CHAPTER VI

HOMOTOPY OPERATOR

6.1. SCOPE OF THE CHAPTER

In this section, we shall attempt to investigate certain fundamental properties of exterior differential forms in depth. The most powerful tool that we can employ for this purpose is the homotopy operator. However, this operator can only be defined on manifolds possessing a particular structure. This structure is treated in Sec. 6.2. A manifold is called locally contractible if every open set in its atlas can be smoothly shrunk to one of its interior points. This situation is realised if the homeomorphic image of that open set is a star-shaped region in the Euclidean space. In Sec. 6.3, the homotopy operator mapping the exterior algebra into itself is defined, its various properties are unravelled and the Poincaré lemma stating that every closed form is locally exact is demonstrated as a very important application of this operator. Sec. 6.4 is concerned with the proof that every exterior form is locally expressible as the sum of an exact form and an antiexact form occupying the kernel of the homotopy operator. Then the basic properties of antiexact forms are studied in detail. This leads to the conclusion that the entire exterior algebra is actually generated by antiexact forms. In Sec. 6.5, we inquire the effect of the change of the centre of contraction on the homotopy operator. We define in Sec. 6.6 the Darboux classes of 1-forms and introduce their canonical forms. Canonical forms of closed 2-forms are elicited by making use of the Poincaré lemma. We obtain the solution of an exterior differential equation in Sec. 6.7 and a system of exterior differential equations in Sec. 6.8 by resorting to properties of antiexact forms and the homotopy operator.

6.2. STAR-SHAPED REGIONS

Let M be a differentiable manifold. Let us take a point $p_0 \in M$ into account. If we can find a *smooth*, i.e., C^∞ function $h : M \times I \to M$ where $I = [0, 1]$ denoted by $h(p;t) = h_t(p) \in M$ on which we shall impose the

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restriction $h(p;0) = p_0$ and $h(p;1) = p$ for each point $p \in M$, then we say that the manifold M is ***contractible*** to the point p_0. Contractibility can also be defined locally. Let us consider a local chart (U, φ). We know that $U \subseteq M$ is an open set and $\varphi : U \to \mathbb{R}^m$ is a homeomorphism so that $V = \varphi(U) \subseteq \mathbb{R}^m$ is also an open set. Let us assume that the set U is contractible to a point $p_0 \in U$. If all charts of an atlas have this property, then the manifold M is called a ***locally contractible manifold***. Such a manifold cannot be shrunk smoothly to a point, but each one of the open sets covering this manifold is contractible to a point inside it. If the open set V, which is the homeomorphic image of the open set U, has a suitable structure in the manifold $\mathbb{R}^m$, then we can easily show that U is contractible. To this end, let us assume that we can find a mapping $h' : V \times I \to V$ and a point $\mathbf{x}_0 \in V$ such that we are able to write $h'(\mathbf{x};t) = h'_t(\mathbf{x}) = (1-t)\mathbf{x}_0 + t\mathbf{x} \in V$ for all points $\mathbf{x} \in V$. This expression signifies that a straight line joining any point $\mathbf{x}$ in V to the ***centre point*** $\mathbf{x}_0$ stays entirely in V. Such a region is called a ***star-shaped region*** (Fig. 6.2.1).

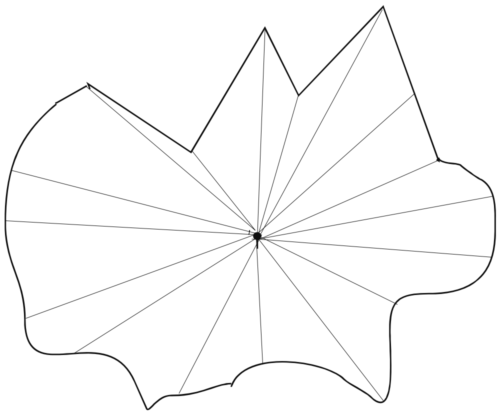

Fig. 6.2.1. Star-shaped region in the Euclidean space.

Evidently, every convex set in $\mathbb{R}^m$ is star-shaped and it is easily shown that open balls in $\mathbb{R}^m$ are convex. Let us consider an open ball in $\mathbb{R}^m$ given by $B_r(\mathbf{x}_0) = \|\mathbf{x} - \mathbf{x}_0\| < r$ where $\mathbf{x}, \mathbf{x}_0 \in \mathbb{R}^m$ and $r > 0$. By using the triangle inequality, we obtain for points $\mathbf{x}, \mathbf{y} \in B_r(\mathbf{x}_0)$ in $\mathbb{R}^m$ and a parameter t satisfying $0 \le t \le 1$

$$\begin{aligned} \|(1-t)\mathbf{x} + t\mathbf{y} - \mathbf{x}_0\| &= \|(1-t)(\mathbf{x} - \mathbf{x}_0) + t(\mathbf{y} - \mathbf{x}_0)\| \\ &\le (1-t)\|(\mathbf{x} - \mathbf{x}_0)\| + t\|(\mathbf{y} - \mathbf{x}_0)\| \\ &< (1-t)r + tr = r \end{aligned}$$

This result shows that $(1-t)\mathbf{x}+t\mathbf{y} \in B_r(\mathbf{x}_0)$. Therefore, any open ball in $\mathbb{R}^m$ is a convex set.

The open set U is homeomorphic to an open set of $\mathbb{R}^m$ that is expressible as some union of open balls. Hence, U itself is the union of inverse images of some open balls implying that a component open subset of U is homeomorphic to a *convex open ball* with centre at a point $\mathbf{x}$. We thus conclude that every manifold is locally contractible and is locally homeomorphic to a star-shaped region. Conversely, when $V \in \mathbb{R}^m$ is a star-shaped open set, if we define on an open set $U = \varphi^{-1}(V)$ of the manifold M a mapping $h_t = \varphi^{-1} \circ h'_t \circ \varphi$ such that $h_t(p) \in U$ for all points $p \in U$ and $t \in [0,1]$, then we immediately observe that the set U can be contracted to the point $p_0 = \varphi^{-1}(\mathbf{x}_0)$ by the mapping h_t.

The entire manifold $\mathbb{R}^m$ is star-shaped with respect to the origin **0**, in fact to every point of $\mathbb{R}^m$. Hence, a manifold M is contractible if it is homeomorphic to the manifold $\mathbb{R}^m$. That the converse statement is not generally true can be demonstrated by constructing a counter example. Three dimensional Whitehead manifold is obtained by embedding a *solid torus* T_1 (a solid torus is a filled-in torus $\mathbb{T}^2$) inside *three dimensional sphere* $\mathbb{S}^3$, then a solid torus T_2 inside T_1 and continuing this way *ad infinitum* [discovered by English mathematician John Henry Constantine Whitehead (1904-1960)]. Hence, we can formally represent the Whitehead manifold by $\bigcap_{i=1}^{\infty} T_i$. A rather small part of the Whitehead manifold is depicted in Fig. 6.2.2. This manifold is contractible but it is not homeomorphic to $\mathbb{R}^3$.

Fig. 6.2.2. Whitehead manifold.

Let a form field $\omega \in \Lambda^k(M)$ be represented in a local chart by

$$\omega(\mathbf{x}) = \frac{1}{k!}\,\omega_{i_1\cdots i_k}(\mathbf{x})dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

This form will of course be defined on an open set U of the manifold M. We can define a new k-form $\overline{\omega}$ depending on a parameter $t \in [0,1]$ in the following manner

$$\overline{\omega}(\mathbf{x};t) = \frac{1}{k!}\,\omega_{i_1\cdots i_k}\big[\mathbf{x}_0 + t(\mathbf{x} - \mathbf{x}_0)\big]dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \tag{6.2.1}$$

If U is contractible, then $\overline{\omega}$ is specified everywhere in U. It is clear that $\overline{\omega}(\mathbf{x};0) = \omega(\mathbf{x}_0)$ and $\overline{\omega}(\mathbf{x};1) = \omega(\mathbf{x})$. Let us now define the new independent variables by $u^i = x_0^i + t(x^i - x_0^i),\ t \in [0,1],\ i = 1,\ldots,m$. If we write $\overline{\omega}_{i_1\cdots i_k}(\mathbf{u}) = \omega_{i_1\cdots i_k}\big[\mathbf{x}_0 + t(\mathbf{x} - \mathbf{x}_0)\big]$, it then follows from (6.2.1) that

$$\begin{aligned} d\overline{\omega} &= \frac{1}{k!}\,d\overline{\omega}_{i_1\cdots i_k}(\mathbf{u}) \wedge\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= \frac{1}{k!}\frac{\partial\overline{\omega}_{i_1\cdots i_k}}{\partial u^i}\,du^i \wedge\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= t\,\frac{1}{k!}\frac{\partial\overline{\omega}_{i_1\cdots i_k}}{\partial u^i}\,dx^i \wedge\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= t\,\overline{d\omega} \end{aligned} \tag{6.2.2}$$

We denote the radius vector in the region V which is the homeomorphic image of U with respect to the point $\mathbf{x}_0$ by the relation

$$\begin{aligned} \mathcal{H}(\mathbf{x}) &= (x^i - x_0^i)\frac{\partial}{\partial x^i} \\ &= \frac{d}{dt}\big[x_0^i + t(x^i - x_0^i)\big]\frac{\partial}{\partial x^i} \end{aligned} \tag{6.2.3}$$

We thus get $\mathcal{H}(\mathbf{x}_0) = 0$. It is clear that one finds

$$\begin{aligned} \overline{\mathcal{H}}(\mathbf{x};t) &= \mathcal{H}\big[\mathbf{x}_0 + t(\mathbf{x} - \mathbf{x}_0)\big] = t(x^i - x_0^i)\frac{\partial}{\partial x^i} \\ &= t\mathcal{H}(\mathbf{x}) \end{aligned} \tag{6.2.4}$$

for $t \in [0,1]$.

6.3. HOMOTOPY OPERATOR

Let a form $\omega \in \Lambda^k(M)$ be defined on an open set $U \subseteq M$ that is contractible to a point $p_0 \in M$. We will assume that the homeomorphic image

$V \subseteq \mathbb{R}^m$ of the set U is a star-shaped region. We define *the linear operator* $H : \Lambda^k(U) \to \Lambda^{k-1}(U)$ by the following expression in local coordinates

$$\begin{aligned} H\omega &= \int_0^1 \mathbf{i}_{\mathcal{H}}\big(\overline{\omega}(\mathbf{x};t)\big)t^{k-1}dt \qquad (6.3.1)\\ &= \frac{1}{(k-1)!}\int_0^1 t^{k-1}(x^{i_1}-x_0^{i_1})\,\omega_{i_1 i_2\cdots i_k}\big[\mathbf{x}_0+t(\mathbf{x}-\mathbf{x}_0)\big]\,dt\,dx^{i_2}\wedge\cdots\wedge dx^{i_k}. \end{aligned}$$

Since V is star-shaped, the form ω is prescribed at every point of the open set U. Therefore, the operator H introduced by (6.3.1) is well defined on the exterior algebra $\Lambda(U)$. H is called ***the homotopy operator***. This definition will automatically lead to the result $Hf = 0$ for $f \in \Lambda^0(M)$.

If we choose $\mathbf{x}_0$ at the origin $\mathbf{0}$ of the local coordinate system without loss of generality, then the homotopy operator takes the form

$$\begin{aligned} H\omega(\mathbf{x}) &= \int_0^1 \mathbf{i}_{\mathcal{H}}\omega(t\mathbf{x})t^{k-1}dt\\ &= \frac{1}{(k-1)!}\int_0^1 t^{k-1}x^{i_1}\,\omega_{i_1 i_2\cdots i_k}(t\mathbf{x})\,dt\,dx^{i_2}\wedge\cdots\wedge dx^{i_k} \end{aligned}$$

Let us now consider vector fields $V_1, V_2, \dots, V_{k-1} \in T(M)$. The above expression implies that

$$H\omega(\mathbf{x})(V_1,\dots,V_{k-1}) = \int_0^1 \omega(t\mathbf{x})(\mathbf{x},V_1,\dots,V_{k-1})t^{k-1}dt$$

The main properties of the homotopy operator are embodied in the following theorem.

Theorem 6.3.1. *The homotopy operator H has the properties listed below*:

(i). $dH + Hd = i_\Lambda$ *if* $k \geq 1$.
$Hdf(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{x}_0)$ *if* $k = 0$.
(ii). $H \circ H = H^2 = 0$ *and* $H\omega(\mathbf{x}_0) = 0$.
(iii). $HdH = H$ *and* $dHd = d$.
(iv). $HdHd = (Hd)^2 = Hd$ *and* $dHdH = (dH)^2 = dH$,
$(dH)(Hd) = dH^2d = 0$ *and* $(Hd)(dH) = Hd^2H = 0$.
(v). $\mathbf{i}_{\mathcal{H}} \circ H = 0$ *and* $H \circ \mathbf{i}_{\mathcal{H}} = 0$.

