. Numerical solutions of the equations (9.5.5). ■

9.6. THE METHOD OF GENERALISED CHARACTERISTICS

We have seen in Sec. 9.2 that the solution of a first order non-linear partial differential equation can be constructed by means of characteristics starting from a given initial submanifold. We shall now try to generalise this method by employing isovectors of the ideal $\mathfrak{I}_m$ of $\Lambda(\mathcal{C}_m)$ associated with a system of partial differential equations. Let us denote the $\mathfrak{n} = n + D$ number of local coordinates $\{x^i, v^\alpha_{i_1 i_2 \cdots i_r} : 0 \le r \le m\}$ of the contact manifold $\mathcal{C}_m$ by $z^\mathfrak{a}$, $\mathfrak{a} = 1, \dots, \mathfrak{n}$. Consider a vector field $V = v^\mathfrak{a}(\mathbf{z})\,\partial/\partial z^\mathfrak{a} \in T(\mathcal{C}_m)$. We know that its integral curves are obtained as solutions of the following ordinary differential equations and initial conditions

$$\frac{d\zeta^\mathfrak{a}}{dt} = v^\mathfrak{a}(\boldsymbol{\zeta}), \quad \zeta^\mathfrak{a}(0) = z^\mathfrak{a}$$

in the form $\zeta^\mathfrak{a} = \phi^\mathfrak{a}(t;\mathbf{z}) = \phi_V(t)z^\mathfrak{a} = e^{tV}(z^\mathfrak{a})$. We wish to get the parameter t acquired a status of a coordinate to appreciate its independent variations. Therefore, we embed the integral curves into the *graph manifold* $L_m = \mathcal{C}_m \times \mathbb{R}$ whose coordinates are prescribed by $\{z^\mathfrak{a}, t\}$. Thus the contact manifold $\mathcal{C}_m$ might be specified as a submanifold of the manifold L_m obtained by $t = 0$. It appears to be advantageous now to extend the mapping ϕ_V describing the flow as $\phi_V : L_m \to L_m$ such that

$$\phi_V(\{z^\mathfrak{a}, t\}) = \{\zeta^\mathfrak{a} = e^{tV}(z^\mathfrak{a}), t\}. \tag{9.6.1}$$

We can naturally define a canonical projection $\pi : L_m \to \mathcal{C}_m$ as follows

$$\pi(\{z^{\mathfrak{a}}, t\}) = \{z^{\mathfrak{a}}\}.$$

The operator $\pi \circ \phi_V : L_m \to \mathcal{C}_m$ induces naturally the pull-back operator $(\pi \circ \phi_V)^* = \phi_V^* \circ \pi^* : \Lambda(\mathcal{C}_m) \to \Lambda(L_m)$ along trajectories of the vector field V. In order to illustrate the properties of this mapping, let us consider a form $\omega \in \Lambda^k(\mathcal{C}_m)$ given by

$$\omega = \frac{1}{k!}\,\omega_{\mathfrak{a}_1\cdots\mathfrak{a}_k}(\mathbf{z})\,dz^{\mathfrak{a}_1} \wedge \cdots \wedge dz^{\mathfrak{a}_k}.$$

Since the flow carries the forms $dz^{\mathfrak{a}}$ in the neighbourhood of $t = 0$ to the forms $d\zeta^{\mathfrak{a}} = \phi_V^* \circ \pi^*(dz^{\mathfrak{a}}) = dz^{\mathfrak{a}} + v^{\mathfrak{a}}(\mathbf{z})\,dt$, the pulled back form can be written at $t = 0$ as follows

$$\begin{aligned}
\omega^*\big|_{t=0} = \pi^*\omega\big|_{t=0} &= \frac{1}{k!}\,\omega_{\mathfrak{a}_1\cdots\mathfrak{a}_k}(\mathbf{z})\,(dz^{\mathfrak{a}_1} + v^{\mathfrak{a}_1}dt) \wedge \cdots \wedge (dz^{\mathfrak{a}_k} + v^{\mathfrak{a}_k}dt)\\
&= \omega + \frac{1}{(k-1)!}\,\omega_{\mathfrak{a}_1\cdots\mathfrak{a}_k}(\mathbf{z})v^{\mathfrak{a}_1}dt \wedge dz^{\mathfrak{a}_2} \wedge \cdots \wedge dz^{\mathfrak{a}_k}\\
&= \omega + dt \wedge \mathbf{i}_V(\omega).
\end{aligned}$$

where we have employed the complete antisymmetries of both the coefficients $\omega_{\mathfrak{a}_1\cdots\mathfrak{a}_k}$ and exterior products. Hence, in view of the relation (5.11.14) the form ω^* can be expressed as

$$\omega^*(t;\mathbf{z}) = \phi_V^* \circ \pi^*\omega(\mathbf{z}) = e^{t\pounds_V}\big(\omega + dt \wedge \mathbf{i}_V(\omega)\big). \tag{9.6.2}$$

Next, we introduce an operator $E_V : \Lambda(\mathcal{C}_m) \to \Lambda(L_m)$ that maps an exterior algebra into a larger exterior algebra by the rule

$$E_V\omega = \omega + dt \wedge \mathbf{i}_V(\omega) \in \Lambda(L_m),\ \ \omega \in \Lambda(\mathcal{C}_m). \tag{9.6.3}$$

The operator E_V has the following properties:

$$\begin{aligned}
&(i).\ E_V(\omega_1 + \omega_2) = E_V\omega_1 + E_V\omega_2, \\
&\qquad\quad E_V(f\omega) = fE_V\omega,\ f \in \Lambda^0(\mathcal{C}_m),\\
&(ii).\ E_V(\omega_1 \wedge \omega_2) = E_V\omega_1 \wedge E_V\omega_2,\\
&(iii).\quad d(E_V\omega) = E_V(d\omega) - dt \wedge \pounds_V\omega,\\
&(iv).\quad \pounds_V(E_V\omega) = E_V(\pounds_V\omega),\\
&(v).\quad (\phi_V^* \circ \pi^*)\,\omega = e^{t\pounds_V}E_V\omega.
\end{aligned} \tag{9.6.4}$$

The relations in (9.6.4) can easily be verified:
(i). This is evident because of the properties of the operator $\mathbf{i}_V$. Hence, E_V is a linear operator on the exterior algebra $\Lambda(\mathcal{C}_m)$.
(ii). To see this, it suffices to note that

$$E_V(\omega_1 \wedge \omega_2) = \omega_1 \wedge \omega_2 + dt \wedge \mathbf{i}_V(\omega_1) \wedge \omega_2 + (-1)^{deg\,\omega_1} dt \wedge \omega_1 \wedge \mathbf{i}_V(\omega_2)$$
$$E_V\omega_1 \wedge E_V\omega_2 = \omega_1 \wedge \omega_2 + (-1)^{deg\,\omega_1} dt \wedge \omega_1 \wedge \mathbf{i}_V(\omega_2) + dt \wedge \mathbf{i}_V(\omega_1) \wedge \omega_2.$$

(iii). This follows from the relation

$$d(E_V\omega) = d\omega - dt \wedge d\mathbf{i}_V(\omega) = d\omega + dt \wedge \mathbf{i}_V(d\omega) - dt \wedge \pounds_V\omega$$

where we have employed the Cartan magic formula.
(iv). This is immediately seen if we take notice of the relations $\pounds_V dt = dV(t) = 0$ and $\pounds_V\big(\mathbf{i}_V(\omega)\big) = \mathbf{i}_V\big(\pounds_V(\omega)\big)$ [*see* (5.11.8)$_2$].
(v). This is in fact just the relation (9.6.2).

According to the properties (9.6.4)$_{1-2}$, we see that the set

$$\Lambda_V(L_m) = \{E_V\omega : \omega \in \Lambda(\mathcal{C}_m)\} \subseteq \Lambda(L_m)$$

becomes an exterior algebra. We can now prove the theorem below.

Theorem 9.6.1. *If a vector field $V \in T(\mathcal{C}_m)$ is an isovector field of a closed ideal $\mathfrak{I}$ of the exterior algebra $\Lambda(\mathcal{C}_m)$, then it is also an isovector field of the ideal $E_V\mathfrak{I}$ of the exterior algebra $\Lambda_V(L_m)$.*

If $\omega_1, \omega_2 \in \mathfrak{I}$, we have of course $\omega_1 + \omega_2 \in \mathfrak{I}$ and then (9.6.4)$_1$ leads to $E_V\omega_1 + E_V\omega_2 = E_V(\omega_1 + \omega_2) \in E_V\mathfrak{I}$. Similarly, if $\omega \in \mathfrak{I}$, we find that $\gamma \wedge \omega \in \mathfrak{I}$ with $\gamma \in \Lambda(\mathcal{C}_m)$. Since a form $\gamma' \in \Lambda_V(L_m)$ must now be written as $\gamma' = E_V\gamma$ where $\gamma \in \Lambda(\mathcal{C}_m)$, we thus get $\gamma' \wedge E_V\omega = E_V\gamma \wedge E_V\omega = E_V(\gamma \wedge \omega) \in E_V\mathfrak{I}$ due to (9.6.4)$_2$. Therefore, the set $E_V\mathfrak{I}$ is an ideal of the exterior algebra $\Lambda_V(L_m)$. However, this ideal is no longer closed since we have the relation $d \circ E_V \neq E_V \circ d$ because of the property (9.6.4)$_3$. If the vector field V is an isovector field of the closed ideal $\mathfrak{I}$, then we get $d\omega \in \mathfrak{I}$ and $\pounds_V\omega \in \mathfrak{I}$ for all forms $\omega \in \mathfrak{I}$. On the other hand, because of (9.6.4)$_4$ and (5.11.9) the relations

$$\pounds_V(E_V\omega) = E_V(\pounds_V\omega) \in E_V\mathfrak{I}$$
$$\pounds_V(E_V d\omega) = E_V(\pounds_V d\omega) = E_V(d\pounds_V\omega) \in E_V\mathfrak{I}$$

are satisfied. Hence V is also an isovector field of the ideal $E_V\mathfrak{I}$. □

Let $D_{n-1} \subseteq \mathbb{R}^{n-1}$ denote a connected, open subset whose local coordinates are provided by $\{s^1, \ldots, s^{n-1}\}$. We next suppose that the mapping $\psi : D_{n-1} \longrightarrow \mathcal{C}_m$ prescribed by smooth functions $z^{\mathfrak{a}} = \psi^{\mathfrak{a}}(s^1, \ldots, s^{n-1})$ specifies an *initial data submanifold* in $\mathcal{C}_m$. Let us then determine the integral curves of an isovector field V of a closed ideal $\mathfrak{I}$ of $\Lambda(\mathcal{C}_m)$ as solutions of the ordinary differential equations

$$\frac{d\zeta^{\mathfrak{a}}}{d\tau} = v^{\mathfrak{a}}(\boldsymbol{\zeta}),\ \zeta^{\mathfrak{a}}(0) = \psi^{\mathfrak{a}}(\mathbf{s}),\ \mathfrak{a} = 1, \ldots, \mathfrak{n} \qquad (9.6.5)$$

where τ is a real parameter. It becomes now possible to introduce a mapping $\Psi : D_n = D_{n-1} \times \mathbb{R} \to \mathcal{C}_m$ through the relations

$$z^{\mathfrak{a}} = \zeta^{\mathfrak{a}}\big(\psi^{\mathfrak{b}}(\mathbf{s}); \tau\big) = \Psi^{\mathfrak{a}}(s^1, \ldots, s^{n-1}; \tau). \tag{9.6.6}$$

We have already mentioned that $\mathcal{C}_m$ is a submanifold of L_m specified by $t = 0$. We now define a simple *extension* $\widehat{\psi} : D_{n-1} \times \mathbb{R} \longrightarrow L_m$ of the mapping ψ as follows

$$\widehat{\psi}\,(s^1, \ldots, s^{n-1}; \tau) = \{z^{\mathfrak{a}} = \psi^{\mathfrak{a}}(s^1, \ldots, s^{n-1}), t = \tau\}. \tag{9.6.7}$$

In this case, the mapping $\Psi : D_n \longrightarrow \mathcal{C}_m$ can be expressed as

$$\Psi = \pi \circ \phi_V \circ \widehat{\psi} = \pi \circ e^{\tau V} \circ \widehat{\psi} \tag{9.6.8}$$

when we recall (9.6.1). In the light of the information acquired so far, the following theorem can be proposed.

Theorem 9.6.2. *Let $V \in T(\mathcal{C}_m)$ be an isovector field of a closed ideal $\mathfrak{I}$ of $\Lambda(\mathcal{C}_m)$ and $D_{n-1} \subseteq \mathbb{R}^{n-1}$ be a connected open set whose local coordinates are given by $\{s^1, \ldots, s^{n-1}\}$. The mapping $\psi : D_{n-1} \longrightarrow \mathcal{C}_m$ determines an initial data submanifold in $\mathcal{C}_m$ through the smooth functions $z^{\mathfrak{a}} = \psi^{\mathfrak{a}}(\mathbf{s}), \mathfrak{a} = 1, \ldots, \mathfrak{n}$. If the extension $\widehat{\psi}$ of ψ holds the condition*

$$(\widehat{\psi})^*(E_V \mathfrak{I}) = 0, \tag{9.6.9}$$

then the mapping $\Psi = \pi \circ \phi_V \circ \widehat{\psi}$ satisfies the relation $\Psi^* \mathfrak{I} = 0$. Hence, the mapping $\Psi : D_{n-1} \times \mathbb{R} \to \mathcal{C}_m$ becomes a solution of the ideal $\mathfrak{I}$.

The proof of this theorem is rather straightforward at the first glance. If we keep in mind the relation $(9.6.4)_5$, the pull-back operator Ψ^* may be expressible as

$$\Psi^* \mathfrak{I} = \big((\widehat{\psi})^* \circ \phi_V^* \circ \pi^*\big)\mathfrak{I} = (\widehat{\psi})^*\big(e^{t\pounds_V}(E_V \mathfrak{I})\big). \tag{9.6.10}$$

But, according to Theorem 9.5.1, the isovector field V is also an isovector field of the ideal $E_V \mathfrak{I}$. We thus find $e^{t\pounds_V}(E_V \mathfrak{I}) \subset E_V \mathfrak{I}$. Hence, it follows from (9.6.9) that $(\widehat{\psi})^*\big(e^{t\pounds_V}(E_V \mathfrak{I})\big) = 0$, and as a consequence $\Psi^* \mathfrak{I} = 0$. It is obvious that the condition (9.6.9) imposes a restriction on admissible forms of initial data. If we pay attention to the definition (9.6.7), we observe that $(\widehat{\psi})^* dz^{\mathfrak{a}} = \psi^* dz^{\mathfrak{a}}$ and $(\widehat{\psi})^* dt = d\tau$. In this case, we can also express the condition (9.6.9) as

$$(\widehat{\psi})^*(E_V \omega) = \psi^* \omega + d\tau \wedge \psi^*\big(\mathbf{i}_V(\omega)\big) = 0 \tag{9.6.11}$$

for all $\omega \in \mathfrak{I}$. On the other hand, one can write

$$\psi^* dz^{\mathfrak{a}} = \frac{\partial \psi^{\mathfrak{a}}}{\partial s^1} ds^1 + \frac{\partial \psi^{\mathfrak{a}}}{\partial s^2} ds^2 + \cdots + \frac{\partial \psi^{\mathfrak{a}}}{\partial s^{n-1}} ds^{n-1}$$

and since 1-forms $ds^1, ds^2, \ldots, ds^{n-1}, d\tau$ are all linearly independent, we then conclude that the condition (9.6.11) is satisfied if and only if one has

$$\psi^* \omega = 0, \quad \psi^*\big(\mathbf{i}_V(\omega)\big) = 0 \tag{9.6.12}$$

for all $\omega \in \mathfrak{I}$. Let us now assume that the closed ideal $\mathfrak{I}$ is generated by $2r$ forms $\{\omega^\alpha, d\omega^\alpha : \alpha = 1, 2, \ldots, r\}$. Then it becomes clear that the conditions (9.6.12) are satisfied if and only if we have

$$\psi^* \omega^\alpha = 0,\ \psi^*\big(\mathbf{i}_V(\omega^\alpha)\big) = 0; \quad \psi^* d\omega^\alpha = 0,\ \psi^*\big(\mathbf{i}_V(d\omega^\alpha)\big) = 0$$

for $\alpha = 1, 2, \ldots, r$. Actually, we immediately see that the second set of equations involving exterior derivative are automatically satisfied in case the first set of conditions are met. Indeed, we obtain $\psi^* d\omega^\alpha = d\psi^* \omega^\alpha = 0$ in accordance with Theorem 5.8.2. On taking into account the Cartan formula, we have $\psi^* \mathbf{i}_V(d\omega^\alpha) = \psi^* £_V \omega^\alpha - \psi^* d\mathbf{i}_V(\omega^\alpha)$. Since V is an isovector field of the ideal, we can write $£_V \omega^\alpha = \lambda^\alpha_\beta \wedge \omega^\beta + \Lambda^\alpha_\beta \wedge d\omega^\beta \in \mathfrak{I}$. We thus obtain $\psi^* £_V \omega^\alpha = 0$ and $\psi^* \mathbf{i}_V(d\omega^\alpha) = -d\psi^* \mathbf{i}_V(\omega^\alpha) = 0$. In this situation, the conditions (9.6.12) concerning the initial data are recovered if and only if we are assured that the relations

$$\psi^* \omega^\alpha = 0,\ \psi^*\big(\mathbf{i}_V(\omega^\alpha)\big) = 0,\ \ \alpha = 1, \ldots, r \tag{9.6.13}$$

are satisfied. □

We now consider a closed fundamental ideal associated with a given system of partial differential equations and an isovector field V of this ideal. Let the mapping $\psi : D_{n-1} \longrightarrow \mathcal{C}_m$ specify again an initial data submanifold. We assume that the mapping ψ is satisfying the ***transversality condition***

$$\psi^*\big(\mathbf{i}_V(\mu)\big) \neq 0$$

for the volume form μ in D_n. The transversality condition implies that the rank of ψ is $n-1$ since its domain is the $(n-1)$-dimensional region D_{n-1}. Moreover, we find $(\widehat{\psi})^* \mu = \psi^* \mu = 0$ on D_{n-1} since μ is an n-form. Therefore, we can write

$$\Psi^* \mu = (\widehat{\psi})^*\big(e^{t£_V}(E_V \mu)\big) = (\widehat{\psi})^*\big[e^{t£_V}(dt \wedge \mathbf{i}_V(\mu))\big].$$

At $t = 0$ we get $\Psi^* \mu|_{t=0} = (\widehat{\psi})^*(dt \wedge \mathbf{i}_V(\mu)\big) = d\tau \wedge \psi^*\big(\mathbf{i}_V(\mu)\big) \neq 0$. Thus, the condition $\Psi^* \mu \neq 0$ is satisfied about $\tau = 0$ on the set $D_n = D_{n-1} \times \mathbb{R}$. Consequently, the mapping Ψ defined by (9.6.5) or (9.6.6) is regular. In this case, we can enounce the following theorem that turns out to be actually a direct descendant of Theorem 9.6.2.

Theorem 9.6.3. *Let V be an isovector field of a closed fundamental ideal $\mathfrak{I}_m$ associated with a given system of partial differential equations. If the mapping $\psi : D_{n-1} \longrightarrow \mathcal{C}_m$ specifying an initial data submanifold holds the conditions*

$$\psi^*\big(\mathbf{i}_V(\mu)\big) \neq 0,\ \ \psi^*\mathfrak{I}_m = 0,\ \ \psi^*\big(\mathbf{i}_V(\mathfrak{I}_m)\big) = 0, \tag{9.6.14}$$

then the mapping $\Psi = \pi \circ \phi_V \circ \widehat{\psi} : D_n \to \mathcal{C}_m$ prescribed by the equations (9.6.5) *and* (9.6.6) *on $D_n = D_{n-1} \times \mathbb{R}$ satisfies the condition $\Psi^*\mathfrak{I}_m = 0$, that is, it becomes a solution of the ideal $\mathfrak{I}_m$.* □

We had called $\psi(D_{n-1}) \subset \mathcal{C}_m$ the *initial data submanifold*. We regard (9.6.5) as the equations determining the *characteristics* corresponding to the pair (V, ψ) satisfying the transversality condition. $(9.6.14)_{2-3}$ represent *restrictions imposed on initial data* on the relevant submanifold. Ψ is then called as a ***generalised characteristic solution*** associated with a chosen V.

Generally, characteristic solutions of a system of partial differential equations have to satisfy some additional conditions.

Theorem 9.6.4. *If Ψ is the generalised characteristic solution generated by an isovector field V of the closed fundamental ideal $\mathfrak{I}_m$ associated with a system of partial differential equations, then the condition*

$$\Psi^*\big(\mathbf{i}_V(\mathfrak{I}_m)\big) = 0 \tag{9.6.15}$$

should be satisfied on the domain of Ψ. Therefore, the generalised characteristic solutions have to fulfil a specific set of additional constraints. However, if $\mathbf{i}_V(\mathfrak{I}_m) \subset \mathfrak{I}_m$, namely, if the isovector field is also a characteristic vector field of the ideal, then these conditions are automatically satisfied.

We can realise at once that, we can write

$$E_V \mathbf{i}_V(\omega) = \mathbf{i}_V(\omega) + dt \wedge \mathbf{i}_V \circ \mathbf{i}_V(\omega) = \mathbf{i}_V(\omega)$$

for a form $\omega \in \mathfrak{I}_m$ due to (5.4.5). Therefore (9.6.10) yields

$$\Psi^*\big(\mathbf{i}_V(\omega)\big) = (\widehat{\psi})^*\big[e^{t\pounds_V}\big(\mathbf{i}_V(\omega)\big)\big].$$

But the relation $\pounds_V\big(\mathbf{i}_V(\omega)\big) = \mathbf{i}_V\big(\pounds_V(\omega)$ implies that for every natural number k, we can write $\pounds_V^k\big(\mathbf{i}_V(\omega)\big) = \mathbf{i}_V\big(\pounds_V^k(\omega)\big)$. Hence, we reach to the result

$$e^{t\pounds_V}\big(\mathbf{i}_V(\omega)\big) = \mathbf{i}_V\big(e^{t\pounds_V}(\omega)\big).$$

We have $e^{t\pounds_V}(\omega) \in \mathfrak{I}_m$ since V is an isovector field. Because of $(9.6.12)_2$, we get $\Psi^*\big(\mathbf{i}_V(\omega)\big) = 0$ as well. We have seen in Sec. 9.5 that it suffices to satisfy the relation

$$\Psi^*\big(\mathbf{i}_V(\sigma^\alpha)\big) = 0$$

in order that the constraint (9.6.15) is fulfilled. In the light of this constraint we can say that, the generalised characteristic solutions are nothing but certain group-invariant solutions. However, if the isovector field V is at the same time a characteristic vector of the fundamental ideal, that is, if one has $\mathbf{i}_V(\mathfrak{I}_m) \subset \mathfrak{I}_m$, then $\Psi^*(\mathfrak{I}_m) = 0$ implies that $\Psi^*\big(\mathbf{i}_V(\mathfrak{I}_m)\big) = 0$. In this case the additional constraint is of course redundant. □

The determination of solution of a given system of partial differential equations satisfying prescribed initial conditions by the method of generalised characteristics seems at the first glance the same as the construction of similarity solutions investigated in Sec. 9.5. But, in order to obtain a tangible benefit from a similarity solution we need to solve first analytically partial differential equations (9.5.1) or (9.5.2). Furthermore, boundary and/or initial conditions have to comply totally with the structure of the similarity solution whereas in the method of generalised characteristics the mapping Ψ is determined by solving a system of ordinary differential equations if the initial data manifold is suitably chosen as to comply with the imposed restrictions. It is of course much easier to find numerical solutions of ordinary differential equations to construct a solution of partial differential equations at least approximately.

Example 9.6.1. We consider the non-linear partial differential equation

$$\Sigma = \frac{\partial u}{\partial x^1}\frac{\partial u}{\partial x^2} - ku = 0, \;\; k = constant$$

where $n = 2, N = 1$. Introducing $v_1 = u_{,1}, v_2 = u_{,2}$, we characterise this equation by the following 2-form

$$\omega = \Sigma\mu = (v_1 v_2 - ku)\, dx^1 \wedge dx^2.$$

As we already mentioned on p. 538 in Sec. 9.4, we can determine an isovector field by taking $F(x^1, x^2, u, v_1, v_2) = -\Sigma = ku - v_1 v_2$. Hence, we get

$$X^1 = v_2,\;\; X^2 = v_1,\;\; U = ku + v_1 v_2,\;\; V_1 = kv_1,\;\; V_2 = kv_2.$$

We define the mapping $\psi : \mathbb{R} \to \mathcal{C}_1$ specifying the initial data submanifold by the relations below depending on a single parameter s

$$x^1 = s,\, x^2 = 0,\, u = \psi_0(s),\, v_1 = \psi_1(s),\, v_2 = \psi_2(s).$$

Since we have $\sigma = du - v_1 dx^1 - v_2 dx^2$, $\mathbf{i}_V(\sigma) = U - v_1 X^1 - v_2 X^2$, the constraints

$$\begin{aligned} \psi^*\big(\mathbf{i}_V(\mu)\big) &= \psi^*(X^1\,dx^2 - X^2\,dx^1) = \,-\,\psi_1(s)\,ds \neq 0 \\ \psi^*\sigma &= (\psi_0' - \psi_1)\,ds = 0, \\ \psi^*\,\mathbf{i}_V(\sigma) &= \psi^*(ku - v_1 v_2) = k\psi_0 - \psi_1\psi_2 = 0 \end{aligned}$$

on the initial data yield

$$\psi_1 \neq 0, \quad \psi_1 = \psi_0', \quad \psi_2 = \frac{k\psi_0}{\psi_0'}.$$

Equations (9.6.5) can now be written as

$$\begin{aligned} &\frac{dx^1}{d\tau} = v_2,\ \frac{dx^2}{d\tau} = v_1,\ \frac{du}{d\tau} = ku + v_1 v_2,\ \frac{dv_1}{d\tau} = kv_1,\ \frac{dv_2}{d\tau} = kv_2, \\ &x^1(0) = s,\ x^2(0) = 0, \qquad u(0) = \psi_0(s), \\ &v_1(0) = \psi_0'(s), \quad v_2(0) = \frac{k\psi_0(s)}{\psi_0'(s)} \end{aligned}$$

whose solution is easily found to be

$$\begin{aligned} &x^1 = s + \frac{\psi_0(s)}{\psi_0'(s)}(e^{k\tau} - 1),\ x^2 = \frac{\psi_0'(s)}{k}(e^{k\tau} - 1), \\ &u = \psi_0(s\,)\,e^{2k\tau} \\ &v_1 = \psi_0'(s)\,e^{k\tau}, \quad v_2 = \frac{k\psi_0(s)}{\psi_0'(s)}\,e^{k\tau}. \end{aligned}$$

These relations create a solution in the form $u = u(x^1, x^2)$ after having expressed the parameters (s, τ) in terms of independent variables x^1 and x^2 by inverting, at least in principle the relations for x^1 and x^2.

As a very simple example, let us suppose that $\psi_0(s) = cs$ where c is a constant. We then obtain

$$x^1 = s + s(e^{k\tau} - 1),\ x^2 = \frac{c}{k}(e^{k\tau} - 1)$$

whence we easily deduce that

$$s = \frac{x^1}{1 + \frac{k}{c}x^2},\ e^{k\tau} = 1 + \frac{k}{c}x^2.$$

Hence the solution becomes simply

$$u(x^1, x^2) = cx^1\Big(1 + \frac{k}{c}x^2\Big).$$

On the other hand, if we take $\psi_0(s) = cs^2$, we have

$$x^1 = s + \frac{1}{2}s(e^{k\tau} - 1),\ \ x^2 = \frac{2c}{k}s(e^{k\tau} - 1)$$

and we find

$$s = x^1 - \frac{k}{4c}x^2,\ \ e^{k\tau} = 1 + \frac{kx^2}{2cx^1 - \frac{k}{2}x^2}$$

so that the corresponding solution becomes

$$u(x^1, x^2) = c\Big(x^1 - \frac{k}{4c}x^2\Big)^2\Big[\frac{2cx^1 + \frac{k}{2}x^2}{2cx^1 - \frac{k}{2}x^2}\Big]^2. \qquad \blacksquare$$

Example 9.6.2. As a more difficult example, let us consider the partial differential equation characterised by the 2-form

$$\omega = df_1 \wedge df_2 \in \Lambda^2(K_1),\ f_1 = (x^1)^2 + v_2^2,\ f_2 = (x^2)^2 + v_1^2$$

where we have again $n = 2, N = 1$ and $v_1 = u_{,1}, v_2 = u_{,2}$. The form ω may explicitly be written as

$$\omega = 4(x^1x^2dx^1 \wedge dx^2 + x^1v_1dx^1 \wedge dv_1 + x^2v_2dv_2 \wedge dx^2 + v_1v_2dv_2 \wedge dv_1)$$

If we choose a mapping $u = \phi(x^1, x^2)$ annihilating the form ω, then $\phi^*\omega = 0$ yields quite a complicated non-linear second order partial differential equation

$$\phi_{,1}\phi_{,2}\phi_{,11}\phi_{,22} - \phi_{,1}\phi_{,2}(\phi_{,12})^2 - (x^1\phi_{,1} + x^2\phi_{,2})\phi_{,12} - x^1x^2 = 0.$$

This partial differential equation is known as the non-homogeneous ***Monge-Ampère equation*** [French mathematicians Gaspard Monge (1746-1818) and André Marie Ampère (1775-1836)]. Let $V \in T(\mathcal{C}_1)$ be an isovector field of the contact ideal. By definition, we get $d\omega = 0$. We thus obtain

$$£_V\omega = d\mathbf{i}_V(\omega) = dV(f_1) \wedge df_2 - dV(f_2) \wedge df_1$$

where

$$V(f_1) = 2x^1X^1 + 2v_2V_2, \quad V(f_2) = 2x^2X^2 + 2v_1V_1.$$

We now wish so specify a simple isovector field as to be $V(f_1) = V(f_2) = 0$ implying $£_V\omega = 0$. If we take the function $F = F(x^1, x^2, u, v_1, v_2)$

into account, then it follows from (9.3.26) that this special function F must satisfy the equations

$$v_2\frac{\partial F}{\partial x^2}+v_2^2\frac{\partial F}{\partial u}-x^1\frac{\partial F}{\partial v_1}=0,\ v_1\frac{\partial F}{\partial x^1}+v_1^2\frac{\partial F}{\partial u}-x^2\frac{\partial F}{\partial v_2}=0.$$

The solution of the first equation is easily found to be as $F=F(\xi,\eta)$ where characteristic variables are $\xi=x^1x^2+v_1v_2,\ \eta=u-x^2v_2$. Inserting this result into the second equation we obtain $\big((x^2)^2+v_1^2\big)\partial F/\partial\eta=0$ implying that F is independent of η. Thus, we see that each smooth function of the form $F=F(x^1x^2+v_1v_2)$ generates an isovector field. As a simple example, let us just take $F=x^1x^2+v_1v_2$. Hence, we find that

$$V=-v_2\frac{\partial}{\partial x^1}-v_1\frac{\partial}{\partial x^2}+(x^1x^2-v_1v_2)\frac{\partial}{\partial u}+x^2\frac{\partial}{\partial v_1}+x^1\frac{\partial}{\partial v_2}$$

We choose the mapping $\psi:\mathbb{R}\to\mathcal{C}_1$ as follows

$$x^1=s,\ x^2=1,\ u=\psi_0(s),\ v_1=\psi_1(s),\ v_2=\psi_2(s).$$

The expression $\psi^*\sigma=0$ yields again $\psi_1=\psi_0'$ while the constraint $\psi^*\mathbf{i}_V(\sigma)=\psi^*(x^1x^2+v_1v_2)=0$ requires that $\psi_2=-s/\psi_0'$. Because of the relation $\mathbf{i}_V(\mu)=-v_2dx^2+v_1dx^1$, the transversality condition $\psi^*\big(\mathbf{i}_V(\mu)\big)\neq 0$ is met if $\psi_1=\psi_0'\neq 0$. To deduce the solution mapping Ψ, we have to solve the ordinary differential equations

$$\begin{aligned}&\frac{dx^1}{d\tau}=-v_2,\ \frac{dx^2}{d\tau}=-v_1,\ \frac{du}{d\tau}=x^1x^2+v_1v_2,\\&\frac{dv_1}{d\tau}=x^2,\quad \frac{dv_2}{d\tau}=x^1\\&x^1(0)=s,\quad x^2(0)=1,\quad u(0)=\psi_0(s),\\&v_1(0)=\psi_0'(s),\quad v_2(0)=-\frac{s}{\psi_0'(s)}\end{aligned}$$

from which we readily obtain the parametric solution

$$\begin{aligned}&x^1=s\cos\tau+\frac{s}{\psi_0'(s)}\sin\tau,\ x^2=\cos\tau-\psi_0'(s)\sin\tau,\\&v_1=\sin\tau+\psi_0'(s)\cos\tau,\ v_2=s\sin\tau-\frac{s}{\psi_0'(s)}\cos\tau\\&u=\psi_0(s)+s\sin 2\tau-\frac{s}{2}\Big(\psi_0'(s)-\frac{1}{\psi_0'(s)}\Big)(1-\cos 2\tau)\end{aligned}$$

If we would be able to eliminate the parameters s and τ, we might obtain the corresponding solution in the form $u = u(x^1, x^2)$. ■

Example 9.6.3. Let us take into account Korteweg-de Vries equation studied in Example 9.4.2. The most general isovector field associated with this equation would be

$$\begin{aligned} V = & -(c_1 x + c_3 t + c_4)\frac{\partial}{\partial x} - (3c_1 t + c_2)\frac{\partial}{\partial t} + (2c_1 u - c_3)\frac{\partial}{\partial u} \\ & + 3c_1 v_1 \frac{\partial}{\partial v_1} + (c_3 v_1 + 5c_1 v_2)\frac{\partial}{\partial v_2}. \end{aligned}$$

We define the mapping $\psi : \mathbb{R} \to \mathcal{C}_2$ specifying the initial data submanifold by the relations

$$\begin{gathered} x = s,\ t = 0,\ u = \psi_0(s),\ v_1 = \psi_1(s),\ v_2 = \psi_2(s), \\ v_{11} = \psi_{11}(s),\ \ v_{12} = \psi_{12}(s),\ \ v_{12} = \psi_{11}(s) \end{gathered}$$

where $v_1 = u_{,1},\ \ v_2 = u_{,2},\ \ v_{11} = u_{,11},\ \ v_{12} = u_{,12},\ \ v_{22} = u_{,22}$. Because $\psi^*\big(\mathbf{i}_V(\mu)\big) = c_2\, ds$, the tranversality condition is satisfied if we take $c_2 \neq 0$. 1-forms generating the contact ideal are

$$\begin{aligned} \sigma &= du - v_1 dx - v_2\, dt, \\ \sigma_1 &= dv_1 - v_{11}\, dx - v_{12}\, dt, \\ \sigma_2 &= dv_2 - v_{12}\, dx - v_{22}\, dt. \end{aligned}$$

Hence, the expressions

$$\begin{aligned} \psi^*\sigma &= (\psi_0' - \psi_1)\, ds = 0, \\ \psi^*(\sigma_1) &= (\psi_1' - \psi_{11}) ds = 0, \\ \psi^*(\sigma_2) &= (\psi_2' - \psi_{12}) ds = 0 \end{aligned}$$

yield $\psi_1(s) = \psi_0'(s),\ \ \psi_{11}(s) = \psi_1'(s) = \psi_0''(s),\ \ \psi_{12}(s) = \psi_2'(s)$. On the other hand, we find $\psi^*\, \mathbf{i}_V(\sigma) = 2c_1\psi_0 - c_3 + (c_1 s + c_4)\psi_1 + c_2\psi_2 = 0$ and, consequently

$$\psi_2(s) = \frac{1}{c_2}\big[c_3 - 2c_1\psi_0(s) - (c_1 s + c_4)\psi_0'(s)\big].$$

The balance form is given by $\omega = dv_{11} \wedge dt + (uv_1 + v_2 - f)\, dx \wedge dt$. Therefore, the relation $\psi^* dt = 0$ leads to $\psi^*\omega \equiv 0$ and it follows from

$$\psi^*\, \mathbf{i}_V(\omega) = c_2\psi^*\big[dv_{11} + (uv_1 + v_2 - f)\, dx\big] = 0$$

that $\psi_{11}' + \psi_0\psi_1 + \psi_2 - f(s, 0, \psi_0) = 0$. Hence, the admissible initial data $\psi_0(s)$ has to satisfy the following non-linear, third order ordinary differential equation

$$\psi_0'''(s) + \big[\psi_0(s) - as - c\big]\psi_0'(s) - 2a\psi_0(s) + b - f\big(s, 0, \psi_0(s)\big) = 0$$

where new constants are defined as $c_1/c_2 = a, c_3/c_2 = b, c_4/c_2 = c$. In order to obtain the mapping Ψ we must solve the linear ordinary differential equations

$$\frac{dx}{d\tau} = -c_2(ax + bt + c),\ \frac{dt}{d\tau} = -c_2(3at + 1),\ \frac{du}{d\tau} = c_2(2au - b)$$

under the initial conditions $x(0) = s, t(0) = 0, u(0) = \psi_0(s)$. The solution is easily found to be

$$x(s,\tau) = \frac{2(b - 3ac) + 3(2a^2 s + 2ac - b)e^{-c_2 a\tau} + be^{-3c_2 a\tau}}{6a^2}$$

$$t(s,\tau) = \frac{e^{-3c_2 a\tau} - 1}{3a}, \quad u(s,\tau) = \frac{\big(2a\psi_0(s) - b\big)e^{2c_2 a\tau} + b}{2a}.$$

As a simple application, we take $a = b = 0$ and $f = 0$. In this case, ψ_0 must satisfy the non-linear differential equation

$$\psi_0'''(s) + \big(\psi_0(s) - c\big)\psi_0'(s) = 0$$

whose solution is known to be

$$\psi_0(s) = 3c\,\mathrm{sech}^2\Big(\frac{c^{1/2}}{2}s + d\Big)$$

Since, in the limit $a \to 0, b \to 0$ we get

$$x = s - c_2 c\tau,\ t = -c_2\tau,\ u = \psi_0(s)$$

we have $s = x - ct$ and the soliton solution

$$u = \psi_0(x - ct) = 3c\,\mathrm{sech}^2\Big(\frac{c^{1/2}}{2}(x - ct) + d\Big)$$

is obtained as a generalised characteristic solution. ■

Example 9.6.4. This time we choose $n = 2, N = 2$ and consider the partial differential equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial t} = 1,\ \ \frac{\partial u}{\partial t}\frac{\partial v}{\partial x} = c^2$$

where c is a real constant. If we eliminate u or v between these equations, we see that u and v dependent variables have to satisfy separate non-linear wave equations

$$\Big(\frac{\partial u}{\partial t}\Big)^2\frac{\partial^2 u}{\partial x^2}=c^2\frac{\partial^2 u}{\partial t^2},\quad \Big(\frac{\partial v}{\partial x}\Big)^2\frac{\partial^2 v}{\partial t^2}=c^2\frac{\partial^2 v}{\partial x^2}.$$