(i). We consider a form $\omega \in \Lambda^k(M), k \geq 1$. We shall try to evaluate explicitly the action of the operator $d \circ H + H \circ d$ on this form. At the first step, we obtain

$$
\begin{aligned}
(dH + Hd)\omega &= d(H\omega) + H(d\omega) \\
&= \int_0^1 d\big(\mathbf{i}_{\mathcal{H}}(\overline{\omega})\big)t^{k-1}dt + \int_0^1 \mathbf{i}_{\mathcal{H}}(\overline{d\omega})t^k dt \\
&= \int_0^1 t^{k-1}\big[d\big(\mathbf{i}_{\mathcal{H}}(\overline{\omega})\big) + \mathbf{i}_{\mathcal{H}}(d\overline{\omega})\big]dt \\
&= \int_0^1 t^{k-1}\pounds_{\mathcal{H}}\overline{\omega}\, dt
\end{aligned}
$$

where we have employed the Cartan magic formula. On the other hand, Lie derivative with respect to the vector $\mathcal{H}$ yields

$$
\pounds_{\mathcal{H}}\overline{\omega} = \frac{1}{k!}\,(\pounds_{\mathcal{H}}\overline{\omega})_{i_1\cdots i_k}\, dx^{i_1}\wedge\cdots\wedge dx^{i_k}
$$

where the coefficients follow from (5.11.4) as

$$
\begin{aligned}
(\pounds_{\mathcal{H}}\overline{\omega})_{i_1\cdots i_k} &= (x^i - x^i_0)\frac{\partial\overline{\omega}_{i_1 i_2\cdots i_k}}{\partial x^i} + \sum_{r=1}^{k}\overline{\omega}_{i_1\cdots i_{r-1} i i_{r+1}\cdots i_k}\frac{\partial(x^i - x^i_0)}{\partial x^{i_r}} \\
&= (x^i - x^i_0)\frac{\partial\overline{\omega}_{i_1\cdots i_k}}{\partial x^i} + \sum_{r=1}^{k}\overline{\omega}_{i_1\cdots i_{r-1} i i_{r+1}\cdots i_k}\delta^i_{i_r} \\
&= t(x^i - x^i_0)\frac{\partial\overline{\omega}_{i_1\cdots i_k}}{\partial u^i} + k\overline{\omega}_{i_1\cdots i_k} = t\frac{d\overline{\omega}_{i_1\cdots i_k}}{dt} + k\overline{\omega}_{i_1\cdots i_k}.
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
(dH + Hd)\omega &= \frac{1}{k!}\int_0^1 \Big[t^k\frac{d\overline{\omega}_{i_1\cdots i_k}}{dt} + kt^{k-1}\overline{\omega}_{i_1\cdots i_k}\Big]dt\, dx^{i_1}\wedge\cdots\wedge dx^{i_k} \\
&= \frac{1}{k!}\int_0^1 \frac{d}{dt}(t^k\overline{\omega}_{i_1\cdots i_k})\, dt\, dx^{i_1}\wedge\cdots\wedge dx^{i_k} \\
&= \frac{1}{k!}\big(t^k\omega_{i_1\cdots i_k}\big[\mathbf{x}_0 + t(\mathbf{x} - \mathbf{x}_0)\big]\big)\Big|_{t=0}^{t=1} dx^{i_1}\wedge\cdots\wedge dx^{i_k} \\
&= \frac{1}{k!}\omega_{i_1\cdots i_k}(\mathbf{x})\, dx^{i_1}\wedge\cdots\wedge dx^{i_k}.
\end{aligned}
$$

This means that for every form $\omega\in\Lambda(M)$ with non-zero degree, we find

$$
(dH + Hd)\omega = \omega \quad \text{or} \quad dH + Hd = i_\Lambda \tag{6.3.2}
$$

where i_Λ denotes the identity mapping on the exterior algebra $\Lambda(M)$. When $k = 0$, on resorting to (6.2.2) for every function $f\in\Lambda^0(M)$ we arrive at the result

$$\begin{aligned} dHf + Hdf &= Hdf \qquad (6.3.3)\\ &= \int_0^1 \mathbf{i}_{\mathcal{H}}(\overline{df})\,dt = \int_0^1 \frac{1}{t}\,\mathbf{i}_{\mathcal{H}}(d\overline{f})\,dt \\ &= \int_0^1 \frac{1}{t}\frac{\partial \overline{f}}{\partial x^i}\,\mathbf{i}_{\mathcal{H}}(dx^i)\,dt = \int_0^1 \frac{x^i - x_0^i}{t}\,t\,\frac{\partial \overline{f}}{\partial u^i}\,dt \\ &= \int_0^1 (x^i - x_0^i)\,\frac{\partial \overline{f}}{\partial u^i}\,dt = \int_0^1 \frac{d\overline{f}}{dt}\,dt = \overline{f}\big|_0^1 = f(\mathbf{x}) - f(\mathbf{x}_0). \end{aligned}$$

(ii). Since $H\omega(\mathbf{x}_0) = \displaystyle\int_0^1 \mathbf{i}_{\mathcal{H}(\mathbf{x}_0)}\big(\omega(\mathbf{x}_0)\big)t^{k-1}dt$ and $\mathcal{H}(\mathbf{x}_0) = 0$, we obtain $H\omega(\mathbf{x}_0) = 0$. On the other hand, for a form $\omega \in \Lambda^k(M)$ we find

$$\begin{aligned} H^2\omega &= \int_0^1 s^{k-2}\,\mathbf{i}_{\mathcal{H}}\Big[\int_0^1 t^{k-1}\,\overline{\mathbf{i}_{\mathcal{H}}(\overline{\omega}(t))}\,dt\Big](s)\,ds \\ &= \int_0^1\int_0^1 t^{k-1}s^{k-2}\,\mathbf{i}_{\mathcal{H}}\big[\mathbf{i}_{\overline{\mathcal{H}}(s)}\big(\overline{\overline{\omega}(t)}\big)(s)\big]\,dt\,ds \\ &= \int_0^1\int_0^1 t^{k-1}s^{k-2}\,\mathbf{i}_{\mathcal{H}}\big[\mathbf{i}_{s\mathcal{H}}\big(\overline{\overline{\omega}(t)}\big)(s)\big]\,dt\,ds \\ &= \int_0^1\int_0^1 t^{k-1}s^{k-1}\,\mathbf{i}_{\mathcal{H}}^2\big(\overline{\overline{\omega}(t)}\big)(s)\,dt\,ds = 0 \end{aligned}$$

where we have employed (6.2.4) and the relation $\mathbf{i}_{\mathcal{H}}^2 = 0$.

(iii). Since $d^2 = 0$ and $H^2 = 0$, the relation (6.3.2) leads right away to $dHd = d$ and $HdH = H$.

(iv). If we make use of the property (iii) in expressions $(Hd)^2 = HdHd$ and $(dH)^2 = dHdH$, we find that $(Hd)^2 = Hd$ and $(dH)^2 = dH$.

(v). This property can also be demonstrated quite easily. If we consider a form $\omega \in \Lambda^k(M)$, we obtain

$$\begin{aligned} \mathbf{i}_{\mathcal{H}}(H\omega) &= \mathbf{i}_{\mathcal{H}}\Big[\int_0^1 \mathbf{i}_{\mathcal{H}}(\overline{\omega})\,t^{k-1}dt\Big] = \int_0^1 \mathbf{i}_{\mathcal{H}}\big(\mathbf{i}_{\mathcal{H}}(\overline{\omega})\big)t^{k-1}dt = 0 \\ H\mathbf{i}_{\mathcal{H}}(\omega) &= \int_0^1 t^{k-2}\,\mathbf{i}_{\mathcal{H}}\big(\overline{\mathbf{i}_{\mathcal{H}}(\omega)}\big)dt = \int_0^1 t^{k-2}\,\mathbf{i}_{\mathcal{H}}\big(\mathbf{i}_{\overline{\mathcal{H}}}(\overline{\omega})\big)dt \\ &= \int_0^1 t^{k-1}\,\mathbf{i}_{\mathcal{H}}\big(\mathbf{i}_{\mathcal{H}}(\overline{\omega})\big)dt = 0 \end{aligned}$$

because of the relation $\mathbf{i}_{\mathcal{H}} \circ \mathbf{i}_{\mathcal{H}} = 0$. □

In case we can define the homotopy operator, the celebrated Poincaré lemma can readily be proven.

Theorem 6.3.2 (The Poincaré Lemma). *An exterior form defined on an open set $U \subseteq M$ contractible to one of its interior points is closed if and only if it is exact on U.*

If a form ω is exact, that is, if one is able to write $\omega = d\Omega$, this form is closed because $d\omega = 0$. Conversely, let us now assume that $\omega \in \Lambda^k(M)$ is a closed form. When the homeomorphic image of U in $\mathbb{R}^m$ is a star-shaped region, we will be free to employ the homotopy operator. Since $d\omega = 0$, we then obtain

$$\omega = dH\omega + Hd\omega = d(H\omega) = d\Omega$$

where we have defined the form $\Omega = H\omega \in \Lambda^{k-1}(M)$. Thus, the closed form ω is likewise an exact form on U. Since every chart of an m-dimensional differentiable manifold M is homeomorphic to an open set of $\mathbb{R}^m$, the Poincaré Lemma is locally valid. Therefore, *every closed form on M is locally, in other words, in an open neighbourhood of every point $p \in M$, is an exact form*. However, this statement is generally not true globally. This means that we cannot be sure in general the existence of a form Ω defined over the entire manifold M so that a closed form is expressed as $\omega = d\Omega$. For instance, if we have prescribed a closed form on the punctured differentiable manifold $\mathbb{R}^m - \{\mathbf{0}\}$, we cannot validate the Poincaré Lemma on any open set containing the point $\{\mathbf{0}\}$. □

If we take the manifold $\mathbb{R}^m, m > 0$ into consideration, we know that the whole manifold can be contracted, say, to the point $\mathbf{0}$. Hence, according to the Poincaré lemma *every closed form defined on the entire $\mathbb{R}^m$ is globally exact*. Similarly, we can say that *every closed form on a contractible manifold M is globally exact.*

Example 6.3.1. A form $\omega \in \Lambda^2(\mathbb{R}^3)$ is given by

$$\omega = -2(x+y)z\,dx \wedge dy + x^2 dy \wedge dz + y^2 dz \wedge dx.$$

We observe at once that $d\omega = 0$. $\mathbb{R}^3$ is star-shaped with respect to the centre $\mathbf{0}$. Thus, the radius vector can be taken as $\mathcal{H} = x\partial_x + y\partial_y + z\partial_z$. We can then evaluate the form $H\omega$ easily as

$$\begin{aligned}\Omega = H\omega = \int_0^1 t\big[-2t^2(x+y)z(x\,dy - y\,dx) + t^2x^2(y\,dz - z\,dy)\\ + t^2y^2(z\,dx - x\,dz)\big]dt\\ = \frac{1}{4}\big[yz(3y+2x)\,dx - xz(3x+2y)\,dy + xy(x-y)\,dz\big] \in \Lambda^1(\mathbb{R}^3).\end{aligned}$$

We can readily verify that the relation $\omega = d\Omega$ holds.

Let us now consider a more general 2-form defined by

$$\omega = R\,dx \wedge dy + P\,dy \wedge dz + Q\,dz \wedge dx \in \Lambda^2(\mathbb{R}^3).$$

If ω is a closed form, that is, if $d\omega = 0$, then we have to impose the following restriction on the functions $P(x,y,z), Q(x,y,z), R(x,y,z)$:

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0.$$

In this situation, on resorting to the homotopy operator, we can determine the form $\Omega = H\omega$ as follows

$$\begin{aligned}\Omega = &\Big(\int_0^1 \big[tzQ(tx,ty,tz) - tyR(tx,ty,tz)\big]\,dt\Big)dx\\ &+\Big(\int_0^1 \big[txR(tx,ty,tz) - tzP(tx,ty,tz)\big]\,dt\Big)dy\\ &+\Big(\int_0^1 \big[tyP(tx,ty,tz) - txQ(tx,ty,tz)\big]\,dt\Big)dz.\end{aligned}$$

If we recall the restriction imposed of the functions P, Q, R, we can verify at once that we get the relation $\omega = d\Omega$. This is of course valid on the entire manifold $\mathbb{R}^3$. ■

It is clear that the form Ω introduced in the foregoing theorem cannot be determined uniquely. Evidently, for an arbitrary form $\sigma \in \Lambda^{k-2}(M)$, the form $\Omega' = \Omega + d\sigma$ will also satisfy the relation $\omega = d\Omega'$.

We had denoted the graded algebra $\mathcal{E}(U)$ of exact forms on an open subset $U \subseteq M$. For a form $\omega \in \Lambda^k(U)$, we get $dH\omega \in \mathcal{E}^k(U)$ implying that $dH : \Lambda(U) \to \mathcal{E}(U)$. But, the restriction $dH|_{\mathcal{E}^k(U)} : \mathcal{E}^k(U) \to \mathcal{E}^k(U)$ satisfies the relation $\omega = dH|_{\mathcal{E}^k(U)}\omega$. Hence, we may regard the operator d as the inverse of the operator H on $\mathcal{E}^k(U)$.