Let us write $x^1=x$, $x^2=t$, $u^1=u$ and $u^2=v$ so that $v^1_1=u^1_{,1}, v^1_2=u^1_{,2}$, $v^2_1=u^2_{,1}, v^2_2=u^2_{,2}$. The contact forms in $\mathcal{C}_1$ are given by

$$\sigma^1=du^1-v^1_1dx^1-v^1_2dx^2,\ \ \sigma^2=du^2-v^2_1dx^1-v^2_2dx^2.$$

The relevant components of the isovector field V of the contact ideal may be extracted from (9.4.27). We know that $\mathbf{X}=\mathbf{X}(\mathbf{x},\mathbf{u})$, $\mathbf{U}=\mathbf{U}(\mathbf{x},\mathbf{u})$ appearing in those expressions are arbitrary functions. 0-forms inducing the differential equations become

$$F^1=v^1_1+v^2_2-1=0,\ \ F^2=v^1_2v^2_1-c^2=0.$$

We shall be looking for a simpler kind of an isovector field. Hence, we want to satisfy the conditions $£_V dF^1=0$ and $£_V dF^2=0$. Since $\mathbf{i}_V(F^1)=\mathbf{i}_V(F^2)=0$, they are reduced to $\mathbf{i}_V(dF^1)=0$ and $\mathbf{i}_V(dF^2)=0$. These relations lead to

$$V^1_1+V^2_2=0,\qquad v^2_1V^1_2+v^1_2V^2_1=0.$$

If we insert the expressions $v^2_2=1-v^1_1$ and $v^2_1=c^2/v^1_2$ into (9.4.27) we get the following polynomial identity in terms of the variables v^1_1 and v^1_2

$$\begin{aligned}&-\Big(\frac{\partial X^1}{\partial u^1}+\frac{\partial X^2}{\partial u^2}\Big)(v^1_1)^2v^1_2+\Big(\frac{\partial U^2}{\partial u^1}-\frac{\partial X^2}{\partial u^1}-\frac{\partial X^2}{\partial x^1}\Big)(v^1_2)^2+\\&\Big(\frac{\partial U^1}{\partial u^1}-\frac{\partial U^2}{\partial u^2}+2\frac{\partial X^2}{\partial u^2}-\frac{\partial X^1}{\partial x^1}+\frac{\partial X^2}{\partial x^2}\Big)v^1_1v^1_2+\Big(\frac{\partial U^2}{\partial u^2}+\frac{\partial U^1}{\partial x^1}+\frac{\partial U^2}{\partial x^2}\\&-(1+c^2)\frac{\partial X^2}{\partial u^2}-c^2\frac{\partial X^1}{\partial u^1}-\frac{\partial X^2}{\partial x^2}\Big)v^1_2+c^2\Big(\frac{\partial U^1}{\partial u^2}-\frac{\partial X^1}{\partial u^2}-\frac{\partial X^1}{\partial x^2}\Big)=0\\&\frac{\partial X^2}{\partial u^1}(v^1_1)^2(v^1_2)^2+\Big(\frac{\partial U^2}{\partial u^1}-\frac{\partial X^2}{\partial u^1}+\frac{\partial X^2}{\partial x^1}\Big)v^1_1(v^1_2)^2+c^2\frac{\partial X^1}{\partial u^2}(v^1_1)^2\\&\quad+\Big(\frac{\partial U^2}{\partial x^1}-c^2\frac{\partial X^2}{\partial u^1}-\frac{\partial X^2}{\partial x^1}\Big)(v^1_2)^2-2c^2\Big(\frac{\partial X^1}{\partial u^1}-\frac{\partial X^2}{\partial u^2}\Big)v^1_1v^1_2\\&\qquad-c^2\Big(\frac{\partial U^1}{\partial u^2}+\frac{\partial X^1}{\partial u^2}+\frac{\partial X^1}{\partial x^2}\Big)v^1_1+c^2\Big(\frac{\partial U^1}{\partial u^1}+\frac{\partial U^2}{\partial u^2}-2\frac{\partial X^2}{\partial u^2}\\&\qquad\quad-\frac{\partial X^1}{\partial x^1}-\frac{\partial X^2}{\partial x^2}\Big)v^1_2+c^2\Big(\frac{\partial U^1}{\partial u^2}+\frac{\partial U^1}{\partial x^2}-c^2\frac{\partial X^1}{\partial u^2}\Big)=0.\end{aligned}$$

Hence, we deduce 13 equations below obtained by setting the coefficients of v^1_1 and v^1_2 to zero:

$$\frac{\partial X^1}{\partial u^1}+\frac{\partial X^2}{\partial u^2}=0^1,\ \frac{\partial U^2}{\partial u^1}-\frac{\partial X^2}{\partial u^1}-\frac{\partial X^2}{\partial x^1}=0^2,\ \frac{\partial U^1}{\partial u^2}-\frac{\partial X^1}{\partial u^2}-\frac{\partial X^1}{\partial x^2}=0^3,$$
$$\frac{\partial U^1}{\partial u^1}-\frac{\partial U^2}{\partial u^2}+2\frac{\partial X^2}{\partial u^2}-\frac{\partial X^1}{\partial x^1}+\frac{\partial X^2}{\partial x^2}=0^4,\quad \frac{\partial U^2}{\partial u^1}-\frac{\partial X^2}{\partial u^1}+\frac{\partial X^2}{\partial x^1}=0^5,$$
$$\frac{\partial U^2}{\partial u^2}+\frac{\partial U^1}{\partial x^1}+\frac{\partial U^2}{\partial x^2}-(1+c^2)\frac{\partial X^2}{\partial u^2}-c^2\frac{\partial X^1}{\partial u^1}-\frac{\partial X^2}{\partial x^2}=0^6,\ \frac{\partial X^2}{\partial u^1}=0^7,$$
$$\frac{\partial X^1}{\partial u^2}=0^8,\quad \frac{\partial U^2}{\partial x^1}-c^2\frac{\partial X^2}{\partial u^1}-\frac{\partial X^2}{\partial x^1}=0^9,\quad \frac{\partial X^1}{\partial u^1}-\frac{\partial X^2}{\partial u^2}=0^{10},$$
$$\frac{\partial U^1}{\partial u^2}+\frac{\partial X^1}{\partial u^2}+\frac{\partial X^1}{\partial x^2}=0^{11},\qquad \frac{\partial U^1}{\partial u^2}+\frac{\partial U^1}{\partial x^2}-c^2\frac{\partial X^1}{\partial u^2}=0^{12},$$
$$\frac{\partial U^1}{\partial u^1}+\frac{\partial U^2}{\partial u^2}-2\frac{\partial X^2}{\partial u^2}-\frac{\partial X^1}{\partial x^1}-\frac{\partial X^2}{\partial x^2}=0^{13}.$$

Equations 1 and 10 together with equations 7 and 8 give rise to

$$X^1=X^1(x^1,x^2),\quad X^2=X^2(x^1,x^2)$$

On employing these relations, it follows from equations 3 and 11 and equations 2 and 5 that

$$X^1=g(x^1),\ X^2=h(x^2),$$
$$U^1=U^1(x^1,x^2,u^1),\ U^2=U^2(x^1,x^2,u^2).$$

Thus equations 12 and 9 yield

$$U^1=U^1(x^1,u^1),\ U^2=U^2(x^2,u^2).$$

If we add and subtract equations 4 and 13, we obtain

$$\frac{\partial U^1}{\partial u^1}=g'(x^1),\ \ \frac{\partial U^2}{\partial u^2}=h'(x^2)$$

whose integrations result in

$$U^1=g'(x^1)u^1+\gamma(x^1),\ U^2=h'(x^2)u^2+\delta(x^2).$$

If we introduce these expressions into the equation 6, we find that

$$g''(x^1)u^1+h''(x^2)u^2+\gamma'(x^1)+\delta'(x^2)=0$$

whence we deduce that

$$g''(x^1)=0,\ h''(x^2)=0,\ \gamma'(x^1)=\,-\,\delta'(x^2)=\mathit{constant}$$

Hence, we are led to the conclusion

$$g = c_1x^1 + c_2,\ h = c_3x^2 + c_4,\ \gamma = c_5x^1 + c_6,\ \delta = -c_5x^2 + c_7.$$

Thus, the functions determining the isovector components become

$$\begin{aligned} X^1 &= c_1x^1 + c_2, & X^2 &= c_3x^2 + c_4, \\ U^1 &= c_1u^1 + c_5x^1 + c_6, & U^2 &= c_3u^2 - c_5x^2 + c_7. \end{aligned}$$

This means that the isovector field in question is the prolongation of the vector field

$$V_G = (c_1x^1 + c_2)\frac{\partial}{\partial x^1} + (c_3x^2 + c_4)\frac{\partial}{\partial x^2} + (c_5x^1 + c_1u^1 + c_6)\frac{\partial}{\partial u^1} + (-c_5x^2 + c_3u^2 + c_7)\frac{\partial}{\partial u^2}.$$

In order to easily produce a characteristic solution, we select a particular form of the vector field V_G by taking $c_1 = a, c_3 = 1, c_5 = b, c_2 = c_4 = c_6 = c_7 = 0$:

$$V_G = ax^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + (bx^1 + au^1)\frac{\partial}{\partial u^1} + (-bx^2 + u^2)\frac{\partial}{\partial u^2}$$

We define the mapping $\psi : \mathbb{R} \to \mathcal{C}_1$ through the given smooth functions

$$\begin{aligned} &x^1 = s, \quad x^2 = 1, \quad u^1 = \psi_0^1(s), \quad u^2 = \psi_0^2(s) \\ &v_1^1 = \psi_1^1(s),\ v_2^1 = \psi_2^1(s),\ v_1^2 = \psi_1^2(s),\ v_2^2 = \psi_2^2(s). \end{aligned}$$

The transversality condition is met if $\psi^*\big(\mathbf{i}_V(\mu)\big) = ax^1dx^2 - x^2dx^1 \neq 0$. The constraints on initial data must satisfy the relations

$$\begin{aligned} &\psi_1^1 = (\psi_0^1)',\ \ \psi_1^2 = (\psi_0^2)', \\ &\psi_2^1 = bs + a\psi_0^1 - as(\psi_0^1)', \\ &\psi_2^2 = -b + \psi_0^2 - as(\psi_0^2)', \\ &\psi^*F^1 = (\psi_0^1)' - b + \psi_0^2 - as(\psi_0^2)' - 1 = 0, \\ &\psi^*F^2 = \big[bs + a\psi_0^1 - as(\psi_0^1)'\big](\psi_0^2)' = c^2. \end{aligned}$$

This amounts to say that to generate the characteristic solution corresponding to our present choice, the initial data ψ_0^1 and ψ_0^2 must satisfy the ordinary non-linear differential equations

$$(\psi_0^1)' + \psi_0^2 - as(\psi_0^2)' = b + 1,\ \big[bs + a\psi_0^1 - as(\psi_0^1)'\big](\psi_0^2)' = c^2.$$

The solution of this non-linear system is obviously not easy to find. But, we can try out to obtain a particular solution. Let us choose

$$\psi_0^1(s) = \alpha s + \beta s \log s, \;\; \psi_0^2\,(s) = \gamma + \delta \log s.$$

Introducing these functions into the differential equations, we see that the coefficients must satisfy the following relations

$$\alpha + \beta + \gamma - a\delta + (\beta + \delta) \log s = b + 1, \;\; (b - a\beta)\delta = c^2.$$

If we take $\delta = \, - \beta$, then the second equation implies that β ought to be chosen as a root of the quadratic equation

$$a\beta^2 - b\beta - c^2 = 0.$$

We therefore reach to the conclusion

$$\psi_0^1(s) = \alpha s + \beta s \log s, \;\; \psi_0^2\,(s) = b + 1 - \beta(1 + a) - \alpha - \beta \log s$$

where α is also an arbitrary constant. To determine the characteristic solution associated with the isovector field taken into consideration, we have to solve the ordinary linear differential equations

$$\begin{aligned} &\frac{dx^1}{d\tau} = ax^1, \; \frac{dx^2}{d\tau} = x^2, \\ &\frac{du^1}{d\tau} = bx^1 + au^1, \; \frac{du^2}{d\tau} = \, - bx^2 + u^2 \end{aligned}$$

under the initial conditions $x^1(0) = s$, $x^2(0) = 1$, $u^1(0) = \psi_0^1(s)$, $u^2(0) = \psi_0^2(s)$. We thus obtain

$$\begin{aligned} &x^1 = se^{a\tau}, \quad x^2 = e^{a\tau}, \quad u^1 = s(\alpha + \beta \log s + b\tau)e^{a\tau}, \\ &u^2 = \big[b + 1 - \beta(1 + a) - \alpha - \beta \log s - b\tau\big]e^{a\tau} \end{aligned}$$

describing the solution parametrically. ■

9.7. HORIZONTAL IDEALS AND THEIR SOLUTIONS

The most general transformation preserving the structure of a contact ideal $\mathcal{I}_m = \overline{\mathcal{I}}(\sigma^\alpha, \sigma^\alpha_{i_1}, \sigma^\alpha_{i_1 i_2}, \ldots \sigma^\alpha_{i_1 i_2 \cdots i_{m-1}}, d\sigma^\alpha_{i_1 i_2 \cdots i_{m-1}})$ has been determined in Sec. 9.3. Especially for $N > 1$, we know that this transformation is found as a prolongation of a point transformation in the graph space. This limitation creates, however, a major obstacle in obtaining solutions of a system of partial differential equations by using transformation methods This obstacle can be overcome to some extent by enlarging the contact ideal in an appropriate way. To this end, we would like first to determine linearly independent vector fields $V_i \in T(\mathcal{C}_m), i = 1, \ldots, n$ as to satisfy the relations

$$\mathbf{i}_{V_i}(dx^j) = \delta_i^j, \qquad (9.7.1)$$
$$\mathbf{i}_{V_{i_1}} \circ \mathbf{i}_{V_{i_2}} \circ \cdots \circ \mathbf{i}_{V_{i_k}}(\omega) = 0,\ 1 \leq k \leq n,\ \forall\, \omega \in \mathcal{I}_m \cap \Lambda^k(\mathcal{C}_m).$$

We call such a set of vector fields as a ***canonical system***. A canonical system generates an n-dimensional submodule of the tangent bundle $T(\mathcal{C}_m)$, so they constitute a basis for the n-dimensional module of ***Cartan annihilators*** of the contact ideal $\mathcal{I}_m$. It is easily seen that the conditions $(9.7.1)_2$ are fulfilled if and only if the following relations are satisfied:

$$\mathbf{i}_{V_i}(\sigma^\alpha_{i_1\cdots i_r}) = 0,\ 0 \leq r \leq m-1; \qquad (9.7.2)$$
$$\mathbf{i}_{V_j} \circ \mathbf{i}_{V_i}(d\sigma^\alpha_{i_1\cdots i_{m-1}}) = 0.$$

Indeed, if the conditions $(9.7.1)_2$ hold, then the conditions (9.7.2) are automatically satisfied since the generators $\sigma^\alpha_{i_1\cdots i_r}$ and $d\sigma^\alpha_{i_1 i_2\cdots i_{m-1}}$ of the ideal $\mathcal{I}_m$ are, respectively 1- and 2- forms. Conversely, let us assume that the conditions (9.7.2) are met. Let $\omega \in \mathcal{I}_m$ be a k-form in the ideal. Therefore, we have to write

$$\omega = \sum_{r=0}^{m-1} \lambda_\alpha^{i_1\cdots i_r} \wedge \sigma^\alpha_{i_1\cdots i_r} + \Lambda_\alpha^{i_1\cdots i_{m-1}} \wedge d\sigma^\alpha_{i_1\cdots i_{m-1}}$$

where $\lambda_\alpha^{i_1\cdots i_r} \in \Lambda^{k-1}(\mathcal{C}_m)$ and $\Lambda_\alpha^{i_1\cdots i_{m-1}} \in \Lambda^{k-2}(\mathcal{C}_m)$. But, we have

$$\mathbf{i}_{V_{i_1}} \circ \cdots \circ \mathbf{i}_{V_{i_k}}(\lambda_\alpha^{i_1\cdots i_r}) = 0, \quad \mathbf{i}_{V_{i_1}} \circ \cdots \circ \mathbf{i}_{V_{i_{k-1}}}(\Lambda_\alpha^{i_1\cdots i_{m-1}}) = 0$$

because of the degrees of those forms. Then we immediately observe that we get

$$\begin{aligned}\mathbf{i}_{V_{i_1}} \circ \cdots \circ \mathbf{i}_{V_{i_k}}(\omega) = &\sum_{r=0}^{m-1} \mathbf{i}_{V_{i_1}} \circ \cdots \circ \mathbf{i}_{V_{i_k}}(\lambda_\alpha^{i_1\cdots i_r}) \wedge \sigma^\alpha_{i_1\cdots i_r}\\ &+ \mathbf{i}_{V_{i_1}} \circ \cdots \circ \mathbf{i}_{V_{i_{k-1}}}(\Lambda_\alpha^{i_1\cdots i_{m-1}}) \wedge d\sigma^\alpha_{i_1\cdots i_{m-1}} = 0\end{aligned}$$

provided the relations (9.7.2) are satisfied. Let us represent a vector field $V_i \in T(\mathcal{C}_m)$ by

$$V_i = X_i^j \frac{\partial}{\partial x^j} + \sum_{r=0}^{m} V^\alpha_{i i_1\cdots i_r} \frac{\partial}{\partial v^\alpha_{i_1\cdots i_r}}$$

where $X_i^j, V^\alpha_{i i_1\cdots i_r} \in \Lambda^0(\mathcal{C}_m)$. *It is clear that the smooth functions* $V^\alpha_{i i_1\cdots i_r}$ *are to be taken as completely symmetric in subscripts* $i_1, \ldots, i_r$ *without loss of generality*. The condition $(9.7.1)_1$ yields simply

$$X_i^j = \delta_i^j$$

whereas we find from $(9.7.2)_1$ that

$$\mathbf{i}_{V_i}(dv^\alpha_{i_1\cdots i_r} - v^\alpha_{i_1\cdots i_r j}\,dx^j) = V^\alpha_{ii_1\cdots i_r} - v^\alpha_{i_1\cdots i_r i} = 0$$

and $V^\alpha_{ii_1\cdots i_r} = v^\alpha_{i_1\cdots i_r i}$ for $0 \le r \le m-1$. On the other hand, we can write $d\sigma^\alpha_{i_1\cdots i_{m-1}} = -\,dv^\alpha_{i_1\cdots i_{m-1}k} \wedge dx^k$ so that we arrive at the following interior products

$$\begin{aligned}\mathbf{i}_{V_i}(d\sigma^\alpha_{i_1\cdots i_{m-1}}) &= -\,V^\alpha_{ii_1\cdots i_{m-1}k}\,dx^k + dv^\alpha_{i_1\cdots i_{m-1}i},\\ \mathbf{i}_{V_j}\circ\mathbf{i}_{V_i}(d\sigma^\alpha_{i_1\cdots i_{m-1}}) &= -\,V^\alpha_{ii_1\cdots i_{m-1}j} + V^\alpha_{ji_1\cdots i_{m-1}i} = 0.\end{aligned}$$

This implies the symmetry property $V^\alpha_{ii_1\cdots i_{m-1}j} = V^\alpha_{ji_1\cdots i_{m-1}i}$ amounting to say that *the coefficients* $V^\alpha_{ii_1\cdots i_m}$ *must be completely symmetric with respect to all their subscripts*. Therefore the general form of a canonical system involving n linearly independent vector fields and satisfying the conditions (9.7.1) is given by

$$V_i = \frac{\partial}{\partial x^i} + \sum_{r=0}^{m-1} v^\alpha_{i_1\cdots i_r i}\frac{\partial}{\partial v^\alpha_{i_1\cdots i_r}} + V^\alpha_{ii_1\cdots i_m}\frac{\partial}{\partial v^\alpha_{i_1\cdots i_m}}. \tag{9.7.3}$$

Thus a contact manifold of order m admits infinitely many canonical systems associated with $N\binom{n+m}{m+1} = \frac{N(n+m)!}{(n-1)!\,(m+1)!}$ number of arbitrary smooth functions $V^\alpha_{ii_1\cdots i_m} \in \Lambda^0(\mathcal{C}_m)$. We next consider another vector V_j of the canonical system by

$$V_j = \frac{\partial}{\partial x^j} + \sum_{s=0}^{m-1} v^\beta_{j_1\cdots j_s j}\frac{\partial}{\partial v^\beta_{j_1\cdots j_s}} + V^\beta_{jj_1\cdots j_m}\frac{\partial}{\partial v^\beta_{j_1\cdots j_m}}$$

Successive application of the operators V_i and V_j results in the following expression after some manipulations

$$\begin{aligned}V_iV_j = \frac{\partial^2}{\partial x^i\partial x^j} &+ V^\alpha_{ii_1\cdots i_m}\frac{\partial^2}{\partial v^\alpha_{i_1\cdots i_m}\partial x^j} + V^\alpha_{ji_1\cdots i_m}\frac{\partial^2}{\partial v^\alpha_{i_1\cdots i_m}\partial x^i}\\ &+ \sum_{r=0}^{m-1}\Big[v^\alpha_{i_1\cdots i_r i}\frac{\partial^2}{\partial v^\alpha_{i_1\cdots i_r}\partial x^j} + v^\alpha_{i_1\cdots i_r j}\frac{\partial^2}{\partial v^\alpha_{i_1\cdots i_r}\partial x^i}\Big]\\ &\quad + \sum_{r=0}^{m-1}\sum_{s=0}^{m-1} v^\alpha_{i_1\cdots i_r i}v^\beta_{j_1\cdots j_s j}\frac{\partial^2}{\partial v^\alpha_{i_1\cdots i_r}\partial v^\beta_{j_1\cdots j_s}}\end{aligned}$$

$$+\sum_{r=0}^{m-1}\big[V^\alpha_{ii_1\cdots i_m}v^\beta_{j_1\cdots j_rj}+V^\alpha_{ji_1\cdots i_m}v^\beta_{j_1\cdots j_ri}\big]\frac{\partial^2}{\partial v^\alpha_{i_1\cdots i_m}\partial v^\beta_{j_1\cdots j_r}}$$
$$+\sum_{r=1}^{m-1}v^\alpha_{i_1\cdots i_{r-1}ij}\frac{\partial}{\partial v^\alpha_{i_1\cdots i_{r-1}}}+V^\alpha_{ii_1\cdots i_{m-1}j}\frac{\partial}{\partial v^\alpha_{i_1\cdots i_{m-1}}}$$
$$+V_i\big(V^\alpha_{ji_1\cdots i_m}\big)\frac{\partial}{\partial v^\alpha_{i_1\cdots i_m}}$$

where we have renamed the dummy indices whenever necessary. Hence, the Lie product of these two vector fields is easily found to be

$$\begin{aligned}[][V_i,V_j]&=V_iV_j-V_jV_i\\&=\big[V_i\big(V^\alpha_{ji_1\cdots i_m}\big)-V_j\big(V^\alpha_{ii_1\cdots i_m}\big)\big]\frac{\partial}{\partial v^\alpha_{i_1\cdots i_m}}.\end{aligned}\tag{9.7.4}$$

The differential of a function $f\in\Lambda^0(\mathcal{C}_m)$ can now be expressed as

$$\begin{aligned}df&=\frac{\partial f}{\partial x^i}\,dx^i+\sum_{r=0}^{m}\frac{\partial f}{\partial v^\alpha_{i_1\cdots i_r}}\,dv^\alpha_{i_1\cdots i_r}\\&=\Big[\frac{\partial f}{\partial x^i}+\sum_{r=0}^{m-1}v^\alpha_{i_1\cdots i_ri}\frac{\partial f}{\partial v^\alpha_{i_1\cdots i_r}}+V^\alpha_{ii_1\cdots i_m}\frac{\partial f}{\partial v^\alpha_{i_1\cdots i_m}}\Big]dx^i\\&\quad+\sum_{r=0}^{m-1}\frac{\partial f}{\partial v^\alpha_{i_1\cdots i_r}}\,\sigma^\alpha_{i_1\cdots i_r}+\frac{\partial f}{\partial v^\alpha_{i_1\cdots i_m}}\,\Sigma^\alpha_{i_1\cdots i_m}\\&=V_i(f)\,dx^i+\sum_{r=0}^{m-1}\frac{\partial f}{\partial v^\alpha_{i_1\cdots i_r}}\,\sigma^\alpha_{i_1\cdots i_r}+\frac{\partial f}{\partial v^\alpha_{i_1\cdots i_m}}\,\Sigma^\alpha_{i_1\cdots i_m}\in\Lambda^1(\mathcal{C}_m).\end{aligned}\tag{9.7.5}$$

In order to obtain this relation we have first replaced 1-forms $dv_{i_1\cdots i_r}$ with 1-forms $\sigma_{i_1\cdots i_r}+v^\alpha_{i_1\cdots i_ri}\,dx^i$ for $0\le r\le m-1$ and then we have further introduced 1-forms $\Sigma^\alpha_{i_1\cdots i_m}$ appearing in the above expression. However, their definition will be given a little bit later in (9.7.7).

The ***vertical ideal*** of the exterior algebra $\Lambda(\mathcal{C}_m)$ is identified as the closed ideal prescribed as follows

$$\mathcal{V}_m=\mathcal{I}(dx^1,dx^2,\dots,dx^n).\tag{9.7.6}$$

On the D-dimensional submanifold $\{x^i=c^i;i=1,\dots,n\}$ where c^i's are constants, this ideal is obviously annihilated. In other words, the ideal $\mathcal{V}_m$ vanishes when restricted to the fibres of the ideal $\mathcal{C}_m$ over $\mathcal{D}_n$. This, of course, justifies our use of the term '*vertical*'. Let us now take into account the forms

$$\Sigma^\alpha_{i_1\cdots i_m} = dv^\alpha_{i_1\cdots i_m} - V^\alpha_{ii_1\cdots i_m}dx^i \in \Lambda^1(\mathcal{C}_m). \tag{9.7.7}$$

We shall call them as ***horizontal* 1*-forms***. A ***horizontal ideal*** of the exterior algebra $\Lambda(\mathcal{C}_m)$ will now be defined as

$$\mathcal{H}_m = \mathcal{I}(\sigma^\alpha_{i_1\cdots i_r}, 0 \le r \le m-1;\ \Sigma^\alpha_{i_1\cdots i_m}). \tag{9.7.8}$$

We know that $d\sigma^\alpha_{i_1\cdots i_r} \in \mathcal{H}_m$ for $0 \le r \le m-2$. For $r = m-1$, due to the relation $d\sigma^\alpha_{i_1\cdots i_{m-1}} = -\,dv^\alpha_{i_1\cdots i_{m-1}i} \wedge dx^i$ we find immediately that

$$\begin{aligned} d\sigma^\alpha_{i_1\cdots i_{m-1}} &= -\,\Sigma^\alpha_{i_1\cdots i_{m-1}i} \wedge dx^i - V^\alpha_{ii_1\cdots i_{m-1}j}\, dx^i \wedge dx^j \\ &= -\,\Sigma^\alpha_{i_1\cdots i_{m-1}i} \wedge dx^i \in \mathcal{H}_m. \end{aligned} \tag{9.7.9}$$

Moreover on using (9.7.5), the relation

$$d\Sigma^\alpha_{i_1\cdots i_m} = -\,dV^\alpha_{ji_1\cdots i_m} \wedge dx^j$$

leads to

$$\begin{aligned} d\Sigma^\alpha_{i_1\cdots i_m} = -\,V_i(V^\alpha_{ji_1\cdots i_m})\, dx^i \wedge dx^j - \sum_{r=0}^{m-1} \frac{\partial V^\alpha_{ji_1\cdots i_m}}{\partial v^\alpha_{i_1\cdots i_r}}\, \sigma_{i_1\cdots i_r} \wedge dx^j \\ -\,\frac{\partial V^\alpha_{ji_1\cdots i_m}}{\partial v^\alpha_{i_1\cdots i_m}}\, \Sigma^\alpha_{i_1\cdots i_m} \wedge dx^j \end{aligned}$$

from which we deduce that

$$d\Sigma^\alpha_{i_1\cdots i_m} + \frac{1}{2}\big[V_i(V^\alpha_{ji_1\cdots i_m}) - V_j(V^\alpha_{ii_1\cdots i_m})\big]dx^i \wedge dx^j \in \mathcal{H}_m. \tag{9.7.10}$$

Therefore, we get $d\Sigma^\alpha_{i_1\cdots i_m} \in \mathcal{H}_m$ *if and only if the conditions*

$$V_i(V^\alpha_{ji_1\cdots i_m}) = V_j(V^\alpha_{ii_1\cdots i_m}) \tag{9.7.11}$$

are satisfied. In this case, the horizontal ideal $\mathcal{H}_m$ turns out to be a closed ideal. In view of (9.7.4), we see at once that the relation (9.7.11) becomes possible if and only if

$$[V_i, V_j] = 0, \ \ 1 \le i, j \le n \tag{9.7.12}$$

that is, *if the canonical system consists of commuting vector fields*.

We denote the characteristic subspace of the ideal $\mathcal{H}_m$ which will be called henceforth as the ***horizontal module*** by $\mathcal{S}_{\mathcal{H}_m}$. Thus, if $U \in T(\mathcal{C}_m)$ is a characteristic vector of $\mathcal{H}_m$, then the relation $\mathbf{i}_U(\mathcal{H}_m) \subset \mathcal{H}_m$ must be satisfied. Since all generators of the horizontal ideal are 1-forms, it becomes possible to comply with this condition if and only if we have

$$\mathbf{i}_U(\sigma^\alpha_{i_1\cdots i_r}) = 0,\ 0 \le r \le m-1;\ \ \mathbf{i}_U(\Sigma^\alpha_{i_1\cdots i_m}) = 0. \tag{9.7.13}$$

Let us take a vector field

$$U = X^i \frac{\partial}{\partial x^i} + \sum_{r=0}^{m} U^\alpha_{i_1\cdots i_r}\frac{\partial}{\partial v^\alpha_{i_1\cdots i_r}}$$

into consideration. Then (9.7.13) requires that

$$U^\alpha_{i_1\cdots i_r} = v^\alpha_{i_1\cdots i_r i}X^i,\ 0 \le r \le m-1;\ \ U^\alpha_{i_1\cdots i_m} = V^\alpha_{ii_1\cdots i_m}X^i.$$

Hence, any vector $U \in \mathcal{S}_{\mathcal{H}_m}$ can be written as

$$U = X^i\Big(\frac{\partial}{\partial x^i} + \sum_{r=0}^{m-1} v^\alpha_{i_1\cdots i_r i}\frac{\partial}{\partial v^\alpha_{i_1\cdots i_r}} + V^\alpha_{ii_1\cdots i_m}\frac{\partial}{\partial v^\alpha_{i_1\cdots i_m}}\Big) = X^i V_i.$$

Thus, *canonical system constitutes a basis of the horizontal module as well.* Consequently, *to each choice of completely symmetric smooth functions $V^\alpha_{ii_1\cdots i_r} \in \Lambda^0(\mathcal{C}_m)$ there corresponds a horizontal ideal of $\Lambda(\mathcal{C}_m)$. The horizontal module of this ideal coincides with both the modules of characteristic vectors of $\mathcal{H}_m$ and Cartan annihilators of the contact ideal $\mathcal{I}_m$*. It is clear that the vectors V_i satisfy naturally the characteristic conditions (9.7.13). In this situation the canonical system produces the distribution $\mathcal{S}_{\mathcal{H}_m}$. In case the conditions (9.7.11) are also met, this distribution proves to be involutive. We can now show the following theorem.

Theorem 9.7.1. *The horizontal module $\mathcal{S}_{\mathcal{H}_m}$ is the module of isovectors of the horizontal ideal $\mathcal{H}_m$ if and only if the ideal $\mathcal{H}_m$ is closed.*

In order that a vector field $V \in \mathcal{S}_{\mathcal{H}_m}$ is to be an isovector field of the horizontal ideal $\mathcal{H}_m$, the conditions $\pounds_V\sigma^\alpha_{i_1\cdots i_r} \in \mathcal{H}_m$ where $0 \le r \le m-1$ and $\pounds_V\Sigma^\alpha_{i_1\cdots i_m} \in \mathcal{H}_m$ should be satisfied. If we note (9.7.13), we get

$$\pounds_V\sigma^\alpha_{i_1\cdots i_r} = \mathbf{i}_V(d\sigma^\alpha_{i_1\cdots i_r}),\quad \pounds_V\Sigma^\alpha_{i_1\cdots i_m} = \mathbf{i}_V(d\Sigma^\alpha_{i_1\cdots i_m}).$$

The relations (9.7.9) and (9.7.10) lead to $\pounds_V\sigma^\alpha_{i_1\cdots i_r} \in \mathcal{H}_m$ for $0 \le r \le m-1$ together with

$$\pounds_V\Sigma^\alpha_{i_1\cdots i_m} + \frac{1}{2}\big[V_i(V^\alpha_{ji_1\cdots i_m}) - V_j(V^\alpha_{ii_1\cdots i_m})\big]\mathbf{i}_V(dx^i \wedge dx^j) \in \mathcal{H}_m.$$

If we write $V = X^iV_i$, then we find $\mathbf{i}_V(dx^i \wedge dx^j) = X^i\,dx^j - X^j\,dx^i$. Therefore, the last terms belong obviously to the vertical ideal. Thus, we find $\pounds_V \Sigma^{\alpha}_{i_1\cdots i_m} \in \mathcal{H}_m$ *if and only if* the conditions

$$V_i(V^{\alpha}_{ji_1\cdots i_m}) = V_j(V^{\alpha}_{ii_1\cdots i_m})$$

are again satisfied. (9.7.11) constitute the necessary and sufficient conditions for a horizontal ideal $\mathcal{H}_m$ to be closed. We had seen that they were equivalent to the conditions $[V_i, V_j] = 0$. □

According to Theorem 5.13.4, the closed horizontal ideals are completely integrable. Let $\mathfrak{H}_m$ denote the set of all closed horizontal ideals. We can readily demonstrate that this set is not empty. For instance, we may consider the smooth functions $f^{\alpha} \in \Lambda^0(M)$ and define the functions

$$V^{\alpha}_{ii_1\cdots i_m} = \frac{\partial^{m+1} f^{\alpha}(\mathbf{x})}{\partial x^i \partial x^{i_1} \cdots \partial x^{i_m}}.$$

These function plainly verify both the condition of complete symmetry and the relations (9.7.11). When we consider a member of $\mathfrak{H}_m$, the vectors V_i, $1 \leq i \leq n$ generate n-dimensional integral manifolds in $\mathcal{C}_m$ annihilating the closed ideal $\mathcal{H}_m$. Since the dimension of $\mathcal{C}_m$ is $n + D$, we know that these manifolds are obtainable from the independent solutions $g^a \in \Lambda^0(\mathcal{C}_m)$, $a = 1, \ldots, D$ of the linear partial differential equations

$$V_i(g) = 0, \ i = 1, \ldots, n$$

by setting $g^a = c^a$ where c^a are real constants. The general solution of the above equations may be written as $g = G(g^1, \ldots, g^D) = G(g^a)$. Hence, the closed ideal $\mathcal{H}_m$ provides an n-dimensional *foliation* on the manifold $\mathcal{C}_m$ [*see* Sec. 2.11]. Each choice of constants characterises a *leaf*.

Next, we shall try to calculate all isovector fields of a horizontal ideal $\mathcal{H}_m \in \mathfrak{H}_m$. If U is an isovector, then the relations $\pounds_U \sigma^{\alpha}_{i_1\cdots i_r} \in \mathcal{H}_m$ for $0 \leq r \leq m-1$ and $\pounds_U \Sigma^{\alpha}_{i_1\cdots i_m} \in \mathcal{H}_m$ must be satisfied. If we make use of the Cartan formula $\pounds_U \omega = d\big(\mathbf{i}_U(\omega)\big) + \mathbf{i}_U(d\omega)$, we get

$$\begin{aligned}
\pounds_U \sigma^{\alpha}_{i_1\cdots i_r} &= d\big(\mathbf{i}_U(\sigma^{\alpha}_{i_1\cdots i_r})\big) - \mathbf{i}_U(\sigma^{\alpha}_{i_1\cdots i_r i})\, dx^i + \mathbf{i}_U(dx^i)\, \sigma^{\alpha}_{i_1\cdots i_r i},\\
\pounds_U \sigma^{\alpha}_{i_1\cdots i_{m-1}} &= d\big(\mathbf{i}_U(\sigma^{\alpha}_{i_1\cdots i_{m-1}})\big) - \mathbf{i}_U(\Sigma^{\alpha}_{i_1\cdots i_{m-1} i})\, dx^i + \mathbf{i}_U(dx^i)\, \Sigma^{\alpha}_{i_1\cdots i_{m-1} i},\\
\pounds_U \Sigma^{\alpha}_{i_1\cdots i_m} &= d\big(\mathbf{i}_U(\Sigma^{\alpha}_{i_1\cdots i_m})\big) - \mathbf{i}_U(dV^{\alpha}_{ii_1\cdots i_{m-1}})\, dx^i + \mathbf{i}_U(dx^i)\, dV^{\alpha}_{i_1\cdots i_m i}
\end{aligned}$$

where $0 \leq r \leq m-2$. Let us represent the isovector U as

$$U = X^i V_i + \sum_{r=0}^{m} U^{\alpha}_{i_1\cdots i_r} \frac{\partial}{\partial v^{\alpha}_{i_1\cdots i_r}}$$

without loss of generality where X^i, $U^{\alpha}_{i_1\cdots i_r} \in \Lambda^0(\mathcal{C}_m)$ for $0 \leq r \leq m$ are smooth functions. Since members of the canonical system are characteristic vectors of $\mathcal{H}_m$, we eventually obtain

$$\begin{aligned}
\pounds_U \sigma^\alpha_{i_1\cdots i_r} &= \big[V_i(U^\alpha_{i_1\cdots i_r}) - U^\alpha_{i_1\cdots i_r i}\big]dx^i + \sum_{s=0}^{m-1}\frac{\partial U^\alpha_{i_1\cdots i_r}}{\partial v^\beta_{i_1\cdots i_s}}\sigma^\beta_{i_1\cdots i_s}\\
&\quad + \frac{\partial U^\alpha_{i_1\cdots i_r}}{\partial v^\beta_{i_1\cdots i_m}}\Sigma^\beta_{i_1\cdots i_m} + X^i\sigma^\alpha_{i_1\cdots i_r i}, \qquad 0 \le r \le m-2\\
\pounds_U \sigma^\alpha_{i_1\cdots i_{m-1}} &= \big[V_i(U^\alpha_{i_1\cdots i_{m-1}}) - U^\alpha_{i_1\cdots i_{m-1} i}\big]dx^i + \sum_{r=0}^{m-1}\frac{\partial U^\alpha_{i_1\cdots i_{m-1}}}{\partial v^\beta_{i_1\cdots i_r}}\sigma^\beta_{i_1\cdots i_r}\\
&\quad + \frac{\partial U^\alpha_{i_1\cdots i_{m-1}}}{\partial v^\beta_{i_1\cdots i_m}}\Sigma^\beta_{i_1\cdots i_m} + X^i\Sigma^\alpha_{i_1\cdots i_{m-1} i}.
\end{aligned}$$

Thus, in order to get $\pounds_U \sigma^\alpha_{i_1\cdots i_r} \in \mathcal{H}_m, 0 \le r \le m-1$ we have to set

$$U^\alpha_{i_1\cdots i_r i} = V_i(U^\alpha_{i_1\cdots i_r}),\ \ 0 \le r \le m-1.$$

The solution of this recurrence relation is clearly given by

$$U^\alpha_{i_1 i_2\cdots i_r} = V_{i_1}V_{i_2}\cdots V_{i_r}(U^\alpha),\ \ 0 \le r \le m \tag{9.7.14}$$

in terms of N functions $U^\alpha \in \Lambda^0(\mathcal{C}_m)$ where we have adopted the convention that $U^\alpha_{i_0} = U^\alpha$. On the other hand, the relation

$$\begin{aligned}
\pounds_U \Sigma^\alpha_{i_1\cdots i_m} &= \big[V_i(U^\alpha_{i_1\cdots i_m}) - U(V^\alpha_{i i_1\cdots i_m}) + X^j V_i(V^\alpha_{j i_1\cdots i_m})\big]dx^i\\
&\quad + \sum_{r=0}^{m-1}\frac{\partial U^\alpha_{i_1\cdots i_m}}{\partial v^\beta_{i_1\cdots i_r}}\sigma^\beta_{i_1\cdots i_r} + \frac{\partial U^\alpha_{i_1\cdots i_m}}{\partial v^\beta_{i_1\cdots i_m}}\Sigma^\beta_{i_1\cdots i_m}\\
&\quad + X^j\Big[\sum_{r=0}^{m-1}\frac{\partial V^\alpha_{j i_1\cdots i_m}}{\partial v^\beta_{i_1\cdots i_r}}\sigma^\beta_{i_1\cdots i_r} + \frac{\partial V^\alpha_{j i_1\cdots i_m}}{\partial v^\beta_{i_1\cdots i_m}}\Sigma^\beta_{i_1\cdots i_m}\Big]
\end{aligned}$$

requires that we have to satisfy the following equations

$$V_i(U^\alpha_{i_1\cdots i_m}) - U(V^\alpha_{i i_1\cdots i_m}) + X^j V_i(V^\alpha_{j i_1\cdots i_m}) = 0$$

in order to get $\pounds_U \Sigma^\alpha_{i_1\cdots i_m} \in \mathcal{H}_m$. Thereby we obtain the expressions

$$V_i(U^\alpha_{i_1\cdots i_m}) = \sum_{r=0}^{m} U^\beta_{i_1\cdots i_r}\frac{\partial V^\alpha_{i i_1\cdots i_m}}{\partial v^\beta_{i_1\cdots i_r}} + X^j\big[V_j(V^\alpha_{i i_1\cdots i_m}) - V_i(V^\alpha_{j i_1\cdots i_m})\big].$$

Because $\mathcal{H}_m$ is closed, the relations (9.7.11) are to be satisfied. Hence, the smooth functions U^α ought to verify the restrictions

$$\begin{aligned} V_i V_{i_1} \cdots V_{i_m}(U^\alpha) &= \sum_{r=0}^{m} V_{i_1} \cdots V_{i_r}(U^\beta) \frac{\partial V^\alpha_{ii_1\cdots i_m}}{\partial v^\beta_{i_1\cdots i_r}} \qquad (9.7.15)\\ &= U^\beta \frac{\partial V^\alpha_{ii_1\cdots i_m}}{\partial u^\beta} + V_{i_1}(U^\beta) \frac{\partial V^\alpha_{ii_1\cdots i_m}}{\partial v^\beta_{i_1}} + \cdots + V_{i_1} \cdots V_{i_m}(U^\beta) \frac{\partial V^\alpha_{ii_1\cdots i_m}}{\partial v^\beta_{i_1\cdots i_m}}. \end{aligned}$$

For $U^\alpha = 0$, (9.7.15) is trivially fulfilled. Therefore we arrive at the following theorem.