Let M and N be, respectively, m- and n-dimensional differentiable manifolds with $n \geq m$. $\phi : M \to N$ is a smooth mapping. We consider an open subset $U \subseteq M$. Let us assume that the mapping ϕ is a diffeomorphism on U. Thus $\phi^{-1} : \phi(U) \to U$ is a smooth mapping. If U is contractible to a point $p_0 \in U$, then the region $\phi(U) \subset N$ can also be contracted to the point $\phi(p_0) \in \phi(U)$ and since ϕ^{-1} is continuous on $\phi(U) \subseteq N$, we see that $\phi(U)$ is also an open subset. The mappings ϕ and ϕ^{-1} give obviously rise to pullback mappings $\phi^* : \Lambda\big(\phi(U)\big) \to \Lambda(U)$ and $(\phi^*)^{-1} : \Lambda(U) \to \Lambda\big(\phi(U)\big)$. Let H be the homotopy operator defined on the region U. If $\omega \in \Lambda\big(\phi(U)\big)$, we have $\phi^*\omega \in \Lambda(U)$ and we can write

$$dH\phi^*\omega + Hd\phi^*\omega = \phi^*\omega.$$

According to Theorem 5.8.2, it is possible to write $Hd\phi^*\omega = H\phi^*d\omega$. Let us now define an operator $H^* : \Lambda^k\big(\phi(U)\big) \to \Lambda^{k-1}\big(\phi(U)\big)$ through the relation

$$\phi^* H^* = H\phi^* \quad \text{or} \quad H^* = (\phi^*)^{-1} H\phi^*. \tag{6.3.4}$$

We thus obtain

$$d\phi^* H^*\omega + H\phi^* d\omega = \phi^* dH^*\omega + \phi^* H^* d\omega = \phi^*(dH^*\omega + H^* d\omega) = \phi^*\omega.$$

By applying the operator $(\phi^*)^{-1}$ on this expression, we find that

$$dH^*\omega + H^* d\omega = \omega \;\text{ or }\; dH^* + H^* d = i_{\Lambda(\phi(U))}.$$

H^* is then called the homotopy operator generated by the mapping ϕ.

6.4. EXACT AND ANTIEXACT FORMS

Let $U \subseteq M$ be a contractible open set on which the homotopy operator can be defined where M is an m-dimensional smooth manifold. Thus, on taking heed of the relation (6.3.2) it becomes possible to express a form $\omega \in \Lambda(U)$ in the following manner

$$\omega = dH\omega + Hd\omega = \omega_e + \omega_a \tag{6.4.1}$$

where we introduce the following forms with degree preserving operations

$$\omega_e = dH\omega, \quad \omega_a = Hd\omega = \omega - \omega_e. \tag{6.4.2}$$

They will be called as the ***exact*** and ***antiexact parts*** of the form ω, respectively. (6.4.2) then leads to the result $H\omega_a = H^2 d\omega = 0$. Hence, antiexact forms are located in the null space or the kernel of the linear operator H. Let us denote the set of all antiexact forms of the module $\Lambda^k(U)$ by $\mathcal{A}^k(U)$. $\mathcal{E}^0(U)$ is of course empty. On the other hand, we can write $f(\mathbf{x}) - f(\mathbf{x}_0) = Hdf = f_a$ for all $f \in \Lambda^0(U)$. So there will be no harm in assuming that $\mathcal{A}^0(U) = \Lambda^0(U)$. We can now easily demonstrate the following lemmas.

Lemma 6.4.1. *The operator dH maps $\mathcal{E}^k(U)$ onto $\mathcal{E}^k(U)$ and $\Lambda(U)$ onto $\mathcal{E}(U)$. Furthermore, the operator d is the inverse of the operator H when the domain of H is restricted to $\mathcal{E}^k(U)$.*

In view of (6.4.2), $dH\omega$ is exact for every $\omega \in \Lambda^k(U)$ thus dH maps $\Lambda^k(U)$ into $\mathcal{E}^k(U)$. If $\omega \in \mathcal{E}^k(U)$, then $\omega = d\alpha$ where $\alpha \in \Lambda^{k-1}(U)$ so we get $dH\omega = dHd\alpha = d\alpha = \omega$. Hence dH restricted to $\mathcal{E}^k(U)$ is the identity operator. This also shows that dH is a surjective mapping.

Lemma 6.4.2. *The necessary and sufficient conditions to completely determine the set $\mathcal{A}^k(U), k \geq 1$ of antiexact forms are given as follows*

$$\mathcal{A}^k(U) = \{\alpha \in \Lambda^k(U) : \mathbf{i}_{\mathcal{H}}(\alpha) = 0, \alpha(\mathbf{x}_0) = 0, k > 0\}.$$

For all $\omega \in \Lambda^k(U)$, according to Theorem 6.3.1 (v) and (ii) we find that antiexact parts satisfy $\mathbf{i}_{\mathcal{H}}(\omega_a) = 0$ and $\omega_a(\mathbf{x}_0) = 0$. Conversely, let us assume that a form $\alpha \in \Lambda^k(U)$ satisfies the relations $\mathbf{i}_{\mathcal{H}}(\alpha) = 0, \alpha(\mathbf{x}_0) = 0$. For an arbitrary form $\beta \in \Lambda^{k-1}(U)$, let us write $\omega = d\beta + \alpha$. However, we have $H\alpha = \int_0^1 t^{k-1}\mathbf{i}_{\mathcal{H}}(\overline{\alpha})\,dt = \int_0^1 t^{k-2}\mathbf{i}_{\overline{\mathcal{H}}}(\overline{\alpha})\,dt = \int_0^1 t^{k-2}\overline{\mathbf{i}_{\mathcal{H}}(\alpha)}\,dt = 0$ so that we get $H\omega = Hd\beta + H\alpha = Hd\beta$ and $\omega_e = dH\omega = dHd\beta = d\beta$. Hence, we obtain $\alpha = \omega - \omega_e = \omega_a = Hd\omega$. This equality does not lead to a contradiction if only $\alpha(\mathbf{x}_0) = 0$. Thus we find $\alpha \in \mathcal{A}^k(U)$. □

Lemma 6.4.3. *The operator Hd maps $\Lambda^k(U)$ onto $\mathcal{A}^k(U)$ and $\mathcal{A}^m(U) = 0$ on the m-dimensional open set U. Furthermore, the operator H is the inverse of the operator d when the domain of H is restricted to $\mathcal{A}^k(U)$.*

We obviously have $Hd : \Lambda^k(U) \to \mathcal{A}^k(U)$. Let us consider the form $\omega_a = Hd\omega \in \mathcal{A}^k(U)$ where $\omega \in \Lambda^k(U)$. We then obtain $Hd\omega_a = (Hd)^2\omega = Hd\omega = \omega_a$. This also shows that Hd restricted to $\mathcal{A}^k(U)$ is the identity operator for $k \geq 1$. If $k = 0$, then the same situation is also realised up to a constant: $f(\mathbf{x}) = Hdf(\mathbf{x}) + f(\mathbf{x}_0)$. If we pay attention to the sequence $\Lambda^k(U) \xrightarrow{d} \Lambda^{k+1}(U) \xrightarrow{H} \Lambda^k(U)$, we observe at once that $\mathcal{A}^m(U) = 0$ on the m-dimensional open set U of the manifold M^m. □

Various properties of antiexact forms are collected in the theorem below.

Theorem 6.4.1. *Antiexact forms possess the following properties*:

(i). $\mathcal{A}^k(U) \subseteq \mathcal{N}(H) = \mathrm{Ker}\,(H), k \geq 0$.

(ii). *If $\alpha \in \mathcal{A}^k(U)$ and $\beta \in \mathcal{A}^l(U)$, then $\alpha \wedge \beta \in \mathcal{A}^{k+l}(U)$.*

(iii). *For $k \geq 1$, $\mathcal{A}^k(U)$ is a module on $\Lambda^0(U)$.*

(i). We have seen above that $H\alpha = 0$ because of $\mathbf{i}_{\mathcal{H}}(\alpha) = 0$. Hence, we find that $\mathcal{A}^k(U) \subseteq \mathcal{N}(H)$.

(ii). For $k = 0$, this statement becomes true automatically. Therefore, we take the case $\min\{k, l\} \geq 1$ into account. Since, the antiexact form factors vanish at the point $\mathbf{x}_0$, we naturally obtain $(\alpha \wedge \beta)(\mathbf{x}_0) = 0$. On the other hand, we get

$$\mathbf{i}_{\mathcal{H}}(\alpha \wedge \beta) = (\mathbf{i}_{\mathcal{H}}\alpha) \wedge \beta + (-1)^k \alpha \wedge (\mathbf{i}_{\mathcal{H}}\beta) = 0 + 0 = 0.$$

We thus conclude that $\alpha \wedge \beta \in \mathcal{A}^{k+l}(U)$.

(iii). The set $\mathcal{A}^k(U)$ is a submodule of the module $\Lambda^k(U)$. If $\alpha, \beta \in \mathcal{A}^k(U)$, we get $\mathbf{i}_{\mathcal{H}}(\alpha + \beta) = \mathbf{i}_{\mathcal{H}}(\alpha) + \mathbf{i}_{\mathcal{H}}(\beta) = 0$,$(\alpha + \beta)(\mathbf{x}_0) = \alpha(\mathbf{x}_0) + \beta(\mathbf{x}_0) = 0$ and $\mathbf{i}_{\mathcal{H}}(f\alpha) = f\mathbf{i}_{\mathcal{H}}(\alpha) = 0$ and $f(\mathbf{x}_0)\alpha(\mathbf{x}_0) = 0$ for all

$f \in \Lambda^0(U)$. We thus have $\alpha + \beta \in \mathcal{A}^k(U)$ and $f\alpha \in \mathcal{A}^k(U)$. □

When $\omega \in \mathcal{A}^k(U)$, if we write $d\omega = \alpha \in \Lambda^{k+1}(U)$ then we are led to $\omega = H\alpha$. Likewise, when $f \in \Lambda^0(U)$, if we write $df = \alpha \in \Lambda^1(U)$ we get $f(\mathbf{x}) = H\alpha + f(\mathbf{x}_0)$. We can immediately observe that $\mathcal{A}(U) = \overset{m}{\underset{k=0}{\oplus}} \mathcal{A}^k(U)$ is a graded algebra that is a subalgebra of $\Lambda(U)$. Furthermore, for any form $\omega \in \Lambda(U)$ we obtain $Hd\omega \in \mathcal{A}(U)$ so that we can symbolically write the relation $\mathcal{A}(U) = Hd\big(\Lambda(U)\big)$. Because of the identity $Hd = (Hd)^2$, we are led to the conclusion that Hd is a ***projection operator***. Hence, we can say that *the algebra $\mathcal{A}(U)$ is a Hd-projection of the algebra $\Lambda(U)$*.

With the information we have acquired so far, we can now manage to better identify the characteristics of the operator H. For $k \geq 0$, it is possible to express $H : \Lambda^{k+1}(U) \to \mathcal{A}^k(U)$ implying that $\mathcal{A}^k(U) = H\big(\Lambda^{k+1}(U)\big)$. Indeed, If $\omega \in \Lambda^{k+1}(U)$, then we find that $H\omega \in \Lambda^k(U)$ and $\mathbf{i}_{\mathcal{H}}(H\omega) = 0$, $H\omega(\mathbf{x}_0) = 0$ because of Theorem (6.3.1) (v) and (ii) and consequently $H\omega \in \mathcal{A}^k(U)$. Conversely, let us suppose that $\alpha \in \mathcal{A}^k(U)$. This means that $\alpha = Hd\alpha$. Next, we introduce the form $\beta = d\alpha \in \Lambda^{k+1}(U)$ so we get $\alpha = H\beta$.

Theorem 6.4.2. *If $\alpha \in \mathcal{A}^k(U)$, there exists a form $\widehat{\alpha} \in \Lambda^{k+1}(U)$ such that α is expressible as $\alpha = \mathbf{i}_{\mathcal{H}}\widehat{\alpha}$.*

When $\alpha \in \mathcal{A}^k(U)$, there is a form $\beta \in \Lambda^{k+1}(U)$ such that one is able to write $\alpha = H\beta$ and thus it has the following expression

$$\begin{aligned}\alpha &= \int_0^1 t^k\, \mathbf{i}_{\mathcal{H}}\overline{\beta}\, dt = \\ &= \mathbf{i}_{\mathcal{H}}\Big(\frac{1}{(k+1)!}\int_0^1 t^k \beta_{i_1\cdots i_{k+1}}\big[\mathbf{x}_0 + t(\mathbf{x}-\mathbf{x}_0)\big]\, dt\, dx^{i_1}\wedge\cdots\wedge dx^{i_{k+1}}\Big) = \mathbf{i}_{\mathcal{H}}\widehat{\alpha}.\end{aligned}$$

Conversely, if $\alpha = \mathbf{i}_{\mathcal{H}}\widehat{\alpha}$, then we find $\mathbf{i}_{\mathcal{H}}\alpha = \mathbf{i}_{\mathcal{H}}^2\widehat{\alpha} = 0$ and $\alpha(\mathbf{x}_0) = 0$ since $\mathcal{H}(\mathbf{x}_0) = 0$. □

Next, as an application of Theorem 6.4.2, let us show once more that *the exterior product of two antiexact forms is again an antiexact form*. If α, β are antiexact forms, then they are expressible as $\alpha = \mathbf{i}_{\mathcal{H}}\widehat{\alpha}$ and $\beta = \mathbf{i}_{\mathcal{H}}\widehat{\beta}$. We thus get

$$\alpha \wedge \beta = \mathbf{i}_{\mathcal{H}}\widehat{\alpha} \wedge \mathbf{i}_{\mathcal{H}}\widehat{\beta} = \mathbf{i}_{\mathcal{H}}(\widehat{\alpha} \wedge \mathbf{i}_{\mathcal{H}}\widehat{\beta}).$$

Recalling that $H \circ \mathbf{i}_{\mathcal{H}} = 0$, we obtain

$$Hd\mathbf{i}_{\mathcal{H}}(\widehat{\alpha} \wedge \mathbf{i}_{\mathcal{H}}\widehat{\beta}) = \mathbf{i}_{\mathcal{H}}(\widehat{\alpha} \wedge \mathbf{i}_{\mathcal{H}}\widehat{\beta}) - dH\mathbf{i}_{\mathcal{H}}(\widehat{\alpha} \wedge \mathbf{i}_{\mathcal{H}}\widehat{\beta}) = \mathbf{i}_{\mathcal{H}}(\widehat{\alpha} \wedge \mathbf{i}_{\mathcal{H}}\widehat{\beta})$$

Introducing the form $\gamma = \mathbf{i}_{\mathcal{H}}(\widehat{\alpha} \wedge \mathbf{i}_{\mathcal{H}}\widehat{\beta})$, we finally find

$$\alpha \wedge \beta = Hd\mathbf{i}_{\mathcal{H}}(\widehat{\alpha} \wedge \mathbf{i}_{\mathcal{H}}\widehat{\beta}) = Hd\gamma \in \mathcal{A}(U).$$

Let us consider a form $\omega \in \Lambda^k(U)$. This form may be expressed as

$$\omega = \alpha + \beta, \;\; \alpha = dH\omega \in \mathcal{E}^k(U), \quad \beta = Hd\omega \in \mathcal{A}^k(U).$$

This implies that one is allowed to write $\Lambda^k(U) = \mathcal{E}^k(U) + \mathcal{A}^k(U)$. But we can readily show that $\mathcal{E}^k(U) \cap \mathcal{A}^k(U) = \{0\}$. Let $\omega \in \mathcal{E}^k(U) \cap \mathcal{A}^k(U)$ so that this form has to satisfy both $\omega = d\sigma$ and $\omega = Hd\omega$. This leads to the result $\omega = Hd^2\sigma = 0$ which amounts to say that we have a direct sum at hand: $\Lambda^k(U) = \mathcal{E}^k(U) \oplus \mathcal{A}^k(U)$. We then conclude that the exterior algebra on U may be represented as the direct sum $\Lambda(U) = \mathcal{E}(U) \oplus \mathcal{A}(U)$ of graded algebras of exact and antiexact forms.