Theorem 9.7.2. *In terms of n functions $X^i \in \Lambda^0(\mathcal{C}_m)$ and N functions $U^\alpha \in \Lambda^0(\mathcal{C}_m)$ satisfying* (9.7.15), *all isovector fields of a closed horizontal ideal $\mathcal{H}_m$ are expressible in the form*

$$U = X^i V_i + \sum_{r=0}^{m} V_{i_1} \cdots V_{i_r}(U^\alpha) \frac{\partial}{\partial v^\alpha_{i_1\cdots i_r}}. \qquad (9.7.16)$$

Canonical system constitutes also the module of isovectors of closed $\mathcal{H}_m$ if only one chooses $U^\alpha = 0$. □

When the ideal $\mathcal{H}_m$ is closed, the canonical system is a Lie algebra and n-dimensional submanifold $\mathfrak{S}$ it produces annihilates this ideal. The mapping $\phi : \mathfrak{S} \to \mathcal{C}_m$ prescribing the manifold $\mathfrak{S}$ is a solution mapping of the ideal, that is, one has $\phi^* \mathcal{H}_m = 0$. Since the Lie products of vectors V_i vanish, they generate a coordinate mesh on $\mathfrak{S}$. We can determine the mapping ϕ by means of congruences that are integral curves of vector fields V_i. Let us denote $\mathfrak{n} = n + D$ number of coordinates of the manifold $\mathcal{C}_m$ by

$$\{x^i, v^\alpha_{i_1\cdots i_r} : 0 \le r \le m\} = \{z^\mathfrak{a} : 1 \le \mathfrak{a} \le \mathfrak{n}\}$$

as in Sec. 9.6. Let us take into consideration characteristic vector fields, or Cartan annihilators $V_i = v^\mathfrak{a}_i(\mathbf{z}) \partial / \partial z^\mathfrak{a} \in T(\mathcal{C}_m)$ of the horizontal ideal $\mathcal{H}_m$. We know that Lie products of these vector fields vanish. Their trajectories are found as usual by integrating the ordinary differential equations

$$\frac{d\zeta^\mathfrak{a}}{dt^i} = v^\mathfrak{a}_i(\boldsymbol{\zeta}), \quad \zeta^\mathfrak{a}(0) = z^\mathfrak{a}$$

where t^i is a real parameter. In order to determine the mapping ϕ, we start with the vector field V_1. We can formally express the solution of the ordinary differential equations

$$\frac{d\zeta^\mathfrak{a}_1}{dt^1} = v^\mathfrak{a}_1(\boldsymbol{\zeta}_1), \quad \zeta^\mathfrak{a}_1(0) = z^\mathfrak{a}$$

as $\zeta^\mathfrak{a}_1(t^1; \mathbf{z}) = e^{t^1 V_1}(z^\mathfrak{a})$. In the second step, the solution of the equations

$$\frac{d\zeta_2^{\mathfrak{a}}}{dt^2} = v_2^{\mathfrak{a}}(\boldsymbol{\zeta_2}), \quad \zeta_2^{\mathfrak{a}}(0) = \zeta_1^{\mathfrak{a}}(t^1; \mathbf{z})$$

can be written as $\zeta_2^{\mathfrak{a}}(t^2; \boldsymbol{\zeta}_1) = e^{t^2 V_2}(\boldsymbol{\zeta}_1) = e^{t^2 V_2} e^{t^1 V_1}(z^{\mathfrak{a}})$. Since the vectors V_1 and V_2 commute, we then find that

$$\zeta_2^{\mathfrak{a}}\big(t^2; \boldsymbol{\zeta}_1(t^1; \mathbf{z})\big) = e^{t^1 V_1 + t^2 V_2}(z^{\mathfrak{a}}).$$

If we continue in this fashion, the mapping $\boldsymbol{\zeta} = \phi(\mathbf{t}; \mathbf{z})$ is specified by the relation

$$\zeta^{\mathfrak{a}} = e^{t^i V_i}(z^{\mathfrak{a}}). \tag{9.7.17}$$

This expression determines n-dimensional solution manifold of a closed ideal $\mathcal{H}_m$ through any point **z** of $\mathcal{C}_m$ or, in other words, a leaf of the foliation annihilating the ideal $\mathcal{H}_m$ passing through a point $z^{\mathfrak{a}}$. The integration parameters $t^1, t^2, \dots, t^n$ form the *natural coordinates* of the solution manifold.

The action of the mapping ϕ on the coordinates x^i of the manifold M can easily be evaluated. Since $v_j^i = \delta_j^i$, the differential equations

$$\frac{dx^i}{dt^j} = \delta_j^i, \;\; x^i(0) = x_0^i$$

yield immediately the simple solution

$$x^i = x_0^i + t^i. \tag{9.7.18}$$

Thus, coordinates of the open set $\mathcal{D}_n$ of $\mathbb{R}^n$ over which differential equations are defined and local coordinates of the solution manifold are connected by a simple translation. In this case, we have $\phi^* dx^i = dt^i$ and as a result of this we obtain

$$\phi^*\mu = \phi^*(dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n) = dt^1 \wedge dt^2 \wedge \cdots \wedge dt^n.$$

On writing $\phi^* u^\alpha = \phi^\alpha(\mathbf{t})$ and noting that $\phi^*\sigma^\alpha_{i_1\cdots i_r} = 0$ and $\phi^*\Sigma^\alpha_{i_1\cdots i_m} = 0$, we draw the conclusion

$$\phi^* v^\alpha_{i_1\cdots i_r} = \frac{\partial^r \phi^\alpha(\mathbf{t})}{\partial t^{i_1}\cdots\partial t^{i_r}}, 0 \le r \le m-1; \; \phi^* V^\alpha_{ii_1\cdots i_m} = \frac{\partial^{m+1}\phi^\alpha(\mathbf{t})}{\partial t^i \partial t^{i_1}\cdots\partial t^{i_m}}.$$

Hence, selected functions $V^\alpha_{ii_1\cdots i_m}$ provide information about $(m+1)$th order partial derivatives of functions u^α on the solution manifold.

If we take notice of the relation (9.7.9), we immediately realise that ϕ is a solution mapping of the contact ideal. Instead of the closed fundamental ideal $\mathfrak{I}_m$ [*see p.* 521], let us now introduce a *new balance ideal* by

$$\mathfrak{B}_m = \mathcal{I}(\sigma^\alpha_{i_1\cdots i_r}, 0\le r\le m-1; \Sigma^\alpha_{i_1\cdots i_m}; \omega^\alpha)$$

where n-forms ω^α are given in (9.4.3). *If* $\mathcal{H}_m \in \mathfrak{H}_m$, *then* $\mathfrak{B}_m$ *is closed.* In fact, since $dx^i \wedge \mu = 0$ then (9.7.5) for the function Σ^α implies that

$$d\omega^\alpha = d\Sigma^\alpha \wedge \mu \in \mathcal{H}_m \subset \mathfrak{B}_m.$$

On the other hand, we can similarly write

$$\begin{aligned}\omega^\alpha &= d\Sigma^{\alpha i} \wedge \mu_i + \Sigma^\alpha \mu = \big[V_i(\Sigma^{\alpha i}) + \Sigma^\alpha\big]\mu \\ &\quad + \Big[\sum_{r=0}^{m-1} \frac{\partial \Sigma^{\alpha i}}{\partial v^\beta_{i_1\cdots i_r}} \sigma^\beta_{i_1\cdots i_r} + \frac{\partial \Sigma^{\alpha i}}{\partial v^\beta_{i_1\cdots i_m}} \Sigma^\beta_{i_1\cdots i_m}\Big] \wedge \mu_i \in \Lambda^n(\mathcal{C}_m).\end{aligned}$$

Introducing the smooth functions

$$\mathcal{F}^\alpha = V_i(\Sigma^{\alpha i}) + \Sigma^\alpha \in \Lambda^0(\mathcal{C}_m), \tag{9.7.19}$$

we readily observe that

$$\omega^\alpha - \mathcal{F}^\alpha \mu \in \mathcal{H}_m. \tag{9.7.20}$$

Therefore, we obtain

$$\phi^*\omega^\alpha = \phi^*\mathcal{F}^\alpha \phi^*\mu = \phi^*\mathcal{F}^\alpha dt^1 \wedge dt^2 \wedge \cdots \wedge dt^n$$

for the solution mapping of the horizontal ideal. Consequently, we conclude that $\phi^*\omega^\alpha = 0$ if and only if $\phi^*\mathcal{F}^\alpha = 0$. Hence, only in this case the solution mapping ϕ of $\mathcal{H}_m$ corresponds to a solution of the new balance ideal as well. Since $\mathcal{F}^\alpha(\mathbf{z}) \in \Lambda^0(\mathcal{C}_m)$ are 0-forms, we get $\phi^*\mathcal{F}^\alpha = 0$ if only if $\mathcal{F}^\alpha = 0$. Let us define the submanifold $\mathcal{P}_m \subseteq \mathcal{C}_m$ by

$$\mathcal{P}_m = \{\mathbf{z} \in \mathcal{C}_m : \mathcal{F}^\alpha(\mathbf{z}) = 0,\ \alpha = 1, \dots, N\}. \tag{9.7.21}$$

Hence, we see that the relations $\phi^*\mathcal{F}^\alpha = 0$ can only be realised on the region $\mathcal{R}_m \subseteq \mathcal{C}_m$ that is determined by non-empty intersections of submanifold $\mathcal{P}_m$ with the leaves of the foliation generated by the mapping ϕ. Moreover, because the set $\mathcal{D}_n \subseteq \mathbb{R}^n$ over which differential equations are defined is open, the set $\phi^*\mathcal{R}_m$ must also be open. Therefore, it is clear that we can obtain such a solution under rather restricting conditions. However, the dependence of the ideal $\mathcal{H}_m$ on functions $V^\alpha_{ii_1\cdots i_m}$ that offer some freedom of choice despite they have to obey certain rules might offer various alternatives. That makes it possible to find some useful solutions by clever choices.

We now attempt to determine isovector fields of the balance ideal $\mathfrak{B}_m$.

Let U be an isovector field of the closed horizontal ideal $\mathcal{H}_m$ given by (9.7.16). It then follows from (9.7.20) that

$$£_U\omega^\alpha - £_U(\mathcal{F}^\alpha\mu) \in \mathcal{H}_m.$$

Furthermore, we can write

$$£_U(\mathcal{F}^\alpha\mu) = £_U(\mathcal{F}^\alpha)\mu + \mathcal{F}^\alpha £_U(\mu) = U(\mathcal{F}^\alpha)\mu + \mathcal{F}^\alpha dX^i \wedge \mu_i$$

whence we deduce

$$£_U(\mathcal{F}^\alpha\mu) = \big[U(\mathcal{F}^\alpha) + \mathcal{F}^\alpha V_i(X^i)\big]\mu + \mathcal{G}^\alpha,\ \ \mathcal{G}^\alpha \in \mathcal{H}_m$$

on utilising (9.7.5). Since $\omega^\alpha \notin \mathcal{H}_m$, in order to obtain $£_U\omega^\alpha \in \mathfrak{B}_m$ we may write $\Lambda^\alpha_\beta\omega^\beta = \Lambda^\alpha_\beta\mathcal{F}^\beta\mu \bmod \mathcal{H}_m$ in view of $\omega^\alpha = \mathcal{F}^\alpha\mu \bmod \mathcal{H}_m$ for functions $\Lambda^\alpha_\beta \in \Lambda^0(\mathcal{C}_m)$ so that the conditions

$$\big[U(\mathcal{F}^\alpha) + \mathcal{F}^\alpha V_i(X^i)\big]\mu = \Lambda^\alpha_\beta\mathcal{F}^\beta\mu$$

lead to the result

$$£_U\omega^\alpha = \Lambda^\alpha_\beta\omega^\beta \bmod \mathcal{H}_m \in \mathfrak{B}_m.$$

In other words, we have to find some functions $\lambda^\alpha_\beta \in \Lambda^0(\mathcal{C}_m)$ so that the relations

$$U(\mathcal{F}^\alpha) = X^iV_i(\mathcal{F}^\alpha) + \sum_{r=0}^{m} V_{i_1}\cdots V_{i_r}(U^\beta)\frac{\partial\mathcal{F}^\alpha}{\partial v^\beta_{i_1\cdots i_r}} = \lambda^\alpha_\beta\mathcal{F}^\beta \quad (9.7.22)$$

must be satisfies. Here, we have defined $\lambda^\alpha_\beta = \Lambda^\alpha_\beta - V_i(X^i)\,\delta^\alpha_\beta$. Equations (9.7.22) help us to determine the admissible functions X^i and U^α complying with the conditions (9.7.15) for isovectors of the closed ideal $\mathcal{H}_m$ to be isovectors of the balance ideal $\mathfrak{B}_m$ as well. Knowing isovectors of the balance ideal makes it possible for us according to Theorem 5.13.7 to elicit new families of solutions if we have a solution at hand. If we take $U^\alpha = 0$, then we have $U = X^iV_i$ and if we write $\lambda^\alpha_\beta = \lambda^\alpha_{\beta i}X^i$ without loss of generality, we must be able to find functions $\lambda^\alpha_{\beta i} \in \Lambda^0(\mathcal{C}_m)$ such that

$$V_i(\mathcal{F}^\alpha) = \lambda^\alpha_{\beta i}\mathcal{F}^\beta$$

in order that canonical system coincides with set of isovectors of the balance ideal. These relations pave the way to produce some solutions of the balance ideal by suitably choosing somewhat arbitrary functions $V^\alpha_{ii_1\cdots i_m}$ characterising the horizontal ideal $\mathcal{H}_m$.

(i). Let us suppose that the completely symmetric functions $V^\alpha_{ii_1\cdots i_m}$ may be so chosen that the conditions (9.7.11) and the relations

$$\begin{aligned}\mathcal{F}^\alpha &= V_i(\Sigma^{\alpha i}) + \Sigma^\alpha \\ &= \frac{\partial \Sigma^{\alpha i}}{\partial x^i} + \sum_{r=0}^{m-1} v^\alpha_{i_1\cdots i_r i}\frac{\partial \Sigma^{\alpha i}}{\partial v^\alpha_{i_1\cdots i_r}} + V^\alpha_{ii_1\cdots i_m}\frac{\partial \Sigma^{\alpha i}}{\partial v^\alpha_{i_1\cdots i_m}} + \Sigma^\alpha = 0\end{aligned}$$

are satisfied. In this case every leaf of $\mathcal{H}_m$ proves to be a solution manifold of the balance ideal, and, consequently, of the system of partial differential equations.

(ii). Let us suppose that the completely symmetric functions $V^\alpha_{ii_1\cdots i_m}$ in subscripts may be so chosen that the conditions (9.7.11) and the relations

$$V_i(\mathcal{F}^\alpha) = V_i V_j(\Sigma^{\alpha j}) + V_i(\Sigma^\alpha) = 0$$

are satisfied by taking $\lambda^\alpha_{\beta i} = 0$. In this case, we know that we can write

$$\mathcal{F}^\alpha = F^\alpha(g^a),\ \ V_i(\,g^a) = 0,\ \ g^a \in \Lambda^0(\mathcal{C}_m),\ \ a = 1, \dots, D.$$

Leaves of the ideal $\mathcal{H}_m$ are obtained by setting $g^a = c^a$ where c^a are real constants. Out of these leaves, those corresponding to solution manifolds can be found by determining the constants satisfying the algebraic equations $F^\alpha(c^1, c^2, \dots, c^D) = 0$ where $1 \le \alpha \le N$.

(iii). Let us suppose that the completely symmetric functions $V^\alpha_{ii_1\cdots i_m}$ may be so chosen that the conditions (9.7.11) and the relations

$$V_i(\mathcal{F}^\alpha) = \lambda^\alpha_{\beta i}\mathcal{F}^\beta$$

are satisfied for functions $\lambda^\alpha_{\beta i} \in \Lambda^0(\mathcal{C}_m)$ that are not all equal to zero. In this case each leaf of the foliation of $\mathcal{H}_m$ intersecting the set $\mathcal{P}_m$ given by (9.7.21) becomes the graph of a solution mapping of the balance ideal.

(iv). Finally, let us assume that the distribution $\mathcal{S}_{\mathcal{H}_m}$ is not involutive, but the restriction $\mathcal{S}_{\mathcal{H}_m}|_{\mathcal{P}_m}$ belongs to the tangent bundle $T(\mathcal{P}_m)$ and is involutive. In this case, although the horizontal ideal $\mathcal{H}_m$ is not completely integrable over the manifold $\mathcal{C}_m$, its restriction on the submanifold $\mathcal{P}_m$ is completely integrable. In order to implement this, we have to choose the completely symmetric functions $V^\alpha_{ii_1\cdots i_m}$ in such a way that we might be able to find functions $\Lambda^\alpha_{\beta iji_1\cdots i_m}, \lambda^\alpha_{\beta i} \in \Lambda^0(\mathcal{C}_m)$ such that the relations

$$[V_i, V_j] = \Lambda^\alpha_{\beta iji_1\cdots i_m}\mathcal{F}^\beta\frac{\partial}{\partial v^\alpha_{i_1\cdots i_m}},\ \ V_i(\mathcal{F}^\alpha) = \lambda^\alpha_{\beta i}\mathcal{F}^\beta$$

or on noting (9.7.4)

$$V_i\big(V^\alpha_{ji_1\cdots i_m}\big) - V_j\big(V^\alpha_{ii_1\cdots i_m}\big) = \Lambda^\alpha_{\beta iji_1\cdots i_m}\mathcal{F}^\beta,\ \ V_i(\mathcal{F}^\alpha) = \lambda^\alpha_{\beta i}\mathcal{F}^\beta,$$

are to be satisfied. The functions $\Lambda^\alpha_{\beta iji_1\cdots i_m}$ are antisymmetric in indices i, j and completely symmetric in indices $i_1, \dots, i_m$. In this situation, some solutions of the balance ideal can be found by determining the integral curves of the canonical system passing through $\mathcal{P}_m$.

Example 9.7.1. As an example to the case $N = 1, n = 2$, let us consider the ***Gordon equation*** [German physicist Walter Gordon (1893-1939)]

$$\frac{\partial^2 u}{\partial x \partial t} = \Phi'(u)$$

where Φ is a smooth function of its argument. The reason why this function is introduced into the equation as a derivative is to facilitate the calculations. Let us take

$$x^1 = x,\ \ x^2 = t,\ \ u_x = v_1,\ \ u_t = v_2,$$
$$\mu = dx \wedge dt,\ \ \mu_1 = dt,\ \ \mu_2 = \, - \, dx.$$

Then the ideal $\mathcal{H}_1$ is generated by the following 1-forms

$$\begin{aligned}\sigma &= du - v_1 dx - v_2\, dt,\\ \Sigma_1 &= dv_1 - V_{11}dx - V_{21}dt,\\ \Sigma_2 &= dv_2 - V_{12}\, dx - V_{22}\, dt.\end{aligned}$$

Symmetry condition is met by taking $V_{21} = V_{12}$. We denote the canonical system by

$$\begin{aligned}V_1 &= \frac{\partial}{\partial x} + v_1\frac{\partial}{\partial u} + V_{11}\frac{\partial}{\partial v_1} + V_{12}\frac{\partial}{\partial v_2},\\ V_2 &= \frac{\partial}{\partial t} + v_2\frac{\partial}{\partial u} + V_{12}\frac{\partial}{\partial v_1} + V_{22}\frac{\partial}{\partial v_2}.\end{aligned}$$

The balance form will now be written as

$$\omega = \, - \, dv_1 \wedge dx - \Phi'(u)\, dx \wedge dt = d\Sigma^2 \wedge \mu_2 + \Sigma\mu$$

so that we have $\Sigma^1 = 0, \Sigma^2 = v_1, \Sigma = \, - \, \Phi'(u)$. Consequently, (9.7.19) takes the form

$$\mathcal{F} = V_2(\Sigma^2) + \Sigma = V_{12} - \Phi'(u).$$

Hence, if we choose $V_{12} = \Phi'(u)$, then we get $\mathcal{F} = 0$. Thus, each leaf of $\mathcal{H}_1$ will constitute a solution manifold. Furthermore, the conditions (9.7.11) are reduced to

$$V_1(V_{12}) = V_2(V_{11}),\ \ V_1(V_{22}) = V_2(V_{12})$$

so that we get the partial differential equations below to determine the functions V_{11} and V_{22}

$$\begin{aligned}\frac{\partial V_{11}}{\partial t} + v_2\frac{\partial V_{11}}{\partial u} + \Phi'(u)\frac{\partial V_{11}}{\partial v_1} + V_{22}\frac{\partial V_{11}}{\partial v_2} &= v_1\Phi''(u),\\ \frac{\partial V_{22}}{\partial x} + v_1\frac{\partial V_{22}}{\partial u} + V_{11}\frac{\partial V_{22}}{\partial v_1} + \Phi'(u)\frac{\partial V_{22}}{\partial v_2} &= v_2\Phi''(u).\end{aligned}$$

Evidently, we will not be able to find the general solution of these non-linear equations for an arbitrary function Φ. However, we may try a particular solution in the form

$$V_{11} = \frac{v_1}{v_2}\Phi'(u),\quad V_{22} = \frac{v_2}{v_1}\Phi'(u).$$

It is a very simple exercise to show that this choice satisfies the above equations identically. Therefore, the canonical system corresponding to this case are given by

$$\begin{aligned}V_1 &= \frac{\partial}{\partial x} + v_1\frac{\partial}{\partial u} + \Phi'(u)\Big(\frac{v_1}{v_2}\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2}\Big),\\ V_2 &= \frac{\partial}{\partial t} + v_2\frac{\partial}{\partial u} + \Phi'(u)\Big(\frac{\partial}{\partial v_1} + \frac{v_2}{v_1}\frac{\partial}{\partial v_2}\Big)\end{aligned}$$

In order to determine the foliation of the closed ideal $\mathcal{H}_1$ we have to solve the following linear partial differential equations

$$\begin{aligned}V_1(f) &= \frac{\partial f}{\partial x} + v_1\frac{\partial f}{\partial u} + \Phi'(u)\frac{v_1}{v_2}\frac{\partial f}{\partial v_1} + \Phi'(u)\frac{\partial f}{\partial v_2} = 0,\\ V_2(f) &= \frac{\partial f}{\partial t} + v_2\frac{\partial f}{\partial u} + \Phi'(u)\frac{\partial f}{\partial v_1} + \Phi'(u)\frac{v_2}{v_1}\frac{\partial f}{\partial v_2} = 0,\end{aligned}$$

To this end, we apply the method of characteristics. From the first equation, we get the ordinary differential equations

$$\begin{gathered}\frac{dx}{ds} = 1,\quad \frac{dt}{ds} = 0,\quad \frac{du}{ds} = v_1,\\ \frac{dv_1}{ds} = \Phi'(u)\frac{v_1}{v_2},\quad \frac{dv_2}{ds} = \Phi'(u).\end{gathered}$$

The trivial characteristic variable is $t = c_0$. From the fourth and fifth equations we obtain the characteristic variable

$$\xi = \frac{v_1}{v_2} = c_1$$

while the third and fourth equations yield

$$\frac{dv_1}{du} = \frac{c_1 \Phi'(u)}{v_1}$$

whose integral provides another characteristic variable

$$\eta = \frac{1}{2}\, v_1^2 - c_1 \Phi(u) = c_2.$$

Finally, the first and third equations lead to

$$\frac{du}{dx} = v_1 = \sqrt{2\big[c_1 \Phi(u) + c_2\big]}$$

whose solution gives the last characteristic variable

$$\zeta = x - \int \frac{du}{\sqrt{2}\sqrt{c_1 \Phi(u) + c_2}} = \overline{c}_3.$$

Consequently, the general solution of the first partial differential equation becomes $f = F(\xi, \eta, \zeta, t)$. On introducing this function into the second partial differential equation, we find

$$\frac{\partial F}{\partial t} - \frac{1}{c_1} \frac{\partial F}{\partial \zeta} = 0$$

whose solution is obviously $f = F(\xi, \eta, \psi)$ where $\psi = t + c_1 \zeta = c_3$. Therefore, the leaves of the horizontal ideal $\mathcal{H}_1$ are characterised by the functions below

$$g^1 = t + c_1 \Big[x - \int \frac{du}{\sqrt{2}\sqrt{c_1 \Phi(u) + c_2}}\Big] = c_3,$$
$$g^2 = \frac{v_1}{v_2} = c_1, \qquad g^3 = \frac{1}{2}\, v_1^2 - c_1 \Phi(u) = c_2.$$

Hence, a solution is given implicitly by

$$\frac{c_1}{\sqrt{2}} \int \frac{du}{\sqrt{c_1 \Phi(u) + c_2}} = t + c_1 x - c_3 \tag{9.7.23}$$

depending on three arbitrary constants. As a simple example, let us take

$$\frac{\partial^2 u}{\partial x \partial t} = u.$$

We thus have $\Phi(u) = \dfrac{u^2}{2}$. If we define new constants by

$$a_1 = \sqrt{c_1}, \quad a_2 = \frac{2c_2}{c_1}, \quad a_3 = \frac{c_3}{\sqrt{c_1}},$$

then the expression (9.7.23) assumes the form

$$\int \frac{du}{\sqrt{u^2 + a_2}} = \log\left[u + \sqrt{u^2 + a_2}\right] = \frac{t}{a_1} + a_1 x - a_3$$

whence we arrive at the solution

$$u(x,t) = \frac{1}{2}\left[e^{\frac{t}{a_1} + a_1 x - a_3} - a_2\, e^{-\left(\frac{t}{a_1} + a_1 x - a_3\right)}\right].$$

Finally, let us consider the *sine-Gordon equation*

$$\frac{\partial^2 u}{\partial x \partial t} + \sin u = 0.$$

In this case, we have $\Phi(u) = \cos u$. If we introduce the new constants by $a_1 = \sqrt{c_1}, a_2 = c_2/c_1, a_3 = c_3/\sqrt{c_1}$, then it follows from (9.7.23) that

$$\int \frac{du}{\sqrt{\cos u + a_2}} = \frac{2}{\sqrt{a_2 + 1}}\, F\Big(\frac{u}{2}, \sqrt{\frac{2}{a_2 + 1}}\Big)\Big) = \frac{t}{a_1} + a_1 x - a_3$$

where F is the *Legendre elliptic integral of the first kind* [French mathematician Adrien-Marie Legendre (1752-1833)] defined by

$$F(\phi, k) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}.$$

The solution of the equation $F(\phi, k) = \psi$ for ϕ in terms of ψ is expressible as $\sin\phi = \operatorname{sn}\psi$ where sn denotes the *Jacobi elliptic function* and the function $\eta = \operatorname{sn}\xi$ is found as the solution of the non-linear ordinary differential equation

$$\frac{d\eta}{d\xi} = \sqrt{(1 - \eta^2)(1 - k^2\eta^2)}.$$

Hence, a particular solution of the sine-Gordon equation may be written as

$$u(x,t) = 2\arcsin \mathrm{sn}\Big[\frac{\sqrt{a_2+1}}{2}\Big(\frac{t}{a_1}+a_1x-a_3\Big)\Big]$$

depending on three parameters. ■

Example 9.7.2. We consider n partial differential equations

$$\frac{\partial u}{\partial x^i} = \phi_i(\mathbf{x}, u)$$

involving a single dependent variable where $\phi_i(\mathbf{x}, u)$ are given functions. We look for the solution $u = u(\mathbf{x})$. But, because of the symmetry relations $u_{,ij} = u_{,ji}$, the functions ϕ_i must satisfy the compatibility conditions

$$\frac{\partial \phi_i}{\partial x^j} + \frac{\partial \phi_i}{\partial u}\phi_j = \frac{\partial \phi_j}{\partial x^i} + \frac{\partial \phi_j}{\partial u}\phi_i \tag{9.7.24}$$

for the existence of a solution. The horizontal ideal $\mathcal{H}_1$ is now be generated solely by 1-forms

$$\sigma = du - v_i\,dx^i,\ \ \Sigma_i = dv_i - V_{ji}\,dx^j.$$

The functions $V_{ji} \in \Lambda^0(\mathcal{C}_1)$ are presently arbitrary except for satisfying the symmetry condition $V_{ij} = V_{ji}$. Despite there exists just one dependent variable, namely, $N = 1$, there are n balance equations $(A = n)$. Therefore, we choose balance n-forms as

$$\omega_i = \Sigma_i\mu = \big(v_i - \phi_i(\mathbf{x}, u)\big)\mu,\ \ \mu = dx^1\wedge\cdots\wedge dx^n.$$

The canonical system will now have the form

$$V_i = \frac{\partial}{\partial x^i} + v_i\frac{\partial}{\partial u} + V_{ij}\frac{\partial}{\partial v_j}.$$

From (9.7.19), we obtain

$$\mathcal{F}_i = v_i - \phi_i(\mathbf{x}, u). \tag{9.7.25}$$

Hence, the submanifold $\mathcal{P}_1$ of $\mathcal{C}_1$ is specified by the relations $\mathcal{F}_i = 0$, or

$$v_i = \phi_i(\mathbf{x}, u),\ \ 1 \le i \le n.$$

Next, let us choose the functions V_{ij} as

$$V_{ij} = \frac{\partial \phi_i}{\partial x^j} + \frac{\partial \phi_i}{\partial u}\phi_j. \tag{9.7.26}$$

Because of the compatibility conditions (9.7.24), the symmetry relations are automatically satisfied. However, if we consider the case (iv), the equality (9.7.11) must be satisfied at least on the submanifold $\mathcal{P}_1$. For this purpose, let us evaluate the expressions $V_i(\mathcal{F}_j)$ and $V_i(V_{kj}) - V_j(V_{ki})$ and employ the relations (9.7.24) to obtain

$$V_i(\mathcal{F}_j) = -\frac{\partial \phi_j}{\partial u}\mathcal{F}_i$$

and

$$\begin{aligned}V_i(V_{kj}) - V_j(V_{ki}) &= V_i\Big(\frac{\partial \phi_k}{\partial x^j} + \frac{\partial \phi_k}{\partial u}\phi_j\Big) - V_j\Big(\frac{\partial \phi_k}{\partial x^i} + \frac{\partial \phi_k}{\partial u}\phi_i\Big)\\ &= \Big(\frac{\partial^2 \phi_k}{\partial x^j \partial u} + \frac{\partial^2 \phi_k}{\partial u^2}\phi_j + \frac{\partial \phi_k}{\partial u}\frac{\partial \phi_j}{\partial u}\Big)\mathcal{F}_i\\ &\quad - \Big(\frac{\partial^2 \phi_k}{\partial x^i \partial u} + \frac{\partial^2 \phi_k}{\partial u^2}\phi_i + \frac{\partial \phi_k}{\partial u}\frac{\partial \phi_i}{\partial u}\Big)\mathcal{F}_j.\end{aligned}$$

Since $\mathcal{F}_i = 0$ on $\mathcal{P}_1$, the relations $V_i(\mathcal{F}_j) = 0$ and $V_i(V_{kj}) - V_j(V_{ki}) = 0$ are also satisfied on the same submanifold. Therefore, the solutions to our system of differential equations are obtained via the integral curves of the vector fields

$$V_i = \frac{\partial}{\partial x^i} + v_i\frac{\partial}{\partial u} + \Big(\frac{\partial \phi_i}{\partial x^j} + \frac{\partial \phi_i}{\partial u}\phi_j\Big)\frac{\partial}{\partial v_j}$$

passing through the submanifold $\mathcal{P}_1$. As a special case, let us take $n = 2$, $x^1 = x, x^2 = t$ and choose the functions ϕ_1 and ϕ_2 as follows

$$\phi_1 = \frac{t-u}{x+e^u}, \quad \phi_2 = \frac{x+t}{x+e^u}.$$

We can easily verify that these functions satisfy the compatibility conditions (9.7.24) [*see* Edelen and Wang (1992), *p.* 144]. Hence, the relations (9.7.26) yield at once

$$\begin{aligned}V_{11} &= \frac{(u-t)\big[e^u(t-u+2) + 2x\big]}{(x+e^u)^3}\\ V_{22} &= \frac{(x+e^u)^2 - (x+t)^2 e^u}{(x+e^u)^3}\\ V_{12} = V_{21} &= \frac{(x+e^u)^2 - (x+e^u)(x+t) + e^u(u-t)(x+t)}{(x+e^u)^3}\end{aligned}$$

In this case, the solution mapping must be found by solving the differential equations $V_1(f) = 0$ and $V_2(f) = 0$ by using the method of characteristics. However, we might reach to a particular solution by a simple observation. Let us define a mapping $\phi : G \to \mathcal{C}_1$ by the relations $x = x,\ t = t,\ u = u,$ $v_1 = \phi_1, v_2 = \phi_2$. We then immediately see that $\phi^*\Sigma_i = 0, i = 1, 2$ whereas the expression $\phi^*\sigma = 0$ gives

$$\begin{aligned}\phi^*\sigma &= du - \frac{t-u}{x+e^u}\,dx - \frac{x+t}{x+e^u}\,dt \\ &= \frac{1}{x+e^u}\big[(x+e^u)\,du + (u-t)\,dx - (x+t)\,dt\big] \\ &= \frac{1}{x+e^u}\,d\Big(e^u + xu - \frac{1}{2}t^2 - xt\Big) = 0.\end{aligned}$$

Therefore, some implicit solutions of the partial differential equations

$$\frac{\partial u}{\partial x} = \frac{t-u}{x+e^u}, \quad \frac{\partial u}{\partial t} = \frac{x+t}{x+e^u}$$

are provided by

$$e^u + xu - \frac{1}{2}t^2 - xt = c$$

where c is an arbitrary constant.

An exemplary plot of this function is depicted in Fig. 9.7.1. ■

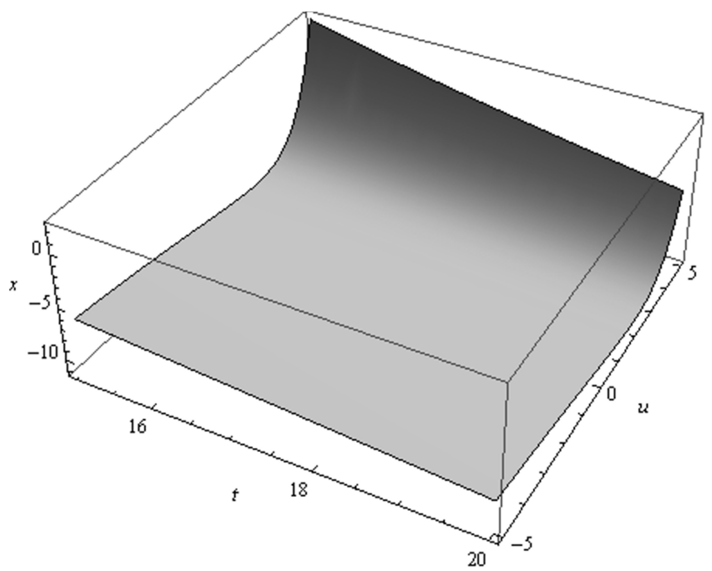

Fig. 9.7.1. A plot of $e^u + xu - \frac{1}{2}t^2 - xt = 10$.

9.8. EQUIVALENCE TRANSFORMATIONS

Several system of partial differential equations, particularly modelling natural laws, contain some arbitrary functions or parameters reflecting physical constitution of materials involved. Almost all field equations of classical continuum physics fall into this category. Thus, such systems are actually family of equations whose fundamental structures remain unchanged but show some differences in their physical constitutions from one material to another through some constitutive functions or parameters. For instance, the field equations of hyperelastic solids are of the same type and only the particular form of the stress potential distinguishes one material from the other. The ***equivalence groups*** are defined as groups of continuous transformations that leave a given family of equations invariant. In contrast to a symmetry transformation that transforms one set of equations into themselves, an equivalence transformation maps an arbitrary member of the family onto another member of the same family which may possess somewhat different physical properties. Meanwhile, it transforms a solution of the one member onto a solution of another member of that family.

In other words, if we manage to determine an equivalence transformation, we can employ a solution corresponding to a certain material to obtain a solution associated with another material of the same sort whose physical properties obey the rules dictated by the appropriate equivalence transformation. Although the concept of equivalence transformations is well-known in the theory of ordinary and partial differential equations, we owe their first systematic treatment within the realm of classical Lie groups to Russian mathematician and engineer Lev Vasil'evich Ovsiannikov (1919) [*see* Ovsiannikov (1982)]. In this section, we will try to treat equivalence transformations of balance equations by employing exterior differential forms.