Actually, we can show that the algebra of antiexact forms generates almost the entire exterior algebra on U.

Theorem 6.4.3. *A form $\omega \in \Lambda^k(U), k \geq 1$ has a unique representation $\omega = d\alpha + \beta$ where $\alpha \in \mathcal{A}^{k-1}(U)$ and $\beta \in \mathcal{A}^k(U)$.*

Since we have assumed that U is contractible, any form ω can be expressed as $\omega = dH\omega + Hd\omega$. We then introduce the antiexact forms $\alpha = H\omega \in \mathcal{A}^{k-1}(U)$ and $\beta = Hd\omega \in \mathcal{A}^k(U)$ to represent ω as $\omega = d\alpha + \beta$. However, it remains now to demonstrate that this representation is unique. To this end, let us suppose that there exists another representation in the shape $\omega = d\alpha_1 + \beta_1$ where $\alpha_1 \in \mathcal{A}^{k-1}(U)$ and $\beta_1 \in \mathcal{A}^k(U)$. We then find $d(\alpha - \alpha_1) + (\beta - \beta_1) = 0$ and the exterior derivative of this form gives $d(\beta - \beta_1) = 0$. Because $\beta - \beta_1 \in \mathcal{A}^k(U)$ and Hd is the identity operator on $\mathcal{A}^k(U)$, we obtain at once $0 = Hd(\beta - \beta_1) = \beta - \beta_1$, or $\beta_1 = \beta$. Therefore, we get $d(\alpha - \alpha_1) = 0$ and the Poincaré lemma leads to $\alpha - \alpha_1 = d\gamma$ where $\gamma \in \Lambda^{k-2}(U)$ whenever $k > 1$. Since $\alpha - \alpha_1 \in \mathcal{A}^{k-1}(U)$, we find that $H(\alpha - \alpha_1) = Hd\gamma = 0$. Hence, the relation $\gamma = dH\gamma + Hd\gamma = dH\gamma$ gives rise to $\alpha - \alpha_1 = d^2H\gamma = 0$, or $\alpha_1 = \alpha$. Thus, this representation is unique.

But, if $k = 1$, then we have $\alpha - \alpha_1 \in \mathcal{A}^0(U) = \Lambda^0(U)$ and the condition $d(\alpha - \alpha_1) = 0$ results in $\alpha_1 = \alpha +$ *constant*. Namely, in this case the form α can only be determined uniquely up to a constant.

This theorem can be symbolically expressed in the form

$$\Lambda^k(U) = d\big(\mathcal{A}^{k-1}(U)\big) \oplus \mathcal{A}^k(U), \;\; k \geq 1 \qquad \square$$

Example 6.4.1. $\omega \in \Lambda^1(\mathbb{R}^3)$ is given by $\omega = 2x\,dx + z\,dy - y^2\,dz$ so that we get $d\omega = -(1 + 2y)\,dy \wedge dz$. If we choose the point $\mathbf{x}_0 = (0,0,0)$ as the centre, the radius vector becomes $\mathcal{H} = x\,\partial_x + y\,\partial_y + z\,\partial_z$. Then, by applying the homotopy operator, we obtain

$$\begin{aligned} H\omega &= \int_0^1 \mathbf{i}_{\mathcal{H}}(2tx\,dx + tz\,dy - t^2y^2\,dz)\,dt \\ &= \int_0^1 (2tx^2 + tyz - t^2y^2z)\,dt = x^2 + \tfrac{1}{2}yz - \tfrac{1}{3}y^2z \\ Hd\omega &= -\int_0^1 t(1+2ty)\mathbf{i}_{\mathcal{H}}(dy \wedge dz)dt \\ &= -\int_0^1 t(1+2ty)(ydz - zdy)dt = \big(\tfrac{1}{2} + \tfrac{2}{3}y\big)(z\,dy - y\,dz) \end{aligned}$$

Hence, the form ω is expressible as

$$\omega = d\big(x^2 + \tfrac{1}{2}yz - \tfrac{1}{3}y^2z\big) + \big(\tfrac{1}{2} + \tfrac{2}{3}y\big)(z\,dy - y\,dz) \qquad \blacksquare$$

Let us now consider two antiexact forms $\alpha \in \mathcal{A}^k(U)$ and $\beta \in \mathcal{A}^l(U)$. Since we know that $\alpha \wedge \beta \in \mathcal{A}^{k+l}(U)$, we can write $Hd(\alpha \wedge \beta) = \alpha \wedge \beta$ whence we deduce that $H(d\alpha \wedge \beta) + (-1)^k H(\alpha \wedge d\beta) = \alpha \wedge \beta$. Hence, we obtain

$$H(d\alpha \wedge \beta) = \alpha \wedge \beta + (-1)^{k+1} H(\alpha \wedge d\beta). \tag{6.4.3}$$

This relation can be interpreted as a sort of integration by parts.

6.5. CHANGE OF CENTRE

The open set $U \subseteq M$ may be contractible with respect to several points. Therefore, its homeomorphic image in $\mathbb{R}^m$ may appear to be star-shaped with respect to various centres. Since the homotopy operator is explicitly dependent on the location of the centre, we shall then try to establish the connection between homotopy operators associated with different centres.

Theorem 6.5.1. *According to a local chart, let $V = \varphi(U)$ be a star-shaped region of the Euclidean space where $U \subseteq M$ is an open set. If H_1 and H_2 are two homotopy operators associated with centres $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively, then they are related by*

$$\begin{aligned} H_1\omega &= H_2\omega + \gamma + d\lambda \quad \text{if } deg\,\omega > 1, \\ H_1\omega &= H_2\omega + \gamma + c \quad\;\; \text{if } deg\,\omega = 1 \end{aligned}$$

where we define $\gamma = -H_2H_1d\omega \in \mathcal{A}_2(U)$ and $\lambda = H_2H_1\omega \in \mathcal{A}_2(U)$. c is a constant.

For a form $\omega \in \Lambda^k(U)$, we can of course write

$$\omega = dH_1\omega + H_1 d\omega = dH_2\omega + H_2 d\omega$$

from which we find that $(H_2 - H_1)d\omega = d(H_1 - H_2)\omega$. If we replace ω by $d\omega$ in this expression, we get $d(H_2 - H_1)d\omega = 0$. In view of the Poincaré lemma, there exists a form $\alpha \in \Lambda^{k-1}(U)$ so that one is able to write $(H_2 - H_1)d\omega = d\alpha$. According to Theorem 6.4.3 the form α is given by $\alpha = d\beta + \gamma$ where $\gamma \in \mathcal{A}_2^{k-1}(U)$. The relation $(H_2 - H_1)\,d\omega = d\gamma$ then leads to $(H_2^2 - H_2H_1)\,d\omega = H_2 d\gamma$ and $-\,H_2H_1 d\omega = \gamma$. On the other hand, the equality $d(H_1 - H_2)\,\omega = d\gamma$ or $d\big[(H_1 - H_2)\,\omega - \gamma\big] = 0$ gives rise to

$$(H_1 - H_2)\,\omega = \gamma + d\sigma, \;\; \sigma \in \Lambda^{k-2}(U)$$

if $k > 1$. If we write $\sigma = d\nu + \lambda$ where $\lambda \in \mathcal{A}_2^{k-2}(U)$, we obtain

$$H_1\omega = H_2\omega + \gamma + d\lambda.$$

On applying the operator H_2 on the above equality, we eventually find that $\lambda = H_2H_1\omega$. However, if $k = 1$, then it follows from the foregoing relation that $(H_1 - H_2)\omega - \gamma = c =$ *constant* and, consequently,

$$H_1\omega = H_2\omega + \gamma + c. \qquad \square$$

Example 6.5.1. $\omega \in \Lambda^1(\mathbb{R}^3)$ is given by $\omega = 2x\,dx + z\,dy - y^2\,dz$. Let us consider two centres $\mathbf{x}_1 = (1,0,1)$ and $\mathbf{x}_2 = (0,0,0)$. With the radius vector $\mathcal{H}_2$ we have already found in Example 6.4.1 that

$$H_2\omega = x^2 + \tfrac{1}{2}yz - \tfrac{1}{3}y^2 z.$$

With the radius vector $\mathcal{H}_1 = (x-1)\,\partial_x + y\,\partial_y + (z-1)\,\partial_z$, we are led to the relation

$$\begin{aligned} H_1\omega &= \int_0^1 \mathbf{i}_{\mathcal{H}_1}\big[2\big(1 + t(x-1)\big)dx + \big(1 + t(z-1)\big)dy - t^2y^2\,dz\big]\,dt \\ &= \int_0^1 \big[2(x-1)\big(1 + t(x-1)\big) + y\big(1 + t(z-1)\big) - t^2y^2(z-1)\,\big]\,dt \\ &= 2(x-1) + (x-1)^2 + y + \tfrac{1}{2}y(z-1) - \tfrac{1}{3}y^2(z-1) \\ &= H_2\omega + \tfrac{1}{2}y + \tfrac{1}{3}y^2 - 1. \end{aligned}$$

We see that $\gamma = \frac{1}{2}y + \frac{1}{3}y^2$ and $c = \,-\,1$. ■

6.6. CANONICAL FORMS OF 1-FORMS, CLOSED 2- FORMS

We consider a form $\omega = \omega_i(\mathbf{x})\,dx^i \in \Lambda^1(M)$ on an m-dimensional smooth manifold M. Starting with this form, let us construct the following sequence of forms of increasing degrees:

$$I_1 = \omega \in \Lambda^1(M),\ \ I_2 = d\omega \in \Lambda^2(M),\ \ I_3 = \omega \wedge d\omega \in \Lambda^3(M),$$
$$I_4 = d\omega \wedge d\omega \in \Lambda^4(M), \dots, I_{2n} = (d\omega)^n = \underbrace{d\omega \wedge \cdots \wedge d\omega}_{n} \in \Lambda^{2n}(M),$$
$$I_{2n+1} = \omega \wedge I_{2n} \in \Lambda^{2n+1}(M), \dots,\ n = 0, 1, 2, \dots.$$

Here, we adopt the convention that $I_0 = 1$. It is clearly observable from this sequence that we can write the recurrence relations

$$\begin{aligned} I_{2n} &= d\omega \wedge I_{2n-2} = I_2 \wedge I_{2n-2}, \\ I_{2n+1} &= I_1 \wedge I_{2n}, \qquad n = 0, 1, \dots. \end{aligned} \tag{6.6.1}$$

By definition, we evidently have

$$dI_{2n} = 0, \quad dI_1 = I_2, \quad dI_{2n+1} = dI_1 \wedge I_{2n} = I_2 \wedge I_{2n} = I_{2n+2}.$$

This sequence must be finite because all forms whose degrees is greater than m are identically zero. We can thus write $I_{m+k} \equiv 0$ when $k = 1, 2, \dots$. However, this sequence may vanish beginning from a number $K \leq m$. Since $d\omega$ is a 2-form, its rank is $r = 2k \leq m$ [*see p.* 39]. This means that we can express $d\omega$ as follows

$$d\omega = f^1 \wedge g^1 + f^2 \wedge g^2 + \cdots + f^k \wedge g^k,\ f^i, g^i \in \Lambda^1(M), i = 1, \dots, k$$

and we get $(d\omega)^k \neq 0, (d\omega)^{k+n} = 0, n = 1, 2, \dots$ [*see p.* 44].