We know that an $(m+1)$th order system of balance equations with n independent variables x^i and N dependent variables u^α are given by (9.4.1)

$$\frac{\partial \Sigma^{\alpha i}}{\partial x^i} + \Sigma^\alpha = 0, \quad i = 1, 2, \dots, n; \quad \alpha = 1, 2, \dots, N.$$

As we have mentioned on $p.\,522$, a difference between numbers of the equations and the dependent variables does not create undue difficulties in our general approach. In order to be able to determine equivalence transformations, we have first to enlarge the manifold $\mathcal{C}_m$ to a much bigger manifold $\mathcal{K}_m$ by adding new auxiliary independent variables to the coordinate cover of $\mathcal{C}_m$ to take into account $\Sigma^{\alpha i}$, Σ^α and their derivatives with respect to their argument in order to identify their functional forms. To this end, we introduce the new auxiliary variables

$$s_j^{\alpha i} = \frac{\partial \Sigma^{\alpha i}}{\partial x^j}, \quad s_\beta^{\alpha i i_1 \cdots i_r} = \frac{\partial \Sigma^{\alpha i}}{\partial v^\beta_{i_1 \cdots i_r}}, \tag{9.8.1}$$
$$t_i^{\alpha} = \frac{\partial \Sigma^{\alpha}}{\partial x^i}, \quad t_\beta^{\alpha i_1 \cdots i_r} = \frac{\partial \Sigma^{\alpha}}{\partial v^\beta_{i_1 \cdots i_r}}, \quad 0 \le r \le m.$$

It is clear that the variables $s_\beta^{\alpha i i_1 \cdots i_r}$ and $t_\beta^{\alpha i_1 \cdots i_r}$ are completely symmetric in the indices $i_1, \cdots, i_r$. Hence, the coordinate cover of the manifold $\mathcal{K}_m$ that is enlarged significantly compared to that of the manifold $\mathcal{C}_m$ are given by

$$\left\{x^i, \Sigma^{\alpha i}, \Sigma^\alpha, s_j^{\alpha i}, t_i^\alpha, \{v^\alpha_{i_1 \cdots i_r}, s_\beta^{\alpha i i_1 \cdots i_r}, t_\beta^{\alpha i_1 \cdots i_r} : 0 \le r \le m\}\right\}.$$

We can easily verify that the dimension of the manifold $\mathcal{K}_m$ is at most

$$n + (1+n)^2 N + [1 + (n+1)N]N \frac{(n+m)!}{n!m!}.$$

Let $\Lambda(\mathcal{K}_m)$ be the exterior algebra on the manifold $\mathcal{K}_m$. Contact 1-forms in the manifold $\mathcal{K}_m$ are now defined by

$$\sigma^\alpha_{i_1 \cdots i_r} = dv^\alpha_{i_1 \cdots i_r} - v^\alpha_{i_1 \cdots i_r i}\, dx^i \in \Lambda^1(\mathcal{K}_m), \ 0 \le r \le m-1, \tag{9.8.2}$$
$$\Omega^{\alpha i} = d\Sigma^{\alpha i} - s_j^{\alpha i} dx^j - \sum_{r=0}^{m} s_\beta^{\alpha i i_1 \cdots i_r} dv^\beta_{i_1 \cdots i_r} \in \Lambda^1(\mathcal{K}_m),$$
$$\Omega^{\alpha} = d\Sigma^{\alpha} - t_i^{\alpha} dx^i - \sum_{r=0}^{m} t_\beta^{\alpha i_1 \cdots i_r} dv^\beta_{i_1 \cdots i_r} \in \Lambda^1(\mathcal{K}_m).$$

Balance n-forms are again given by

$$\omega^\alpha = d\Sigma^{\alpha i} \wedge \mu_i + \Sigma^\alpha \mu \in \Lambda^n(\mathcal{K}_m).$$

Let $\mathfrak{D}_m$ denote the balance ideal of $\Lambda(\mathcal{K}_m)$ generated by forms ω^α, $\Omega^{\alpha i}$, Ω^α, $d\Omega^{\alpha i}$, $d\Omega^\alpha$, $\{\sigma^\alpha_{i_1 \cdots i_r} : \ 0 \le r \le m-1\}$ and $d\sigma^\alpha_{i_1 \cdots i_{m-1}}$. Exterior derivatives of 1-forms are

$$d\sigma^\alpha_{i_1 \cdots i_r} = -\, dv^\alpha_{i_1 \cdots i_r i} \wedge dx^i, \quad 0 \le r \le m-1,$$
$$d\Omega^{\alpha i} = -\, ds_j^{\alpha i} \wedge dx^j - \sum_{r=0}^{m} ds_\beta^{\alpha i i_1 \cdots i_r} \wedge dv^\beta_{i_1 \cdots i_r},$$
$$d\Omega^{\alpha} = -\, dt_i^{\alpha} \wedge dx^i - \sum_{r=0}^{m} dt_\beta^{\alpha i_1 \cdots i_r} \wedge dv^\beta_{i_1 \cdots i_r}.$$

The ideal $\mathfrak{D}_m$ is closed. Indeed, we can easily verify that

$$d\omega^\alpha = d\Sigma^\alpha \wedge \mu = \Omega^\alpha \wedge \mu + \sum_{r=0}^{m-1} t_\beta^{\alpha i_1\cdots i_r} \sigma^\beta_{i_1\cdots i_r} \wedge \mu \\ - t_\beta^{\alpha i_1\cdots i_{m-1} i_m} d\sigma^\beta_{i_1\cdots i_{m-1}} \wedge \mu_{i_m} \in \mathfrak{D}_m.$$

If a mapping $\phi : M \to \mathcal{K}_m$ annihilates the ideal $\mathfrak{D}_m$, that is, if the pull-back operator $\phi^* : \Lambda(\mathcal{K}_m) \to \Lambda(M)$ satisfies the relations

$$\phi^* \sigma^\alpha_{i_1\cdots i_r} = 0, 0 \le r \le m-1,\ \phi^*\Omega^{\alpha i} = 0,\ \phi^*\Omega^\alpha = 0,\ \phi^*\omega^\alpha = 0,$$

then we easily observe that ϕ is a solution mapping of the given system of partial differential equations. In order to determine the equivalence transformations, we have to find isovector fields that leave the closed balance ideal $\mathfrak{D}_m$ invariant. A vector field $V \in T(\mathcal{K}_m)$ can now be represented by the expression

$$V = X^i \frac{\partial}{\partial x^i} + S^{\alpha i} \frac{\partial}{\partial \Sigma^{\alpha i}} + T^\alpha \frac{\partial}{\partial \Sigma^\alpha} + S^{\alpha i}_j \frac{\partial}{\partial s^{\alpha i}_j} + T^\alpha_i \frac{\partial}{\partial t^\alpha_i} \tag{9.8.3}$$
$$+ \sum_{r=0}^{m} \Big(V^\alpha_{i_1\cdots i_r} \frac{\partial}{\partial v^\alpha_{i_1\cdots i_r}} + S_\beta^{\alpha i i_1\cdots i_r} \frac{\partial}{\partial s_\beta^{\alpha i i_1\cdots i_r}} + T_\beta^{\alpha i_1\cdots i_r} \frac{\partial}{\partial t_\beta^{\alpha i_1\cdots i_r}} \Big).$$

All coefficients in the vector field (9.8.3) are smooth functions of the coordinates of the manifold $\mathcal{K}_m$. Let us first take into consideration the ideal $\mathcal{I}(\{\sigma^\alpha_{i_1\cdots i_r} : 0 \le r \le m-1\}, \Omega^{\alpha i}, \Omega^\alpha)$. Since this ideal is produced by 1-forms only, Theorem 5.12.4 secures us that the isovectors of this ideal and its closure

$$\mathfrak{C}_m = \mathcal{I}(\{\sigma^\alpha_{i_1\cdots i_r} : 0 \le r \le m-1\}, d\sigma_{i_1\cdots i_{m-1}}, \Omega^{\alpha i}, \Omega^\alpha, d\Omega^{\alpha i}, d\Omega^\alpha)$$

are the same. Therefore, to determine the isovector fields of the closed contact ideal $\mathfrak{C}_m$, we have to show the existence of smooth functions ,

$$\lambda^{\alpha j_1\cdots j_r}_{\beta i_1\cdots i_k},\ K^\alpha_{\beta i i_1\cdots i_k},\ L^\alpha_{\beta i_1\cdots i_k},\ \mu_\beta^{\alpha i i_1\cdots i_r},\ M^{\alpha i}_{\beta j},\ M^{\alpha i}_\beta,\ \nu_\beta^{\alpha i_1\cdots i_r},\ N^\alpha_{\beta i},\ N^\alpha_\beta$$

belonging to $\Lambda^0(\mathcal{K}_m)$ such that the following relations are satisfied

$$£_V\, \sigma^\alpha_{i_1\cdots i_k} = \sum_{r=0}^{m-1} \lambda^{\alpha j_1\cdots j_r}_{\beta i_1\cdots i_k} \sigma^\beta_{j_1\cdots j_r} + K^\alpha_{\beta i i_1\cdots i_k} \Omega^{\beta i} + L^\alpha_{\beta i_1\cdots i_k} \Omega^\beta, \tag{9.8.4}$$
$$£_V\, \Omega^{\alpha i} = \sum_{r=0}^{m-1} \mu_\beta^{\alpha i i_1\cdots i_r} \sigma^\beta_{i_1\cdots i_r} + M^{\alpha i}_{\beta j}\, \Omega^{\beta j} + M^{\alpha i}_\beta\, \Omega^\beta,$$

$$\pounds_V \,\Omega^\alpha = \sum_{r=0}^{m-1} \nu_\beta^{\alpha i_1\cdots i_r}\sigma^\beta_{i_1\cdots i_r} + N^\alpha_{\beta i}\,\Omega^{\beta i} + N^\alpha_\beta\,\Omega^\beta$$

where $0 \le k \le m-1$. To this end, we start with the equations $(9.8.4)_1$. For $0 \le k \le m-1$, the relations

$$\begin{aligned} F^\alpha_{i_1\cdots i_k} &= \mathbf{i}_V\,\sigma^\alpha_{i_1\cdots i_k} = V^\alpha_{i_1\cdots i_k} - v^\alpha_{i_1\cdots i_k i}X^i, \\ \mathbf{i}_V\,d\sigma^\alpha_{i_1\cdots i_k} &= -\,V^\alpha_{i_1\cdots i_k i}dx^i + X^i dv^\alpha_{i_1\cdots i_k i} \end{aligned} \tag{9.8.5}$$

lead to

$$\begin{aligned} \pounds_V\,\sigma^\alpha_{i_1\cdots i_k} &= dF_{i_1\cdots i_k} - V^\alpha_{i_1\cdots i_k i}dx^i + X^i dv^\alpha_{i_1\cdots i_k i} \\ &= dV^\alpha_{i_1\cdots i_k} - V^\alpha_{i_1\cdots i_k i}dx^i - v^\alpha_{i_1\cdots i_k i}dX^i \end{aligned}$$

If we introduce the above expressions into $(9.8.4)_1$ after having calculated the differentials $dF_{i_1\cdots i_k}$ and $dV_{i_1\cdots i_k}$, and arrange the resulting expression, we then find that

$$\begin{aligned} &\Big[-V^\alpha_{i_1\cdots i_k i} + \frac{\partial F^\alpha_{i_1\cdots i_k}}{\partial x^i} + \sum_{r=0}^{m-1}\lambda^{\alpha j_1\cdots j_r}_{\beta i_1\cdots i_k} v^\beta_{j_1\cdots j_r i} + K^\alpha_{\beta j i_1\cdots i_k}s^{\beta j}_i \\ &\quad + L^\alpha_{\beta i_1\cdots i_k}t^\beta_i\Big]dx^i + \Big[\frac{\partial F^\alpha_{i_1\cdots i_k}}{\partial \Sigma^{\beta i}} - K^\alpha_{\beta i i_1\cdots i_k}\Big]d\Sigma^{\beta i} \\ &\quad + \Big[\frac{\partial F^\alpha_{i_1\cdots i_k}}{\partial \Sigma^\beta} - L^\alpha_{\beta i_1\cdots i_k}\Big]d\Sigma^\beta \\ &\quad + \frac{\partial F^\alpha_{i_1\cdots i_k}}{\partial s^{\beta i}_j}ds^{\beta i}_j + \frac{\partial F^\alpha_{i_1\cdots i_k}}{\partial t^\beta_i}dt^\beta_i \\ &\quad + \sum_{r=0}^{m-1}\Big[\frac{\partial V^\alpha_{i_1\cdots i_k}}{\partial v^\beta_{j_1\cdots j_r}} - v^\alpha_{i_1\cdots i_k i}\frac{\partial X^i}{\partial v^\beta_{j_1\cdots j_r}} - \lambda^{\alpha j_1\cdots j_r}_{\beta i_1\cdots i_k} \\ &\quad + K^\alpha_{\gamma i i_1\cdots i_k}s^{\gamma i j_1\cdots j_r}_\beta + L^\alpha_{\gamma i_1\cdots i_k}t^{\gamma j_1\cdots j_r}_\beta\Big]dv^\beta_{j_1\cdots j_r} \\ &\quad + \sum_{r=0}^{m}\frac{\partial F^\alpha_{i_1\cdots i_k}}{\partial s^{\beta i j_1\cdots j_r}_\gamma}ds^{\beta i j_1\cdots j_r}_\gamma \\ &\quad + \sum_{r=0}^{m}\frac{\partial F^\alpha_{i_1\cdots i_k}}{\partial t^{\beta j_1\cdots j_r}_\gamma}dt^{\beta j_1\cdots j_r}_\gamma + \Big[\frac{\partial V^\alpha_{i_1\cdots i_k}}{\partial v^\beta_{j_1\cdots j_m}} - v^\alpha_{i_1\cdots i_k i}\frac{\partial X^i}{\partial v^\beta_{j_1\cdots j_m}} \\ &\quad + K^\alpha_{\gamma i i_1\cdots i_k}s^{\gamma i j_1\cdots j_m}_\beta + L^\alpha_{\gamma i_1\cdots i_k}t^{\gamma j_1\cdots j_m}_\beta\Big]dv^\beta_{j_1\cdots j_m} = 0 \end{aligned}$$

from which we get the following relations for $k = 0, 1, \ldots, m-1$ and

$r = 0, 1, \dots, m-1$

$$K^{\alpha}_{\beta i i_1\cdots i_k} = \frac{\partial F^{\alpha}_{i_1\cdots i_k}}{\partial \Sigma^{\beta i}}, \qquad L^{\alpha}_{\beta i_1\cdots i_k} = \frac{\partial F^{\alpha}_{i_1\cdots i_k}}{\partial \Sigma^{\beta}}, \tag{9.8.6}$$

$$\frac{\partial F^{\alpha}_{i_1\cdots i_k}}{\partial s^{\beta i}_j} = 0, \frac{\partial F^{\alpha}_{i_1\cdots i_k}}{\partial t^{\beta}_i} = 0, \frac{\partial F^{\alpha}_{i_1\cdots i_k}}{\partial s^{\beta i j_1\cdots j_r}_{\gamma}} = 0, \frac{\partial F^{\alpha}_{i_1\cdots i_k}}{\partial t^{\beta j_1\cdots j_r}_{\gamma}} = 0, 0 \le r \le m,$$

$$V^{\alpha}_{i_1\cdots i_k i} = \frac{\partial F^{\alpha}_{i_1\cdots i_k}}{\partial x^i} + \sum_{r=0}^{m-1} \lambda^{\alpha j_1\cdots j_r}_{\beta i_1\cdots i_k} v^{\beta}_{j_1\cdots j_r i} + \frac{\partial F^{\alpha}_{i_1\cdots i_k}}{\partial \Sigma^{\beta j}}\, s^{\beta j}_i + \frac{\partial F^{\alpha}_{i_1\cdots i_k}}{\partial \Sigma^{\beta}}\, t^{\beta}_i,$$

$$\lambda^{\alpha j_1\cdots j_r}_{\beta i_1\cdots i_k} = \frac{\partial V^{\alpha}_{i_1\cdots i_k}}{\partial v^{\beta}_{j_1\cdots j_r}} - v^{\alpha}_{i_1\cdots i_k i} \frac{\partial X^i}{\partial v^{\beta}_{j_1\cdots j_r}} + \frac{\partial F^{\alpha}_{i_1\cdots i_k}}{\partial \Sigma^{\gamma i}} s^{\gamma i j_1\cdots j_r}_{\beta} + \frac{\partial F^{\alpha}_{i_1\cdots i_k}}{\partial \Sigma^{\gamma}} t^{\gamma j_1\cdots j_r}_{\beta},\ r = 0, 1, \dots, m-1,$$

$$\frac{\partial V^{\alpha}_{i_1\cdots i_k}}{\partial v^{\beta}_{j_1\cdots j_m}} - v^{\alpha}_{i_1\cdots i_k i} \frac{\partial X^i}{\partial v^{\beta}_{j_1\cdots j_m}} + \frac{\partial F^{\alpha}_{i_1\cdots i_k}}{\partial \Sigma^{\gamma i}} s^{\gamma i j_1\cdots j_m}_{\beta} + \frac{\partial F^{\alpha}_{i_1\cdots i_k}}{\partial \Sigma^{\gamma}} t^{\gamma j_1\cdots j_m}_{\beta} = 0.$$

The equations $(9.8.6)_{3\text{-}6}$ indicate that the functions $F^{\alpha}_{i_1\cdots i_k}$, $0 \le k \le m-1$ cannot depend on the variables $s^{\beta i}_j, t^{\beta}_i, s^{\beta i j_1\cdots j_r}_{\gamma}, t^{\beta j_1\cdots j_r}_{\gamma}, 0 \le r \le m$. If $k < m-1$, then it follows from $(9.8.5)_1$ that

$$\frac{\partial V^{\alpha}_{i_1\cdots i_k}}{\partial v^{\beta}_{j_1\cdots j_m}} - v^{\alpha}_{i_1\cdots i_k i} \frac{\partial X^i}{\partial v^{\beta}_{j_1\cdots j_m}} = \frac{\partial F^{\alpha}_{i_1\cdots i_k}}{\partial v^{\beta}_{j_1\cdots j_m}}.$$

Hence $(9.8.6.)_5$ takes the form

$$\frac{\partial F^{\alpha}_{i_1\cdots i_k}}{\partial v^{\beta}_{j_1\cdots j_m}} + \frac{\partial F^{\alpha}_{i_1\cdots i_k}}{\partial \Sigma^{\gamma i}} s^{\gamma i j_1\cdots j_m}_{\beta} + \frac{\partial F^{\alpha}_{i_1\cdots i_k}}{\partial \Sigma^{\gamma}} t^{\gamma j_1\cdots j_m}_{\beta} = 0, 0 \le k < m-2$$

whence we deduce, respectively,

$$\frac{\partial F^{\alpha}_{i_1\cdots i_k}}{\partial \Sigma^{\gamma i}} = 0,\ \frac{\partial F^{\alpha}_{i_1\cdots i_k}}{\partial \Sigma^{\gamma}} = 0,\ \frac{\partial F^{\alpha}_{i_1\cdots i_k}}{\partial v^{\beta}_{j_1\cdots j_m}} = 0,\ k = 0, 1, \dots, m-2.$$

Let us now take $k = m-1$. This time $(9.8.5)_1$ gives rise to the relation

$$\frac{\partial F^{\alpha}_{i_1\cdots i_{m-1}}}{\partial v^{\beta}_{j_1\cdots j_m}} = \frac{\partial V^{\alpha}_{i_1\cdots i_{m-1}}}{\partial v^{\beta}_{j_1\cdots j_m}} - v^{\alpha}_{i_1\cdots i_{m-1} i} \frac{\partial X^i}{\partial v^{\beta}_{j_1\cdots j_m}} - \delta^{\alpha}_{\beta} \delta^{j_1}_{i_1} \delta^{j_2}_{i_2} \cdots \delta^{j_{m-1}}_{i_{m-1}} \delta^{j_m}_i X^i.$$

We then insert this expression into $(9.8.6)_5$ for $k = m-1$ to obtain

$$\delta^\alpha_\beta \delta^{j_1}_{i_1}\delta^{j_2}_{i_2}\cdots\delta^{j_{m-1}}_{i_{m-1}}\delta^{j_m}_{i} X^i + \frac{\partial F^\alpha_{i_1\cdots i_{m-1}}}{\partial v^\beta_{j_1\cdots j_m}} + \frac{\partial F^\alpha_{i_1\cdots i_{m-1}}}{\partial \Sigma^{\gamma i}} s^{\gamma i j_1\cdots j_m}_\beta + \frac{\partial F^\alpha_{i_1\cdots i_{m-1}}}{\partial \Sigma^{\gamma}} t^{\gamma j_1\cdots j_m}_\beta = 0.$$

When $N > 1$, we can always choose the indices α and β in such a way that $\alpha \neq \beta$. In that case, we find that

$$\frac{\partial F^\alpha_{i_1\cdots i_{m-1}}}{\partial \Sigma^{\gamma i}} = 0,\ \frac{\partial F^\alpha_{i_1\cdots i_{m-1}}}{\partial \Sigma^{\gamma}} = 0;\ \frac{\partial F^\alpha_{i_1\cdots i_{m-1}}}{\partial v^\beta_{j_1\cdots j_m}} = 0,\ \alpha \neq \beta.$$

Thus if $0 \leq k \leq m-1$, then the functions $F^\alpha_{i_1\cdots i_k}$ and consequently the functions X^i and $V^\alpha_{i_1\cdots i_k} = F^\alpha_{i_1\cdots i_k} + v^\alpha_{i_1\cdots i_k i}X^i$ cannot depend on variables $\Sigma^{\gamma i}$, Σ^γ and $v^\beta_{j_1\cdots j_m}$ whenever $\alpha \neq \beta$. Therefore, $(9.8.6)_{3,5}$ reduce to the expressions

$$\begin{aligned} V^\alpha_{i_1\cdots i_k i} &= \frac{\partial V^\alpha_{i_1\cdots i_k}}{\partial x^i} - v^\alpha_{i_1\cdots i_k j}\frac{\partial X^j}{\partial x^i} \\ &\quad + \sum_{r=0}^{m-1}\Big[\frac{\partial V^\alpha_{i_1\cdots i_k}}{\partial v^\beta_{j_1\cdots j_r}} - v^\alpha_{i_1\cdots i_k j}\frac{\partial X^j}{\partial v^\beta_{j_1\cdots j_r}}\Big] v^\beta_{j_1\cdots j_r i}, \\ &\frac{\partial V^\alpha_{i_1\cdots i_k}}{\partial v^\beta_{j_1\cdots j_m}} - v^\alpha_{i_1\cdots i_k i}\frac{\partial X^i}{\partial v^\beta_{j_1\cdots j_m}} = 0,\ k = 0,1,\dots,m-1. \end{aligned}$$

The above equations are exactly the same as the equations $(9.3.5)_3$ and (9.3.6). Hence, their solutions are again given by the recurrence relations

$$\begin{aligned} X^i &= X^i(\mathbf{x},\mathbf{u}),\ \ U^\alpha = U^\alpha(\mathbf{x},\mathbf{u}), \\ V^\alpha_{i_1\cdots i_k i} &= \frac{\partial V^\alpha_{i_1\cdots i_k}}{\partial x^i} - v^\alpha_{i_1\cdots i_k j}\frac{\partial X^j}{\partial x^i} + \Big[\frac{\partial V^\alpha_{i_1\cdots i_k}}{\partial u^\beta} - v^\alpha_{i_1\cdots i_k j}\frac{\partial X^j}{\partial u^\beta}\Big] v^\beta_i \\ &\quad + \sum_{r=1}^{k} v^\beta_{j_1\cdots j_r i}\frac{\partial V^\alpha_{i_1\cdots i_k}}{\partial v^\beta_{j_1\cdots j_r}},\quad k = 0,1,\dots,m-1 \end{aligned} \tag{9.8.7}$$

[*see* (9.3.16) and (9.3.17)]. Let us now take the relations $(9.8.4)_2$ into account. By employing the relations

$$\mathbf{i}_V\Omega^{\alpha i} = S^{\alpha i} - s^{\alpha i}_j X^j - \sum_{r=0}^{m} s^{\alpha i i_1\cdots i_r}_\beta V^\beta_{i_1\cdots i_r} = F^{\alpha i} \tag{9.8.8}$$

$$\mathbf{i}_V d\Omega^{\alpha i} = -S^{\alpha i}_j dx^j + X^j ds^{\alpha i}_j - \sum_{r=0}^{m}\big(S^{\alpha i i_1\cdots i_r}_\beta dv^\beta_{i_1\cdots i_r} - V^\beta_{i_1\cdots i_r} ds^{\alpha i i_1\cdots i_r}_\beta\big)$$

we can evaluate Lie derivatives through the Cartan formula. Let us then make the transformation $dv^\beta_{i_1\cdots i_r} = \sigma^\beta_{i_1\cdots i_r} + v^\beta_{i_1\cdots i_r i} dx^i$ for $0 \le r \le m-1$, arrange the resulting expression and equate the coefficients of independent 1-forms to zero to obtain

$$M^{\alpha i}_{\beta j} = \frac{\partial F^{\alpha i}}{\partial \Sigma^{\beta j}},\ \ M^{\alpha i}_\beta = \frac{\partial F^{\alpha i}}{\partial \Sigma^\beta},\ \ \frac{\partial F^{\alpha i}}{\partial t^\beta_j} = 0,\ \ \delta^\alpha_\beta \delta^i_k X^j + \frac{\partial F^{\alpha i}}{\partial s^{\beta k}_j} = 0.$$

$$\frac{\partial F^{\alpha i}}{\partial t^{\gamma i_1\cdots i_r}_\beta} = 0, \quad \delta^\alpha_\gamma \delta^i_j V^\beta_{i_1\cdots i_r} + \frac{\partial F^{\alpha i}}{\partial s^{\gamma j i_1\cdots i_r}_\beta} = 0, \quad 0 \le r \le m,$$

$$\mu^{\alpha i i_1\cdots i_r}_\beta = \overline{\mu}^{\alpha i i_1\cdots i_r}_\beta - S^{\alpha i i_1\cdots i_r}_\beta,$$

$$\overline{\mu}^{\alpha i i_1\cdots i_r}_\beta = \frac{\partial F^{\alpha i}}{\partial v^\beta_{i_1\cdots i_r}} + \frac{\partial F^{\alpha i}}{\partial \Sigma^{\gamma j}} s^{\gamma j i_1\cdots i_r}_\beta + \frac{\partial F^{\alpha i}}{\partial \Sigma^\gamma} t^{\gamma i_1\cdots i_r}_\beta, \ \ 0 \le r \le m-1,$$

$$S^{\alpha i}_j = \overline{S}^{\alpha i}_j - \sum_{r=0}^{m-1} S^{\alpha i i_1\cdots i_r}_\beta v^\beta_{i_1\cdots i_r j},$$

$$\overline{S}^{\alpha i}_j = \frac{\partial F^{\alpha i}}{\partial x^j} + \sum_{r=0}^{m-1} \overline{\mu}^{\alpha i i_1\cdots i_r}_\beta v^\beta_{i_1\cdots i_r j} + \frac{\partial F^{\alpha i}}{\partial \Sigma^{\beta k}} s^{\beta k}_j + \frac{\partial F^{\alpha i}}{\partial \Sigma^\beta} t^\beta_j,$$

$$S^{\alpha i i_1\cdots i_m}_\beta = \frac{\partial F^{\alpha i}}{\partial v^\beta_{i_1\cdots i_m}} + \frac{\partial F^{\alpha i}}{\partial \Sigma^{\gamma j}} s^{\gamma j i_1\cdots i_m}_\beta + \frac{\partial F^{\alpha i}}{\partial \Sigma^\gamma} t^{\gamma i_1\cdots i_m}_\beta = \overline{\mu}^{\alpha i i_1\cdots i_m}_\beta.$$

As is clearly seen from the above relations, there are no restrictions on functions $S^{\alpha i i_1\cdots i_r}_\beta, r = 0, 1, \ldots, m-1$, hence they can be selected totally arbitrarily. Finally, let us take the equations $(9.8.4)_3$ into account. The Lie derivatives evaluated by using the relations

$$\mathbf{i}_V \Omega^\alpha = T^\alpha - t^\alpha_i X^i - \sum_{r=0}^{m} t^{\alpha i_1\cdots i_r}_\beta V^\beta_{i_1\cdots i_r} = G^\alpha \tag{9.8.9}$$

and

$$\mathbf{i}_V d\Omega^\alpha = -T^\alpha_i dx^i + X^i dt^\alpha_i - \sum_{r=0}^{m} T^{\alpha i_1\cdots i_r}_\beta dv^\beta_{i_1\cdots i_r} + \sum_{r=0}^{m} V^\beta_{i_1\cdots i_r} dt^{\alpha i_1\cdots i_r}_\beta$$

lead eventually to the results

$$
\begin{aligned}
&N^{\alpha}_{\beta i} = \frac{\partial G^{\alpha}}{\partial \Sigma^{\beta i}},\ \ N^{\alpha}_{\beta} = \frac{\partial G^{\alpha}}{\partial \Sigma^{\beta}},\ \ \frac{\partial G^{\alpha}}{\partial s^{\beta i}_{j}} = 0,\ \ \delta^{\alpha}_{\beta} X^{i} + \frac{\partial G^{\alpha}}{\partial t^{\beta}_{i}} = 0;\\
&\frac{\partial G^{\alpha}}{\partial s^{\gamma i i_1\cdots i_r}_{\beta}} = 0, 0 \le r \le m,\ \delta^{\alpha}_{\gamma} V^{\beta}_{i_1\cdots i_r} + \frac{\partial G^{\alpha}}{\partial t^{\gamma i_1\cdots i_r}_{\beta}} = 0,\ 0 \le r \le m,\\
&\nu^{\alpha i_1\cdots i_r}_{\beta} = \overline{\nu}^{\alpha i_1\cdots i_r}_{\beta} - T^{\alpha i_1\cdots i_r}_{\beta},\\
&\overline{\nu}^{\alpha i_1\cdots i_r}_{\beta} = \frac{\partial G^{\alpha}}{\partial v^{\beta}_{i_1\cdots i_r}} + \frac{\partial G^{\alpha}}{\partial \Sigma^{\gamma i}} s^{\gamma i i_1\cdots i_r}_{\beta} + \frac{\partial G^{\alpha}}{\partial \Sigma^{\gamma}} t^{\gamma i_1\cdots i_r}_{\beta}, 0 \le r \le m-1,\\
&T^{\alpha}_{i} = \overline{T}^{\alpha}_{i} - \sum_{r=0}^{m-1} T^{\alpha i_1\cdots i_r}_{\beta} v^{\beta}_{i_1\cdots i_r i},\\
&\overline{T}^{\alpha}_{i} = \frac{\partial G^{\alpha}}{\partial x^{i}} + \sum_{r=0}^{m-1} \overline{\nu}^{\alpha i_1\cdots i_r}_{\beta} v^{\beta}_{i_1\cdots i_r i} + \frac{\partial G^{\alpha}}{\partial \Sigma^{\beta j}} s^{\beta j}_{i} + \frac{\partial G^{\alpha}}{\partial \Sigma^{\beta}} t^{\beta}_{i},\\
&T^{\alpha i_1\cdots i_m}_{\beta} = \frac{\partial G^{\alpha}}{\partial v^{\beta}_{i_1\cdots i_m}} + \frac{\partial G^{\alpha}}{\partial \Sigma^{\gamma i}} s^{\gamma i i_1\cdots i_m}_{\beta} + \frac{\partial G^{\alpha}}{\partial \Sigma^{\gamma}} t^{\gamma i_1\cdots i_m}_{\beta} = \overline{\nu}^{\alpha i_1\cdots i_m}_{\beta}.
\end{aligned}
$$

where the functions $T^{\alpha i_1\cdots i_r}_{\beta}, r = 0, 1, \ldots, m-1$ may be chosen totally arbitrarily. Thus, isovector fields of the contact ideal are characterised in the following manner

$$
\begin{aligned}
V = {}& X^{i}\frac{\partial}{\partial x^{i}} + \sum_{r=0}^{m} V^{\alpha}_{i_1\cdots i_r}\frac{\partial}{\partial v^{\alpha}_{i_1\cdots i_r}} + S^{\alpha i}\frac{\partial}{\partial \Sigma^{\alpha i}} + T^{\alpha}\frac{\partial}{\partial \Sigma^{\alpha}} + \overline{S}^{\alpha i}_{j}\frac{\partial}{\partial s^{\alpha i}_{j}}\\
&+ \overline{T}^{\alpha}_{i}\frac{\partial}{\partial t^{\alpha}_{i}} + S^{\alpha i i_1\cdots i_m}_{\beta}\frac{\partial}{\partial s^{\alpha i i_1\cdots i_m}_{\beta}} + T^{\alpha i_1\cdots i_m}_{\beta}\frac{\partial}{\partial t^{\alpha i_1\cdots i_m}_{\beta}}\\
&+ V_1 + V_2
\end{aligned}
$$

where the vectors fields V_1 and V_2 are defined by

$$
\begin{aligned}
V_1 &= \sum_{r=0}^{m-1} S^{\alpha i i_1\cdots i_r}_{\beta}\Big(\frac{\partial}{\partial s^{\alpha i i_1\cdots i_r}_{\beta}} - v^{\beta}_{i_1\cdots i_r j}\frac{\partial}{\partial s^{\alpha i}_{j}}\Big),\\
V_2 &= \sum_{r=0}^{m-1} T^{\alpha i_1\cdots i_r}_{\beta}\Big(\frac{\partial}{\partial t^{\alpha i_1\cdots i_r}_{\beta}} - v^{\beta}_{i_1\cdots i_r i}\frac{\partial}{\partial t^{\alpha}_{i}}\Big)
\end{aligned}
$$

depending on indeterminate coefficients $S^{\alpha i i_1\cdots i_r}_{\beta}$ and $T^{\alpha i_1\cdots i_r}_{\beta}, r = 0, 1, \ldots, m-1$. On the other hand, we can easily demonstrate that the Lie derivatives of the generators of the contact ideal with respect to the vector fields V_1 and V_2 satisfy the relations

$$
\begin{aligned}
\pounds_{V_1}\sigma^{\alpha}_{i_1\cdots i_r} &= 0,\ \ \pounds_{V_1}\Omega^{\alpha} = 0,\ \ \pounds_{V_1}\omega^{\alpha} = 0,\\
\pounds_{V_1}\Omega^{\alpha i} &= \sum_{r=0}^{m-1} S_{\beta}^{\alpha i i_1\cdots i_r}(v^{\beta}_{i_1\cdots i_r j}dx^j - dv^{\beta}_{i_1\cdots i_r})\\
&= -\sum_{r=0}^{m-1} S_{\beta}^{\alpha i i_1\cdots i_r}\sigma^{\beta}_{i_1\cdots i_r},\\
\pounds_{V_2}\sigma^{\alpha}_{i_1\cdots i_r} &= 0,\quad \pounds_{V_2}\Omega^{\alpha i} = 0,\ \ \pounds_{V_2}\omega^{\alpha} = 0,\\
\pounds_{V_2}\Omega^{\alpha} &= \sum_{r=0}^{m-1} T_{\beta}^{\alpha i_1\cdots i_r}(v^{\beta}_{i_1\cdots i_r i}dx^i - dv^{\beta}_{i_1\cdots i_r})\\
&= -\sum_{r=0}^{m-1} T_{\beta}^{\alpha i_1\cdots i_r}\sigma^{\beta}_{i_1\cdots i_r}
\end{aligned}
$$

so that they are automatically isovectors of the contact ideal without imposing any restriction on the coefficient functions. Hence, these trivial isovector fields can be discarded without loss of generality because they will not be operative in determining the equivalence groups. The rather simple differential equations for the functions $F^{\alpha i}$ and G^{α} appearing in the first and second lines in the foregoing sets of relations concerned with the coefficients in equations (9.8.4) can readily be integrated to yield

$$
\begin{aligned}
F^{\alpha i} &= -s^{\alpha i}_{j}X^j - \sum_{r=0}^{m} s_{\beta}^{\alpha i i_1\cdots i_r}V^{\beta}_{i_1\cdots i_r} + \mathcal{F}^{\alpha i},\\
G^{\alpha} &= -t^{\alpha}_{i}X^i - \sum_{r=0}^{m} t_{\beta}^{\alpha i_1\cdots i_r}V^{\beta}_{i_1\cdots i_r} + \mathcal{G}^{\alpha}
\end{aligned}
$$

where the functions $\mathcal{F}^{\alpha i}$ and $\mathcal{G}^{\alpha}$ depend only on the variables

$$x^j, \Sigma^{\beta j}, \Sigma^{\beta},\ \{v^{\beta}_{i_1\cdots i_r} : r = 0, 1, \ldots, m\}.$$

When we insert the above expressions into the equations $(9.8.8)_1$ and $(9.8.9)_1$, we see at once that we can write

$$S^{\alpha i} = \mathcal{F}^{\alpha i}, \qquad T^{\alpha} = \mathcal{G}^{\alpha}.$$

Therefore, the isovector fields of the contact ideal are entirely specified by $n + 2N + nN$ functions X^i, U^{α} depending only on x^i, u^{α} and $S^{\alpha i}, T^{\alpha}$ only on $x^j, \Sigma^{\beta j}, \Sigma^{\beta}, v^{\beta}_{i_1\cdots i_r}, 0 \leq r \leq m$ They are presently chosen arbitrarily.

In order to determine the isovector fields of the balance ideal, we next have to consider the following relations

$$\begin{aligned}\pounds_V\,\omega^\alpha &= \nu^\alpha_\beta\omega^\beta + \sum_{r=0}^{m-1}\sigma^\beta_{i_1\cdots i_r}\wedge A^{\alpha i_1\cdots i_r}_\beta + \sum_{r=0}^{m-1}d\sigma^\beta_{i_1\cdots i_r}\wedge B^{\alpha i_1\cdots i_r}_\beta\\ &\quad + \Omega^{\beta i}\wedge C^\alpha_{\beta i} + d\Omega^{\beta i}\wedge D^\alpha_{\beta i} + \Omega^\beta\wedge C^\alpha_\beta + d\Omega^\beta\wedge D^\alpha_\beta\end{aligned}$$

and show that the forms $\nu^\alpha_\beta\in\Lambda^0(\mathcal{K}_m)$; $A^{\alpha i_1\cdots i_r}_\beta, C^\alpha_{\beta i}, C_\beta\in\Lambda^{n-1}(\mathcal{K}_m)$; and $B^{\alpha i_1\cdots i_r}_\beta, D^\alpha_{\beta i}, D^\alpha_\beta\in\Lambda^{n-2}(\mathcal{K}_m)$ can be so found as to satisfy the above expressions. On inserting the expressions

$$\begin{aligned}\pounds_V\,\omega^\alpha &= \mathbf{i}_V(d\,\omega^\alpha) + d(\mathbf{i}_V\omega^\alpha)\\ &= T^\alpha\mu + (dS^{\alpha i} + \Sigma^\alpha dX^i)\wedge\mu_i + d\Sigma^{\alpha i}\wedge dX^j\wedge\mu_{ji}\\ &= \Big(T^\alpha + \frac{\partial S^{\alpha i}}{\partial x^i} + \Sigma^\alpha\frac{\partial X^i}{\partial x^i}\Big)\mu + \Big(\frac{\partial S^{\alpha i}}{\partial u^\beta} + \Sigma^\alpha\frac{\partial X^i}{\partial u^\beta}\Big)du^\beta\wedge\mu_i\\ &\quad + \sum_{r=1}^{m}\frac{\partial S^{\alpha i}}{\partial v^\beta_{i_1\cdots i_r}}dv^\beta_{i_1\cdots i_r}\wedge\mu_i + \frac{\partial S^{\alpha i}}{\partial\Sigma^{\beta j}}d\Sigma^{\beta j}\wedge\mu_i\\ &\quad + \frac{\partial S^{\alpha i}}{\partial\Sigma^\beta}d\Sigma^\beta\wedge\mu_i - \frac{\partial X^j}{\partial x^i}d\Sigma^{\alpha i}\wedge\mu_j + \frac{\partial X^j}{\partial x^j}d\Sigma^{\alpha i}\wedge\mu_i\\ &\quad - \frac{\partial X^j}{\partial u^\beta}du^\beta\wedge d\Sigma^{\alpha i}\wedge\mu_{ji}\end{aligned}$$

into $\pounds_V\,\omega^\alpha$, we immediately see that we have to take $D^\alpha_{\beta i}=0$ and $D^\alpha_\beta=0$. Let us note that we can write

$$\begin{aligned}dv^\alpha_{i_1\cdots i_r} &= \sigma^\alpha_{i_1\cdots i_r} + v^\alpha_{i_1\cdots i_r i}\,dx^i,\ 0\le r\le m-1,\\ d\sigma^\alpha_{i_1\cdots i_r} &= -\,dv^\alpha_{i_1\cdots i_r i}\wedge dx^i\\ &= -\,\sigma^\alpha_{i_1\cdots i_r i}\wedge dx^i,\ \ 0\le r\le m-2,\\ d\sigma^\alpha_{i_1\cdots i_{m-1}} &= -\,dv^\alpha_{i_1\cdots i_{m-1}i}\wedge dx^i\\ d\Sigma^{\alpha i} &= \Omega^{\alpha i} + \Big(s^{\alpha i}_j + \sum_{r=0}^{m-1}s^{\alpha i i_1\cdots i_r}_\beta v^\beta_{i_1\cdots i_r j}\Big)dx^j\\ &\quad + \sum_{r=0}^{m-1}s^{\alpha i i_1\cdots i_r}_\beta\sigma^\beta_{i_1\cdots i_r} + s^{\alpha i i_1\cdots i_m}_\beta dv^\beta_{i_1\cdots i_m}\\ d\Sigma^\alpha &= \Omega^\alpha + \Big(t^\alpha_i + \sum_{r=0}^{m-1}t^{\alpha i_1\cdots i_r}_\beta v^\beta_{i_1\cdots i_r i}\Big)dx^i\\ &\quad + \sum_{r=0}^{m-1}t^{\alpha i_1\cdots i_r}_\beta\sigma^\beta_{i_1\cdots i_r} + t^{\alpha i_1\cdots i_m}_\beta dv^\beta_{i_1\cdots i_m}\end{aligned}$$

and then introduce these transformations into proper places in the invariance relations connected with Lie derivatives of forms ω^α with respect to an isovector field V and arrange the resulting expressions suitably to arrive at the equations below

$$\begin{aligned}
&\Big[\Gamma^\alpha - \nu^\alpha_\beta\Big(\Sigma^\beta + s^{\beta i}_i + \sum_{r=0}^{m-1} s^{\beta i i_1\cdots i_r}_\gamma v^\gamma_{i_1\cdots i_r i}\Big)\Big]\mu \\
&\qquad + \sigma^\beta \wedge \Big[\Gamma^{\alpha i}_\beta \mu_i - \frac{\partial X^j}{\partial u^\beta} s^{\alpha i}_\gamma \sigma^\gamma \wedge \mu_{ji} \\
&\qquad\qquad - \nu^\alpha_\gamma s^{\gamma i}_\beta \mu_i - \frac{\partial X^j}{\partial u^\beta} s^{\alpha i i_1\cdots i_m}_\gamma dv^\gamma_{i_1\cdots i_m} \wedge \mu_{ji} - A^\alpha_\beta\Big] \\
&+ \sum_{r=1}^{m-1} \sigma^\beta_{i_1\cdots i_r} \wedge \Big[\Gamma^{\alpha i i_1\cdots i_r}_\beta \mu_i - \nu^\alpha_\gamma s^{\gamma i i_1\cdots i_r}_\beta \mu_i - A^{\alpha i_1\cdots i_r}_\beta + dx^{i_r} \wedge B^{\alpha i_1\cdots i_{r-1}}_\beta\Big] \\
&\quad + dv^\beta_{i_1\cdots i_{m-1} i_m} \wedge \big(\Gamma^{\alpha i i_1\cdots i_m}_\beta \mu_i - \nu^\alpha_\gamma s^{\gamma i i_1\cdots i_m}_\beta \mu_i + dx^{i_m} \wedge B^{\alpha i_1\cdots i_{m-1}}_\beta\big) \\
&\qquad + \Omega^{\beta j} \wedge \Big(\Gamma^{\alpha i}_{\beta j}\mu_i - \frac{\partial X^i}{\partial u^\gamma}\delta^\alpha_\beta \sigma^\gamma \wedge \mu_{ji} - \nu^\alpha_\beta \delta^i_j \mu_i - C^\alpha_{\beta j}\Big) \\
&\qquad\qquad + \Omega^\beta \wedge \Big(\frac{\partial S^{\alpha i}}{\partial \Sigma^\beta}\mu_i - C^\alpha_\beta\Big) = 0.
\end{aligned}$$