Let us now suppose that we have succeeded in determining an integer $K(\mathbf{x}, \omega) > 0$ such that

$$I_{K(\mathbf{x},\omega)} \neq 0, \quad I_{K(\mathbf{x},\omega)+n} = 0,\ n = 1, 2, \dots. \tag{6.6.2}$$

This number may generally be dependent on the points of the manifold. We assume that forms are defined on an open set $U \subseteq M$. We denote the homeomorphic image of U through appropriate charts by $V \subseteq \mathbb{R}^m$. The positive integer $K(\omega) = \sup_{\mathbf{x} \in V} K(\mathbf{x}, \omega)$ is called the ***Darboux class*** of the 1-form ω relative to the set U [French mathematician Jean Gaston Darboux (1842-1917)]. The points at which $K(\mathbf{x}) < K(\omega)$ is said to be *critical points* of the form ω relative to the set U while the points at which $K(\mathbf{x}) = K(\omega)$ are called *regular points* relative to U. *We shall here assume that all points in the region U are regular*. The ***rank*** $r(\omega)$ of the form ω relative to U is defined as the greatest *even integer* less than or equal to $K(\omega)$ whereas the number $\epsilon(\omega) = K(\omega) - r(\omega)$ is called the ***index*** of ω relative to U. According to this definition $\epsilon(\omega)$ is either 0 or 1.

Let us first prove the following lemma.

Lemma 6.6.1. *Let $\Omega \in \Lambda^k(M)$ be a closed simple k-form. Then the form Ω is expressible as exterior products of so called gradients, that is, as $\Omega = dg^1 \wedge dg^2 \wedge \cdots \wedge dg^k$ where $g^i \in \Lambda^0(M)$, $i = 1, \ldots, k$.*

Let us take the closed form $\Omega = \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^k \neq 0$ where the forms $\omega^i \in \Lambda^1(M)$, $i = 1, \ldots, k$ are linearly independent. We now consider the ideal $\mathcal{I}(\omega^1, \ldots, \omega^k)$. We will show that this ideal is also closed. Since $d\Omega = 0$, we readily obtain

$$\begin{aligned} d\Omega &= \sum_{i=1}^{k} (-1)^{i-1} \omega^1 \wedge \cdots \wedge \omega^{i-1} \wedge d\omega^i \wedge \omega^{i+1} \wedge \cdots \wedge \omega^k \\ &= \sum_{i=1}^{k} d\omega^i \wedge \omega^1 \wedge \cdots \wedge \omega^{i-1} \wedge \omega^{i+1} \wedge \cdots \wedge \omega^k = 0 \end{aligned}$$

where we have adopted the convention $\omega^0 = 1$. It follows from the above expression that we arrive at the relation

$$\begin{aligned} 0 = \omega^j \wedge d\Omega &= \sum_{i=1}^{k} \omega^j \wedge d\omega^i \wedge \omega^1 \wedge \cdots \wedge \omega^{i-1} \wedge \omega^{i+1} \wedge \cdots \wedge \omega^k \\ &= \omega^j \wedge d\omega^j \wedge \omega^1 \wedge \cdots \wedge \omega^{j-1} \wedge \omega^{j+1} \wedge \cdots \wedge \omega^k \\ &= (-1)^{j+1} d\omega^j \wedge \omega^1 \wedge \cdots \wedge \omega^{j-1} \wedge \omega^j \wedge \omega^{j+1} \wedge \cdots \wedge \omega^k \end{aligned}$$

for every form $\omega^j \in \Lambda^1(M)$. This implies that

$$d\omega^i \wedge \Omega = 0, \;\; i = 1, \ldots, k.$$

These relations show according to Theorem 5.10.2 that the ideal $\mathcal{I}(\omega^1, \ldots, \omega^k)$ is closed. Hence, as we have demonstrated in Theorem 5.13.5 we can find *independent* functions $f^i \in \Lambda^0(M)$, $i = 1, \ldots, k$ so that this ideal is equivalent to the ideal $\mathcal{I}(df^1, \ldots, df^k)$. Furthermore we have to write

$$\omega^i = A^i_j \, df^j, \;\; A^i_j \in \Lambda^0(M)$$

whence we deduce that

$$\Omega = \omega^1 \wedge \cdots \wedge \omega^k = \alpha df^1 \wedge \cdots \wedge df^k, \; \alpha = \det{(A^i_j)} \in \Lambda^0(M).$$

On the other hand, we must have $d\Omega = d\alpha \wedge df^1 \wedge \cdots \wedge df^k = 0$. Let us now choose new local coordinates for the manifold M as f^i, x^a where $i = 1, \ldots, k$ and $a = k+1, \ldots, m$. Because the function α may presently be written as

$$\alpha = \alpha(f^1, \dots, f^k, x^{k+1}, \dots, x^m),$$

the expression $d\Omega = 0$ can be cast into the form

$$\Big(\frac{\partial\alpha}{\partial f^i}\,df^i + \frac{\partial\alpha}{\partial x^a}\,dx^a\Big) \wedge df^1 \wedge \cdots \wedge df^k$$
$$= \frac{\partial\alpha}{\partial x^a}\,dx^a \wedge df^1 \wedge \cdots \wedge df^k = 0.$$

These equations yield

$$\frac{\partial\alpha}{\partial x^a} = 0,$$

where $a = k+1, \dots, m$ so that we find $\alpha = \alpha(f^1, \dots, f^k)$. Let us now define functions

$$g^1 = g^1(f^1, \dots, f^k),\ g^2 = f^2,\ \dots,\ g^k = f^k$$

such that we can express α without loss of generality as

$$\alpha = \frac{\partial g^1}{\partial f^1} \neq 0.$$

Since we now have

$$\det\Big(\frac{\partial g^i}{\partial f^j}\Big) = \begin{vmatrix} \dfrac{\partial g^1}{\partial f^1} & \dfrac{\partial g^1}{\partial f^2} & \cdots & \dfrac{\partial g^1}{\partial f^k} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} = \frac{\partial g^1}{\partial f^1} = \alpha \neq 0,$$

the functions $g^1, \dots, g^k$ are independent and we get

$$dg^1 \wedge \cdots \wedge dg^k = \det\,(\partial g^i/\partial f^j)\,df^1 \wedge \cdots \wedge df^k.$$

Hence, we are led to the conclusion

$$\Omega = dg^1 \wedge \cdots \wedge dg^k,\ \ g^i \in \Lambda^0(M),\ i = 1, \dots, k. \qquad \square$$

We can now prove the following theorem.

Theorem 6.6.1 (The Darboux Theorem). *If the Darboux class of the form $\omega \in \Lambda^1(M)$ is K, then the canonical form of ω is given by*

$$\omega = u^\alpha dv_\alpha + \epsilon(\omega)\,dv_{k+1},\ \ u^\alpha, v_\alpha, v_{k+1} \in \Lambda^0(M),\ \alpha = 1, \dots, k$$

where $k = [K/2]$ denotes the greatest integer that is less than or equal to $K/2$. If K is even, then $\epsilon(\omega) = 0$ whereas $\epsilon(\omega) = 1$ if K is odd.

We shall prove this theorem in two steps.

(i). *Darboux class is an even integer*. Thus, we can write $K = 2k$ so that we have to take $k = [K/2] = K/2$. According to the definition of the Darboux class, we obviously find

$$I_{2k} \neq 0,\ I_{2k+1} = 0,\ I_{2k+2} = 0,\ \dots\ .$$

Hence, the rank of the form $d\omega \in \Lambda^2(M)$ is $2k$ implying that $I_{2k} \in \Lambda^{2k}(M)$ is a simple form. Since $dI_{2k} = 0$, in view of Lemma 6.6.1 the form I_{2k} is a gradient product, i.e., by means of independent functions $u^1, v_1, u^2, v_2, \dots, u^k, v_k$ the form I_{2k} can be depicted as follows

$$I_{2k} = k!\, du^1 \wedge dv_1 \wedge du^2 \wedge dv_2 \wedge \cdots \wedge du^k \wedge dv_k.$$

Accordingly the form $d\omega$ has the following structure [*see* Sec. 1.6]

$$d\omega = du^\alpha \wedge dv_\alpha,\ \ u^\alpha, v_\alpha \in \Lambda^0(M),\ \alpha = 1, \dots, k.$$

On the other hand, the satisfaction of the condition

$$I_{2k+1} = \omega \wedge I_{2k} = k!\, \omega \wedge du^1 \wedge dv_1 \wedge du^2 \wedge dv_2 \wedge \cdots \wedge du^k \wedge dv_k = 0$$

suggests due to Theorem 5.3.1 that the 1-forms $\omega, du^1, dv_1, \dots, du^k, dv_k$ are linearly dependent. Therefore, there exist functions $f_\alpha, g^\alpha \in \Lambda^0(M)$ that enable us to write

$$\omega = f_\alpha\, du^\alpha + g^\alpha\, dv_\alpha.$$

Hence, the form ω belongs to the ideal $\mathcal{I}(du^\alpha, dv_\alpha)$. In this situation, we naturally get

$$d\omega = du^\alpha \wedge dv_\alpha = df_\alpha \wedge du^\alpha + dg^\alpha \wedge dv_\alpha,$$

Since $2k \leq m$, we are free to choose new local coordinates as follows: u^α, $v_\alpha, \alpha = 1, \dots, k; x^a, a = 2k+1, \dots, m$. In the equation just above, let us evaluate differentials of functions f_α, g^α with respect to new coordinates. On comparing both sides, we find that

$$\frac{\partial f_\alpha}{\partial x^a} = 0, \quad \frac{\partial g^\alpha}{\partial x^a} = 0 \tag{6.6.3}$$

since there are no terms like $dx^a \wedge du^\alpha$ and $dx^a \wedge dv_\alpha$ at the left hand side of that expression. We thus conclude that these function must have the forms $f_\alpha = f_\alpha(u^\beta, v_\beta)$ and $g^\alpha = g^\alpha(u^\beta, v_\beta)$. Remaining terms then conduce to the relations

$$\frac{\partial f_\alpha}{\partial u^\beta}\,du^\beta \wedge du^\alpha + \frac{\partial f_\alpha}{\partial v_\beta}\,dv_\beta \wedge du^\alpha + \frac{\partial g^\alpha}{\partial u^\beta}\,du^\beta \wedge dv_\alpha + \frac{\partial g^\alpha}{\partial v_\beta}\,dv_\beta \wedge dv_\alpha = du^\alpha \wedge dv_\alpha.$$

However, on utilising the antisymmetry of the exterior product we can transform this expression into the form

$$\frac{1}{2}\Big(\frac{\partial f_\alpha}{\partial u^\beta} - \frac{\partial f_\beta}{\partial u^\alpha}\Big)du^\beta \wedge du^\alpha + \frac{1}{2}\Big(\frac{\partial g^\alpha}{\partial v_\beta} - \frac{\partial g^\beta}{\partial v_\alpha}\Big)dv_\beta \wedge dv_\alpha + \Big(\frac{\partial g^\beta}{\partial u^\alpha} - \frac{\partial f_\alpha}{\partial v_\beta}\Big)du^\alpha \wedge dv_\beta = \delta^\beta_\alpha\,du^\alpha \wedge dv_\beta$$

which result in the equations

$$\frac{\partial f_\alpha}{\partial u^\beta} = \frac{\partial f_\beta}{\partial u^\alpha}, \quad \frac{\partial g^\alpha}{\partial v_\beta} = \frac{\partial g^\beta}{\partial v_\alpha}, \quad \frac{\partial g^\beta}{\partial u^\alpha} - \frac{\partial f_\alpha}{\partial v_\beta} = \delta^\beta_\alpha. \tag{6.6.4}$$

The equations $(6.6.4)_{1-2}$ ensure the existence of functions ϕ_1 and ψ_1 that make it possible for us to write

$$f_\alpha = \frac{\partial \phi_1}{\partial u^\alpha},\ g^\alpha = \frac{\partial \psi_1}{\partial v_\alpha}, \quad \phi_1, \psi_1 = \phi_1, \psi_1(u^\alpha, v_\alpha, x^a). \tag{6.6.5}$$

But, because of (6.6.3) we get

$$\frac{\partial^2 \phi_1}{\partial u^\alpha \partial x^a} = \frac{\partial^2 \psi_1}{\partial v_\alpha \partial x^a} = 0$$

whence we obtain

$$\begin{aligned}\phi_1(\mathbf{u},\mathbf{v}) &= \phi(\mathbf{u},\mathbf{v}) + \overline{\phi}(\mathbf{v}, x^a),\\ \psi_1(\mathbf{u},\mathbf{v}) &= \psi(\mathbf{u},\mathbf{v}) + \overline{\psi}(\mathbf{u}, x^a).\end{aligned} \tag{6.6.6}$$

If we insert the expressions (6.6.5) and (6.6.6) into the equations $(6.6.4)_3$, we find that

$$\frac{\partial^2 \psi_1}{\partial v_\beta \partial u^\alpha} - \frac{\partial^2 \phi_1}{\partial u^\alpha \partial v_\beta} = \frac{\partial^2}{\partial v_\beta \partial u^\alpha}(\psi_1 - \phi_1) = \frac{\partial^2}{\partial v_\beta \partial u^\alpha}(\psi - \phi) = \delta^\beta_\alpha.$$

The integration of these equations yields readily

$$\psi - \phi = u^\alpha v_\alpha + \Phi(\mathbf{u}) + \Psi(\mathbf{v}) \tag{6.6.7}$$

and it follows from (6.6.5) that

$$f_\alpha = \frac{\partial \phi_1}{\partial u^\alpha} = \frac{\partial \phi}{\partial u^\alpha}, \quad g^\alpha = \frac{\partial \psi_1}{\partial v_\alpha} = \frac{\partial \psi}{\partial v_\alpha}$$

On the other hand, we obtain from the expression (6.6.7) that

$$\frac{\partial \psi}{\partial v_\alpha} = u^\alpha + \frac{\partial \phi}{\partial v_\alpha} + \frac{\partial \Psi}{\partial v_\alpha},$$

Hence, the form ω is expressible as

$$\begin{aligned}\omega &= \frac{\partial \phi}{\partial u^\alpha}\, du^\alpha + \frac{\partial \psi}{\partial v_\alpha}\, dv_\alpha = u^\alpha dv_\alpha + \frac{\partial \phi}{\partial u^\alpha}\, du^\alpha + \frac{\partial \phi}{\partial v_\alpha}\, dv_\alpha + \frac{\partial \Psi}{\partial v_\alpha} dv_\alpha \\ &= u^\alpha dv_\alpha + d\widetilde{\phi}\,,\ \widetilde{\phi}\ = \phi + \Psi\end{aligned}$$

where the function $\widetilde{\phi}\,(\mathbf{u}, \mathbf{v})$ can be selected arbitrarily. Therefore, if we take $\widetilde{\phi}\ = constant$, the canonical form of ω is found to be

$$\omega = u^\alpha dv_\alpha = u^1 dv_1\ + u^2 dv_2\ + \cdots + u^k dv_k. \tag{6.6.8}$$

It is clear that this representation is not unique. For instance, because of the identity

$$u^\alpha dv_\alpha = d(u^\alpha v_\alpha) - v_\alpha du^\alpha$$

we get $\omega =\ - v_\alpha du^\alpha$ if we choose $\widetilde{\phi}\ =\ - u^\alpha v_\alpha$.