The coefficient functions Γ^α, $\Gamma^{\alpha i}_\beta$, $\Gamma^{\alpha i}_{\beta j}$ and $\Gamma^{\alpha i i_1\cdots i_r}_\beta, r = 1, \ldots, m$ appearing in these equations and derived from essentially unknown functions $X^i, S^{\alpha i}$, T^α are listed below

$$\begin{aligned}
\Gamma^\alpha &= T^\alpha + \frac{\partial S^{\alpha i}}{\partial x^i} + \Sigma^\alpha \frac{\partial X^i}{\partial x^i} + \Big(\frac{\partial S^{\alpha i}}{\partial u^\beta} + \Sigma^\alpha \frac{\partial X^i}{\partial u^\beta}\Big) v^\beta_i + \sum_{r=1}^{m-1} \frac{\partial S^{\alpha i}}{\partial v^\beta_{i_1\cdots i_r}} v^\beta_{i_1\cdots i_r i} \\
&\quad + \Big\{\frac{\partial S^{\alpha i}}{\partial \Sigma^{\beta j}} + \Big[\Big(\frac{\partial X^k}{\partial x^k} + \frac{\partial X^k}{\partial u^\gamma} v^\gamma_k\Big)\delta^i_j - \Big(\frac{\partial X^i}{\partial x^j} + \frac{\partial X^i}{\partial u^\gamma} v^\gamma_j\Big)\Big]\delta^\alpha_\beta\Big\} \times \\
&\qquad \Big(s^{\beta j}_i + \sum_{r=0}^{m-1} s^{\beta j i_1\cdots i_r}_\gamma v^\gamma_{i_1\cdots i_r i}\Big) + \frac{\partial S^{\alpha i}}{\partial \Sigma^\beta}\Big(t^\beta_i + \sum_{r=0}^{m-1} t^{\beta i_1\cdots i_r}_\gamma v^\gamma_{i_1\cdots i_r i}\Big) \\
\Gamma^{\alpha i}_\beta &= \frac{\partial S^{\alpha i}}{\partial u^\beta} + \Sigma^\alpha \frac{\partial X^i}{\partial u^\beta} + \Big\{\frac{\partial S^{\alpha i}}{\partial \Sigma^{\gamma j}} + \Big[\Big(\frac{\partial X^k}{\partial x^k} + \frac{\partial X^k}{\partial u^\delta} v^\delta_k\Big)\delta_j \\
&- \Big(\frac{\partial X^i}{\partial x^j} + \frac{\partial X^i}{\partial u^\delta} v^\delta_j\Big)\Big]\delta^\alpha_\gamma\Big\} s^{\gamma j}_\beta + \frac{\partial S^{\alpha i}}{\partial \Sigma^\gamma} t^\gamma_\beta + \frac{\partial X^i}{\partial u^\beta}\Big(s^{\alpha j}_j + \sum_{r=0}^{m-1} s^{\alpha j i_1\cdots i_r}_\gamma v^\gamma_{i_1\cdots i_r j}\Big) \\
&\quad - \frac{\partial X^j}{\partial u^\beta}\Big(s^{\alpha i}_j + \sum_{r=0}^{m-1} s^{\alpha i i_1\cdots i_r}_\gamma v^\gamma_{i_1\cdots i_r j}\Big)
\end{aligned} \tag{9.8.10}$$

$$\Gamma_\beta^{\alpha i i_1\cdots i_r} = \frac{\partial S^{\alpha i}}{\partial v^\beta_{i_1\cdots i_r}} + \Big\{\frac{\partial S^{\alpha i}}{\partial \Sigma^{\gamma j}} + \Big[\Big(\frac{\partial X^k}{\partial x^k} + \frac{\partial X^k}{\partial u^\delta}v^\delta_k\Big)\delta^i_j - \Big(\frac{\partial X^i}{\partial x^j} + \frac{\partial X^i}{\partial u^\delta}v^\delta_j\Big)\Big]\delta^\alpha_\gamma\Big\} s_\beta^{\gamma j i_1\cdots i_r} + \frac{\partial S^{\alpha i}}{\partial \Sigma^\gamma} t_\beta^{\gamma i_1\cdots i_r},\ 1\le r\le m$$

$$\Gamma^{\alpha i}_{\beta j} = \frac{\partial S^{\alpha i}}{\partial \Sigma^{\beta j}} + \Big[\Big(\frac{\partial X^k}{\partial x^k} + \frac{\partial X^k}{\partial u^\gamma}v^\gamma_k\Big)\delta^i_j - \Big(\frac{\partial X^i}{\partial x^j} + \frac{\partial X^i}{\partial u^\gamma}v^\gamma_j\Big)\Big]\delta^\alpha_\beta$$

To satisfy the foregoing equations, the coefficient functions must verify the conditions

$$\begin{aligned}
&\Gamma^\alpha - \nu^\alpha_\beta\Big(\Sigma^\beta + s_i^{\beta i} + \sum_{r=0}^{m-1} s_\gamma^{\beta i i_1\cdots i_r} v^\gamma_{i_1\cdots i_r i}\Big) = 0,\\
&A^\alpha_\beta = (\Gamma^{\alpha i}_\beta - \nu^\alpha_\gamma s^{\gamma i}_\beta)\mu_i - \frac{\partial X^j}{\partial u^\beta}(s^{\alpha i}_\gamma \sigma^\gamma + s_\gamma^{\alpha i i_1\cdots i_m} dv^\gamma_{i_1\cdots i_m})\wedge\mu_{ji},\\
&A^{\alpha i_1\cdots i_r}_\beta = (\Gamma^{\alpha i i_1\cdots i_r}_\beta - \nu^\alpha_\gamma s_\beta^{\gamma i i_1\cdots i_r})\mu_i + dx^{i_r}\wedge B_\beta^{\alpha i_1\cdots i_{r-1}},\\
&C^\alpha_{\beta j} = (\Gamma^{\alpha i}_{\beta j} - \nu^\alpha_\beta \delta^i_j)\mu_i - \frac{\partial X^i}{\partial u^\gamma}\delta^\alpha_\beta \sigma^\gamma\wedge\mu_{ji},\ C^\alpha_\beta = \frac{\partial S^{\alpha i}}{\partial \Sigma^\beta}\mu_i,\\
&(\Gamma^{\alpha i i_1\cdots i_m}_\beta - \nu^\alpha_\gamma s_\beta^{\gamma i i_1\cdots i_m})\mu_i + dx^{i_m}\wedge B_\beta^{\alpha i_1\cdots i_{m-1}} = 0.
\end{aligned}$$

To fully exploit the last relation above, let us take

$$B_\beta^{\alpha i_1\cdots i_{m-1}} = B_\beta^{\alpha i_1\cdots i_{m-1} i j}\mu_{ij}$$

where $B_\beta^{\alpha i_1\cdots i_{m-1} i j} = -\,B_\beta^{\alpha i_1\cdots i_{m-1} j i} \in \Lambda^0(\mathcal{K}_m)$ are arbitrary functions. On employing the relation

$$\begin{aligned}
dx^{i_m}\wedge B_\beta^{\alpha i_1\cdots i_{m-1}} &= B_\beta^{\alpha i_1\cdots i_{m-1} i j} dx^{i_m}\wedge \mu_{ij}\\
&= B_\beta^{\alpha i_1\cdots i_{m-1} i j}(\delta^{i_m}_i \mu_j - \delta^{i_m}_j \mu_i)\\
&= B_\beta^{\alpha i_1\cdots i_{m-1} i_m j}\mu_j - B_\beta^{\alpha i_1\cdots i_{m-1} i i_m}\mu_i\\
&= 2B_\beta^{\alpha i_1\cdots i_{m-1} i_m i}\mu_i
\end{aligned}$$

we get

$$2B_\beta^{\alpha i_1\cdots i_{m-1} i_m i} = -\,(\Gamma_\beta^{\alpha i i_1\cdots i_m} - \nu^\alpha_\gamma s_\beta^{\gamma i i_1\cdots i_m}).$$

However, the left hand side above is antisymmetric with respect to its last two superscripts. Hence, the symmetric part of the right hand side with respect to indices i and i_m must vanish. Therefore, the determining equations for the isovector components X^i, $S^{\alpha i}$, T^α and the functions ν^α_β are found eventually to be

$$\Gamma^\alpha = \nu^\alpha_\beta\Big(\Sigma^\beta + s^{\beta i}_i + \sum_{r=0}^{m-1} s^{\beta i i_1\cdots i_r}_\gamma v^\gamma_{i_1\cdots i_r i}\Big), \tag{9.8.11}$$
$$\Gamma^{\alpha i i_1\cdots i_m}_\beta + \Gamma^{\alpha i_m i_1\cdots i}_\beta = \nu^\alpha_\gamma(s^{\gamma i i_1\cdots i_m}_\beta + s^{\gamma i_m i_1\cdots i}_\beta).$$

These equations do not impose any restriction on the isovector components U^α. *Nonetheless, if some variables do not appear in the coordinate cover of the manifold $\mathcal{K}_m$ due to a particular structure of the balance equations, that might entail some new restrictions on the isovector components because the corresponding isovector components must then be set to zero*. Equivalence transformations are now obtained by solving the following autonomous ordinary differential equations

$$\frac{d\overline{x}^i}{d\epsilon} = X^i(\overline{x}^j, \overline{u}^\beta),\ \frac{d\overline{u}^\alpha}{d\epsilon} = U^\alpha(\overline{x}^j, \overline{u}^\beta),$$
$$\frac{d\overline{v}^\alpha_{i_1\cdots i_r}}{d\epsilon} = V^\alpha_{i_1\cdots i_r}(\overline{x}^j, \overline{v}^\beta_{j_1\cdots j_s}),\ \ 0\le s\le r,\ \ 1\le r\le m,$$
$$\frac{d\overline{\Sigma}^{\alpha i}}{d\epsilon} = \mathcal{F}^{\alpha i}(\overline{x}^j, \overline{\Sigma}^{\beta j}, \overline{\Sigma}^\beta, \overline{v}^\beta_{j_1\cdots j_r}),\ \ 0\le r\le m,$$
$$\frac{d\overline{\Sigma}^{\alpha}}{d\epsilon} = \mathcal{G}^{\alpha}(\overline{x}^j, \overline{\Sigma}^{\beta j}, \overline{\Sigma}^\beta, \overline{v}^\beta_{j_1\cdots j_r}),\ \ 0\le r\le m$$

under the initial conditions

$$\overline{x}^i(0) = x^i,\ \overline{v}^\alpha_{i_1\cdots i_r}(0) = v^\alpha_{i_1\cdots i_r},\ 0\le r\le m;$$
$$\overline{\Sigma}^{\alpha i}(0) = \Sigma^{\alpha i},\quad \overline{\Sigma}^{\alpha}(0) = \Sigma^{\alpha}.$$

For $m = 0$, that is, for the first order balance equations we have to take $\Sigma^{\alpha i}(\mathbf{x},\mathbf{u})$ and $\Sigma^\alpha(\mathbf{x},\mathbf{u})$, and we have to modify our analysis substantially. In this case, the coordinates of $\mathcal{K}_m$ are merely $\{x^i, u^\alpha, \Sigma^{\alpha i}, \Sigma^\alpha, s^{\alpha i}_j, s^{\alpha i}_\beta, t^\alpha_i, t^\alpha_\beta\}$ [*see* (9.8.1)]. The contact ideal is now constructed by the following 1-forms

$$\Omega^{\alpha i} = d\Sigma^{\alpha i} - s^{\alpha i}_j dx^j - s^{\alpha i}_\beta du^\beta,\ \ \Omega^\alpha = d\Sigma^\alpha - t^\alpha_i dx^i - t^\alpha_\beta du^\beta.$$

An isovector field of the balance ideal must then be taken as

$$V = X^i\frac{\partial}{\partial x^i} + U^\alpha\frac{\partial}{\partial u^\alpha} + S^{\alpha i}\frac{\partial}{\partial\Sigma^{\alpha i}} + T^\alpha\frac{\partial}{\partial\Sigma^\alpha} + S^{\alpha i}_j\frac{\partial}{\partial s^{\alpha i}_j} + S^{\alpha i}_\beta\frac{\partial}{\partial s^{\alpha i}_\beta}$$
$$+ T^\alpha_i\frac{\partial}{\partial t^\alpha_i} + \mathcal{T}^\alpha_\beta\frac{\partial}{\partial t^\alpha_\beta}.$$

In order that this vector turns out to be an isovector of the contact ideal, one has to satisfy the relations

$$\mathbf{i}_V\Omega^{\alpha i} = L^{\alpha i}_{\beta j}\Omega^{\beta j} + L^{\alpha i}_{\beta}\Omega^{\beta}, \quad \mathbf{i}_V\Omega^{\alpha} = M^{\alpha}_{\beta i}\Omega^{\beta i} + M^{\alpha}_{\beta}\Omega^{\beta}$$

from which we easily deduce that

$$\begin{aligned}
&L^{\alpha i}_{\beta j} = \frac{\partial F^{\alpha i}}{\partial \Sigma^{\beta j}},\ L^{\alpha i}_{\beta} = \frac{\partial F^{\alpha i}}{\partial \Sigma^{\beta}},\ M^{\alpha}_{\beta i} = \frac{\partial G^{\alpha}}{\partial \Sigma^{\beta i}},\ M^{\alpha}_{\beta} = \frac{\partial G^{\alpha}}{\partial \Sigma^{\beta}},\\
&\frac{\partial F^{\alpha i}}{\partial t^{\beta}_{j}} = 0,\ \frac{\partial F^{\alpha i}}{\partial t^{\beta}_{\gamma}} = 0,\ \frac{\partial G^{\alpha}}{\partial s^{\beta i}_{j}} = 0,\ \frac{\partial G^{\alpha}}{\partial s^{\beta i}_{\gamma}} = 0,\\
&S^{\alpha i}_{j} = \frac{\partial F^{\alpha i}}{\partial x^{j}} + \frac{\partial F^{\alpha i}}{\partial \Sigma^{\beta k}}s^{\beta k}_{j} + \frac{\partial F^{\alpha i}}{\partial \Sigma^{\beta}}t^{\beta}_{j}, S^{\alpha i}_{\beta} = \frac{\partial F^{\alpha i}}{\partial u^{\beta}} + \frac{\partial F^{\alpha i}}{\partial \Sigma^{\gamma j}}s^{\gamma j}_{\beta} + \frac{\partial F^{\alpha i}}{\partial \Sigma^{\gamma}}t^{\gamma}_{\beta},\\
&T^{\alpha}_{i} = \frac{\partial G^{\alpha}}{\partial x^{i}} + \frac{\partial G^{\alpha}}{\partial \Sigma^{\beta j}}s^{\beta j}_{i} + \frac{\partial G^{\alpha}}{\partial \Sigma^{\beta}}t^{\beta}_{i}, \quad \mathcal{T}^{\alpha}_{\beta} = \frac{\partial G^{\alpha}}{\partial u^{\beta}} + \frac{\partial G^{\alpha}}{\partial \Sigma^{\gamma i}}s^{\gamma i}_{\beta} + \frac{\partial G^{\alpha}}{\partial \Sigma^{\gamma}}t^{\gamma}_{\beta},\\
&X^{j}\delta^{\alpha}_{\beta}\delta^{i}_{k} + \frac{\partial F^{\alpha i}}{\partial s^{\beta k}_{j}} = X^{i}\delta^{\alpha}_{\beta} + \frac{\partial G^{\alpha}}{\partial t^{\beta}_{i}} = 0, U^{\beta}\delta^{\alpha}_{\gamma}\delta^{i}_{j} + \frac{\partial F^{\alpha i}}{\partial s^{\gamma j}_{\beta}} = U^{\beta}\delta^{\alpha}_{\gamma} + \frac{\partial G^{\alpha}}{\partial t^{\gamma}_{\beta}} = 0.
\end{aligned}$$

Here, we have defined

$$\mathbf{i}_V\Omega^{\alpha i} = F^{\alpha i} = S^{\alpha i} - X^{j}s^{\alpha i}_{j} - U^{\beta}s^{\alpha i}_{\beta},\ \mathbf{i}_V\Omega^{\alpha} = G^{\alpha} = T^{\alpha} - X^{i}t^{\alpha}_{i} - U^{\beta}t^{\alpha}_{\beta}$$

If we scrutinise carefully the above equations, we immediately reach to the conclusion

$$X^{i} = X^{i}(x^{j}, u^{\beta}, \Sigma^{\beta j}, \Sigma^{\beta}),\ \ U^{\alpha} = U^{\alpha}(x^{j}, u^{\beta}, \Sigma^{\beta j}, \Sigma^{\beta})$$

and

$$F^{\alpha i} = -X^{j}s^{\alpha i}_{j} - U^{\beta}s^{\alpha i}_{\beta} + \mathcal{F}^{\alpha i},\ G^{\alpha} = -X^{i}t^{\alpha}_{i} - U^{\beta}t^{\alpha}_{\beta} + \mathcal{G}^{\alpha}$$

where $\mathcal{F}^{\alpha i}$ and $\mathcal{G}^{\alpha}$ depend only on the variables $x^{j}, u^{\beta}, \Sigma^{\beta j}, \Sigma^{\beta}$. Ultimately, let us consider the relations

$$\pounds_V\,\omega^{\alpha} = \nu^{\alpha}_{\beta}\omega^{\beta} + \Omega^{\beta i}\wedge C^{\alpha}_{\beta i} + d\Omega^{\beta i}\wedge D^{\alpha}_{\beta i} + \Omega^{\beta}\wedge C^{\alpha}_{\beta} + d\Omega^{\beta}\wedge D^{\alpha}_{\beta}.$$

We readily observe that $D^{\alpha}_{\beta i} = 0, D^{\alpha}_{\beta} = 0$. We thus obtain

$$\begin{aligned}
&\big[\Gamma^{\alpha} - \nu^{\alpha}_{\beta}(\Sigma^{\beta} + s^{\beta i}_{i})\big]\mu + (G^{\alpha i}_{\beta} + \Gamma^{\alpha i}_{\gamma}t^{\gamma}_{\beta} - \nu^{\alpha}_{\gamma}s^{\gamma i}_{\beta})\,du^{\beta}\wedge\mu^{\alpha}_{i}\\
&\quad + \Omega^{\beta j}\wedge\Big[-C_{\beta j} + (\Gamma^{\alpha i}_{\beta j} - \nu^{\alpha}_{\beta}\delta^{i}_{j})\mu_{i} - \Big(\Gamma^{\alpha ik}_{\gamma\beta j}\,du^{\gamma} + \frac{\partial X^{k}}{\partial \Sigma^{\beta j}}\Omega^{\alpha i}\Big)\wedge\mu_{ki}\Big]\\
&\qquad + \Gamma^{\alpha ij}_{\beta\gamma}du^{\beta}\wedge du^{\gamma}\wedge\mu_{ji}
\end{aligned}$$

$$+\,\Omega^\beta\wedge\Big[-C^\alpha_\beta+\Gamma^{\alpha i}_\beta\mu_i-\frac{\partial X^j}{\partial\Sigma^\beta}(\Omega^{\alpha i}+s^{\alpha i}_\gamma\,du^\gamma)\wedge\mu_{ji}\Big]=0.$$

Smooth functions Γ^α, $\Gamma^{\alpha i}_\beta$, $G^{\alpha i}_\beta$, $\Gamma^{\alpha i}_{\beta j}$, $\Gamma^{\alpha ij}_{\beta\gamma k}$ and $\Gamma^{\alpha ij}_{\beta\gamma}$ in the module $\Lambda^0(\mathcal{K}_0)$ appearing in the above equations are given as follows

$$\begin{aligned}
\Gamma^\alpha &= T^\alpha+\frac{\partial S^{\alpha i}}{\partial x^i}+\Sigma^\alpha\frac{\partial X^i}{\partial x^i}+\Big(\frac{\partial S^{\alpha i}}{\partial\Sigma^{\beta j}}+\Sigma^\alpha\frac{\partial X^i}{\partial\Sigma^{\beta j}}+\frac{\partial X^k}{\partial x^k}\delta^\alpha_\beta\delta^i_j\\
&\quad-\frac{\partial X^i}{\partial x^j}\delta^\alpha_\beta\Big)s^{\beta j}_i+\Big(\frac{\partial S^{\alpha i}}{\partial\Sigma^\beta}+\Sigma^\alpha\frac{\partial X^i}{\partial\Sigma^\beta}\Big)t^\beta_i\\
&\quad-\frac{\partial X^j}{\partial\Sigma^{\beta k}}(s^{\beta k}_i s^{\alpha i}_j-s^{\beta k}_j s^{\alpha i}_i)-\frac{\partial X^j}{\partial\Sigma^\beta}(t^\beta_i s^{\alpha i}_j-t^\beta_j s^{\alpha i}_i),\\
\Gamma^{\alpha i}_\beta &= \frac{\partial S^{\alpha i}}{\partial\Sigma^\beta}+\Sigma^\alpha\frac{\partial X^i}{\partial\Sigma^\beta}-\frac{\partial X^j}{\partial\Sigma^\beta}(s^{\alpha i}_j-s^{\alpha k}_k\delta^i_j),\\
G^{\alpha i}_\beta &= \frac{\partial S^{\alpha i}}{\partial u^\beta}+\Sigma^\alpha\frac{\partial X^i}{\partial u^\beta}\\
&\quad+\Big(\frac{\partial S^{\alpha i}}{\partial\Sigma^{\gamma j}}+\Sigma^\alpha\frac{\partial X^i}{\partial\Sigma^{\gamma j}}+\frac{\partial X^k}{\partial x^k}\delta^\alpha_\gamma\delta^i_j-\frac{\partial X^i}{\partial x^j}\delta^\alpha_\gamma\Big)s^{\gamma j}_\beta\\
&\quad-\frac{\partial X^j}{\partial u^\beta}(s^{\alpha i}_j-s^{\alpha k}_k\delta^i_j)-\frac{\partial X^i}{\partial\Sigma^{\gamma k}}(s^{\alpha j}_\beta s^{\gamma k}_j-s^{\gamma k}_\beta s^{\alpha j}_j)\\
&\quad+\frac{\partial X^j}{\partial\Sigma^{\gamma k}}(s^{\alpha i}_\beta s^{\gamma k}_j-s^{\gamma k}_\beta s^{\alpha i}_j)+\frac{\partial X^j}{\partial\Sigma^\gamma}(t^\gamma_j s^{\alpha i}_\beta-t^\gamma_k s^{\alpha k}_\beta\delta^i_j),\\
\Gamma^{\alpha i}_{\beta j} &= \frac{\partial S^{\alpha i}}{\partial\Sigma^{\beta j}}+\Sigma^\alpha\frac{\partial X^i}{\partial\Sigma^{\beta j}}-\frac{\partial X^k}{\partial x^l}(\delta^i_k\delta^l_j-\delta^l_k\delta^i_j)\delta^\alpha_\beta\\
&\quad-\frac{\partial X^k}{\partial\Sigma^{\beta j}}(s^{\alpha i}_k-s^{\alpha l}_l\delta^i_k)\\
&\quad+\Big(\frac{\partial X^l}{\partial\Sigma^{\gamma k}}s^{\gamma k}_l\delta^i_j-\frac{\partial X^i}{\partial\Sigma^{\gamma k}}s^{\gamma k}_j+\frac{\partial X^k}{\partial\Sigma^\gamma}t^\gamma_k\delta^i_j-\frac{\partial X^i}{\partial\Sigma^\gamma}t^\gamma_j\Big)\delta^\alpha_\beta,\\
\Gamma^{\alpha ij}_{\beta\gamma k} &= \frac{\partial X^j}{\partial\Sigma^{\gamma k}}s^{\alpha i}_\beta-\Big(\frac{\partial X^j}{\partial\Sigma^{\delta l}}s^{\delta l}_\beta+\frac{\partial X^j}{\partial\Sigma^\delta}t^\delta_\beta+\frac{\partial X^j}{\partial u^\beta}\Big)\delta^\alpha_\gamma\delta^i_k,\\
\Gamma^{\alpha ij}_{\beta\gamma} &= -\frac{\partial X^j}{\partial u^\beta}s^{\alpha i}_\gamma+\Big(\frac{\partial X^j}{\partial\Sigma^{\delta k}}s^{\delta k}_\gamma+\frac{\partial X^j}{\partial\Sigma^\delta}t^\delta_\gamma\Big)s^{\alpha i}_\beta.
\end{aligned}$$

Hence, we have to take

$$\begin{aligned}
C^\alpha_{\beta j} &= (\Gamma^{\alpha i}_{\beta j}-\nu^\alpha_\beta\delta^i_j)\mu_i-\Big(\Gamma^{\alpha ik}_{\gamma\beta j}\,du^\gamma+\frac{\partial X^k}{\partial\Sigma^{\beta j}}\Omega^{\alpha i}\Big)\wedge\mu_{ki},\\
C^\alpha_\beta &= \Gamma^{\alpha i}_\beta\mu_i-\frac{\partial X^j}{\partial\Sigma^\beta}(\Omega^{\alpha i}+s^{\alpha i}_\gamma\,du^\gamma)\wedge\mu_{ji}
\end{aligned}$$

in order to satisfy the above equations. Thus, we reach to the following determining equations

$$\begin{aligned} &\Gamma^{\alpha} - \nu^{\alpha}_{\beta}(\Sigma^{\beta} + s^{\beta i}_{i}) = 0, \\ &G^{\alpha i}_{\beta} + \Gamma^{\alpha i}_{\gamma} t^{\gamma}_{\beta} - \nu^{\alpha}_{\gamma} s^{\gamma i}_{\beta} = 0, \\ &\Gamma^{\alpha[ij]}_{[\beta\gamma]} = 0 \end{aligned} \tag{9.8.12}$$

The foregoing results will loose their validity in the case $N = 1$, that is, if there is only a single dependent variable. Evidently, we do not need to employ the Greek indices anymore since they all take only the value 1. In that case, we have to consider obviously just one balance equation which is represented by

$$\frac{\partial \Sigma^{i}}{\partial x^{i}} + \Sigma = 0.$$

The coordinate cover of the contact ideal $\mathcal{K}_m$ then becomes

$$\big\{x^{i},\ \Sigma^{i},\ \Sigma,\ s^{i}_{j},\ t_{i},\ \{v_{i_1\cdots i_r},\ s^{ii_1\cdots i_r},\ t^{i_1\cdots i_r} : 0 \le r \le m\}\big\}$$

where we have naturally defined the auxiliary variables by

$$\begin{aligned} s^{i}_{j} &= \frac{\partial \Sigma^{i}}{\partial x^{j}}, \\ s^{ii_1\cdots i_r} &= \frac{\partial \Sigma^{i}}{\partial v_{i_1\cdots i_r}}, \\ t_{i} &= \frac{\partial \Sigma}{\partial x^{i}}, \\ t^{i_1\cdots i_r} &= \frac{\partial \Sigma}{\partial v_{i_1\cdots i_r}},\ 0 \le r \le m. \end{aligned}$$

Consequently, an isovector field of $\mathcal{K}_m$ that is a member of the tangent bundle $T(\mathcal{K}_m)$ must be represented by

$$\begin{aligned} V = X^{i}\frac{\partial}{\partial x^{i}} + S^{i}\frac{\partial}{\partial \Sigma^{i}} + T\frac{\partial}{\partial \Sigma} + S^{i}_{j}\frac{\partial}{\partial s^{i}_{j}} + T_{i}\frac{\partial}{\partial t_{i}} \\ + \sum_{r=0}^{m}\Big(V_{i_1\cdots i_r}\frac{\partial}{\partial v_{i_1\cdots i_r}} + S^{ii_1\cdots i_r}\frac{\partial}{\partial s^{ii_1\cdots i_r}} + T^{i_1\cdots i_r}\frac{\partial}{\partial t^{i_1\cdots i_r}}\Big). \end{aligned}$$

On making use of exactly the same sort of operations as we had done in the case of $N > 1$, we obtain in this situation the following expressions for $0 \le k \le m - 1$

$$
\begin{aligned}
V_{i_1\cdots i_k i} &= \frac{\partial F_{i_1\cdots i_k}}{\partial x^i} + \sum_{r=0}^{m-1}\lambda^{j_1\cdots j_r}_{i_1\cdots i_k} v_{j_1\cdots j_r i} + \frac{\partial F_{i_1\cdots i_k}}{\partial \Sigma^j}\, s^j_i \\
&\qquad + \frac{\partial F_{i_1\cdots i_k}}{\partial \Sigma}\, t_i, \\
\lambda^{j_1\cdots j_r}_{i_1\cdots i_k} &= \frac{\partial V_{i_1\cdots i_k}}{\partial v_{j_1\cdots j_r}} - v_{i_1\cdots i_k i}\frac{\partial X^i}{\partial v_{j_1\cdots j_r}} + \frac{\partial F_{i_1\cdots i_k}}{\partial \Sigma^i} s^{ij_1\cdots j_r} \\
&\qquad + \frac{\partial F_{i_1\cdots i_k}}{\partial \Sigma} t^{j_1\cdots j_r},\ 0 \le r \le m-1, \qquad (9.8.13) \\
&\frac{\partial V_{i_1\cdots i_k}}{\partial v_{j_1\cdots j_m}} - v_{i_1\cdots i_k i}\frac{\partial X^i}{\partial v_{j_1\cdots j_m}} + \frac{\partial F_{i_1\cdots i_k}}{\partial \Sigma^i} s^{ij_1\cdots j_m} \\
&\qquad + \frac{\partial F_{i_1\cdots i_k}}{\partial \Sigma} t^{j_1\cdots j_m} = 0, \\
&\delta^i_j V_{i_1\cdots i_r} + \frac{\partial F^i}{\partial s^{j i_1\cdots i_r}} = V_{i_1\cdots i_r} + \frac{\partial G}{\partial t^{i_1\cdots i_r}} = 0, \qquad 0 \le r \le m, \\
&\delta^i_k X^j + \frac{\partial F^i}{\partial s^k_j} = X^i + \frac{\partial G}{\partial t_i} = 0
\end{aligned}
$$

where the functions $F_{i_1\cdots i_k}, F^i, G \in \Lambda^0(\mathcal{K}_m)$ are defined by the relations

$$
\begin{aligned}
F_{i_1\cdots i_k} &= V_{i_1\cdots i_k} - v_{i_1\cdots i_k i} X^i, \qquad (9.8.14) \\
F^i &= S^i - s^i_j X^j - \sum_{r=0}^{m} s^{i i_1\cdots i_r} V_{i_1\cdots i_r}, \\
G &= T - t_i X^i - \sum_{r=0}^{m} t^{i_1\cdots i_r} V_{i_1\cdots i_r}.
\end{aligned}
$$

Furthermore, we immediately observe that $F_{i_1\cdots i_k}$ cannot depend on the variables s^i_j, t_i and $\{s^{ii_1\cdots i_r}, t^{i_1\cdots i_r} : 0 \le r \le m\}$ while F^i is independent of the variables t_i and $\{t^{i_1\cdots i_r} : 0 \le r \le m\}$, and G is independent of the variables s^i_j and $\{s^{ii_1\cdots i_r} : 0 \le r \le m\}$. For $k < m-1$, it follows from $(9.8.13)_3$ and $(9.8.14)_1$ that

$$
\frac{\partial F_{i_1\cdots i_k}}{\partial v_{j_1\cdots j_m}} + \frac{\partial F_{i_1\cdots i_k}}{\partial \Sigma^i} s^{ij_1\cdots j_m} + \frac{\partial F_{i_1\cdots i_k}}{\partial \Sigma} t^{j_1\cdots j_m} = 0
$$

which gives rise to

$$
\frac{\partial F_{i_1\cdots i_k}}{\partial \Sigma^i} = 0,\ \ \frac{\partial F_{i_1\cdots i_k}}{\partial \Sigma} = 0,\ \ \frac{\partial F_{i_1\cdots i_k}}{\partial v_{j_1\cdots j_m}} = 0,\ \ 0 \le k \le m-2.
$$

By taking $k = m-1$ in the relation $(9.8.13)_3$, we get

$$\delta^{j_1}_{i_1}\delta^{j_2}_{i_2}\cdots\delta^{j_{m-1}}_{i_{m-1}}\delta^{j_m}_{i}X^i + \frac{\partial F_{i_1\cdots i_{m-1}}}{\partial v_{j_1\cdots j_m}} + \frac{\partial F_{i_1\cdots i_{m-1}}}{\partial \Sigma^i}s^{ij_1\cdots j_m} + \frac{\partial F_{i_1\cdots i_{m-1}}}{\partial \Sigma}t^{j_1\cdots j_m} = 0.$$

On the other hand, according to the relations $(9.8.13)_{4,5}$, the functions X^i and $V_{i_1\cdots i_r}$ cannot depend on the variables s^i_j, t_i and $\{s^{ii_1\cdots i_r}, t^{i_1\cdots i_r}\}$ where $0 \le r \le m$. But, this leads us to the conclusion

$$\frac{\partial F_{i_1\cdots i_k}}{\partial \Sigma^i} = 0, \quad \frac{\partial F_{i_1\cdots i_k}}{\partial \Sigma} = 0, \quad 0 \le k \le m-1.$$

Therefrom, we arrive at the previously derived solutions (9.3.22), (9.3.23) and (9.3.25) for the components X^i, U and $V_{i_1\cdots i_k}$. In a similar fashion, we deduce the following expressions

$$\begin{aligned} F^i &= -s^i_jX^j - \sum_{r=0}^{m} s^{ii_1\cdots i_r}V_{i_1\cdots i_r} + \mathcal{F}^i, \\ G &= -t_iX^i - \sum_{r=0}^{m} t^{i_1\cdots i_r}V_{i_1\cdots i_r} + \mathcal{G} \end{aligned}$$

from $(9.8.13)_{4,5}$ where the functions $\mathcal{F}^i$ and $\mathcal{G}$ depend only on the variables x^j, Σ^j, Σ and $\{v_{i_1\cdots i_r} : 0 \le r \le m\}$. In conclusion, isovector fields of the contact ideal are to be expressed as

$$\begin{aligned} V = {} & X^i\frac{\partial}{\partial x^i} + \sum_{r=0}^{m} V_{i_1\cdots i_r}\frac{\partial}{\partial v_{i_1\cdots i_r}} + S^i\frac{\partial}{\partial \Sigma^i} + T\frac{\partial}{\partial \Sigma} + \overline{S}^i_j\frac{\partial}{\partial s^i_j} + \overline{T}_i\frac{\partial}{\partial t_i} \\ & + S^{ii_1\cdots i_m}\frac{\partial}{\partial s^{ii_1\cdots i_m}} + T^{i_1\cdots i_m}\frac{\partial}{\partial t^{i_1\cdots i_m}} + V_1 + V_2. \end{aligned}$$

Trivial isovector fields that can be discarded without loss of generality are given by

$$\begin{aligned} V_1 &= \sum_{r=0}^{m-1} S^{ii_1\cdots i_r}\Big(\frac{\partial}{\partial s^{ii_1\cdots i_r}} - v_{i_1\cdots i_r j}\frac{\partial}{\partial s^i_j}\Big), \\ V_2 &= \sum_{r=0}^{m-1} T^{i_1\cdots i_r}\Big(\frac{\partial}{\partial t^{i_1\cdots i_r}} - v_{i_1\cdots i_r i}\frac{\partial}{\partial t_i}\Big) \end{aligned}$$

as dependent upon arbitrarily selected functions $S^{ii_1\cdots i_r}$ and $T^{i_1\cdots i_r}$ with

$0 \le r \le m-1$. The remaining isovector components are listed below

$$
\begin{aligned}
&S^i = \mathcal{F}^i, \quad T = \mathcal{G}, \qquad (9.8.15)\\
&\overline{S}^i_j = \frac{\partial F^i}{\partial x^j} + \sum_{r=0}^{m-1}\Big(\frac{\partial F^i}{\partial v_{i_1\cdots i_r}} + \frac{\partial F^i}{\partial \Sigma^j} s^{ji_1\cdots i_r} + \frac{\partial F^i}{\partial \Sigma} t^{i_1\cdots i_r}\Big) v_{i_1\cdots i_r j}\\
&\qquad + \frac{\partial F^i}{\partial \Sigma^k} s^k_j + \frac{\partial F^i}{\partial \Sigma} t_j,\\
&\overline{T}_i = \frac{\partial G}{\partial x^i} + \sum_{r=0}^{m-1}\Big(\frac{\partial G}{\partial v_{i_1\cdots i_r}} + \frac{\partial G}{\partial \Sigma^i} s^{ji_1\cdots i_r} + \frac{\partial G}{\partial \Sigma} t^{i_1\cdots i_r}\Big) v_{i_1\cdots i_r i}\\
&\qquad + \frac{\partial G}{\partial \Sigma^j} s^j_i + \frac{\partial G}{\partial \Sigma} t_i,\\
S^{ii_1\cdots i_m} &= \frac{\partial F^i}{\partial v_{i_1\cdots i_m}} + \frac{\partial F^i}{\partial \Sigma^j} s^{ji_1\cdots i_m} + \frac{\partial F^i}{\partial \Sigma} t^{i_1\cdots i_m},\\
T^{i_1\cdots i_m} &= \frac{\partial G}{\partial v_{i_1\cdots i_m}} + \frac{\partial G}{\partial \Sigma^i} s^{ii_1\cdots i_m} + \frac{\partial G}{\partial \Sigma} t^{i_1\cdots i_m}.
\end{aligned}
$$

They are entirely determined in terms of $n+2$ functions

$$F(x^i,u,v_i),\ S^i = \mathcal{F}^i(x^j,\Sigma^j,\Sigma,v_{i_1\cdots i_r}),\ T = \mathcal{G}(x^j,\Sigma^j,\Sigma,v_{i_1\cdots i_r})$$

with $0 \le r \le m$. In order that this vector field becomes also an isovector field of the balance ideal, we have to show the existence of the forms $\nu \in \Lambda^0(\mathcal{K}_m);\ A^{i_1\cdots i_r},\ B_i,\ B \in \Lambda^{n-1}(\mathcal{K}_m);\ C^{i_1\cdots i_r},\ D_i,\ D \in \Lambda^{n-2}(\mathcal{K}_m)$, $0 \le r \le m-1$ such that the following relation is satisfied

$$
\begin{aligned}
\pounds_V\omega &= T\mu - X^i d\Sigma\wedge\mu_i + d(S^i\mu_i - X^j d\Sigma^i\wedge\mu_{ji} + \Sigma X^i\mu_i)\\
&= T\mu + (dS^i + \Sigma dX^i)\wedge\mu_i - dX^j\wedge d\Sigma^i\wedge\mu_{ji} = T\mu\\
&\quad + \Big(\frac{\partial S^i}{\partial x^j} + \Sigma\frac{\partial X^i}{\partial x^j}\Big)dx^j\wedge\mu_i + \Big(\frac{\partial S^i}{\partial u} + \Sigma\frac{\partial X^i}{\partial u}\Big)du\wedge\mu_i\\
&\quad + \Big(\frac{\partial S^i}{\partial v_j} + \Sigma\frac{\partial X^i}{\partial v_j}\Big)dv_j\wedge\mu_i\\
&\quad + \sum_{r=2}^{m}\frac{\partial S^i}{\partial v_{i_1\cdots i_r}}dv_{i_1\cdots i_r}\wedge\mu_i + \frac{\partial S^i}{\partial \Sigma^j}d\Sigma^j\wedge\mu_i\\
&\quad + \frac{\partial S^i}{\partial \Sigma}d\Sigma\wedge\mu_i - \Big(\frac{\partial X^j}{\partial x^k}dx^k + \frac{\partial X^j}{\partial u}du + \frac{\partial X^j}{\partial v_k}dv_k\Big)\wedge d\Sigma^i\wedge\mu_{ji}\\
&= \nu\omega + \sum_{r=0}^{m-1}\sigma_{i_1\cdots i_r}\wedge A^{i_1\cdots i_r} + \Omega^i\wedge B_i + \Omega\wedge B
\end{aligned}
$$

$$+\sum_{r=0}^{m-1} d\sigma_{i_1\cdots i_r} \wedge C^{i_1\cdots i_r} + d\Omega^i \wedge D_i + d\Omega \wedge D.$$

If we follow a path similar to what we have followed in the case of $N > 1$, we easily obtain the expression

$$\begin{aligned}\Big[\Gamma - \nu\Big(\Sigma + s_i^i + \sum_{r=0}^{m-1} s^{ii_1\cdots i_r} v_{i_1\cdots i_r i}\Big)\Big]\mu + \sigma \wedge (\gamma - A - \nu\, s^i \mu_i) \quad (9.8.16)\\ + \sigma_i \wedge (\gamma^i - A^i - \nu s^{ji}\mu_j dx^i \wedge C) + \sum_{r=2}^{m-1} \sigma_{i_1\cdots i_r} \wedge \big(\gamma^{i_1\cdots i_r} - A^{i_1\cdots i_r}\\ - \nu s^{ii_1\cdots i_r}\mu_i + dx^{i_r} \wedge C^{i_1\cdots i_{r-1}}\big) + \Omega^i \wedge (\lambda_i - B_i - \nu\mu_i)\\ + \Omega\wedge\Big(\frac{\partial S^i}{\partial \Sigma}\mu_i - B\Big)\\ + (\Gamma^{ii_1\cdots i_m} - \nu s^{ii_1\cdots i_m})\, dv_{i_1\cdots i_m} \wedge \mu_i\\ + dv_{i_1\cdots i_m} \wedge dx^{i_m} \wedge C^{i_1\cdots i_{m-1}} = 0,\end{aligned}$$