(ii). *Darboux class is an odd integer*. Thus we can write $K = 2k + 1$. Hence, we can take $k = [K/2] = (K-1)/2$ which requires that

$$I_{2k} \neq 0,\ \ I_{2k+1} \neq 0,\ \ I_{2k+2} = 0,\ \ldots.$$

Since $I_{2k} \neq 0$ and $I_{2k+2} = 0$, the rank of the form $d\omega \in \Lambda^2(M)$, that must be an even number, is again $2k$. This implies that I_{2k} is still a simple form and a gradient product. On the other hand, we can write $I_{2k+1} = \omega \wedge I_{2k}$. Since I_{2k} is a simple form and ω is a 1-form, the form $I_{2k+1} \in \Lambda^{2k+1}(M)$ is also a simple form. I_{2k+1} is a closed form because $dI_{2k+1} = I_{2k+2} = 0$. Consequently, I_{2k+1} is likewise a gradient product. Since $I_{2k+1} = \omega \wedge I_{2k} \neq 0$, the form $\omega \in \Lambda^1(M)$ cannot be expressed as a linear combination of factor forms of the simple form I_{2k}. Therefore the form I_{2k} is a divisor of the form I_{2k+1}. Since I_{2k+1} is a gradient product, this form is expressible as

$$I_{2k+1} = d\eta \wedge I_{2k}, \quad \eta \in \Lambda^0(M).$$

Thus, just like in the part (i), we can write

$$I_{2k} = k!\,du^1 \wedge dv_1 \wedge \cdots \wedge du^k \wedge dv_k,$$
$$d\omega = du^\alpha \wedge dv_\alpha,\ \ u^\alpha, v_\alpha \in \Lambda^0(M),\ \ \alpha = 1, \dots, k.$$

Let us now introduce a form $\omega' \in \Lambda^1(M)$ through a gradient transformation given as

$$\omega' = \omega - d\lambda$$

where $\lambda \in \Lambda^0(M)$. We obviously get $d\omega' = d\omega$ so that we obtain $I'_{2k} = I_{2k}$ while I'_{2k+1} is found to be

$$I'_{2k+1} = \omega' \wedge I'_{2k} = \omega \wedge I_{2k} - d\lambda \wedge I_{2k} = I_{2k+1} - d\lambda \wedge I_{2k}.$$

We thus arrive at the relation

$$I'_{2k+1} = d\eta \wedge I_{2k} - d\lambda \wedge I_{2k} = d(\eta - \lambda) \wedge I_{2k}.$$

On choosing the arbitrary function λ as $\lambda = \eta$, we conclude that

$$I'_{2k} \neq 0 \quad \text{and} \quad I'_{2k+1} = 0.$$

This amounts to say that the Darboux class of the form ω' is $K = 2k$ and its canonical form turns out to be $\omega' = u^\alpha dv_\alpha$ as in (i). Hence, we obtain $\omega = u^\alpha dv_\alpha + d\eta$. Since the functions u^α, v_α are independent, we can write $\eta = \eta(\mathbf{u}, \mathbf{v}, x^a)$ by a choice of local coordinates as above. Thus the form $d\lambda$ is not expressible as a linear combination of the forms du^α, dv_α so it does not belong to the ideal generated by these forms. Hence, the function $\eta = v_{k+1}$ is independent of the functions u^α, v_α. Ultimately, in terms of $2k+1$ independent functions the canonical form of ω now becomes

$$\omega = u^\alpha dv_\alpha + dv_{k+1} = u^1 dv_1 + \cdots + u^k dv_k + dv_{k+1}. \tag{6.6.9}$$

This finishes the proof of the theorem showing that we are now able to write

$$\omega = u^\alpha dv_\alpha + \epsilon(\omega) dv_{k+1} = u^1 dv_1 + \cdots + u^k dv_k + \epsilon(\omega) dv_{k+1}. \qquad \square$$

Example 6.6.1. We take the form $\omega = 2x\,dx + z\,dy - y^2\,dz \in \Lambda^1(\mathbb{R}^3)$ into consideration. Let us construct the sequence

$$I_1 = \omega, I_2 = -(1+2y)dy \wedge dz, I_3 = -2x(1+2y)dx \wedge dy \wedge dz, I_4 = 0.$$

We thus find $K = 3$ and $k = 1$. Let us choose $v_2 = x^2$ and assume that the functions u^1 and v_1 depend only on y and z. Then the relations

$$u^1 \frac{\partial v_1}{\partial y} = z, \quad u^1 \frac{\partial v_1}{\partial z} = -\,y^2$$

yield the partial differential equation $y^2(\partial v_1/\partial y) + z(\partial v_1/\partial z) = 0$ whose solution is $v_1 = f(y^{-1} + \log z)$. For simplicity, let us choose the function as $v_1 = y^{-1} + \log z$. Then we find that $u^1 = -y^2 z$. Hence ω can now be expressed in the following canonical form

$$\omega = -y^2 z\, d(y^{-1} + \log z) + d(x^2) \qquad \blacksquare$$

Example 6.6.2. Let us consider the form

$$\omega = (x - y^2)dx + (y^3 - z^2)dy + t^2 dt \in \Lambda^1(\mathbb{R}^4).$$

Since we obtain

$$\begin{aligned} I_1 &= \omega, \qquad I_2 = -2y dy \wedge dx - 2z dz \wedge dy, \\ I_3 &= -2z(x - y^2)\, dx \wedge dz \wedge dy, \qquad I_4 = 0 \end{aligned}$$

we find that $K = 3$ and $k = 1$. Hence one can write $\omega = u^1 dv_1 + dv_2$. We then readily show that

$$u^1 = 2xy - z^2,\ v_1 = y,\ v_2 = \frac{x^2}{2} + \frac{y^4}{4} - xy^2 + \frac{t^3}{3}. \qquad \blacksquare$$

The number K can be equal at most to the dimension m of the manifold. Therefore, a form in $\Lambda^1(M)$ is expressible at most $k = [m/2] + \epsilon$ number of terms. For instance, we can write

$$\begin{aligned} m = 1 &: \omega = dv_1, & v_1 &\in \Lambda^0(\mathbb{R}), \\ m = 2 &: \omega = u^1 dv_1, & u^1, v_1 &\in \Lambda^0(\mathbb{R}^2), \\ m = 3 &: \omega = u^1 dv_1 + dv_2, & u^1, v_1, v_2 &\in \Lambda^0(\mathbb{R}^3). \end{aligned}$$

We can now discuss the second Darboux theorem concerning closed 2-forms which is in fact an almost trivial corollary of the Darboux theorem.

Theorem 6.6.2. *Let U be an open set of m-dimensional manifold M contractible to one of its points. The homeomorphic image of this set in $\mathbb{R}^m$ through an appropriate chart is a star-shaped region. The canonical form of a closed form $\omega \in \Lambda^2(U)$ is given by*

$$\omega = du^\alpha \wedge dv_\alpha \tag{6.6.10}$$

where the functions $u^\alpha, v_\alpha \in \Lambda^0(U), \alpha = 1, \dots, k$ are independent and $k = [K/2]$.

Poincaré lemma states that there exists a form $\Omega \in \Lambda^1(U)$ such that the relation $\omega = d\Omega$ is satisfied. If the Darboux class of Ω is K, then we can write $\Omega = u^\alpha dv_\alpha + \epsilon dv_{k+1}$ where $\alpha = 1, \dots, k, k = [K/2]$. We thus find $\omega = du^\alpha \wedge dv_\alpha$. Since every differentiable manifold is locally contractible, then every form $\omega \in \Lambda^2(M)$ becomes locally expressible in the *canonical form* (6.6.10). □

We know that a form $\omega \in \Lambda^1(M)$ is completely integrable if only it can be written as $\omega = \xi\, d\eta$ where $\xi, \eta \in \Lambda^0(M)$. This is only possible if $k = 0$ and $k = 1$ or Darboux classes $K = 1$ and $K = 2$. In those cases, we get $\omega = dv_1$ and $\omega = u^1 dv_1$, respectively. We thus conclude that *1-forms whose Darboux classes are $K \geq 3$ are not completely integrable*. This result coincides with the concept of accessibility propounded by Greek-German mathematician Constantin Carathéodory (1873-1950). Let us consider a form $\omega \in \Lambda^1(M)$. We say that a form ω has the ***inaccessibility*** property if and only if a sufficiently small neighbourhood of any point $p \in M$ contains a point $q \in M$ that cannot be reached by a path (a curve) through p entirely on M satisfying the exterior equation $\omega = 0$. If a 1-form ω does not have the inaccessibility property, namely, if in a neighbourhood of any point p there is no point that cannot be reached by such paths, then the form ω possesses the ***accessibility*** property.

Theorem 6.6.3 (The Carathéodory Theorem). *A form $\omega \in \Lambda^1(M)$ that is not identically zero has the inaccessibility property if and only if its Darboux class is less than three. If its Darboux class is greater than or equal to three, then ω has the accessibility property.*

If $K = 1$, then $\omega = dv_1$ and the exterior equation $\omega = 0$ holds only on $(m-1)$-dimensional submanifolds $\mathcal{S}_c$ described by $v_1(\mathbf{x}) = c$ where c's are arbitrary constants. Let us assume that a point $p \in M$ is located on $\mathcal{S}_{c_1}$. For a sufficiently small number δ, we immediately see that there is a point q in any neighbourhood of the point p that belongs to the submanifold $S_{c_1+\delta}$. Since these two hypersurfaces cannot intersect, no curve through the point p lying on S_{c_1} thereby satisfying the equation $\omega = 0$ can reach to the point q. If $K = 2$, then $\omega = u^1 dv_1$ and the exterior equation $\omega = 0$ holds on $(m-1)$-dimensional submanifolds S_c described by $v_1(\mathbf{x}) = c$. Thus, we naturally arrive at the same conclusion about inaccessibility.

Let us now consider the case $K \geq 3$. In this situation we have the representations

$$\omega = u^\alpha dv_\alpha \text{ if } K \text{ is even; } \omega = u^\alpha dv_\alpha + dv_{k+1} \text{ if } K \text{ is odd.}$$

where $\alpha = 1, \dots, k, k = [K/2]$. Since the form ω is not identically zero, at least one of the functions u^α does not vanish in a neighbourhood of a point $p \in M$. On dividing the form ω by this function and renaming the indices if necessary, we can cast the exterior equation $\omega = 0$ into the form

$$u^1 dv_1 + \cdots + u^{r-1} dv_{r-1} + dv_r = 0 \tag{6.6.11}$$

where $r = k$ if K is even and $r = k + 1$ if K is odd. A solution of the equation (6.6.11) can evidently be taken as

$$v_\alpha(\mathbf{x}) = c_\alpha,\ \alpha = 1, \dots, r-1;\ v_r(\mathbf{x}) = c_r. \tag{6.6.12}$$

The equations (6.6.12) prescribe obviously a family of $(m-r)$-dimensional submanifolds of M.