The functions and forms entering into the single equation (9.8.16) are essentially associated with unknown functions X^i, S^i, T and are given by

$$\begin{aligned}\Gamma = T + \frac{\partial S^i}{\partial x^i} + \frac{\partial S^i}{\partial u} v_i + \frac{\partial S^i}{\partial v_j} v_{ji} + \Sigma\Big(\frac{\partial X^i}{\partial x^i} + \frac{\partial X^i}{\partial u} v_i + \frac{\partial X^i}{\partial v_j} v_{ji}\Big)\\ + \sum_{r=2}^{m-1} \frac{\partial S^i}{\partial v_{i_1\cdots i_r}} v^i_{i_1\cdots i_r i}\\ + \Big[\frac{\partial S^i}{\partial \Sigma^j} + \Big(\frac{\partial X^k}{\partial x^k} + \frac{\partial X^k}{\partial u} v_k + \frac{\partial X^k}{\partial v_l} v_{lk}\Big)\delta_j\\ - \Big(\frac{\partial X^i}{\partial x^j} + \frac{\partial X^i}{\partial u} v_j + \frac{\partial X^i}{\partial v_k} v_{kj}\Big)\Big]\Big(s_i^j + \sum_{r=0}^{m-1} s^{ji_1\cdots i_r} v_{i_1\cdots i_r i}\Big)\\ + \frac{\partial S^i}{\partial \Sigma}\Big(t_i + \sum_{r=0}^{m-1} t^{i_1\cdots i_r} v_{i_1\cdots i_r i}\Big) \in \Lambda^0(\mathcal{K}_m),\end{aligned}$$

$$\begin{aligned}\gamma = \Big[\frac{\partial S^i}{\partial u} + \Sigma\,\frac{\partial X^i}{\partial u} + \frac{\partial S^i}{\partial \Sigma^j} s^j + \frac{\partial S^i}{\partial \Sigma}\, t + \Big(\frac{\partial X^j}{\partial x^j} + \frac{\partial X^j}{\partial u} v_j + \frac{\partial X^j}{\partial v_k} v_{kj}\Big) s^i\\ - \Big(\frac{\partial X^i}{\partial x^j} + \frac{\partial X^i}{\partial u} v_j + \frac{\partial X^i}{\partial v_k} v_{kj}\Big) s^j + \frac{\partial X^i}{\partial u}\Big(s_j^j + \sum_{r=0}^{m-1} s^{ji_1\cdots i_r} v_{i_1\cdots i_r j}\Big)\\ - \frac{\partial X^j}{\partial u}\Big(s_j^i + \sum_{r=0}^{m-1} s^{ii_1\cdots i_r} v_{i_1\cdots i_r j}\Big)\Big]\mu_i + \Big(\frac{\partial X^j}{\partial v_k} s^i - \frac{\partial X^j}{\partial u} s^{ik}\Big)\sigma_k \wedge \mu_{ji}\end{aligned}$$

$$- \frac{\partial X^j}{\partial u}\Big(\sum_{r=2}^{m-1} s^{ii_1\cdots i_r}\sigma_{i_1\cdots i_r} \wedge \mu_{ji}\Big) - \frac{\partial X^j}{\partial u} s^{ii_1\cdots i_m} dv_{i_1\cdots i_m} \wedge \mu_{ji} \in \Lambda^{n-1}(\mathcal{K}_m),$$

$$\gamma^j = \Big[\frac{\partial S^i}{\partial v_j} + \Sigma\,\frac{\partial X^i}{\partial v_j} + \frac{\partial S^i}{\partial \Sigma^k} s^{kj} + \frac{\partial S^i}{\partial \Sigma}\, t^j + \Big(\frac{\partial X^k}{\partial x^k} + \frac{\partial X^k}{\partial u} v_k + \frac{\partial X^k}{\partial v_l} v_{lk}\Big) s^{ij}$$
$$- \Big(\frac{\partial X^i}{\partial x^k} + \frac{\partial X^i}{\partial u} v_k + \frac{\partial X^i}{\partial v_l} v_{lk}\Big)\, s^{kj} + \frac{\partial X^i}{\partial v_j}\Big(s^k_k + \sum_{r=0}^{m-1} s^{ki_1\cdots i_r} v_{i_1\cdots i_r k}\Big)$$
$$- \frac{\partial X^k}{\partial v_j}\Big(s^i_k + \sum_{r=0}^{m-1} s^{ii_1\cdots i_r} v_{i_1\cdots i_r k}\Big)\Big]\mu_i - \frac{\partial X^k}{\partial v_j} s^{il}\sigma_l \wedge \mu_{ki}$$
$$- \frac{\partial X^k}{\partial v_j}\Big(\sum_{r=2}^{m-1} s^{ii_1\cdots i_r}\sigma_{i_1\cdots i_r} \wedge \mu_{ki}\Big)$$
$$- \frac{\partial X^k}{\partial v_j} s^{ii_1\cdots i_m} dv_{i_1\cdots i_m} \wedge \mu_{ki} \in \Lambda^{n-1}(\mathcal{K}_m),$$

$$\lambda_j = \Big[\frac{\partial S^i}{\partial \Sigma^j} + \Big(\frac{\partial X^k}{\partial x^k} + \frac{\partial X^k}{\partial u}\, v_k + \frac{\partial X^k}{\partial v_l} v_{lk}\Big)\delta^i_j - \Big(\frac{\partial X^i}{\partial x^j} + \frac{\partial X^i}{\partial u}\, v_j$$
$$+ \frac{\partial X^i}{\partial v_k} v_{kj}\Big)\Big]\mu_i + \frac{\partial X^i}{\partial u}\sigma \wedge \mu_{ij} + \frac{\partial X^i}{\partial v_k}\sigma_k \wedge \mu_{ij} \in \Lambda^{n-1}(\mathcal{K}_m),$$

$$\gamma^{i_1\cdots i_r} = \Big[\frac{\partial S^i}{\partial v_{i_1\cdots i_r}} + \frac{\partial S^i}{\partial \Sigma^j} s^{ji_1\cdots i_r} + \frac{\partial S^i}{\partial \Sigma} t^{i_1\cdots i_r}$$
$$+ \Big(\frac{\partial X^j}{\partial x^j} + \frac{\partial X^j}{\partial u}\, v_j + \frac{\partial X^j}{\partial v_k} v_{kj}\Big) s^{ii_1\cdots i_r}$$
$$- \Big(\frac{\partial X^i}{\partial x^j} + \frac{\partial X^i}{\partial u}\, v_j + \frac{\partial X^i}{\partial v_k} v_{kj}\Big) s^{ji_1\cdots i_r}\Big]\mu_i \in \Lambda^{n-1}(\mathcal{K}_m), 2 \le r \le m-1,$$

$$\Gamma^{ii_1\cdots i_m} = \frac{\partial S^i}{\partial v_{i_1\cdots i_m}} + \frac{\partial S^i}{\partial \Sigma^j} s^{ji_1\cdots i_m} + \frac{\partial S^i}{\partial \Sigma} t^{i_1\cdots i_m}$$
$$+ \Big(\frac{\partial X^j}{\partial x^j} + \frac{\partial X^j}{\partial u}\, v_j + \frac{\partial X^j}{\partial v_k} v_{kj}\Big) s^{ii_1\cdots i_m}$$
$$- \Big(\frac{\partial X^i}{\partial x^j} + \frac{\partial X^i}{\partial u}\, v_j + \frac{\partial X^i}{\partial v_k} v_{kj}\Big) s^{ji_1\cdots i_m} \in \Lambda^0(\mathcal{K}_m).$$

To satisfy the equation (9.8.16), it would suffice now to select

$$A = \gamma - \nu\, s^i \mu_i, \qquad A^i = \gamma^i - \nu s^{ji}\mu_j + dx^i \wedge C,$$
$$B_i = \lambda_i - \nu\mu_i, \qquad B = \frac{\partial S^i}{\partial \Sigma}\mu_i,$$
$$A^{i_1\cdots i_r} = \gamma^{i_1\cdots i_r} - \nu s^{ii_1\cdots i_r}\mu_i + dx^{i_r} \wedge C^{i_1\cdots i_{r-1}},\ \ 0 \le r \le m-1$$

In this case, (9.8.16) reduces to

$$\Big[\Gamma - \nu\Big(\Sigma + s^i_i + \sum_{r=0}^{m-1} s^{ii_1\cdots i_r} v_{i_1\cdots i_r i}\Big)\Big]\mu \\ + \left(\Gamma^{ii_1\cdots i_m} - \nu s^{ii_1\cdots i_m}\right) dv_{i_1\cdots i_m} \wedge \mu_i \\ + dv_{i_1\cdots i_m} \wedge dx^{i_m} \wedge C^{i_1\cdots i_{m-1}} = 0.$$

If we introduce the forms

$$C^{i_1\cdots i_{m-1}} = C^{i_1\cdots i_{m-1}ij}\mu_{ij} \in \Lambda^{n-2}(\mathcal{K}_m), \\ C^{i_1\cdots i_{m-1}ij} = -\,C^{i_1\cdots i_{m-1}ji} \in \Lambda^0(\mathcal{K}_m)$$

we find

$$\begin{aligned} dv_{i_1\cdots i_m} \wedge dx^{i_m} \wedge C^{i_1\cdots i_{m-1}ij}\mu_{ij} \\ &= -\,dv_{i_1\cdots i_m} \wedge dx^{i_m} \wedge C^{i_1\cdots i_{m-1}ij}\mu_{ji} \\ &= -C^{i_1\cdots i_{m-1}ij} dv_{i_1\cdots i_m} \wedge dx^{i_m} \wedge \mu_{ij} \\ &= -C^{i_1\cdots i_{m-1}ij} dv_{i_1\cdots i_m} \wedge (\delta^{i_m}_j \mu_i - \delta^{i_m}_i \mu_j) \\ &= -\,2C^{i_1\cdots i_{m-1}ii_m} dv_{i_1\cdots i_m} \wedge \mu_i. \end{aligned}$$

We thus obtain

$$\Big[\Gamma - \nu\Big(\Sigma + s^i_i + \sum_{r=0}^{m-1} s^{ii_1\cdots i_r} v_{i_1\cdots i_r i}\Big)\Big]\mu \\ + (\Gamma^{ii_1\cdots i_m} - \nu s^{ii_1\cdots i_m} - 2C^{i_1\cdots i_{m-1}ii_m}) dv_{i_1\cdots i_m} \wedge \mu_i = 0$$

implying that the coefficients of independent forms μ and $dv_{i_1\cdots i_m} \wedge \mu_i$ must vanish. Obviously the antisymmetric parts of the terms $\Gamma^{ii_1\cdots i_m} - \nu s^{ii_1\cdots i_m}$ with respect to indices i and i_m will determine the coefficients $2C^{i_1\cdots i_{m-1}ii_m}$ that are antisymmetric in those two indices. Thus the symmetric parts of these terms with respect to the same indices should vanish. Hence, the determining equations for isovector components take eventually the forms below

$$\Gamma - \nu\Big(\Sigma + s^i_i + \sum_{r=0}^{m-1} s^{ii_1\cdots i_r} v_{i_1\cdots i_r i}\Big) = 0, \tag{9.8.17}$$
$$\Gamma^{ii_1\cdots i_m} + \Gamma^{i_m i_1\cdots i} - \nu\big(s^{ii_1\cdots i_m} + s^{i_m i_1\cdots i}\big) = 0.$$

We cannot extract the determining equations corresponding to the case $m = 1$ directly from (9.8.17). The relation

$$\pounds_V \omega = \nu\omega + \sigma \wedge A + d\sigma \wedge B + \Omega^i \wedge C_i + \Omega \wedge C$$

can now be written as

$$
\begin{aligned}
&\big[\Gamma-\nu(\Sigma+s^i_i+s^iv_i)\big]\mu+\sigma\wedge(\Gamma^i\mu_i-A-\nu\, s^i\mu_i)\\
&+\Omega^j\wedge(\Gamma^i_j\,\mu_i-C_j-\nu\delta^i_j\,\mu_i)+\Omega\wedge\Big(\frac{\partial S^i}{\partial\Sigma}\mu_i-C\Big)+\frac{\partial X^j}{\partial u}\Omega^i\wedge\sigma\wedge\mu_{ji}\\
&+\Big(\frac{\partial X^j}{\partial v_k}s^i-\frac{\partial X^j}{\partial u}s^{ik}\Big)\sigma\wedge dv_k\wedge\mu_{ji}+\frac{\partial X^j}{\partial v_k}\Omega^i\wedge dv_k\wedge\mu_{ji}\\
&+(\Gamma^{ij}-\nu s^{ij})\,dv_j\wedge\mu_i+dv_i\wedge dx^i\wedge B+\Gamma^{ijkl}dv_l\wedge dv_k\wedge\mu_{ji}=0
\end{aligned}
$$

where we defined

$$
\begin{aligned}
\Gamma&=T+\frac{\partial S^i}{\partial x^i}+\Sigma\frac{\partial X^i}{\partial x^i}+\Big(\frac{\partial S^i}{\partial u}+\Sigma\frac{\partial X^i}{\partial u}\Big)v_i+\frac{\partial S^i}{\partial\Sigma}(t_i+tv_i)\\
&\quad+\Big(\frac{\partial S^i}{\partial\Sigma^j}+\frac{\partial X^k}{\partial x^k}\delta^i_j-\frac{\partial X^i}{\partial x^j}\Big)\big(s^j_i+s^jv_i\big)+\frac{\partial X^j}{\partial u}\big(s^i_iv_j-s^i_jv_i\big),\\
\Gamma^{ij}&=\frac{\partial S^i}{\partial v_j}+\Sigma\frac{\partial X^i}{\partial v_j}+\Big(\frac{\partial S^i}{\partial\Sigma^k}+\frac{\partial X^l}{\partial x^l}\delta^i_k-\frac{\partial X^i}{\partial x^k}\Big)s^{kj}+\frac{\partial S^i}{\partial\Sigma}t^j\\
&\quad\frac{\partial X^k}{\partial v_j}\big(s^i_k+s^iv_k\big)+\frac{\partial X^i}{\partial v_j}\big(s^k_k+s^k\,v_k\big)+\Big(\frac{\partial X^k}{\partial u}s^{ij}-\frac{\partial X^i}{\partial u}s^{kj}\Big)v_k,\\
\Gamma^{ijkl}&=-\Gamma^{jikl}=-\Gamma^{ijlk}=\frac{1}{4}\Big(s^{il}\frac{\partial X^j}{\partial v_k}-s^{ik}\frac{\partial X^j}{\partial v_l}+s^{jk}\frac{\partial X^i}{\partial v_l}-s^{jl}\frac{\partial X^i}{\partial v_k}\Big)\\
&=\frac{1}{4}\,\delta^{ij}_{mn}\delta^{kl}_{pq}s^{mq}\frac{\partial X^n}{\partial v_p}.
\end{aligned}
$$

To combine the form $B\in\Lambda^{n-2}(\mathcal{K}_1)$ with other terms, we take it in the form $B=B^{jk}\mu_{jk}+B^{jklm}dv_m\wedge\mu_{jkl}$. Antisymmetry properties impose the restrictions $B^{jk}=-B^{kj}$, $B^{jklm}=B^{[jkl]m}\in\Lambda^0(\mathcal{K}_1)$. We thereby find

$$-dv_i\wedge dx^i\wedge B=2B^{ij}dv_j\wedge\mu_i+3B^{ijlk}dv_l\wedge dv_k\wedge\mu_{ji}.$$

On the other hand, by selecting

$$
\begin{aligned}
A&=(\Gamma^i-\nu\sigma^i)\mu_i+\Big(\frac{\partial X^j}{\partial v_k}s^i-\frac{\partial X^j}{\partial u}s^{ik}\Big)dv_k\wedge\mu_{ji},\\
C&=\frac{\partial S^i}{\partial\Sigma}\mu_i,\\
C_i&=(\Gamma^j_i-\nu\delta^j_i)\mu_j+\frac{\partial X^j}{\partial u}\sigma\wedge\mu_{ji}+\frac{\partial X^j}{\partial v_k}dv_k\wedge\mu_{ji},
\end{aligned}
$$

we arrive at the relation

$$\big[\Gamma - \nu(\Sigma + s^i_i + s^i v_i)\big]\mu + (\Gamma^{ij} - \nu s^{ij} - 2B^{ij})\, dv_j \wedge \mu_i + (\Gamma^{ijkl} - 3B^{ijkl}) dv_l \wedge dv_k \wedge \mu_{ji} = 0$$

which requires that $2B^{ij} = \Gamma^{[ij]} - \nu s^{[ij]}$ and $3B^{ijkl} = \Gamma^{i[jk]l}$. Hence, the determining equations are reduced to

$$\Gamma = \nu(\Sigma + s^i_i + \sigma^i\, v_i), \qquad \Gamma^{ij} + \Gamma^{ji} = \nu(s^{ij} + s^{ji}), \tag{9.8.18}$$
$$\Gamma^{ijkl} + \Gamma^{ikjl} = (\delta^{ij}_{mn}\delta^{kl}_{pq} + \delta^{ik}_{mn}\delta^{jl}_{pq}) s^{mq} \frac{\partial X^n}{\partial v_p} = 0.$$

The third set of equations in (9.8.18) govern the dependence of functions $X^i(x^j, u, v_j)$ on variables v_j. In fact, these equations imply that the coefficients of the variables s^{mq} vanish if only the relations

$$(\delta^{ij}_{mn}\delta^{kl}_{pq} + \delta^{ik}_{mn}\delta^{jl}_{pq}) \frac{\partial X^n}{\partial v_p} = 0$$

are satisfied. Contractions on indices (i, m) and (l, q) above yield

$$(n-1)^2(\delta^j_n\delta^k_p + \delta^k_n\delta^j_p) \frac{\partial X^n}{\partial v_p} = 0$$

whence we obtain the equations

$$\frac{\partial X^j}{\partial v_k} + \frac{\partial X^k}{\partial v_j} = 0$$

for $n > 1$. The solution of this set of partial differential equations can be written simply as

$$X^i = a^{ij}(x^k, u)v_j + b^i(x^k, u), \;\; a^{ij} = \,- a^{ji}.$$

We then see that a^{ij} must also satisfy the conditions

$$(\delta^{ij}_{mn}\delta^{kl}_{pq} + \delta^{ik}_{mn}\delta^{jl}_{pq}) a^{np} = 0$$

that can be expanded into

$$a^{ik}\delta^j_m\delta^l_q + a^{jl}\delta^i_m\delta^k_q - a^{il}\delta^j_m\delta^k_q + a^{ij}\delta^k_m\delta^l_q + a^{kl}\delta^i_m\delta^j_q - a^{il}\delta^k_m\delta^j_q = 0.$$

By contractions on the indices (k, m) and (l, q), we get

$$(n^2 - 1)a^{ij} = 0.$$

Therefore, we have to take $a^{ij} = 0$ for $n > 1$. Consequently, we finally reach to the conclusion $X^i = b^i(x^k, u)$.

For $m = 0$, the case $N = 1$ does not need a special care. Since all Greek indices are equal to 1, the determining equations are directly deduced from (9.8.12) as

$$\Gamma - \nu(\Sigma + s^i_i) = 0, \qquad G^i + \Gamma^i t - \nu s^i = 0.$$

We can easily reproduce the determining equations for the symmetry groups discussed in Sec. 9.4 from the determining equations for the equivalence groups. To this end, it suffices to note that in symmetry transformations $\Sigma^{\alpha i}$ and Σ^α $(N > 1)$, and their derivatives with respect to their arguments can no longer be chosen as independent variables as we have done in equivalence transformations. Therefore, the only surviving isovector components should be X^i and $V^\alpha_{i_1\cdots i_r}$, $0 \le r \le m$. Moreover, we have to keep in mind that the basis vectors $\partial/\partial x^i$ and $\partial/\partial v^\alpha_{i_1\cdots i_r}$, $0 \le r \le m$ appearing in (9.8.3) are calculated by holding $\Sigma^{\alpha i}$ and Σ^α as constants. On taking into notice of the functional forms of $\Sigma^{\alpha i}$ and Σ^α, we immediately conclude that we have to write by using the chain rule

$$\frac{\partial}{\partial x^i}\bigg|_{\Sigma^{\alpha i},\Sigma^\alpha} = \frac{\partial}{\partial x^i} - \frac{\partial \Sigma^{\alpha j}}{\partial x^i}\,\frac{\partial}{\partial \Sigma^{\alpha j}} - \frac{\partial \Sigma^\alpha}{\partial x^i}\,\frac{\partial}{\partial \Sigma^\alpha}$$

$$\frac{\partial}{\partial v^\alpha_{i_1\cdots i_r}}\bigg|_{\Sigma^{\alpha i},\Sigma^\alpha} = \frac{\partial}{\partial v^\alpha_{i_1\cdots i_r}} - \frac{\partial \Sigma^{\beta j}}{\partial v^\alpha_{i_1\cdots i_r}}\,\frac{\partial}{\partial \Sigma^{\beta j}} - \frac{\partial \Sigma^\beta}{\partial v^\alpha_{i_1\cdots i_r}}\,\frac{\partial}{\partial \Sigma^\beta},\ 0 \le r \le m.$$

Thus an isovector field is now expressible as

$$V = X^i\frac{\partial}{\partial x^i} + \sum_{r=0}^{m} V^\alpha_{i_1\cdots i_r}\frac{\partial}{\partial v^\alpha_{i_1\cdots i_r}} + \big[S^{\alpha i} - V(\Sigma^{\alpha i})\big]\frac{\partial}{\partial \Sigma^{\alpha i}} + \big[T^\alpha - V(\Sigma^\alpha)\big]\frac{\partial}{\partial \Sigma^\alpha}.$$

However, the coefficients of the vectors $\partial/\partial\Sigma^{\beta j}$ and $\partial/\partial\Sigma^\beta$ must vanish. Hence, it is obvious that we are compelled to take $S^{\alpha i} = V(\Sigma^{\alpha i})$ and $T^\alpha = V(\Sigma^\alpha)$. As is well-known, we define

$$V(f) = \frac{\partial f}{\partial x^i}X^i + \sum_{r=0}^{m}\frac{\partial f}{\partial v^\alpha_{i_1\cdots i_r}}V^\alpha_{i_1\cdots i_r}$$

for a function $f \in \Lambda^0(\mathcal{C}_m)$. If we also recall the definitions (9.8.1) and insert all we have found so far into the determining equations (9.8.11), we readily observe that we can recover the determining equations (9.4.11). The case $N = 1$ can be treated in exactly similar fashion. We find $S^i = V(\Sigma^i)$, $T =$

$V(\Sigma)$ and it is straightforward to verify that the equations (9.8.17) give rise to the equations (9.4.17).

We shall now try to determine the general solutions of the determining equations (9.8.11) or (9.8.17).

The case $N > 1$: The explicit form of the equations (9.8.11) can be written as

$$\begin{aligned} &T^\alpha + \frac{\partial S^{\alpha i}}{\partial x^i} + \Sigma^\beta\Big(\frac{\partial X^i}{\partial x^i}\delta^\alpha_\beta - \nu^\alpha_\beta\Big) + \Big(\frac{\partial S^{\alpha i}}{\partial u^\beta} + \Sigma^\alpha\frac{\partial X^i}{\partial u^\beta}\Big)v^\beta_i \\ &\quad + \sum_{r=1}^{m-1}\frac{\partial S^{\alpha i}}{\partial v^\beta_{i_1\cdots i_r}}v^\beta_{i_1\cdots i_r i} + \Big\{\frac{\partial S^{\alpha i}}{\partial \Sigma^{\beta j}} + \Big[\Big(\frac{\partial X^k}{\partial x^k} + \frac{\partial X^k}{\partial u^\gamma}v^\gamma_k\Big)\delta^i_j \\ &\quad - \Big(\frac{\partial X^i}{\partial x^j} + \frac{\partial X^i}{\partial u^\gamma}v^\gamma_j\Big)\Big]\delta^\alpha_\beta - \nu^\alpha_\beta\delta^i_j\Big\}\Big(s^{\beta j}_i + \sum_{r=0}^{m-1}s^{\beta j i_1\cdots i_r}_\gamma v^\gamma_{i_1\cdots i_r i}\Big) \\ &\quad + \frac{\partial S^{\alpha i}}{\partial \Sigma^\beta}\Big(t^\beta_i + \sum_{r=0}^{m-1}t^{\beta i_1\cdots i_r}_\gamma v^\gamma_{i_1\cdots i_r i}\Big) = 0 \\ &\Lambda^{\alpha i i_1\cdots i_{m-1} i_m}_\beta + \Lambda^{\alpha i_m i_1\cdots i_{m-1} i}_\beta = 0 \end{aligned} \tag{9.8.19}$$

Here, we have introduced the following functions

$$\begin{aligned} \Gamma^{\alpha i i_1\cdots i_m}_\beta - \nu^\alpha_\gamma s^{\gamma i i_1\cdots i_m}_\beta &= \frac{\partial S^{\alpha i}}{\partial v^\beta_{i_1\cdots i_m}} + \Big\{\frac{\partial S^{\alpha i}}{\partial \Sigma^{\gamma j}} + \Big[\Big(\frac{\partial X^k}{\partial x^k} + \frac{\partial X^k}{\partial u^\delta}v^\delta_k\Big)\delta^i_j \\ - \Big(\frac{\partial X^i}{\partial x^j} + \frac{\partial X^i}{\partial u^\delta}v^\delta_j\Big)\Big]\delta^\alpha_\gamma - \nu^\alpha_\gamma\delta^i_j\Big\}s^{\gamma j i_1\cdots i_m}_\beta &+ \frac{\partial S^{\alpha i}}{\partial \Sigma^\gamma}t^{\gamma i_1\cdots i_m}_\beta = \Lambda^{\alpha i i_1\cdots i_m}_\beta. \end{aligned}$$

If we carefully examine the equations $(9.8.19)_1$, we realise that the coefficients ν^α_β cannot depend on the variables $s^{\alpha i i_1\cdots i_m}_\beta$ and $t^{\alpha i_1\cdots i_m}_\beta$ whereas the equations $(9.8.19)_2$ imply that they are independent of the variables $s^{\alpha i}_j, t^\alpha_i$ and $\{s^{\alpha i i_1\cdots i_r}_\beta, t^{\alpha i_1\cdots i_r}_\beta, 0 \le r \le m-1\}$. We thus obtain

$$\nu^\alpha_\beta = \nu^\alpha_\beta\big(x^i, \{v^\gamma_{i_1\cdots i_r}, 0 \le r \le m\}, \Sigma^{\gamma i}, \Sigma^\gamma\big).$$

Therefore, on recalling arguments of $X^i, S^{\alpha i}$ and T^α, we realise at once that $(9.8.19)_1$ makes way for the following relations

$$\begin{aligned} &\frac{\partial S^{\alpha i}}{\partial \Sigma^\beta} = 0, \\ &\nu^\alpha_\beta\delta^i_j = \frac{\partial S^{\alpha i}}{\partial \Sigma^{\beta j}} + \Big[\Big(\frac{\partial X^k}{\partial x^k} + \frac{\partial X^k}{\partial u^\gamma}v^\gamma_k\Big)\delta^i_j - \Big(\frac{\partial X^i}{\partial x^j} + \frac{\partial X^i}{\partial u^\gamma}v^\gamma_j\Big)\Big]\delta^\alpha_\beta, \end{aligned} \tag{9.8.20}$$

$$T^\alpha + \frac{\partial S^{\alpha i}}{\partial x^i} + \Sigma^\beta\Big(\frac{\partial X^i}{\partial x^i}\delta^\alpha_\beta - \nu^\alpha_\beta\Big) + \Big(\frac{\partial S^{\alpha i}}{\partial u^\beta} + \Sigma^\alpha \frac{\partial X^i}{\partial u^\beta}\Big)v^\beta_i$$
$$+\sum_{r=1}^{m-1}\frac{\partial S^{\alpha i}}{\partial v^\beta_{i_1\cdots i_r}}v^\beta_{i_1\cdots i_r i} = 0.$$

Let us now define the functions

$$\mathfrak{S}^{\alpha i}_{\beta j} = \frac{\partial S^{\alpha i}}{\partial \Sigma^{\beta j}} - \Big(\frac{\partial X^i}{\partial x^j} + \frac{\partial X^i}{\partial u^\gamma}v^\gamma_j\Big)\delta^\alpha_\beta.$$

If we contract the indices i and j in $(9.8.20)_2$ and take into consideration the equations $(9.8.20)_1$, we reach to the relation

$$\nu^\alpha_\beta - \Big(\frac{\partial X^k}{\partial x^k} + \frac{\partial X^k}{\partial u^\gamma}v^\gamma_k\Big)\delta^\alpha_\beta = \frac{1}{n}\,\mathfrak{S}^{\alpha k}_{\beta k}$$
$$= f^\alpha_\beta(x^i, \{v^\gamma_{i_1\cdots i_r}, 0 \le r \le m\}, \Sigma^{\gamma i}).$$

where f^α_β are arbitrary functions of their arguments. Eventually, $(9.8.20)_2$ would lead to

$$\mathfrak{S}^{\alpha i}_{\beta j} - \frac{1}{n}\,\mathfrak{S}^{\alpha k}_{\beta k}\delta^i_j = 0 \quad \text{or} \quad \frac{\partial S^{\alpha i}}{\partial \Sigma^{\beta j}} = f^\alpha_\beta \delta^i_j + \Big(\frac{\partial X^i}{\partial x^j} + \frac{\partial X^i}{\partial u^\gamma}v^\gamma_j\Big)\delta^\alpha_\beta.$$

Differentiating the above expression with respect to $\Sigma^{\gamma k}$, we get

$$\frac{\partial^2 S^{\alpha i}}{\partial \Sigma^{\beta j}\partial \Sigma^{\gamma k}} = \frac{\partial f^\alpha_\beta}{\partial \Sigma^{\gamma k}}\delta^i_j = \frac{\partial f^\alpha_\gamma}{\partial \Sigma^{\beta j}}\delta^i_k.$$

The extreme right hand side in the foregoing relation arises from the symmetry of second order derivatives with respect to the variables $\Sigma^{\beta j}$ requiring that this expression must be invariant under interchanges (β,γ) and (j,k). A contraction on indices i and j yields

$$n\frac{\partial f^\alpha_\beta}{\partial \Sigma^{\gamma k}} = \frac{\partial f^\alpha_\gamma}{\partial \Sigma^{\beta k}}.$$

Hence, we can write

$$\frac{\partial f^\alpha_\beta}{\partial \Sigma^{\gamma k}}\delta^i_j = n\frac{\partial f^\alpha_\beta}{\partial \Sigma^{\gamma j}}\delta^i_k$$

and on contracting the indices i and k, we arrive at the result

$$(n^2-1)\frac{\partial f^\alpha_\beta}{\partial \Sigma^{\gamma j}} = 0.$$

Since $n > 1$ in partial differential equations, we conclude that the functions f^α_β must be independent of $\Sigma^{\beta j}$, that is, their explicit dependence should be given as follows

$$f^\alpha_\beta = f^\alpha_\beta(x^i, u^\gamma, v^\gamma_{i_1}, v^\gamma_{i_1 i_2}, \dots, v^\gamma_{i_1\cdots i_m}).$$

Then, we easily obtain

$$S^{\alpha i} = f^\alpha_\beta \Sigma^{\beta i} + \Big(\frac{\partial X^i}{\partial x^j} + \frac{\partial X^i}{\partial u^\beta} v^\beta_j\Big)\Sigma^{\alpha j} \\ + \chi^{\alpha i}(x^j, u^\beta, v^\beta_{i_1}, \dots, v^\beta_{i_1\cdots i_m}) \tag{9.8.21}$$

where $\chi^{\alpha i}$ are arbitrary functions. With these relations at hand, we find from $(9.8.20)_3$ that

$$T^\alpha = f^\alpha_\beta \Sigma^\beta - \frac{\partial S^{\alpha i}}{\partial x^i} - \sum_{r=0}^{m-1} \frac{\partial S^{\alpha i}}{\partial v^\beta_{i_1\cdots i_r}} v^\beta_{i_1\cdots i_r i} \tag{9.8.22}$$

On the other hand, we can write

$$\Lambda^{\alpha i i_1\cdots i_m}_\beta = \frac{\partial S^{\alpha i}}{\partial v^\beta_{i_1\cdots i_m}} = \frac{\partial f^\alpha_\gamma}{\partial v^\beta_{i_1\cdots i_m}}\Sigma^{\gamma i} + \frac{\partial \sigma^{\alpha i}}{\partial v^\beta_{i_1\cdots i_m}}.$$

where we assume that $m \neq 1$. Then $(9.8.19)_2$ gives

$$\frac{\partial f^\alpha_\gamma}{\partial v^\beta_{i_1\cdots i_m}}\Sigma^{\gamma i} + \frac{\partial f^\alpha_\gamma}{\partial v^\beta_{i_1\cdots i_{m-1} i}}\Sigma^{\gamma i_m} = 0,\ \frac{\partial \chi^{\alpha i}}{\partial v^\beta_{i_1\cdots i_m}} + \frac{\partial \chi^{\alpha i_m}}{\partial v^\beta_{i_1\cdots i_{m-1} i}} = 0$$

On differentiating the first set of equations with respect to $\Sigma^{\delta k}$, we get

$$\frac{\partial f^\alpha_\gamma}{\partial v^\beta_{i_1\cdots i_m}}\delta^i_k + \frac{\partial f^\alpha_\gamma}{\partial v^\beta_{i_1\cdots i_{m-1} i}}\delta^{i_m}_k = 0 \quad \text{or} \quad (n+1)\frac{\partial f^\alpha_\gamma}{\partial v^\beta_{i_1\cdots i_m}} = 0$$

where we have performed contraction on the indices i and k. Thus, the functions f^α_β possess the following form

$$f^\alpha_\beta = f^\alpha_\beta(x^i, u^\gamma, v^\gamma_{i_1}, v^\gamma_{i_1 i_2}, \dots, v^\gamma_{i_1\cdots i_{m-1}}).$$

If we recall that the variables $v^\alpha_{i_1\cdots i_m}$ are completely symmetric in their subscripts, the partial differential equations satisfied by the functions $\chi^{\alpha i}$

may be cast into the form

$$\frac{\partial \chi^{\alpha i}}{\partial v^{\beta}_{j i_1 \cdots i_{m-1}}} + \frac{\partial \chi^{\alpha j}}{\partial v^{\beta}_{i i_1 \cdots i_{m-1}}} = 0. \tag{9.8.23}$$

The number of variables in this system is $N\binom{n+m-1}{m}$ whereas the number of equations is $N^2\frac{n(n+1)}{2}\binom{n+m-2}{m-1}$. Hence, the number of equations is larger by a factor $N\frac{mn(n+1)}{2(n+m-1)}$ than the number of variables. In order to find the solution of (9.8.23), let us start by taking $i = j$ to get

$$\frac{\partial \chi^{\alpha \underline{i}}}{\partial v^{\beta}_{\underline{i} i_1 \cdots i_{m-1}}} = 0$$

for all $1 \le i, i_1, \cdots, i_{m-1} \le n$. Let us recall that the summation convention will be suspended on underlined indices. The above equations mean that the functions $\chi^{\alpha i}$ cannot depend on $v^{\beta}_{i i_1 \cdots i_{m-1}}$. To simplify the notation, let us introduce the sets $\mathbf{v}_{i i_1 \cdots i_{m-1}} = \{v^1_{\underline{i} i_1 \cdots i_{m-1}}, \ldots, v^N_{\underline{i} i_1 \cdots i_{m-1}}\}$ and $\mathbf{W}_{i_1 \cdots i_{m-1}} = \{\mathbf{v}_{1 i_1 \cdots i_{m-1}}, \mathbf{v}_{2 i_1 \cdots i_{m-1}}, \ldots, \mathbf{v}_{n i_1 \cdots i_{m-1}}\}$. We can thus write

$$\chi^{\alpha i} = f^{\alpha i}(\mathbf{W}_{i_1 \cdots i_{m-1}} \backslash \{\mathbf{v}_{\underline{i} i_1 \cdots i_{m-1}}\}), \ 1 \le \alpha \le N$$

where the symbol $\backslash$ denotes the set difference. For notational simplicity, we omit the dependence of $f^{\alpha i}$ on the variables $x^i, u^\alpha, v^\alpha_{i_1}, v^\alpha_{i_1 i_2}, \ldots, v^\alpha_{i_1 \cdots i_{m-1}}$. We now differentiate the equations (9.8.23) with respect to $v^{\gamma}_{j j_1 \cdots j_{m-1}}$ with $j \ne i$ to obtain

$$\frac{\partial^2 \chi^{\alpha i}}{\partial v^{\beta}_{\underline{j} i_1 \cdots i_{m-1}} \partial v^{\gamma}_{\underline{j} j_1 \cdots j_{m-1}}} + \frac{\partial^2 \chi^{\alpha \underline{j}}}{\partial v^{\beta}_{i i_1 \cdots i_{m-1}} \partial v^{\gamma}_{\underline{j} j_1 \cdots j_{m-1}}} = 0.$$

Because of the relations $\partial \chi^{\alpha \underline{j}} / \partial v^{\gamma}_{\underline{j} j_1 \cdots j_{m-1}} = 0$, we find

$$\frac{\partial^2 \chi^{\alpha i}}{\partial v^{\beta}_{\underline{j} i_1 \cdots i_{m-1}} \partial v^{\gamma}_{\underline{j} j_1 \cdots j_{m-1}}} = 0$$

implying that

$$\frac{\partial \chi^{\alpha i}}{\partial v^{\beta}_{j i_1 \cdots i_{m-1}}} = f^{\alpha i j i_1 \cdots i_{m-1}}_{\beta}(\mathbf{W}_{i_1 \cdots i_{m-1}} \backslash \{\mathbf{v}_{i i_1 \cdots i_{m-1}}, \mathbf{v}_{j i_1 \cdots i_{m-1}}\}).$$

Obviously, the functions $f^{\alpha i j i_1 \cdots i_{m-1}}_{\beta}$ are subject to the restrictions

$$f_{\beta}^{\alpha i j i_1\cdots i_{m-1}} = -\, f_{\beta}^{\alpha j i i_1\cdots i_{m-1}}.$$

On the other hand, because of the symmetry of mixed derivatives, we have

$$\frac{\partial^2 \chi^{\alpha i}}{\partial v^{\beta}_{j i_1\cdots i_{m-1}} \partial v^{\gamma}_{k j_1\cdots j_{m-1}}} = \frac{\partial f_{\beta}^{\alpha i j i_1\cdots i_{m-1}}}{\partial v^{\gamma}_{k j_1\cdots j_{m-1}}} = \frac{\partial f_{\gamma}^{\alpha i k j_1\cdots j_{m-1}}}{\partial v^{\beta}_{j i_1\cdots i_{m-1}}}.$$

Since the functions $f_{\gamma}^{\alpha i k j_1\cdots j_{m-1}}$ cannot depend on $\mathbf{v}_{k j_1\cdots j_{m-1}}$, we get at once

$$\frac{\partial^2 f_{\beta}^{\alpha i j i_1\cdots i_{m-1}}}{\partial v^{\gamma}_{\underline{k} j_1\cdots j_{m-1}} \partial v^{\delta}_{\underline{k} k_1\cdots k_{m-1}}} = 0$$

leading to the relations

$$\frac{\partial^2 \chi^{\alpha i}}{\partial v^{\beta}_{j i_1\cdots i_{m-1}} \partial v^{\gamma}_{k j_1\cdots j_{m-1}}} = \frac{\partial f_{\beta}^{\alpha i j i_1\cdots i_{m-1}}}{\partial v^{\gamma}_{k j_1\cdots j_{m-1}}} =$$
$$f_{\beta\gamma}^{\alpha i j k i_1\cdots i_{m-1} j_1\cdots j_{m-1}}(\mathbf{W}_{i_1\cdots i_{m-1}} \backslash \{\mathbf{v}_{i i_1\cdots i_{m-1}}, \mathbf{v}_{j i_1\cdots i_{m-1}}, \mathbf{v}_{k i_1\cdots i_{m-1}}\}),\ i \neq j \neq k$$

where the functions $f_{\beta\gamma}^{\alpha i j k i_1\cdots i_{m-1} j_1\cdots j_{m-1}}$ must satisfy the following symmetry conditions

$$f_{\beta\gamma}^{\alpha i j k i_1\cdots i_{m-1} j_1\cdots j_{m-1}} = -\, f_{\beta\gamma}^{\alpha j i k i_1\cdots i_{m-1} j_1\cdots j_{m-1}} = f_{\gamma\beta}^{\alpha i k j i_1\cdots i_{m-1} j_1\cdots j_{m-1}}.$$

Continuing this way, we can readily reach to the recurrence relations

$$\frac{\partial^{r+1} \chi^{\alpha i}}{\partial v^{\alpha_1}_{i_1 i_1^{(1)}\cdots i_{m-1}^{(1)}} \cdots \partial v^{\alpha_{r+1}}_{i_{r+1} i_1^{(r+1)}\cdots i_{m-1}^{(r+1)}}} = \frac{\partial f_{\alpha_1\cdots\alpha_r}^{\alpha i i_1\cdots i_r i_1^{(1)}\cdots i_{m-1}^{(1)}\cdots i_1^{(r)}\cdots i_{m-1}^{(r)}}}{\partial v^{\alpha_{r+1}}_{i_{r+1} i_1^{(r+1)}\cdots i_{m-1}^{(r+1)}}} =$$
$$f_{\alpha_1\cdots\alpha_{r+1}}^{\alpha i i_1\cdots i_{r+1} i_1^{(1)}\cdots i_{m-1}^{(1)}\cdots i_1^{(r+1)}\cdots i_{m-1}^{(r+1)}}(\mathbf{W}_{j_1\cdots j_{m-1}} \backslash \{\mathbf{v}_{i \underline{j}_1\cdots j_{m-1}}, \mathbf{v}_{i_1 \underline{j}_1\cdots j_{m-1}}, \ldots, \mathbf{v}_{i_{r+1} \underline{j}_1\cdots j_{m-1}}\})$$

whence by taking $r = n-2$ we draw the conclusion

$$\frac{\partial^{n-1} \chi^{\alpha i}}{\partial v^{\alpha_1}_{i_1 i_1^{(1)}\cdots i_{m-1}^{(1)}} \cdots \partial v^{\alpha_{n-1}}_{i_{n-1} i_1^{(n-1)}\cdots i_{m-1}^{(n-1)}}} = f_{\alpha_1\cdots\alpha_{n-1}}^{\alpha i i_1\cdots i_{n-1} i_1^{(1)}\cdots i_{m-1}^{(1)}\cdots i_1^{(n-1)}\cdots i_{m-1}^{(n-1)}}(\emptyset)$$

since $\mathbf{W}_{j_1\cdots j_{m-1}} \backslash \{\mathbf{v}_{i \underline{j}_1\cdots j_{m-1}}, \mathbf{v}_{i_1 \underline{j}_1\cdots j_{m-1}}, \ldots, \mathbf{v}_{i_{n-1} \underline{j}_1\cdots j_{m-1}}\} = \emptyset$. This is tantamount to say that the functions $f_{\alpha_1\cdots\alpha_{n-1}}^{\alpha i i_1\cdots i_{n-1} i_1^{(1)}\cdots i_{m-1}^{(1)}\cdots i_1^{(n-1)}\cdots i_{m-1}^{(n-1)}}$ are independent of the variables $v^{\alpha}_{i_1 i_2\cdots i_m}$. Thus it becomes rather straightforward to integrate those hierarchical system of partial differential equations for $\chi^{\alpha i}$ in the backward direction starting from the last equations above. We then readily obtain the polynomial expressions

$$
\begin{aligned}
\chi^{\alpha i} &= \sum_{k=1}^{n-1} f_{\alpha_1\cdots\alpha_k}^{\alpha i i_1\cdots i_k j_1^{(1)}\cdots j_{m-1}^{(1)}\cdots j_1^{(k)}\cdots j_{m-1}^{(k)}} v^{\alpha_1}_{i_1 j_1^{(1)}\cdots j_{m-1}^{(1)}} \cdots v^{\alpha_k}_{i_k j_1^{(k)}\cdots j_{m-1}^{(k)}} + f^{\alpha i} \qquad (9.8.24)\\
&= f^{\alpha i} + f_{\alpha_1}^{\alpha i i_1 j_1^{(1)}\cdots j_{m-1}^{(1)}} v^{\alpha_1}_{i_1 j_1^{(1)}\cdots j_{m-1}^{(1)}} \\
&\qquad + f_{\alpha_1\alpha_2}^{\alpha i i_1 i_2 j_1^{(1)}\cdots j_{m-1}^{(1)} j_1^{(2)}\cdots j_{m-1}^{(2)}} v^{\alpha_1}_{i_1 j_1^{(1)}\cdots j_{m-1}^{(1)}} v^{\alpha_2}_{i_2 j_1^{(2)}\cdots j_{m-1}^{(2)}} \\
&\quad + f_{\alpha_1\alpha_2\alpha v^{\alpha_1}_{i_1 j_1^{(1)}\cdots j_{m-1}^{(1)}} \cdots v_{i_{n-1} j_1^{(n-1)}\cdots j_{m-1}^{(n-1)}} 3}^{\alpha i i_1 i_2 i_3 j_1^{(1)}\cdots j_{m-1}^{(1)} j_1^{(2)}\cdots j_{m-1}^{(2)} j_1^{(3)}\cdots j_{m-1}^{(3)}} v^{\alpha_1}_{i_1 j_1^{(1)}\cdots j_{m-1}^{(1)}} v^{\alpha_2}_{i_2 j_1^{(2)}\cdots j_{m-1}^{(2)}} v^{\alpha_3}_{i_3 j_1^{(3)}\cdots j_{m-1}^{(3)}} \\
&\qquad + \cdots + f_{\alpha_1\cdots\alpha_{n-1}}^{\alpha i i_1\cdots i_{n-1} j_1^{(1)}\cdots j_{m-1}^{(1)}\cdots j_1^{(n-1)}\cdots j_{m-1}^{(n-1)}} v^{\alpha_1}_{i_1 j_1^{(1)}\cdots j_{m-1}^{(1)}} \cdots v^{\alpha_{n-1}}_{i_{n-1} j_1^{(n-1)}\cdots j_{m-1}^{(n-1)}}
\end{aligned}
$$

The functions $f_{\alpha_1\cdots\alpha_k}^{\alpha i i_1\cdots i_k j_1^{(1)}\cdots j_{m-1}^{(1)}\cdots j_1^{(k)}\cdots j_{m-1}^{(k)}}$ where $1 \le k \le n-1$ and $f^{\alpha i}$ are arbitrary and depend only on the variables $x^i, u^\alpha,\ v^\alpha_{i_1}, v^\alpha_{i_1 i_2}, \ldots, v^\alpha_{i_1\cdots i_{m-1}}$. We can easily verify that the functions $f_{\alpha_1\cdots\alpha_r}^{\alpha i i_1\cdots i_r j_1^{(1)}\cdots j_{m-1}^{(1)}\cdots j_1^{(k)}\cdots j_{m-1}^{(k)}}$ where $r = 1, \ldots, n-1$ enjoy several symmetry requirements. There are antisymmetry with respect to first two roman superscripts and complete symmetry within the groups of indices, $(i_1, j_1^{(1)}, \cdots, j_{m-1}^{(1)}), \ldots, (i_{n-1}, j_1^{(n-1)}, \cdots, j_{m-1}^{(n-1)})$. Furthermore, we immediately observe that block symmetries with respect to the groups of indices $(\alpha_l, i_l, j_1^{(l)}, \cdots, j_{m-1}^{(l)}), \ldots\ , (\alpha_k, i_k, j_1^{(k)}, \cdots, j_{m-1}^{(k)})$ must be obeyed.

We thus see that all components of the isovector fields characterising equivalence transformations of balance equations are determined by means of arbitrary functions $X^i, U^\alpha, f^\alpha_\beta, f^{\alpha i}$ and $f_{\alpha_1\cdots\alpha_k}^{\alpha i i_1\cdots i j_1^{(1)}\cdots j_{m-1}^{(1)}\cdots j_1^{(k)}\cdots j_{m-1}^{(k)}}$ where $1 \le k \le n-1$ depending on certain coordinates of $\mathcal{K}_m$ through the relations (9.8.7), (9.8.21), (9.8.22) and (9.8.24). When $\Sigma^{\alpha i}$ and/or Σ^α are independent of some coordinates, the components of isovector fields corresponding to them must of course vanish. That kind of restrictions removes naturally to some extent the arbitrariness in the determining functions X^i, $U^\alpha, f^\alpha_\beta, \chi^{\alpha i}$.

When $m = 1$, namely, when we take into account second order balance equations, we have to modify slightly the previous analysis. In this case, we have again

$$
\Lambda^{\alpha i j}_\beta = \Gamma^{\alpha i j}_\beta - \nu^\alpha_\gamma s^{\gamma i j}_\beta = \frac{\partial S^{\alpha i}}{\partial v^\beta_j}
$$

But, this time, (9.8.21) yields

$$\Lambda_\beta^{\alpha ij} = \frac{\partial f_\gamma^\alpha}{\partial v_j^\beta}\Sigma^{\gamma i} + \frac{\partial X^i}{\partial u^\beta}\Sigma^{\alpha j} + \frac{\partial \chi^{\alpha i}}{\partial v_j^\beta}$$

Hence, the solution of the equations $\Lambda_\beta^{\alpha ij} + \Lambda_\beta^{\alpha ji} = 0$ is found has a distinct structure from above as

$$f_\beta^\alpha(x^i, u^\gamma, v_j^\gamma) = -\frac{\partial X^i}{\partial u^\gamma} v_i^\gamma \delta_\beta^\alpha + g_\beta^\alpha(x^i, u^\gamma)$$

where g_β^α are arbitrary functions. The solution of the equations

$$\frac{\partial \chi^{\alpha i}}{\partial v_j^\beta} + \frac{\partial \chi^{\alpha j}}{\partial v_i^\beta} = 0$$

can be extracted from the foregoing general solution as follows

$$\chi^{\alpha i} = \sum_{k=1}^{n-1} f_{\alpha_1\cdots\alpha_k}^{\alpha i i_1\cdots i_k}(\mathbf{x},\mathbf{u}) v_{i_1}^{\alpha_1}\cdots v_{i_k}^{\alpha_k} + f^{\alpha i}(\mathbf{x},\mathbf{u}).$$

Thus, the relevant isovector components are found as

$$\begin{aligned}
X^i &= X^i(\mathbf{x},\mathbf{u}),\ \ U^\alpha = U^\alpha(\mathbf{x},\mathbf{u}),\\
V_i^\alpha &= \frac{\partial U^\alpha}{\partial x^i} - \frac{\partial X^j}{\partial x^i} v_j^\alpha + \frac{\partial U^\alpha}{\partial u^\beta} v_i^\beta - \frac{\partial X^j}{\partial u^\beta} v_j^\alpha v_i^\beta,\\
S^{\alpha i} &= g_\beta^\alpha \Sigma^{\beta i} + \left(\frac{\partial X^i}{\partial x^j} + \frac{\partial X^i}{\partial u^\beta} v_j^\beta\right)\Sigma^{\alpha j} - \frac{\partial X^j}{\partial u^\beta} v_j^\beta \Sigma^{\alpha i} + \chi^{\alpha i},\\
T^\alpha &= g_\beta^\alpha \Sigma^\beta - \frac{\partial X^i}{\partial u^\gamma} v_i^\gamma \Sigma^\alpha - \frac{\partial g_\beta^\alpha}{\partial x^i}\Sigma^{\beta i} - \frac{\partial g_\beta^\alpha}{\partial u^\gamma} v_i^\gamma \Sigma^{\beta i}\\
&\quad - \left(\frac{\partial^2 X^i}{\partial x^i \partial x^j} + \frac{\partial^2 X^i}{\partial x^i \partial u^\beta} v_j^\beta\right)\Sigma^{\alpha j} - \frac{\partial \chi^{\alpha i}}{\partial x^i} - \frac{\partial \chi^{\alpha i}}{\partial u^\beta} v_i^\beta.
\end{aligned} \tag{9.8.25}$$

If $m = 0$, that is, when we consider first order balance equations, then we have to search for the solution of the equations (9.8.12). The equations $(9.8.12)_3$ lead to

$$\frac{\partial X^j}{\partial u^\beta} = 0,\ \ \frac{\partial X^j}{\partial \Sigma^{\delta k}} = 0,\ \ \frac{\partial X^j}{\partial \Sigma^\delta} = 0.$$

We thus get $X^i = X^i(\mathbf{x})$. The form of the equations $(9.8.12)_1$ reflects the fact that the functions ν_β^α must be independent of the variables $s_\beta^{\alpha i}$. Hence, the relations $(9.8.12)_2$ reduce to the equations

$$\frac{\partial S^{\alpha i}}{\partial u^\beta}+\frac{\partial S^{\alpha i}}{\partial \Sigma^\gamma}t^\gamma_\beta+\Big(\frac{\partial S^{\alpha i}}{\partial \Sigma^{\gamma j}}+\frac{\partial X^k}{\partial x^k}\delta^\alpha_\gamma\delta^i_j-\frac{\partial X^i}{\partial x^j}\delta^\alpha_\gamma-\nu^\alpha_\gamma\delta^i_j\Big)s^{\gamma j}_\beta=0$$

the satisfaction of which requires that

$$\frac{\partial S^{\alpha i}}{\partial u^\beta}=0,\ \frac{\partial S^{\alpha i}}{\partial \Sigma^\beta}=0,\ \nu^\alpha_\gamma\delta^i_j=\frac{\partial S^{\alpha i}}{\partial \Sigma^{\gamma j}}+\Big(\frac{\partial X^k}{\partial x^k}\delta^i_j-\frac{\partial X^i}{\partial x^j}\Big)\delta^\alpha_\gamma.$$

We therefore find $S^{\alpha i}=S^{\alpha i}(x^j,\Sigma^{\beta j})$ and, consequently, $\nu^\alpha_\beta=\nu^\alpha_\beta(x^j,\Sigma^{\gamma j})$. Similar to the approach employed previously, we immediately obtain after some calculations

$$\frac{\partial S^{\alpha i}}{\partial \Sigma^{\beta j}}=f^\alpha_\beta(x^k,\Sigma^{\gamma k})\,\delta^i_j+\frac{\partial X^i}{\partial x^j}\delta^\alpha_\beta$$

from which we deduce that $\partial f^\alpha_\beta/\partial\Sigma^{\gamma j}=0$ for $n>1$ by considering second order derivatives of functions $S^{\alpha i}$ with respect to the variables $\Sigma^{\gamma k}$. This of course means that $f^\alpha_\beta=f^\alpha_\beta(\mathbf{x})$. We then obtain $S^{\alpha i}$ by simply integrating the above equations. If we introduce these expressions into $(9.8.12)_1$, we get the isovector components T^α. The results so obtained are listed below:

$$\begin{aligned}
X^i&=X^i(\mathbf{x}),\\
U^\alpha&=U^\alpha(\mathbf{x},\mathbf{u},\Sigma^{\alpha i},\Sigma^\alpha),\\
S^{\alpha i}&=f^\alpha_\beta(\mathbf{x})\Sigma^{\beta i}+\frac{\partial X^i}{\partial x^j}\Sigma^{\alpha j}+g^{\alpha i}(\mathbf{x}),\\
T^\alpha&=f^\alpha_\beta(\mathbf{x})\Sigma^\beta-\frac{\partial f^\alpha_\beta(\mathbf{x})}{\partial x^i}\Sigma^{\beta i}-\frac{\partial^2 X^i}{\partial x^i\partial x^j}\Sigma^{\alpha j}-\frac{\partial g^{\alpha i}(\mathbf{x})}{\partial x^i}
\end{aligned}\tag{9.8.26}$$

where $X^i,U^\alpha,f^\alpha_\beta$ and $g^{\alpha i}(\mathbf{x})$ are arbitrary functions.

The case $N=1$: In the case of only one dependent variable, we have to look for the solution of the equations (9.8.17). The explicit form of these equations are given below

$$\begin{aligned}
&T+\frac{\partial S^i}{\partial x^i}+\frac{\partial S^i}{\partial u}v_i+\frac{\partial S^i}{\partial v_j}v_{ji}+\sum_{r=2}^{m-1}\frac{\partial S^i}{\partial v_{i_1\cdots i_r}}v_{i_1\cdots i_r i}\\
&\quad+\Sigma\Big(\frac{\partial X^i}{\partial x^i}+\frac{\partial X^i}{\partial u}v_i+\frac{\partial X^i}{\partial v_j}v_{ji}-\nu\Big)+\Big[\frac{\partial S^i}{\partial \Sigma^j}+\Big(\frac{\partial X^k}{\partial x^k}+\frac{\partial X^k}{\partial u}v_k\\
&+\frac{\partial X^k}{\partial v_l}v_{lk}-\nu\Big)\delta^i_j-\Big(\frac{\partial X^i}{\partial x^j}+\frac{\partial X^i}{\partial u}v_j+\frac{\partial X^i}{\partial v_k}v_{kj}\Big)\Big]\Big(s^j_i+\sum_{r=0}^{m-1}s^{ji_1\cdots i_r}v_{i_1\cdots i_r i}\Big)
\end{aligned}\tag{9.8.27}$$

$$+\frac{\partial S^i}{\partial \Sigma}\Big(t_i + \sum_{r=0}^{m-1} t^{i_1\cdots i_r} v_{i_1\cdots i_r i}\Big) = 0$$

$$\Lambda^{ii_1\cdots i_{m-1}i_m} + \Lambda^{i_m i_1\cdots i_{m-1}i} = 0$$

where we have defined

$$\begin{aligned}\Lambda^{ii_1\cdots i_m} &= \frac{\partial S^i}{\partial v_{i_1\cdots i_m}} + \Big[\frac{\partial S^i}{\partial \Sigma^j} + \Big(\frac{\partial X^k}{\partial x^k} + \frac{\partial X^k}{\partial u} v_k + \frac{\partial X^k}{\partial v_l} v_{lk} - \nu\Big)\delta^i_j \\ &\quad - \Big(\frac{\partial X^i}{\partial x^j} + \frac{\partial X^i}{\partial u} v_j + \frac{\partial X^i}{\partial v_k} v_{kj}\Big)\Big] s^{ji_1\cdots i_m} + \frac{\partial S^i}{\partial \Sigma} t^{i_1\cdots i_m} \\ &= \Gamma^{ii_1\cdots i_m} - \nu s^{ii_1\cdots i_m.}\end{aligned}$$

When we carefully scrutinise the equations (9.8.27), we realise that ν is independent of the variables s_i^j, t_i, $s^{ii_1\cdots i_m}$ and $t^{i_1\cdots i_m}$ so that one writes

$$\nu = \nu\big(x^i, \{v_{i_1\cdots i_r}, 0 \le r \le m\}, \Sigma^i, \Sigma\big).$$

It then follows from $(9.8.27)_1$ that

$$\frac{\partial S^i}{\partial \Sigma} = 0, \qquad (9.8.28)$$

$$\nu\delta^i_j = \frac{\partial S^i}{\partial \Sigma^j} + \Big(\frac{\partial X^k}{\partial x^k} + \frac{\partial X^k}{\partial u} v_k + \frac{\partial X^k}{\partial v_l} v_{lk}\Big)\delta^i_j - \Big(\frac{\partial X^i}{\partial x^j} + \frac{\partial X^i}{\partial u} v_j + \frac{\partial X^i}{\partial v_k} v_{kj}\Big),$$

$$\begin{aligned}T &+ \frac{\partial S^i}{\partial x^i} + \frac{\partial S^i}{\partial u} v_i + \frac{\partial S^i}{\partial v_j} v_{ji} + \sum_{r=2}^{m-1} \frac{\partial S^i}{\partial v_{i_1\cdots i_r}} v_{i_1\cdots i_r i} \\ &+ \Sigma\Big(\frac{\partial X^i}{\partial x^i} + \frac{\partial X^i}{\partial u} v_i + \frac{\partial X^i}{\partial v_j} v_{ji} - \nu\Big) = 0.\end{aligned}$$

Hence, the functions S^i are independent of Σ. With the definition

$$\mathfrak{S}^i_j = \frac{\partial S^i}{\partial \Sigma^j} - \Big(\frac{\partial X^i}{\partial x^j} + \frac{\partial X^i}{\partial u} v_j + \frac{\partial X^i}{\partial v_k} v_{kj}\Big)$$

we deduce from $(9.8.27)_2$ that

$$\nu - \Big(\frac{\partial X^i}{\partial x^i} + \frac{\partial X^i}{\partial u} v_i + \frac{\partial X^i}{\partial v_j} v_{ji}\Big) = \frac{1}{n}\,\mathfrak{S}^i_i = f$$

where $f(x^i, u, v_{i_1}, \dots, v_{i_1\cdots i_m}, \Sigma^i)$ is an arbitrary function. Thus $(9.8.28)_2$ takes the form

$$\frac{\partial S^i}{\partial \Sigma^j} = f\delta^i_j + \Big(\frac{\partial X^i}{\partial x^j} + \frac{\partial X^i}{\partial u}v_j + \frac{\partial X^i}{\partial v_k}v_{kj}\Big) \tag{9.8.29}$$

whence we get

$$\frac{\partial^2 S^i}{\partial \Sigma^j \partial \Sigma^m} = \frac{\partial f}{\partial \Sigma^m}\delta^i_j = \frac{\partial f}{\partial \Sigma^j}\delta^i_m$$

and consequently

$$(n-1)\frac{\partial f}{\partial \Sigma^m} = 0.$$

This equation signifies that the function f is independent of Σ^i if $n > 1$. Then, the integration of the simple partial differential equations (9.8.29) leads to the result

$$S^i = f\Sigma^i + \Big(\frac{\partial X^i}{\partial x^j} + \frac{\partial X^i}{\partial u}v_j + \frac{\partial X^i}{\partial v_k}v_{kj}\Big)\Sigma^j + \chi^i \tag{9.8.30}$$

where $\chi^i(x^j, u, v_{i_1}, \dots, v_{i_1\cdots i_m})$ are arbitrary functions. Hence, the functions $\Lambda^{ii_1\cdots i_m}$ can be written in the form

$$\Lambda^{ii_1\cdots i_m} = \frac{\partial S^i}{\partial v_{i_1\cdots i_m}} = \frac{\partial f}{\partial v_{i_1\cdots i_m}}\Sigma^i + \frac{\partial \chi^i}{\partial v_{i_1\cdots i_m}}$$

and the equations $(9.8.27)_2$ lead to the conclusion

$$\frac{\partial f}{\partial v_{i_1\cdots i_m}}\Sigma^i + \frac{\partial f}{\partial v_{i_1\cdots i}}\Sigma^{i_m} + \frac{\partial \chi^i}{\partial v_{i_1\cdots i_m}} + \frac{\partial \chi^{i_m}}{\partial v_{i_1\cdots i}} = 0.$$

Therefrom, we easily arrive at the partial differential equations

$$\frac{\partial f}{\partial v_{ii_1\cdots i_{m-1}}} = 0, \quad \frac{\partial \chi^i}{\partial v_{ji_1\cdots i_{m-1}}} + \frac{\partial \chi^j}{\partial v_{ii_1\cdots i_{m-1}}} = 0.$$

Hence, we have $f = f(x^i, u, v_{i_1}, \dots, v_{i_1\cdots i_{m-1}})$ and the integration of the set of equations for χ^i yield

$$\chi^i = \alpha^{iji_1\cdots i_{m-1}}\, v_{ji_1\cdots i_{m-1}} + \beta^i, \quad \alpha^{iji_1\cdots i_{m-1}} = -\,\alpha^{jii_1\cdots i_{m-1}}$$

where the functions $\alpha^{iji_1\cdots i_{m-1}}$ and β^i depend on the variables $x^i, u, v_{i_1}, \dots,$ $v_{i_1\cdots i_{m-1}}$. $\alpha^{iji_1\cdots i_{m-1}}$ are completely symmetric with respect to indices $j, i_1,$ $i_2, \dots, i_{m-1}$. Finally, we obtain from $(9.8.28)_3$ that

$$T = f\Sigma - \frac{\partial S^i}{\partial x^i} - \sum_{r=0}^{m-1} \frac{\partial S^i}{\partial v_{i_1\cdots i_r}} v_{i_1\cdots i_r i}. \tag{9.8.31}$$

Consequently, isovector components characterising equivalence transformations of a single balance equation are determined through arbitrary functions $F, f, \alpha^{iji_1\cdots i_{m-1}}$ and β^i. Naturally, particular structures of the functions Σ^i and Σ may limit arbitrariness on these functions.

In case $m = 1$, we have to modify slightly the analysis above. The determining equations for isovector components are now given by (9.8.18). If we closely examine these equations, we realise at once that the function ν must be independent of the variables t_i and t. This, in turn, implies that we get $\partial S^i/\partial\Sigma = 0$, namely, $S^i = S^i(x^j, u, v_j, \Sigma^j)$. If we eliminate the function ν between the first two equations in (9.8.18), we then obtain

$$\big(\Gamma^{ij} + \Gamma^{ji}\big)\big(\Sigma + s^k_k + \sigma^k v_k\big) = \Gamma\big(s^{ij} + s^{ji}\big).$$

The explicit form of the above relations become

$$\begin{aligned}
&\Sigma^2\mathcal{X}^{ij} + \Sigma\Big[\mathcal{S}^{ij} + (\delta^j_n\mathcal{A}^i_m + \delta^i_n\mathcal{A}^j_m)s^{mn} + \\
&\Big\{2\mathcal{X}^{ij}\delta^k_l - \frac{\partial X^k}{\partial v_j}\delta^i_l - \frac{\partial X^k}{\partial v_i}\delta^j_l\Big\}\big(s^l_k + s^l v_k\big)\Big] \\
&\quad + \mathcal{S}^{ij}\big(s^k_k + s^k v_k\big) + (\delta^j_n\mathcal{A}^i_m + \delta^i_n\mathcal{A}^j_m)s^{mn}\big(s^k_k + s^k v_k\big) \\
&\qquad + \Big[\mathcal{X}^{ij}\delta^k_l - \frac{\partial X^k}{\partial v_j}\delta^i_l - \frac{\partial X^k}{\partial v_i}\delta^j_l\Big]\big(s^l_k + s^l v_k\big)(s^m_m + s^m v_m) \\
&= \Big[T + D_k S^k + \Sigma D_k X^k + \mathcal{A}^k_l\big(s^l_k + s^l v_k\big)\Big](\delta^i_m\delta^j_n + \delta^i_n\delta^j_m)s^{mn}
\end{aligned}$$

where we defined

$$\begin{aligned}
D_i &= \frac{\partial}{\partial x^i} + v_i\frac{\partial}{\partial u}, \qquad \mathcal{X}^{ij} = \frac{\partial X^i}{\partial v_j} + \frac{\partial X^j}{\partial v_i}, \\
\mathcal{S}^{ij} &= \frac{\partial S^i}{\partial v_j} + \frac{\partial S^j}{\partial v_i}, \\
\mathcal{A}^i_j &= \frac{\partial S^i}{\partial \Sigma^j} + \delta^i_j D_k X^k - D_j X^i
\end{aligned}$$

If we recall that

$$X^i = -\frac{\partial F}{\partial v_i},\ U = F - v_i\frac{\partial F}{\partial v_i},\ S^i = \mathcal{F}^i\big(x^j, u, v_j, \Sigma^j\big)$$

where $F = F(x^i, u, v_i)$, we immediately observe that the following equations must be satisfied when $\Sigma \neq 0$:

$$\begin{gathered}\frac{\partial S^i}{\partial v_j} + \frac{\partial S^j}{\partial v_i} = 0 \qquad (9.8.32)\\ 2\Big(\frac{\partial X^i}{\partial v_j} + \frac{\partial X^j}{\partial v_i}\Big)\,\delta_l^k - \frac{\partial X^k}{\partial v_j}\,\delta_l^i - \frac{\partial X^k}{\partial v_i}\,\delta_l^j = 0\\ \Big(\frac{\partial X^i}{\partial v_j} + \frac{\partial X^j}{\partial v_i}\Big)\,\delta_l^k - \frac{\partial X^k}{\partial v_j}\,\delta_l^i - \frac{\partial X^k}{\partial v_i}\,\delta_l^j = 0\end{gathered}$$

From the last two equations in (9.8.32), we get

$$\frac{\partial X^i}{\partial v_j} + \frac{\partial X^j}{\partial v_i} = -\,2\frac{\partial^2 F}{\partial v_i \partial v_j} = 0$$

whence we obtain

$$F(\mathbf{x}, u, \mathbf{v}) = \phi^i(\mathbf{x}, u) v_i + \gamma(\mathbf{x}, u)$$

where ϕ^i and γ are arbitrary functions. We thus find $X^i = -\,\phi^i(\mathbf{x}, u)$. In these circumstances, the remaining terms give, after some rather simple manipulations

$$\begin{aligned}&\big(\delta_m^i\delta_n^j + \delta_n^i\delta_m^j\big)\big(T + D_k S^k\big) = \\ &\qquad \Sigma\Big[\delta_n^j\Big(\frac{\partial S^i}{\partial \Sigma^m} - D_m X^i\Big) + \delta_n^i\Big(\frac{\partial S^j}{\partial \Sigma^m} - D_m X^j\Big)\Big],\\ &\big(\delta_m^i\delta_n^j + \delta_n^i\delta_m^j\big)\Big(\frac{\partial S^k}{\partial \Sigma^l} - D_l X^k\Big) = \\ &\qquad \Big[\delta_n^j\Big(\frac{\partial S^i}{\partial \Sigma^m} - D_m X^i\Big) + \delta_n^i\Big(\frac{\partial S^j}{\partial \Sigma^m} - D_m X^j\Big)\Big]\delta_l^k.\end{aligned}$$

On contracting the indices (k, l) and (j, n) in the second set of equations above, we obtain

$$\frac{\partial S^i}{\partial \Sigma^m} - D_m X^i = \frac{1}{n}\Big(\frac{\partial S^k}{\partial \Sigma^k} - D_k X^k\Big)\delta_m^i = \psi(x^j, u, v_j, \Sigma^j)\,\delta_m^i$$

while the relations

$$\frac{\partial^2 S^i}{\partial \Sigma^j \partial \Sigma^k} = \frac{\partial \psi}{\partial \Sigma^k}\delta_j^i = \frac{\partial \psi}{\partial \Sigma^j}\delta_k^i$$

result in $\partial\psi/\partial\Sigma^k = 0$ for $n > 1$. We thus have $\psi = \psi(x^j, u, v_j)$. But, in this case the integration of the equations

$$\frac{\partial S^i}{\partial \Sigma^j} = \psi \delta^i_j + D_j X^i$$

yields simply

$$S^i = \psi \Sigma^i - \Big(\frac{\partial \phi^i}{\partial x^j} + \frac{\partial \phi^i}{\partial u} v_j\Big) \Sigma^j + \chi^i(x^j, u, v_j)$$

where χ^i are arbitrary functions. The equations $(9.8.32)_1$ are now expressed as follows

$$\Big[\Big(\frac{\partial \psi}{\partial v_j} - \frac{\partial \phi^j}{\partial u}\Big)\delta^i_k + \Big(\frac{\partial \psi}{\partial v_i} - \frac{\partial \phi^i}{\partial u}\Big)\delta^j_k\Big]\Sigma^k + \frac{\partial \chi^i}{\partial v_j} + \frac{\partial \chi^j}{\partial v_i} = 0$$

so that we find

$$\Big(\frac{\partial \psi}{\partial v_j} - \frac{\partial \phi^j}{\partial u}\Big)\delta^i_k + \Big(\frac{\partial \psi}{\partial v_i} - \frac{\partial \phi^i}{\partial u}\Big)\delta^j_k = 0,$$
$$\frac{\partial \chi^i}{\partial v_j} + \frac{\partial \chi^j}{\partial v_i} = 0.$$

Contraction on indices (j, k) in the first set of equations above gives

$$\frac{\partial \psi}{\partial v_i} = \frac{\partial \phi^i}{\partial u} \quad \text{and} \quad \psi = \frac{\partial \phi^i}{\partial u} v_i + \omega(\mathbf{x}, u)$$

whereas the solution of the second set is known to be

$$\chi^i = \alpha^{ij}(\mathbf{x}, u) v_j + \beta^i(\mathbf{x}, u), \;\; \alpha^{ij}(\mathbf{x}, u) = -\alpha^{ji}(\mathbf{x}, u).$$

The isovector component T is determined by the relation

$$T = \psi \Sigma - D_i S^i.$$

Hence, the relevant isovector components are given as follows

$$\begin{aligned}
X^i &= -\phi^i(\mathbf{x}, u), \\
U &= \gamma(\mathbf{x}, u), \\
V_i &= \frac{\partial \gamma}{\partial x^i} + v_i \frac{\partial \gamma}{\partial u} + \Big(\frac{\partial \phi^j}{\partial x^i} + v_i \frac{\partial \phi^j}{\partial u}\Big) v_j, \\
S^i &= \Big(\omega + \frac{\partial \phi^j}{\partial u} v_j\Big)\Sigma^i - \Big(\frac{\partial \phi^i}{\partial x^j} + v_j \frac{\partial \phi^i}{\partial u}\Big)\Sigma^j + \alpha^{ij} v_j + \beta^i, \\
T &= \Big(\omega + \frac{\partial \phi^i}{\partial u} v_i\Big)\Sigma - \Big(\frac{\partial \omega}{\partial x^i} + v_i \frac{\partial \omega}{\partial u}\Big)\Sigma^i + \Big(\frac{\partial^2 \phi^i}{\partial x^i \partial x^j} + v_j \frac{\partial^2 \phi^i}{\partial x^i \partial u}\Big)\Sigma^j
\end{aligned} \tag{9.8.33}$$

$$-\Big(\frac{\partial\alpha^{ij}}{\partial x^i}+v_i\frac{\partial\alpha^{ij}}{\partial u}\Big)v_j-\frac{\partial\beta^i}{\partial x^i}+v_i\frac{\partial\beta^i}{\partial u}.$$

When $\Sigma=0$, we are compelled to take $T=0$. This imposes additional conditions on the foregoing solution leading to the relations

$$\omega=\frac{\partial\phi^i}{\partial x^i}+c,\quad \frac{\partial\alpha^{ij}}{\partial x^i}=\frac{\partial\beta^j}{\partial u},\quad \frac{\partial\beta^i}{\partial x^i}=0$$

where c is a constant. A detailed discussion of this case is left to the reader as an exercise.

The case $m=0$ presents no difficulty in determining the isovector components which can be found as

$$\begin{aligned}
X^i &= X^i(\mathbf{x}), \\
U &= U(\mathbf{x},\mathbf{u},\Sigma^i,\Sigma), \\
S^i &= f(\mathbf{x})\Sigma^i+\frac{\partial X^i}{\partial x^j}\Sigma^j+g^i(\mathbf{x}), \\
T &= f(\mathbf{x})\Sigma-\frac{\partial f(\mathbf{x})}{\partial x^i}\Sigma^i-\frac{\partial^2 X^i}{\partial x^i\partial x^j}\Sigma^j-\frac{\partial g^i(\mathbf{x})}{\partial x^i}.
\end{aligned}\tag{9.8.34}$$

Any reader who wish to get more detailed information about calculations concerning this section may be referred to the works below[1].

Example 9.8.1. ***Non-linear wave equation.*** Let us consider a second order non-linear partial differential equation

$$[f(x,t,u,u_x,u_t)]_x-u_{tt}+g(x,t,u,u_x,u_t)=0.$$

Since $n=2,\ N=1$ and $m=1$, let us write

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[1] Şuhubi, E. S., Equivalence Groups for Second Order Balance Equations, *International Journal of Engineering Science*, **37**, 1901-1925, 1999.
Şuhubi, E. S., Explicit Determination of Isovector Fields of Equivalence Groups for Second Order Balance Equations, *International Journal of Engineering Science*, **38**, 715-736, 2000.
Özer, S. and E. S. Şuhubi, E. S, Equivalence Transformations for First Order Balance Equations, *International Journal of Engineering Science*, **42**, 1305-1324, 2004.
Şuhubi, E. S., Equivalence Groups for Balance Equations of Arbitrary Order - Part I, *International Journal of Engineering Science*, **42**, 1729-1751, 2004.
Şuhubi, E. S., Explicit Determination of Isovector Fields of Equivalence Groups for Balance Equations of Arbitrary Order - Part II, *International Journal of Engineering Science*, **43**, 1-15, 2005.

$$x^1 = x,\ x^2 = t,\ v_1 = u_x = p,\ v_2 = u_t = v$$

so that we have

$$\Sigma^1 = f(x,t,u,p,v),\ \Sigma^2 = -v,\ \Sigma = g(x,t,u,p,v).$$

Hence, we have to take into account the submanifold of the manifold $\mathcal{K}_1$ specified by

$$s^2_1 = 0, \quad s^2_2 = 0, \quad s^2 = 0, \quad s^{21} = 0, \quad s^{22} = -1.$$

In the relations (9.8.33), let us denote

$$\phi^1 = \alpha(x,t,u), \quad \phi^2 = \beta(x,t,u), \quad \alpha^{12} = -\alpha^{21} = \lambda(x,t,u)$$
$$\beta^1 = \Psi(x,t,u),\ \beta^2 = \Phi(x,t,u).$$

As is clearly observed, S^2 is no longer an independent component of the isovector field, but it is equal to $-V_2$. After having resorted to $(9.8.33)_3$ and $(9.8.33)_4$, we find that

$$\begin{aligned}
&V_2 = \gamma_t + \alpha_t p + (\gamma_u + \beta_t)v + \alpha_u pv + \beta_u v^2,\\
&S^2 = -(\omega + \alpha_u p + \beta_u v)v - (\beta_x + \beta_u p)f + (\beta_t + \beta_u v)v - \lambda p + \Phi.
\end{aligned}$$

Thus the relation $S^2 = -V_2$ leads to

$$\beta_u v^2 + (\alpha_t - \lambda)p + (2\beta_t + \gamma_u - \omega)v + \Phi + \gamma_t - (\beta_x + \beta_u p)f = 0.$$

Whenever f is an arbitrary function, it follows from this equality that the following equations must be satisfied

$$\begin{aligned}
\beta_x + \beta_u p = 0,\ \beta_u = 0,\ \alpha_t - \lambda = 0,\\
2\beta_t + \gamma_u - \omega = 0, \qquad\qquad \Phi + \gamma_t = 0
\end{aligned}$$

whence we obtain $\beta_x = 0, \beta_u = 0$ and consequently

$$\beta = \beta(t),\ \ \lambda = \alpha_t,\ \ \omega = 2\beta_t + \gamma_u,\ \ \Phi = -\gamma_t.$$

The components $\overline{S}^2_j$ and S^{2j} of the isovector field must also vanish. But, because of the relation $F^2 = -s^2_j X^j - s^2 U - s^{2j}V_j + S^2 = 0$ these two conditions are satisfied identically. Therefore, the relevant components of the isovector field are determined as follows

$$\begin{aligned}
&X^1 = -\alpha(x,t,u),\ \ X^2 = -\beta(t),\\
&U = \gamma(x,t,u),
\end{aligned}$$

$$
\begin{aligned}
V_1 &= \gamma_x + (\alpha_x + \gamma_u)p + \alpha_u p^2,\\
V_2 &= -S^2 = \gamma_t + \alpha_t p + (\beta_t + \gamma_u)v + \alpha_u pv,\\
S^1 &= (2\beta_t - \alpha_x + \gamma_u)f + \alpha_u v^2 + 2\alpha_t v + \Psi,\\
T &= [\alpha_{xx} - \gamma_{xu} + p(\alpha_{xu} - \gamma_{uu})]f + (2\beta_t + \gamma_u + \alpha_u p)g\\
&\quad + v^2(\gamma_{uu} - \alpha_{xu}) + p(\alpha_{tt} - \Psi_u) + v(\beta_{tt} - 2\alpha_{xt} + 2\gamma_{tu}) + \gamma_{tt} - \Psi_x
\end{aligned}
$$

where $\alpha(x,t,u), \beta(t), \gamma(x,t,u)$ and $\Psi(x,t,u)$ are functions which may be chosen arbitrarily.