Let us now consider the points $p = \varphi^{-1}(\mathbf{x}_1)$ and $q = \varphi^{-1}(\mathbf{x}_2)$ on the manifold M and we introduce the constants

$$v_\alpha^1 = v_\alpha(\mathbf{x}_1),\ v_\alpha^2 = v_\alpha(\mathbf{x}_2),\ v_r^1 = v_r(\mathbf{x}_1),\ v_r^2 = v_r(\mathbf{x}_2);\ u_1^\alpha = u^\alpha(\mathbf{x}_1).$$

Then let us assume that we can choose a path $\mathbf{x} = \boldsymbol{\xi}(t)$ on M from the point p to the point q such that the following relations are satisfied

$$\begin{aligned} v_\alpha\big(\boldsymbol{\xi}(t)\big) &= v_\alpha(t) = v_\alpha^1 + (v_\alpha^2 - v_\alpha^1)t, \\ u^\alpha\big(\boldsymbol{\xi}(t)\big) &= u^\alpha(t) = u_1^\alpha + h^\alpha t \end{aligned}$$

where $\boldsymbol{\xi}(0) = \mathbf{x}_1$ and $\boldsymbol{\xi}(1) = \mathbf{x}_2$. $h^\alpha, \alpha = 1, \dots, r-1$ are presently arbitrary constants. Such a path can be chosen, for instance, by first introducing the functions

$$\xi^{\mathfrak{a}}(t) = x_1^{\mathfrak{a}} + t(x_2^{\mathfrak{a}} - x_1^{\mathfrak{a}}),\ \mathfrak{a} = 2r-1, \dots, m$$

and then employing the $2r-2$ number of equations

$$v_\alpha\big(\xi^a(t), \xi^{\mathfrak{a}}(t)\big) = v_\alpha^1 + (v_\alpha^2 - v_\alpha^1)t,\ \ u^\alpha\big(\xi^a(t), \xi^{\mathfrak{a}}(t)\big) = u_1^\alpha + h^\alpha t$$

to determine the remaining functions $\xi^a(t), a = 1, \dots, 2r-2$ in a neighbourhood of the point p. On the path from p to q, the equation (6.6.11) takes the form

$$\frac{dv_r}{dt} + u^\alpha \frac{dv_\alpha}{dt} = \frac{dv_r}{dt} + (u_1^\alpha + h^\alpha t)(v_\alpha^2 - v_\alpha^1) = 0.$$

The integration of this simple differential equation for v_r with the initial condition $v_r(0) = v_r^1$ gives

$$v_r(t) = v_r^1 + (v_\alpha^1 - v_\alpha^2)(u_1^\alpha + \frac{1}{2}h^\alpha t)t.$$

We now need to select the constants h^α in such a way that the relation

$$v_r^2 = v_r^1 + (v_\alpha^1 - v_\alpha^2)(u_1^\alpha + \frac{1}{2}h^\alpha)$$

will hold at $t = 1$. If $v_{\alpha_0}^1 \neq v_{\alpha_0}^2$ for an index α_0, then we reach to our objective by taking $h^\alpha = 0$ for all $\alpha \neq \alpha_0$ and

$$h^{\alpha_0} = 2\frac{v_r^2 - v_r^1 - (v_\alpha^1 - v_\alpha^2)u_1^\alpha}{v_{\alpha_0}^1 - v_{\alpha_0}^2}.$$

If $v_\alpha^1 = v_\alpha^2$ for all $\alpha = 1, \ldots, r-1$, we first determine a path such that

$$v_r(t) = v_r^1 + at, \;\; v_\alpha(t) = v_\alpha^1 + b_\alpha t, \;\; u^\alpha(t) = u_1^\alpha.$$

Since the form ω does not vanish, $u_1^{\alpha_0} \neq 0$ at least for an index α_0. Along the path, the exterior equation $\omega = 0$ can only be satisfied if $a + b_\alpha u_1^\alpha = 0$. Let us choose $b_{\alpha_0} = -a/u_1^{\alpha_0}$ and $b_\alpha = 0$ for all $\alpha \neq \alpha_0$. In this case, we obtain $v_\alpha^1 = v_\alpha^2$ for all $\alpha \neq \alpha_0, v_{\alpha_0}(1) = v_{\alpha_0}^1 + b_{\alpha_0} \neq v_{\alpha_0}^1, v_r(1) = v_r^1 + a$. If we now choose $\big(v_r(1), v_\alpha(1) = v_\alpha^1\,(\alpha \neq \alpha_0), v_{\alpha_0}(1), u^\alpha(1) = u_1\big)$ as the new initial point, we can find a path on which $\omega = 0$ from this point to the point (v_r^2, v_α^2) because $v_{\alpha_0}(1) \neq v_{\alpha_0}^1$. Therefore, when $K \geq 3$ we can always find a path from the point $p \in M$ to reach to a point in a neighbourhood of p such that the exterior equation $\omega = 0$ is satisfied along this path. □

6.7. AN EXTERIOR DIFFERENTIAL EQUATION

Exterior equations involving exterior derivatives of exterior forms will be called *exterior differential equations*. This section is devoted to finding the solution of the exterior differential equation

$$d\Omega = \Gamma \wedge \Omega + \Sigma \tag{6.7.1}$$

defined on a *contractible open set* U of a manifold M by making use of the homotopy operator. $\Gamma \in \Lambda^1(U)$ and $\Sigma \in \Lambda^{k+1}(U)$ are given exterior forms. We look for all forms $\Omega \in \Lambda^k(U)$ satisfying the equation (6.7.1). For the existence of a solution, it is clear that the forms Γ and Σ cannot be assigned arbitrarily. The closure condition $d^2\Omega = 0$ requires clearly that the equality $d\Gamma \wedge \Omega - \Gamma \wedge d\Omega + d\Sigma = 0$ or

$$d\Sigma = \Gamma \wedge \Gamma \wedge \Omega + \Gamma \wedge \Sigma - d\Gamma \wedge \Omega = \Gamma \wedge \Sigma - d\Gamma \wedge \Omega$$

must be satisfied since $\Gamma \in \Lambda^1(U)$. By introducing the form $\Theta = d\Gamma$ where we obviously have $d\Theta = 0$, we can transform the system to be solved into

$$d\Omega = \Gamma \wedge \Omega + \Sigma, \;\; d\Sigma = \Gamma \wedge \Sigma - \Theta \wedge \Omega, \;\; \Theta = d\Gamma, \;\; d\Theta = 0.$$

Our aim is to determine all forms Ω, Σ, Γ and Θ satisfying the above relations. According to the Poincaré lemma, we can locally write $\Theta = d\theta$ where $\theta \in \Lambda^1(U)$. Since the form θ is incorporated in the above equations through only its exterior derivative, we can choose $\theta \in \mathcal{A}^1(U)$ without loss of generality according to Theorem 6.4.3. Let H be the homotopy operator on the contractible open set U. Then we get $H\theta = 0$ and find that

$$\theta = dH\theta + Hd\theta = H\Theta = Hd\Gamma.$$

On the other hand, the form $\Gamma \in \Lambda^1(U)$ can be expressed as

$$\Gamma = dH\Gamma + Hd\Gamma = dH\Gamma + \theta = d\gamma + \theta,\ \gamma = H\Gamma \in \mathcal{A}^0(U).$$

Let us now consider the transformations $\Omega = e^\rho\omega$ and $\Sigma = e^\rho\sigma$ where $\rho \in \Lambda^0(U)$. The relations

$$\begin{aligned} d\Omega &= e^\rho d\rho \wedge \omega + e^\rho d\omega = e^\rho(d\gamma + \theta) \wedge \omega + e^\rho \sigma \\ d\Sigma &= e^\rho d\rho \wedge \sigma + e^\rho d\sigma = e^\rho(d\gamma + \theta) \wedge \sigma - e^\rho d\theta \wedge \omega \end{aligned}$$

lead to

$$d\omega = (d\gamma - d\rho + \theta) \wedge \omega + \sigma,\ d\sigma = (d\gamma - d\rho + \theta) \wedge \sigma - d\theta \wedge \omega.$$

We now choose the arbitrary function ρ as $\rho = \gamma = H\Gamma \in \mathcal{A}^0(U)$. We then reach to the equations

$$d\omega = \theta \wedge \omega + \sigma, \quad d\sigma = \theta \wedge \sigma - d\theta \wedge \omega.$$

With the definition $\beta = \theta \wedge \omega \in \Lambda^{k+1}(U)$, we obtain

$$d\beta = d\theta \wedge \omega - \theta \wedge d\omega = d\theta \wedge \omega - \theta \wedge (\theta \wedge \omega + \sigma) = d\theta \wedge \omega - \theta \wedge \sigma.$$

Hence, our system is reduced to a much simpler system

$$d\omega = \beta + \sigma, \quad d\sigma = -d\beta.$$

The identities $\omega = dH\omega + Hd\omega$, $\sigma = dH\sigma + Hd\sigma$ then yield

$$\omega = dH\omega + H(\beta + \sigma),\ \sigma = dH\sigma - Hd\beta = dH\sigma - \beta + dH\beta.$$

If we define the forms $\phi = H\omega \in \mathcal{A}^{k-1}(U)$, $\eta = H\sigma \in \mathcal{A}^k(U)$, we get

$$\omega = d\phi + \eta + H\beta, \quad \sigma = d\eta - \beta + dH\beta.$$

On using the above relations, we can easily determine the form β. If we write

$$\beta = \theta \wedge \omega = \theta \wedge d\phi + \theta \wedge \eta + \theta \wedge H\beta$$

and note that $\theta \wedge \eta + \theta \wedge H\beta \in \mathcal{A}^{k+1}(U)$, we then find

$$H\beta = H(\theta \wedge d\phi) + H(\theta \wedge \eta + \theta \wedge H\beta) = H(\theta \wedge d\phi).$$

We thus obtain

$$\beta = \theta \wedge d\phi + \theta \wedge \eta + \theta \wedge H(\theta \wedge d\phi)$$

and consequently

$$\begin{aligned}\omega &= d\phi + \eta + H(\theta \wedge d\phi),\\ \sigma &= d\big(\eta + H(\theta \wedge d\phi)\big) - \theta \wedge \big(d\phi + \eta + H(\theta \wedge d\phi)\big).\end{aligned}$$

Hence, the solution of the exterior differential equation is found to be

$$\begin{aligned}\Omega &= e^{\gamma}\big[d\phi + \eta + H(\theta \wedge d\phi)\big],\\ \Sigma &= e^{\gamma}\big[d\big(\eta + H(\theta \wedge d\phi)\big) - \theta \wedge \big(d\phi + \eta + H(\theta \wedge d\phi)\big)\big]\\ \Gamma &= d\gamma + \theta, \quad \Theta = d\Gamma = d\theta\end{aligned} \tag{6.7.2}$$

where $\gamma \in \mathcal{A}^0(U)$, $\theta \in \mathcal{A}^1(U)$, $\phi \in \mathcal{A}^{k-1}(U)$ and $\eta \in \mathcal{A}^k(U)$ are arbitrary forms. Now, introducing the arbitrary form $\chi = \eta + H(\theta \wedge d\phi) \in \mathcal{A}^k(U)$, we can express the above solution in a much simpler fashion as

$$\Omega = e^{\gamma}(d\phi + \chi), \;\; \Sigma = e^{\gamma} d\chi - \theta \wedge \big(d\phi + \chi\big). \tag{6.7.3}$$

If we take $\Sigma = 0$ in (6.7.1), we arrive at the following exterior differential equation

$$d\Omega = \Gamma \wedge \Omega \tag{6.7.4}$$

together with compatibility conditions $\Theta \wedge \Omega = 0, \Theta = d\Gamma$. Since we now have $\sigma = 0$, we find of course $\eta = H\sigma = 0$ and it follows from $(6.7.2)_2$ that $dH(\theta \wedge d\phi) - \theta \wedge \big(d\phi + H(\theta \wedge d\phi)\big) = 0$. We first calculate the exterior derivative of this expression, then consider its exterior product with the form $\theta \in \mathcal{A}^1(U)$ to get

$$d\big[\theta \wedge \big(d\phi + H(\theta \wedge d\phi)\big)\big] = d^2 H(\theta \wedge d\phi) = 0, \theta \wedge dH(\theta \wedge d\phi) = 0,$$

respectively. But, because we can write

$$d\big[\theta \wedge \big(d\phi + H(\theta \wedge d\phi)\big)\big] = d\theta \wedge \big(d\phi + H(\theta \wedge d\phi)\big) - \theta \wedge dH(\theta \wedge d\phi)$$

we must conclude that

$$d\theta \wedge \big(d\phi + H(\theta \wedge d\phi)\big) = 0. \tag{6.7.5}$$

Therefore, the solution of (6.7.4) is represented by

$$\Omega = e^{\gamma}\big[d\phi + H(\theta \wedge d\phi)\big], \;\gamma = H\Gamma, \;\theta = Hd\Gamma \tag{6.7.6}$$

subject to the condition (6.7.5). The forms $\Omega \in \Lambda^k(U)$ satisfying the equation (6.7.4) are called *recursive forms* with coefficient forms Γ.