As a simple example, let us take

$$\alpha = 0,\ \beta = 0,\ \gamma = au^2,\ \Psi = 0$$

so that we obtain

$$
\begin{aligned}
&X^1 = X^2 = 0,\ \ U = au^2,\ \ V_1 = 2aup,\ \ V_2 = 2auv,\\
&S^1 = 2auf,\ \ T = -2apf + 2aug + 2av^2.
\end{aligned}
$$

In order to determine the equivalence group associated with this isovector, we have to solve the following ordinary differential equations

$$
\begin{aligned}
&\frac{d\overline{x}}{d\epsilon} = 0,\ \ \frac{d\overline{t}}{d\epsilon} = 0,\ \ \frac{d\overline{u}}{d\epsilon} = a\overline{u}^2,\ \ \frac{d\overline{p}}{d\epsilon} = 2a\overline{u}\,\overline{p},\ \ \frac{d\overline{v}}{d\epsilon} = 2a\overline{u}\,\overline{v}\\
&\frac{d\overline{f}}{d\epsilon} = 2a\overline{u}\overline{f},\ \frac{d\overline{g}}{d\epsilon} = -2a\overline{p}\overline{f} + 2a\overline{u}\overline{g} + 2a\overline{v}^2
\end{aligned}
$$

under the initial conditions $\overline{x}(0) = x, \overline{t}(0) = t, \overline{u}(0) = u, \overline{p}(0) = p$, $\overline{v}(0) = v, \overline{f}(0) = f, \overline{g}(0) = g$. We can then easily reach to the particular equivalence transformation in which independent variables remain unchanged

$$
\begin{aligned}
&\overline{x} = x,\ \ \overline{t} = t,\ \ \overline{u}(\epsilon) = \frac{u}{1 - \epsilon au},\\
&\overline{p}(\epsilon) = \frac{p}{(1 - \epsilon au)^2},\ \overline{v}(\epsilon) = \frac{v}{(1 - \epsilon au)^2}\\
&\overline{f}(\epsilon) = \frac{f}{(1 - \epsilon au)^2},\ \overline{g}(\epsilon) = \frac{g - \epsilon a(2pf + ug - 2v^2)}{(1 - \epsilon au)^3}.
\end{aligned}
$$

As a simple application to equivalence transformations, let us apply this group of diffeomorphisms to the linear wave equation

$$u_{xx} - u_{tt} = 0$$

where $f = u_x$ and $g = 0$. We know that the general solution of this equation is given by

$$u(x,t) = \phi(x+t) + \psi(x-t).$$

ϕ and ψ are arbitrary functions of their arguments. If we employ the inverse transformation, we can write

$$u = \frac{\overline{u}}{1 + k\,\overline{u}}$$

where $k = \epsilon a$, then the foregoing linear partial differential equation is cast into a family of quasilinear second order equations

$$\overline{u}_{xx} - \overline{u}_{tt} - 2k\frac{\overline{u}_x^2 - \overline{u}_t^2}{1 + k\,\overline{u}} = 0.$$

by this particular equivalence transformation. We can then readily verify by inspection that a solution of that non-linear, second order partial differential equation is indeed given by

$$\overline{u}(x,t) = \frac{\phi(x+t) + \psi(x-t)}{1 - k[\phi(x+t) + \psi(x-t)]}$$

where $\phi(x+t)$ and $\psi(x-t)$ are arbitrary functions.

As a slightly more general case, let us assume that

$$\alpha = 0,\ \beta = 0,\ \gamma = au^2 + bu,\ \Psi = 0.$$

We thus obtain

$$\begin{aligned} &X^1 = X^2 = 0,\ \ U = au^2 + bu,\ \ V_1 = (2au+b)p,\ \ V_2 = (2au+b)v, \\ &S^1 = (2au+b)f,\ \ T = -2apf + (2au+b)g + 2av^2. \end{aligned}$$

To find the corresponding equivalence transformation we have to integrate the differential equations below

$$\begin{aligned} &\frac{d\overline{x}}{d\epsilon} = 0,\ \frac{d\overline{t}}{d\epsilon} = 0,\ \frac{d\overline{u}}{d\epsilon} = a\overline{u}^2 + b\overline{u},\ \frac{d\overline{p}}{d\epsilon} = (2a\overline{u}+b)\overline{p},\ \frac{d\overline{v}}{d\epsilon} = (2a\overline{u}+b)\overline{v} \\ &\frac{d\overline{f}}{d\epsilon} = (2a\overline{u}+b)\overline{f},\ \frac{d\overline{g}}{d\epsilon} = -2a\overline{p}\overline{f} + (2a\overline{u}+b)\overline{g} + 2a\overline{v}^2 \end{aligned}$$

under the initial conditions $\overline{x}(0) = x, \overline{t}(0) = t, \overline{u}(0) = u, \overline{p}(0) = p$, $\overline{v}(0) = v, \overline{f}(0) = f, \overline{g}(0) = g$. We then easily find that

$$\overline{x} = x, \quad \overline{t} = t, \quad \overline{u}(\epsilon) = \frac{be^{b\epsilon}u}{b - a(e^{b\epsilon}-1)u},$$

$$\begin{aligned}
\overline{p}(\epsilon) &= \frac{b^2 e^{b\epsilon} p}{[b - a(e^{b\epsilon} - 1)u]^2},\\
\overline{v}(\epsilon) &= \frac{b^2 e^{b\epsilon} v}{[b - a(e^{b\epsilon} - 1)u]^2}\\
\overline{f}(\epsilon) &= \frac{b^2 e^{b\epsilon} f}{[b - a(e^{b\epsilon} - 1)u]^2},\\
\overline{g}(\epsilon) &= \frac{b^2 e^{b\epsilon}[bg - a(e^{b\epsilon} - 1)(2pf + ug - 2v^2)]}{[b - a(e^{b\epsilon} - 1)u]^3}.
\end{aligned}$$

If we write

$$u = \frac{b\overline{u}}{be^{b\epsilon} + a(e^{b\epsilon} - 1)\,\overline{u}}$$

then the one-dimensional homogeneous wave equation is transformed into

$$\overline{u}_{xx} - \overline{u}_{tt} + \frac{2a(e^{b\epsilon} - 1)(\overline{u}_x^2 - \overline{u}_t^2)}{[be^{b\epsilon} - a(e^{b\epsilon} - 1)\,\overline{u}]} = 0.$$

Therefore a solution for this family of quasilinear second order differential equations is expressible in the form

$$\overline{u}(x, t; \epsilon) = \frac{b[\phi(x + t) + \psi(x - t)]}{be^{b\epsilon} + a(e^{b\epsilon} - 1)[\phi(x + t) + \psi(x - t)]}$$

depending on the parameter ϵ and constants a and b. ■

Example 9.8.2. ***Homogeneous hyperelasticity.*** We have discussed the symmetry transformations of the equations of motion of a homogeneous hyperelastic material in Example 9.4.4. The equations of motion depend heavily on the stress potential $\Sigma = \Sigma(\mathbf{F})$ characterising the physical constitution of the material, thus differing for different types of materials. Hence, they constitute a family of balance equations. We shall now try to determine the equivalence transformations associated with that family. Since, we will be employing the notations introduced earlier, we abstain from repeating them here. The equations of motion corresponding to the case $m = 1,\ n = 4, N = 3$ can be now written

$$\frac{\partial \Sigma_{kK}}{\partial X_K} + \frac{\partial \Sigma_{k4}}{\partial X_4} = 0$$

with $X_4 = t$ and k, K take the values $1, 2, 3$. Then, the coordinates of the manifold $\mathcal{K}_1$ is easily identified as

$$\Sigma_{kK}(\mathbf{X},\mathbf{F}) = \frac{\partial \Sigma}{\partial F_{kK}},\ \Sigma_{k4} = -v_k,\ \Sigma^k = 0,\ v_{kK} = F_{kK},\ v_{k4} = v_k,$$
$$s_{kKlL} = \frac{\partial \Sigma_{kK}}{\partial F_{lL}} = \frac{\partial^2 \Sigma}{\partial F_{kK}\partial F_{lL}},\ s_{k4l4} = -\frac{\partial v_k}{\partial v_l} = -\delta_{kl},\ s_{kKl4} = 0,$$
$$s_{k4lL} = 0,\ s_{kKL} = 0,\ s_{kK4} = 0,\ s_{k4K} = 0,\ s_{k44} = 0.$$

In this circumstance, the coordinate cover of the manifold $\mathcal{K}_1$ should be taken as $\{X_K, t, x_k, F_{kK}, v_k, \Sigma_{kK}, s_{kKlL}\}$. Obviously, the variables s_{kKlL} enjoy the block symmetry $s_{kKlL} = s_{lLkK}$. We denote the isovector field by

$$V = -\phi_K \frac{\partial}{\partial X_K} - \psi \frac{\partial}{\partial t} + U_k \frac{\partial}{\partial x_k} + V_{kK} \frac{\partial}{\partial F_{kK}} + V_k \frac{\partial}{\partial v_k} + S_{kK} \frac{\partial}{\partial \Sigma_{kK}} + S_{kKlL} \frac{\partial}{\partial s_{kKlL}}.$$

We may assume without loss of generality that $S_{kKlL} = S_{lLkK}$. Negative signs above are inserted for convenience, We know that the isovector components ϕ_K, ψ, U_k are functions only of the variables X_K, t, x_k. Furthermore, we have to impose the restrictions

$$\begin{aligned}&S_{k4} = -V_{k4} = -V_k, T_k = 0, S_{kKl4} = 0, S_{k4lL} = 0, \overline{S}_{kKL} = 0\\ &\overline{S}_{kK4} = 0,\ \overline{S}_{k4K} = 0,\ \overline{S}_{k44} = 0\end{aligned} \tag{9.8.35}$$

on the isovector components. In order not to confuse the functions $F^{\alpha i}$ defined in (9.8.8) with the components of the deformation gradients F_{kK}, we will replace the functions F^{Kk} by G_{kK}. We also have to take $F^{k4} = 0$. We thus get

$$G_{kK} = -s_{kKlL} V_{lL} + S_{kK} \tag{9.8.36}$$

and the conditions

$$\overline{S}_{k4K} = 0, \quad \overline{S}_{k44} = 0, \quad S_{k4lL} = 0$$

are satisfied identically. We can then directly deduce from (9.8.25) that

$$\begin{aligned}V_{kK} &= U_{k,K} + \phi_{L,K} F_{kL} + \psi_{,K} v_k + U_{k,l} F_{lK} + \phi_{L,l} F_{kL} F_{lK} + \psi_{,l} v_k F_{lK},\\ V_k &= \dot{U}_k + \dot{\phi}_K F_{kK} + \dot{\psi}\, v_k + U_{k,l} v_l + \phi_{K,l} F_{kK} v_l + \psi_{,l} v_k v_l,\\ S_{kK} &= g_{kl}\Sigma_{lK} - (\phi_{K,L} + \phi_{K,l} F_{lL})\Sigma_{kL} + (\dot{\phi}_K + \phi_{K,l} v_l) v_k\\ &\quad + (\phi_{L,l} F_{lL} + \psi_{,l} v_l)\Sigma_{kK} + f_{kKLM4lmn} F_{lL} F_{mM} v_n\\ &\quad + f_{kKLMlm} F_{lL} F_{mM} + f_{kKL4lm} F_{lL} v_m\\ &\quad + f_{kKLl} F_{lL} + f_{kK4l} v_l + f_{kK},\end{aligned}$$

$$\begin{aligned} S_{k4} = & - g_{kl}v_l - (\psi_{,K} + \psi_{,l}F_{lK})\Sigma_{kK} + \dot{\psi} + \psi_{,l}v_l)v_k \\ & - (\phi_{L,l}F_{lL} + \psi_{,l}v_l)v_k + f_{k4LMNlmn}F_{lL}F_{mM}F_{nN} \\ & + f_{k4LMlm}F_{lL}F_{mM} + f_{k4Ll}F_{lL} + f_{k4}, \\ T_k = & - S_{kK,K} - \dot{S}_{kK} - S_{kK,l}\,F_{lK} - S_{k4,l}\,v_l. \end{aligned} \tag{9.8.37}$$

g_{kl} and all multi-indexed functions f depend only on the variables $\mathbf{X}$, t and $\mathbf{x}$. The functions f must be so chosen as to obey the symmetry requirements on capital and small indices (Greek and roman superscripts and subscripts in general expressions) for $n = 4$ and $N = 3$. An overdot represent the derivative with respect to the time variable t.

Let us first deal with the relation $(9.8.35)_1$. It follows from (9.8.37) that we have to satisfy the equations

$$\begin{aligned} &\psi_{,K} = 0,\ \ \psi_{,l} = 0,\ \ g_{kl} = 2\dot{\psi}\,\delta_{kl} + U_{k,l},\ \ \phi_{M,n}\,\delta_{km} - \phi_{M,m}\,\delta_{kn} = 0, \\ &f_{k4Kl} = -\dot{\phi}_K\,\delta_{kl},\ \ f_{k4} = -\dot{U}_k,\ \ f_{k4LMNlmn} = 0,\ \ f_{k4LMlm} = 0. \end{aligned}$$

The contraction on indices (k, m) in the fourth equation above yields $\phi_{M,n} = 0$. We thus obtain

$$\phi_K = \phi_K(\mathbf{X}, t),\quad \psi = \psi(t).$$

On the other hand, we get from p. 610 that

$$\begin{aligned} &\overline{S}_{kKL} = G_{kK,L} + G_{kK,l}F_{lL} = 0, \overline{S}_{kK4} = \dot{G}_K + G_{kK,l}v_l = 0, \\ &S_{kKl4} = \frac{\partial G_{kK}}{\partial v_l} = 0. \end{aligned}$$

Hence, the functions G_{kK} must be independent of v_l. We then further deduce from above the relations

$$G_{kK,\,l} = 0,\ \ G_{kK,\,L} = 0,\ \ \dot{G}_K = 0.$$

On the other hand, we get

$$V_{kK} = U_{k,K} + \phi_{L,K}F_{kL} + U_{k,l}\,F_{lK}.$$

This implies that the functions V_{kK} do not depend on v_l. When we take the relation (9.8.36) into consideration, we reach to the conclusion

$$\begin{aligned} &V_{kK,\,l} = 0,\ \ V_{kK,\,L} = 0,\ \ \dot{V}_{kK} = 0, \\ &\frac{\partial S_{kK}}{\partial v_l} = 0,\ \ S_{kK,\,l} = 0,\ \ S_{kK,\,L} = 0,\ \ \dot{S}_{kK} = 0. \end{aligned} \tag{9.8.38}$$

The isovector components S_{kK} can now be written as

$$S_{kK} = (2\dot{\psi}\,\delta_{kl} + U_{k,l})\Sigma_{lK} - \phi_{K,L}\Sigma_{kL} + \dot{\phi}_K v_k + f_{kKLM4lmn}F_{lL}F_{mM}v_n + f_{kKLMlm}F_{lL}F_{mM} + f_{kKL4lm}F_{lL}v_m + f_{kKLl}F_{lL} + f_{kK4l}v_l + f_{kK}$$

and $(9.8.38)_4$ leads to

$$\dot{\phi}_K\,\delta_{kp} + f_{kK4p} + f_{kKLM4lmp}\,F_{lL}\,F_{mM} + f_{kKL4lp}\,F_{lL} = 0$$

from which we find that

$$f_{kKLM4lmn} = 0, \;\; f_{kKL4lm} = 0, \;\; f_{kK4l} = -\,\dot{\phi}_K\,\delta_{kl}.$$

However, because of the relation $f_{kK4l} = -\,f_{k4Kl}$ we get $\dot{\phi}_K = 0$, hence we obtain

$$\phi_K = \phi_K(\mathbf{X}).$$

Derivatives of V_{kK} with respect to t, x_m and X_M give, respectively,

$$\dot{U}_{k,K} = 0,\; \dot{U}_{k,l} = 0,\; U_{k,Km} = 0,\; U_{k,lm} = 0,\; U_{k,KM} = 0,\; \phi_{K,LM} = 0.$$

Similar expressions for S_{kK} leads to the equation $\ddot{\psi}(t) = 0$ implying further that all non-zero functions f must be constants. Thus, if we recall the anti-symmetry properties, we are able to write

$$\psi = a_1 t + a_2, \quad f_{kKLMlm} = e_{KLM}\,e_{lmn}\,c_{kn}, \\ f_{kKLl} = e_{KLM}\,c_{Mkl}, \quad f_{kK} = c_{kK}$$

where c_{kn}, c_{Mkl} and c_{kK} are constants. e_{KLM}, e_{klm} are, of course, three-dimensional permutation symbols. The solutions of the differential equations satisfied by the functions ϕ_K and U_k are readily obtained as

$$\phi_K = B_{KL}X_L + B_K, \quad U_k = a_{kl}x_l + A_{kK}X_K + A_k(t)$$

where $B_{KL}, B_K, a_{kl}, A_{kK}$ are constants. Finally, we find from $(9.8.37)_5$ that

$$T_k = \dot{V}_k + V_{k,l}v_l = 0.$$

This equations yield $\ddot{U}_k = 0$ or $\ddot{A}_k = 0$ and

$$A_k(t) = \alpha_k t + A_k.$$

Hence, the relevant isovector components take the form

$$\phi_K = B_{KL}X_L + B_K,\; \psi = a_1 t + a_2,\; U_k = a_{kl}x_l + A_{kK}X_K + \alpha_k t + A_k, \\ V_{kK} = B_{LK}F_{kL} + a_{kl}F_{lK} + A_{kK}, \qquad V_k = (a_1\delta_{kl} + a_{kl})v_l + \alpha_k,$$

$$S_{kK} = (2a_1\delta_{kl} + a_{kl})\Sigma_{lK} - B_{KL}\Sigma_{kL} + e_{KLM}\, e_{lmn}\, c_{kn}\, F_{lL}\, F_{mM} \\ + e_{KLM}\, c_{Mkl}\, F_{lL} + c_{kK}.$$

The equivalence transformation is then found by integrating the ordinary differential equations

$$\begin{aligned} \frac{d\overline{X}_K}{d\epsilon} &= -(B_{KL}\overline{X}_L + B_K), \qquad \frac{d\overline{t}}{d\epsilon} = -(a_1\overline{t} + a_2), \\ \frac{d\overline{x}_k}{d\epsilon} &= a_{kl}\overline{x}_l + A_{kK}\overline{X}_K + \alpha_k\overline{t} + A_k, \\ \frac{d\overline{F}_{kK}}{d\epsilon} &= B_{LK}\overline{F}_{kL} + a_{kl}\overline{F}_{lK} + A_{kK}, \qquad \frac{d\overline{v}_k}{d\epsilon} = (a_1\delta_{kl} + a_{kl})\overline{v}_l + \alpha_k \\ \frac{d\overline{\Sigma}_{kK}}{d\epsilon} &= (2a_1\delta_{kl} + a_{kl})\overline{\Sigma}_{lK} - B_{KL}\overline{\Sigma}_{kL} + e_{KLM}\, e_{lmn}\, c_{kn}\, \overline{F}_{lL}\, \overline{F}_{mM} \\ &\quad + e_{KLM}\, c_{Mkl}\, \overline{F}_{lL} + c_{kK}. \end{aligned}$$

under the initial conditions $\overline{X}_K(0) = X_K, \overline{t}(0) = t, \overline{x}_k(0) = x_k, \overline{F}_{kK}(0) = F_{kK}, \overline{v}_k(0) = v_k$ and $\overline{\Sigma}_{kK}(0) = \Sigma_{kK}$. ϵ is the group parameter. That Σ_{kK} are actually dependent on the deformation tensor $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ instead of $\mathbf{F}$ may impose additional restrictions on some constants appearing in the above expressions. Since $\Sigma_{kK} = \partial\Sigma/\partial F_{kK}$ the transformed expressions help us to determine the stress potential $\overline{\Sigma}$. ■

Example 9.8.3. As a last example, we consider the third order non-linear partial differential equation

$$\big[u_{xx} + \phi(x,t,u)\big]_x + u_t = u_{xxx} + \phi_u u_x + \phi_x + u_t = 0. \quad (9.8.39)$$

We may regard this equation as a kind of generalised Korteweg-de Vries equation. If we take $\phi = u^2/2$, we obtain the known form of the Korteweg-de Vries equation. In this case, it is clear that we have $m = 2$, $N = 1$ and $n = 2$. The manifold $\mathcal{K}_2$ is now generated by taking

$$\begin{aligned} &x^1 = x,\ x^2 = t,\ v_1 = u_x,\ v_2 = u_t,\ v_{11} = u_{xx},\ v_{12} = v_{21} = u_{xt}, \\ &v_{22} = u_{tt},\ \ \Sigma^1 = v_{11} + \phi,\ \ \Sigma^2 = u,\ \ \Sigma = 0, \\ &s^1_1 = \phi_x,\ s^1_2 = \phi_t,\ s^1 = \phi_u,\ s^2 = 1,\ s^{111} = 1. \end{aligned}$$

Therefore, the coordinate cover of the enlarged manifold $\mathcal{K}_2$ is specified by the following list

$$\{x, t, v_1, v_2, v_{11}, v_{12}, v_{22}, \phi, s^1_1, s^1_2, s^1\}.$$

Hence, an isovector field should be represented by

$$V = X^1\frac{\partial}{\partial x} + X^2\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + V_1\frac{\partial}{\partial v_1} + V_2\frac{\partial}{\partial v_2} + V_{11}\frac{\partial}{\partial v_{11}} + V_{12}\frac{\partial}{\partial v_{12}}$$
$$+ V_{22}\frac{\partial}{\partial v_{22}} + \Phi\frac{\partial}{\partial \phi} + \overline{S}^1_1\frac{\partial}{\partial s^1_1} + \overline{S}^1_2\frac{\partial}{\partial s^1_2} + S^{111}\frac{\partial}{\partial s^{111}}$$

However, if we note the forms of Σ^1 and Σ^2, we see that we can write

$$\frac{\partial}{\partial u} \to \frac{\partial}{\partial u} + \frac{\partial}{\partial \Sigma^2}\frac{\partial \Sigma^2}{\partial u} = \frac{\partial}{\partial u} + \frac{\partial}{\partial \Sigma^2},$$
$$\frac{\partial}{\partial v_{11}} \to \frac{\partial}{\partial v_{11}} + \frac{\partial}{\partial \Sigma^1}\frac{\partial \Sigma^1}{\partial v_{11}} = \frac{\partial}{\partial v_{11}} + \frac{\partial}{\partial \Sigma^1},$$
$$\frac{\partial}{\partial \phi} \to \frac{\partial}{\partial \Sigma^1}\frac{\partial \Sigma^1}{\partial \phi} = \frac{\partial}{\partial \Sigma^1}$$

so that the isovector field is expressed in the standard form in terms of quantities entering into the balance equation as follows

$$V = X\frac{\partial}{\partial x} + \mathcal{T}\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + V_1\frac{\partial}{\partial v_1} + V_2\frac{\partial}{\partial v_2} + V_{11}\frac{\partial}{\partial v_{11}} + V_{12}\frac{\partial}{\partial v_{12}}$$
$$+ V_{22}\frac{\partial}{\partial v_{22}} + S^1\frac{\partial}{\partial \Sigma^1} + S^2\frac{\partial}{\partial \Sigma^2} + \overline{S}^1_1\frac{\partial}{\partial s^1_1} + \overline{S}^1_2\frac{\partial}{\partial s^1_2} + S^{111}\frac{\partial}{\partial s^{111}}.$$

It is straightforward to notice that

$$S^1 = V_{11} + \Phi,$$
$$S^2 = U.$$

We know that some of the isovector components are determined by a presently arbitrary function $F = F(x,t,u,v_1,v_2)$ through relations given below [*see* (9.3.26)]:

$$X^1 = X = -\frac{\partial F}{\partial v_1},\; X^2 = \mathcal{T} = -\frac{\partial F}{\partial v_2},\; U = F - v_1\frac{\partial F}{\partial v_1} - v_2\frac{\partial F}{\partial v_2}$$
$$V_1 = \frac{\partial F}{\partial x} + v_1\frac{\partial F}{\partial u},\quad V_2 = \frac{\partial F}{\partial t} + v_2\frac{\partial F}{\partial u}, \tag{9.8.40}$$
$$V_{11} = \frac{\partial^2 F}{\partial x^2} + 2v_1\frac{\partial^2 F}{\partial x\partial u} + v_1^2\frac{\partial^2 F}{\partial u^2} + v_{11}\Big(2\frac{\partial^2 F}{\partial x\partial v_1} + 2v_1\frac{\partial^2 F}{\partial u\partial v_1} + \frac{\partial F}{\partial u}\Big)$$
$$+ 2v_{12}\Big(\frac{\partial^2 F}{\partial x\partial v_2} + v_1\frac{\partial^2 F}{\partial u\partial v_2}\Big) + v_{11}^2\frac{\partial^2 F}{\partial v_1^2} + 2v_{11}v_{12}\frac{\partial^2 F}{\partial v_1\partial v_2} + v_{12}^2\frac{\partial^2 F}{\partial v_2^2},$$
$$V_{12} = \frac{\partial^2 F}{\partial x\partial t} + v_1\frac{\partial^2 F}{\partial t\partial u} + v_2\frac{\partial^2 F}{\partial x\partial u} + v_1v_2\frac{\partial^2 F}{\partial u^2} + v_{11}\Big(\frac{\partial^2 F}{\partial t\partial v_1} + v_2\frac{\partial^2 F}{\partial u\partial v_1}\Big)$$

$$
\begin{aligned}
&+ v_{12}\Big(\frac{\partial^2 F}{\partial t\partial v_2} + v_2\frac{\partial^2 F}{\partial u\partial v_2} + \frac{\partial^2 F}{\partial x\partial v_1} + v_1\frac{\partial^2 F}{\partial u\partial v_1} + \frac{\partial F}{\partial u}\Big) + v_{22}\Big(\frac{\partial^2 F}{\partial x\partial v_2} + \\
&\quad v_1\frac{\partial^2 F}{\partial u\partial v_2}\Big) + v_{11}v_{12}\frac{\partial^2 F}{\partial v_1^2} + (v_{11}v_{22} + v_{12}^2)\frac{\partial^2 F}{\partial v_1\partial v_2} + v_{12}v_{22}\frac{\partial^2 F}{\partial v_2^2},\\
V_{22} &= \frac{\partial^2 F}{\partial t^2} + 2v_2\frac{\partial^2 F}{\partial t\partial u} + v_2^2\frac{\partial^2 F}{\partial u^2} + 2v_{12}\Big(\frac{\partial^2 F}{\partial t\partial v_1} + v_2\frac{\partial^2 F}{\partial u\partial v_1}\Big) + v_{22}\Big(\frac{\partial F}{\partial u}\\
&\quad + 2\frac{\partial^2 F}{\partial t\partial v_2} + 2v_2\frac{\partial^2 F}{\partial u\partial v_2}\Big) + v_{12}^2\frac{\partial^2 F}{\partial v_1^2} + 2v_{12}v_{22}\frac{\partial^2 F}{\partial v_1\partial v_2} + v_{22}^2\frac{\partial^2 F}{\partial v_2^2}.
\end{aligned}
$$

On the other hand, the relations (9.8.8) indicates that we may introduce the functions

$$
\begin{aligned}
F^1 &= -\,s_1^1 X - s_2^1\mathcal{T} - s^1 U - V_{11} + S^1\\
&= -\,s_1^1 X - s_2^1\mathcal{T} - s^1 U + \Phi,\\
F^2 &= -\,U + S^2 = 0.
\end{aligned}
$$

Moreover, the relations

$$
S^{112} = \frac{\partial F^1}{\partial v_{12}} = \frac{\partial \Phi}{\partial v_{12}} = 0, \quad S^{122} = \frac{\partial F^1}{\partial v_{22}} = \frac{\partial \Phi}{\partial v_{22}} = 0
$$

imply that the function Φ must be independent of the variables v_{12} and v_{22}. The isovector components that are obtained from the zero function F^2 will naturally become zero. The isovector components S^1 and S^2 follow from the general definitions as

$$
\begin{aligned}
S^1 &= -\Big(\frac{\partial^2 F}{\partial x\partial v_1} + \frac{\partial^2 F}{\partial u\partial v_1}v_1 + \frac{\partial^2 F}{\partial v_1^2}v_{11} + \frac{\partial^2 F}{\partial v_1\partial v_2}v_{12}\Big)(v_{11}+\phi)\\
&\quad + f(v_{11}+\phi) - \Big(\frac{\partial^2 F}{\partial t\partial v_1} + \frac{\partial^2 F}{\partial u\partial v_1}v_2 + \frac{\partial^2 F}{\partial v_1^2}v_{12} + \frac{\partial^2 F}{\partial v_1\partial v_2}v_{22}\Big)u\\
&\quad + \alpha^{121}v_{12} + \alpha^{122}v_{22} + \beta^1, \qquad (9.8.41)\\
S^2 &= fu - \Big(\frac{\partial^2 F}{\partial x\partial v_2} + \frac{\partial^2 F}{\partial u\partial v_2}v_1 + \frac{\partial^2 F}{\partial v_1\partial v_2}v_{11} + \frac{\partial^2 F}{\partial v_2^2}v_{12}\Big)(v_{11}+\phi)\\
&\quad - \Big(\frac{\partial^2 F}{\partial t\partial v_2} + \frac{\partial^2 F}{\partial u\partial v_2}v_{21} + \frac{\partial^2 F}{\partial v_1\partial v_2}v_{12} + \frac{\partial^2 F}{\partial v_2^2}v_{22}\Big)u\\
&\quad - \alpha^{121}v_{11} - \alpha^{122}v_{12} + \beta^2
\end{aligned}
$$

where $f, \alpha^{121} = -\,\alpha^{211}, \alpha^{122} = -\,\alpha^{212}, \beta^1$ and β^2 are arbitrary functions of the variables x, t, u, v_1 and v_2. But we can readily observe that to satisfy the constraint

$$S^2 = U = F - v_1 \frac{\partial F}{\partial v_1} - v_2 \frac{\partial F}{\partial v_2} \tag{9.8.42}$$

we are required to take

$$\frac{\partial^2 F}{\partial v_1 \partial v_2} = 0, \frac{\partial^2 F}{\partial v_2^2} = 0, \frac{\partial^2 F}{\partial x \partial v_2} + \frac{\partial^2 F}{\partial u \partial v_2} v_1 = 0, \alpha^{121} = \alpha^{122} = 0.$$

These equations lead obviously to

$$F = \alpha(x, t, u, v_1) + \beta(t) v_2.$$

Then (9.8.42) reduces to the equation

$$\beta^2 = (\dot{\beta} - f)u + \alpha - v_1 \frac{\partial \alpha}{\partial v_1}.$$

An overdot denotes again the derivative with respect to the variable t. On the other hand, since the expression

$$\begin{aligned}\Phi = &\Big(f - \frac{\partial^2 \alpha}{\partial x \partial v_1} - \frac{\partial^2 \alpha}{\partial u \partial v_1} v_1 - \frac{\partial^2 \alpha}{\partial v_1^2} v_{11}\Big)(v_{11} + \phi) \\ &- \Big(\frac{\partial^2 \alpha}{\partial t \partial v_1} + \frac{\partial^2 \alpha}{\partial u \partial v_1} v_2 + \frac{\partial^2 \alpha}{\partial v_1^2} v_{12}\Big) u + \beta^1 - \frac{\partial^2 \alpha}{\partial x^2} - 2 v_1 \frac{\partial^2 \alpha}{\partial x \partial u} \\ &- v_1^2 \frac{\partial^2 \alpha}{\partial u^2} - \Big(2 \frac{\partial^2 \alpha}{\partial x \partial v_1} + 2 v_1 \frac{\partial^2 \alpha}{\partial u \partial v_1} + \frac{\partial \alpha}{\partial u}\Big) v_{11} - \frac{\partial^2 \alpha}{\partial v_1^2} v_{11}^2\end{aligned}$$

does not depend on v_{12}, we get $\partial^2 \alpha / \partial v_1^2 = 0$ the integration of which yields simply

$$\alpha = \lambda(x, t, u) v_1 + \mu(x, t, u).$$

We thus conclude that

$$\beta^2 = \big[\dot{\beta}(t) - f(x, t, u, v_1, v_2)\big] u + \mu(x, t, u), U = \mu(x, t, u). \tag{9.8.43}$$

Finally, the condition $T = 0$ leads us to the equation

$$\frac{\partial S^1}{\partial x} + \frac{\partial S^2}{\partial t} + \frac{\partial S^1}{\partial u} v_1 + \frac{\partial S^2}{\partial u} v_2 + \frac{\partial S^1}{\partial v_1} v_{11} + \frac{\partial S^1}{\partial v_2} v_{12} = 0$$

whose explicit form can be written as

$$-(\lambda_{uu} u + \lambda_u) v_1 v_2 - (\lambda_{tu} u + \lambda_t) v_1 + (\mu_u - \lambda_{ux} u) v_2 - \lambda_{xt} u + \mu_t +$$

$$+ (v_{11} + \phi)\Big[f_x - \lambda_{xx} - \lambda_{ux}v_1 + (f_u - \lambda_{ux} - \lambda_{uu}v_1)v_1 + \Big(\frac{\partial f}{\partial v_1} - \lambda_u\Big)v_{11}$$
$$\frac{\partial f}{\partial v_2}v_{12}\Big] + \frac{\partial \beta^1}{\partial v_1}v_{11} + \Big(\frac{\partial \beta^1}{\partial v_2} - \lambda_u u\Big)v_{12} + \frac{\partial \beta^1}{\partial u}v_1 + \frac{\partial \beta^1}{\partial x} = 0$$

whence we evidently deduce the following equations

$$f_x - \lambda_{xx} - \lambda_{ux}v_1 + (f_u - \lambda_{ux} - \lambda_{uu}v_1)v_1 = 0, \tag{9.8.44}$$
$$\frac{\partial f}{\partial v_1} - \lambda_u = 0,\ \frac{\partial f}{\partial v_2} = 0,\ \frac{\partial \beta^1}{\partial v_1} = 0,\ \frac{\partial \beta^1}{\partial v_2} - \lambda_u u = 0,$$
$$-(\lambda_u u)_u v_1 v_2 + \big[\beta^1_u - (\lambda_t u)_u\big]v_1 + (\mu_u - \lambda_{ux}u)v_2 + \beta^1_x - \lambda_{xt}u + \mu_t = 0.$$

We first obtain from $(9.8.44)_{2-3}$ that

$$f = \lambda_u v_1 + \varphi(x, t, u).$$

Then $(9.8.44)_1$ gives rise to

$$\varphi_x - \lambda_{xx} + (\varphi_u - \lambda_{ux})v_1 = 0 \quad \text{or} \quad \varphi_x = \lambda_{xx},\ \ \varphi_u = \lambda_{ux}.$$

These equations determine, in turn, the function φ in the form

$$\varphi = \lambda_x + \psi(t).$$

Once again, $(9.8.44)_{4-5}$ yields easily

$$\beta^1 = \lambda_u u v_2 + b(x, t, u).$$

Thus, the equation $(9.8.44)_6$ is reduced to the form

$$\big[b_u - (\lambda_t u)_u\big]v_1 + \mu_u v_2 + b_x - \lambda_{xt}u - \mu_t = 0$$

whence we find that

$$b_u - (\lambda_t u)_u = 0,\ \ \mu_u = 0,\ \ b_x - \lambda_{xt}u - \mu_t = 0$$

and, respectively,

$$b = \lambda_t u + \nu(x, t),\ \ \mu = \mu(x, t),\ \ \nu_x + \mu_t = 0.$$

Therefore, we obtain $\mu = m_x$ and $\nu = -m_t$ where $m = m(x, t)$ is an arbitrary function. We can thus write

$$\beta^1 = \lambda_u u v_2 + \lambda_t u - m_t,\ \ f = \lambda_u v_1 + \lambda_x + \psi(t)$$

On the other hand, (9.8.43) leads us to the relation

$$\beta^2 = (\dot{\beta} - \lambda_x - \lambda_u v_1)u + m_x$$

where the arbitrary function $\psi(t)$ is absorbed into the function $\dot{\beta}(t)$ which is arbitrary as well. Consequently, the relevant isovector components that will be used in determining equivalence transformations are obtained as follows

$$\begin{aligned}
X^1 &= -\lambda(x,t,u), \quad X^2 = -\beta(t), \quad U = m_x(x,t), \\
V_1 &= \lambda_u v_1^2 + \lambda_x v_1 + m_{xx}, \\
V_{11} &= \lambda_{uu} v_1^3 + 2\lambda_{ux} v_1^2 + \lambda_{xx} v_1 + (2\lambda_x + 3\lambda_u v_1)v_{11} + m_{xxx}, \\
S^1 &= \psi(t)(v_{11} + \phi) - m_t = \psi(t)\Sigma^1 - m_t
\end{aligned}$$

where the functions $\lambda(x,t,u)$, $\beta(t)$, $m(x,t)$ and $\psi(t)$ are arbitrary. The equivalence transformations are then obtained as the solution of the following ordinary differential equations

$$\begin{aligned}
\frac{d\overline{x}}{d\epsilon} &= -\lambda(\overline{x},\overline{t},\overline{u}), \ \frac{d\overline{t}}{d\epsilon} = -\beta(\overline{t}), \ \frac{d\overline{u}}{d\epsilon} = m_{\overline{x}}(\overline{x},\overline{t}), \\
\frac{d\overline{v}_1}{d\epsilon} &= \lambda_{\overline{u}}\overline{v}_1^2 + \lambda_{\overline{x}}\overline{v}_1 + m_{\overline{x}\overline{x}}, \\
\frac{d\overline{v}_{11}}{d\epsilon} &= \lambda_{\overline{u}\overline{u}}\overline{v}_1^3 + 2\lambda_{\overline{u}\overline{x}}\overline{v}_1^2 + \lambda_{\overline{x}\overline{x}}v_1 + (2\lambda_{\overline{x}} + 3\lambda_{\overline{u}}\overline{v}_1)\overline{v}_{11} + m_{\overline{x}\overline{x}\overline{x}}, \\
\frac{d\overline{\Sigma}^1}{d\epsilon} &= \psi(\overline{t})\overline{\Sigma}^1 - \overline{m}_t.
\end{aligned}$$

under the initial conditions $\overline{x}(0) = x, \overline{t}(0) = t, \overline{u}(0) = u, \overline{v}_1(0) = v_1, \overline{v}_{11}(0) = v_{11}$ and $\overline{\Sigma}^1(0) = \Sigma^1$. ■

IX. EXERCISES

9.1. Discuss the solutions of the equation below:

$$\sum_{i=1}^{n}(u_{,i})^2 - u^2 = 0.$$

9.2. Discuss the solutions of the equation below:

$$uu_x + u_y - 1 = 0$$

9.3. Discuss the solutions of the set of equations below:

$$(u+v)u_x + (u-v)u_y - u^2 = 0, \quad (u+v)v_x + (u-v)v_y - 2v^2 = 0$$

9.4. Find the symmetry groups of *Laplace equation* $u_{xx} + u_{yy} = 0$ and explore its similarity solutions.

9.5. Find the symmetry groups of the wave equation $u_{xx} - u_{yy} = 0$ and explore its similarity solutions.

9.6. Find the symmetry groups of the equation $u_{xx} + u_{yy} = f(u)$ and admissible forms of the function f. Explore its similarity solutions.

9.7. Find the symmetry groups of the equation $u_{xx} - u_{yy} = f(u)$ and admissible forms of the function f. Explore its similarity solutions.

9.8. Find the symmetry groups of non-dimensionalised *Fokker-Planck equation* $u_t = u_{xx} + xu_x + u$ [after Dutch physicist Adriaan Daniël Fokker (1887-1972) and German physicist Max Karl Ernst Ludwig Planck (1858-1947)] encountered in statistical mechanics and explore its similarity solutions.

9.9. Find the symmetry groups of non-dimensionalised *Burgers equation* $u_y + uu_x + u_{xx} = 0$ [after Dutch physicist Johannes Martinus Burgers (1895-1981)] encountered in fluid mechanics and modelling of traffic flow and explore its similarity solutions.

9.10. Find the symmetry groups of the n-dimensional heat conduction equation $u_t = u_{,ii}$ where $\mathbf{x} \in \mathbb{R}^n$.

9.11. Discuss the symmetry groups of the biharmonic equation $\Delta^2 u = u_{,iijj} = 0$ in the manifold $\mathbb{R}^n$.

9.12. Find the symmetry groups of *Helmholtz equation* $u_{xx} + u_{yy} + u_{zz} + \lambda u = 0$ encountered in the propagation of waves in $\mathbb{R}^3$. λ is a constant.

9.13. Let us consider the first order, homogeneous partial differential equation

$$V(u) = v^i(\mathbf{x})u_{,i} = 0, \quad V = v^i(\mathbf{x})\partial_i \in T(\mathbb{R}^n)$$

in $\mathbb{R}^n$. Show that a vector field $U = u^i(\mathbf{x})\,\partial_i \in T(\mathbb{R}^n)$ generates a symmetry group of this differential equation if and only if it satisfies the condition $[V, U] = \lambda(\mathbf{x})\,V$. $\lambda(\mathbf{x})$ is a scalar-valued function.

9.14. Find the symmetry groups of *Euler equations*

$$(\partial v_i/\partial t) + v_j v_{i,j} = -p_{,i}, \quad v_{i,i} = 0, \ \ i = 1, 2, 3$$

governing the motion of incompressible fluids where v_i are components of the velocity vector and p is the pressure.

9.15. Determine the symmetry groups of non-dimensional *Navier-Stokes equations*

$$(\partial v_i/\partial t) + v_j v_{i,j} = -p_{,i} + (1/R_e)v_{i,jj}, \quad v_{i,i} = 0, \ \ i = 1, 2, 3$$

[after Stokes and French engineer and mathematician Claude Louis Henri Navier (1785-1836)] governing the motion of incompressible viscous fluids. The constant R_e is called *Reynolds number* [after English engineer and mathematician Osborne Reynolds (1842-1912)].

9.16. Determine equivalence transformations of the non-linear, 1-dimensional heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(\kappa(u)\frac{\partial u}{\partial x}\Big) + h(x, t, u)$$