A recursive form Ω is called a *gradient recursive form* if the coefficient form Γ is exact. In this case, we have $\theta = 0$ and the solution reduces to

$$\Omega = e^{\gamma}\, d\phi,\ \gamma = H\Gamma,\ \ d\Omega = d\gamma \wedge \Omega. \tag{6.7.7}$$

6.8. A SYSTEM OF EXTERIOR DIFFERENTIAL EQUATIONS

We shall now try to deal with a significantly more difficult problem. Let us consider a system of exterior differential equations

$$d\Omega^i = \ -\Gamma^i_j \wedge \Omega^j + \Sigma^i,\ \ i, j = 1, 2, \dots, r \tag{6.8.1}$$

prescribed on a contractible open set U of a manifold M. Here $\Omega^i \in \Lambda^k(U)$ are forms to be determined, and $\Gamma^i_j \in \Lambda^1(U)$ and $\Sigma^i \in \Lambda^{k+1}(U)$ are assumed to be given forms. Minus sign in (6.8.1) is chosen for convenience. It would be rather advantageous to employ a matrix notation in order to discuss this problem more efficiently. Let us denote a *matrix form* whose all entries consist of forms Φ^p_q of the same degree by $\mathbf{\Phi} = [\Phi^p_q]$. If another matrix form with different degree is $\mathbf{\Psi} = [\Psi^p_q]$, we can define the exterior product of these two matrix forms by applying the usual rule of matrix multiplication, but replacing the ordinary multiplications by exterior products as follows

$$\mathbf{\Phi} \wedge \mathbf{\Psi} = [\Phi^p_k \wedge \Psi^k_q].$$

Obviously, we have $deg\,(\mathbf{\Phi} \wedge \mathbf{\Psi}) = deg\,(\mathbf{\Phi}) + deg\,(\mathbf{\Psi})$. We can easily verify that the transpose relation $(\mathbf{\Phi} \wedge \mathbf{\Psi})^{\mathrm{T}} = (-1)^{deg\,(\mathbf{\Phi})\,deg\,(\mathbf{\Psi})}\mathbf{\Psi}^{\mathrm{T}} \wedge \mathbf{\Phi}^{\mathrm{T}}$ will be satisfied. Therefore, we are now able to write the equations (6.8.1) in the form

$$d\mathbf{\Omega} = \ -\mathbf{\Gamma} \wedge \mathbf{\Omega} + \mathbf{\Sigma}. \tag{6.8.2}$$

As matrix forms, we shall use the notations $\mathbf{\Omega} \in \mathbf{\Lambda}^k(U)$, $\mathbf{\Gamma} \in \mathbf{\Lambda}^1(U)$ and $\mathbf{\Sigma} \in \mathbf{\Lambda}^{k+1}(U)$. The compatibility equations are naturally found by taking the exterior derivative of (6.8.2). We get $\mathbf{0} = \ -d\mathbf{\Gamma} \wedge \mathbf{\Omega} + \mathbf{\Gamma} \wedge d\mathbf{\Omega} + d\mathbf{\Sigma}$ that induces the relation

$$d\mathbf{\Sigma} = (d\mathbf{\Gamma} + \mathbf{\Gamma} \wedge \mathbf{\Gamma}) \wedge \mathbf{\Omega} - \mathbf{\Gamma} \wedge \mathbf{\Sigma}. \tag{6.8.3}$$

Let us now define the following forms

$$\mathbf{\Theta} = d\mathbf{\Gamma} + \mathbf{\Gamma} \wedge \mathbf{\Gamma},\ \ \Theta^i_j = d\Gamma^i_j + \Gamma^i_k \wedge \Gamma^k_j \tag{6.8.4}$$

where $\mathbf{\Theta} = [\Theta^i_j] \in \mathbf{\Lambda}^2(U)$. Thus, the relation

$$d\mathbf{\Sigma} = \mathbf{\Theta} \wedge \mathbf{\Omega} - \mathbf{\Gamma} \wedge \mathbf{\Sigma} \tag{6.8.5}$$

must be satisfied. On the other hand, the exterior derivative of (6.8.4) yields

$$\begin{aligned} d\mathbf{\Theta} &= d\mathbf{\Gamma} \wedge \mathbf{\Gamma} - \mathbf{\Gamma} \wedge d\mathbf{\Gamma} = -\mathbf{\Gamma} \wedge \mathbf{\Gamma} \wedge \mathbf{\Gamma} + \mathbf{\Theta} \wedge \mathbf{\Gamma} + \mathbf{\Gamma} \wedge \mathbf{\Gamma} \wedge \mathbf{\Gamma} - \mathbf{\Gamma} \wedge \mathbf{\Theta} \\ &= \mathbf{\Theta} \wedge \mathbf{\Gamma} - \mathbf{\Gamma} \wedge \mathbf{\Theta}. \end{aligned}$$

It is readily checked that the exterior derivative of the above expression vanishes identically. Consequently, the exterior differential equations to be treated take finally the shapes

$$\begin{aligned} d\mathbf{\Omega} &= -\mathbf{\Gamma} \wedge \mathbf{\Omega} + \mathbf{\Sigma}, \;\; d\mathbf{\Sigma} = \mathbf{\Theta} \wedge \mathbf{\Omega} - \mathbf{\Gamma} \wedge \mathbf{\Sigma}, \\ d\mathbf{\Gamma} &= -\mathbf{\Gamma} \wedge \mathbf{\Gamma} + \mathbf{\Theta}, \;\; d\mathbf{\Theta} = \mathbf{\Theta} \wedge \mathbf{\Gamma} - \mathbf{\Gamma} \wedge \mathbf{\Theta}. \end{aligned} \tag{6.8.6}$$

Our task is to find the admissible forms of $\mathbf{\Gamma}$ and $\mathbf{\Sigma}$, and to determine $\mathbf{\Omega}$. According to Theorem 6.4.3, we can take

$$\mathbf{\Gamma} = d\boldsymbol{\gamma} + \mathbf{\Gamma}_a, \;\; \boldsymbol{\gamma} \in \boldsymbol{\mathcal{A}}^0(U), \;\; \mathbf{\Gamma}_a \in \boldsymbol{\mathcal{A}}^1(U) \tag{6.8.7}$$

We can of course represent (6.8.7) explicitly as

$$\Gamma^i_j = d\gamma^i_j + (\Gamma_a)^i_j$$

where $\gamma^i_j \in \mathcal{A}^0(U)$ and $(\Gamma_a)^i_j \in \mathcal{A}^1(U)$. If the centre of the star-shaped homeomorphic image of the region U in $\mathbb{R}^m$ is $\mathbf{x}_0$, then we can take without loss of generality $\boldsymbol{\gamma}(\mathbf{x}_0) = \mathbf{0}$ since $\boldsymbol{\gamma}$ enters the equations through its differential. If we insert the expression (6.8.7) into the equation $(6.8.6)_3$, we find

$$d\mathbf{\Gamma}_a = -d\boldsymbol{\gamma} \wedge d\boldsymbol{\gamma} - \mathbf{\Gamma}_a \wedge d\boldsymbol{\gamma} - d\boldsymbol{\gamma} \wedge \mathbf{\Gamma}_a - \mathbf{\Gamma}_a \wedge \mathbf{\Gamma}_a + \mathbf{\Theta}. \tag{6.8.8}$$

Let us assume that $\mathbf{B} \in \boldsymbol{\mathcal{A}}^0(U)$ is an arbitrary regular $r \times r$ matrix, that is, $\det \mathbf{B} \neq 0$. We define the forms $\overline{\mathbf{\Gamma}}$ and $\widetilde{\mathbf{\Theta}}$ by

$$\overline{\mathbf{\Gamma}} = \mathbf{B}^{-1}\mathbf{\Gamma}_a\mathbf{B}, \qquad \widetilde{\mathbf{\Theta}} = \mathbf{B}^{-1}\mathbf{\Theta}\mathbf{B}. \tag{6.8.9}$$

On inverting these expressions, we get $\mathbf{\Gamma}_a = \mathbf{B}\overline{\mathbf{\Gamma}}\mathbf{B}^{-1}$ and $\mathbf{\Theta} = \mathbf{B}\widetilde{\mathbf{\Theta}}\mathbf{B}^{-1}$. It is clear that $\overline{\mathbf{\Gamma}} \in \boldsymbol{\mathcal{A}}^1(U)$. The exterior derivative of $(6.8.9)_1$ gives

$$d\overline{\mathbf{\Gamma}} = d\mathbf{B}^{-1} \wedge \mathbf{\Gamma}_a\mathbf{B} + \mathbf{B}^{-1}(d\mathbf{\Gamma}_a)\mathbf{B} - \mathbf{B}^{-1}\mathbf{\Gamma}_a \wedge d\mathbf{B}.$$

Differentiating $\mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$, we find that $d\mathbf{B}^{-1}\mathbf{B} + \mathbf{B}^{-1}d\mathbf{B} = \mathbf{0}$ from which we deduce that $d\mathbf{B}^{-1} = -\mathbf{B}^{-1}(d\mathbf{B})\mathbf{B}^{-1}$. On using $(6.8.9)_1$, we finally get

$$d\overline{\mathbf{\Gamma}} = \mathbf{B}^{-1}(d\mathbf{\Gamma}_a)\mathbf{B} - \mathbf{B}^{-1}d\mathbf{B} \wedge \overline{\mathbf{\Gamma}} - \overline{\mathbf{\Gamma}}\mathbf{B}^{-1} \wedge d\mathbf{B}.$$

On the other hand, it follows from (6.8.8) that

$$\begin{aligned}\mathbf{B}^{-1}(d\boldsymbol{\Gamma}_a)\mathbf{B} = &-\mathbf{B}^{-1}(d\boldsymbol{\gamma}\wedge d\boldsymbol{\gamma})\mathbf{B} - \mathbf{B}^{-1}\mathbf{B}\overline{\boldsymbol{\Gamma}}\mathbf{B}^{-1}\wedge(d\boldsymbol{\gamma})\mathbf{B}\\ &-\mathbf{B}^{-1}d\boldsymbol{\gamma}\wedge\mathbf{B}\overline{\boldsymbol{\Gamma}}\mathbf{B}^{-1}\mathbf{B} - \mathbf{B}^{-1}\mathbf{B}\overline{\boldsymbol{\Gamma}}\mathbf{B}^{-1}\wedge\mathbf{B}\overline{\boldsymbol{\Gamma}}\mathbf{B}^{-1}\mathbf{B} + \widetilde{\boldsymbol{\Theta}}.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}d\overline{\boldsymbol{\Gamma}} = &-\mathbf{B}^{-1}(d\boldsymbol{\gamma}\wedge d\boldsymbol{\gamma})\mathbf{B} - \overline{\boldsymbol{\Gamma}}\wedge\mathbf{B}^{-1}(d\boldsymbol{\gamma})\mathbf{B} - \mathbf{B}^{-1}(d\boldsymbol{\gamma})\mathbf{B}\wedge\overline{\boldsymbol{\Gamma}} - \overline{\boldsymbol{\Gamma}}\wedge\overline{\boldsymbol{\Gamma}}\\ &+\boldsymbol{\Theta} - \mathbf{B}^{-1}d\mathbf{B}\wedge\overline{\boldsymbol{\Gamma}} - \overline{\boldsymbol{\Gamma}}\wedge\mathbf{B}^{-1}d\mathbf{B} = -\overline{\boldsymbol{\Gamma}}\wedge\mathbf{B}^{-1}(d\boldsymbol{\gamma}\,\mathbf{B} + d\mathbf{B})\\ &-\mathbf{B}^{-1}(d\boldsymbol{\gamma}\,\mathbf{B} + d\mathbf{B})\wedge\overline{\boldsymbol{\Gamma}} - \mathbf{B}^{-1}(d\boldsymbol{\gamma}\wedge d\boldsymbol{\gamma})\mathbf{B} - \overline{\boldsymbol{\Gamma}}\wedge\overline{\boldsymbol{\Gamma}} + \widetilde{\boldsymbol{\Theta}}.\end{aligned}$$

We shall now try to remove the arbitrariness of the matrix $\mathbf{B}$ in such a way that the relation

$$d\boldsymbol{\gamma}\,\mathbf{B} + d\mathbf{B} = -\mathbf{B}\boldsymbol{\mu} \quad \text{or} \quad d\gamma^i_k\,B^k_j + dB^i_j = B^i_k\mu^k_j \tag{6.8.10}$$

is satisfied and the matrix form $\boldsymbol{\mu}\in\boldsymbol{\Lambda}^1(U)$ belongs to the set $\boldsymbol{\mathcal{A}}^1(U)$. The exterior derivative of (6.8.10) yields

$$-d\boldsymbol{\gamma}\wedge d\mathbf{B} + d\mathbf{B}\wedge\boldsymbol{\mu} + \mathbf{B}\,d\boldsymbol{\mu} = \mathbf{0} \text{ and } d\boldsymbol{\mu} = \mathbf{B}^{-1}d\boldsymbol{\gamma}\wedge d\mathbf{B} - \mathbf{B}^{-1}d\mathbf{B}\wedge\boldsymbol{\mu}.$$

Therefore, on employing (6.8.10) we conclude that

$$\begin{aligned}d\boldsymbol{\mu} &= -\mathbf{B}^{-1}(d\boldsymbol{\gamma}\wedge d\boldsymbol{\gamma})\mathbf{B} - \mathbf{B}^{-1}d\boldsymbol{\gamma}\mathbf{B}\wedge\boldsymbol{\mu} + \mathbf{B}^{-1}d\boldsymbol{\gamma}\mathbf{B}\wedge\boldsymbol{\mu} + \boldsymbol{\mu}\wedge\boldsymbol{\mu}\\ &= -\mathbf{B}^{-1}(d\boldsymbol{\gamma}\wedge d\boldsymbol{\gamma})\mathbf{B} + \boldsymbol{\mu}\wedge\boldsymbol{\mu}.\end{aligned}$$

If $\boldsymbol{\mu}\in\boldsymbol{\mathcal{A}}^1(U)$, then one has $\boldsymbol{\mu}\wedge\boldsymbol{\mu}\in\boldsymbol{\mathcal{A}}^2(U)$ so the relations $\boldsymbol{\mu} = H d\boldsymbol{\mu}$ and $H(\boldsymbol{\mu}\wedge\boldsymbol{\mu}) = \mathbf{0}$ hold. Thus, we can choose

$$\begin{aligned}\boldsymbol{\mu} &= H d\boldsymbol{\mu}\\ &= -H\big(\mathbf{B}^{-1}(d\boldsymbol{\gamma}\wedge d\boldsymbol{\gamma})\mathbf{B}\big)\end{aligned}$$

so that we obtain

$$d\mathbf{B} = -d\boldsymbol{\gamma}\mathbf{B} + \mathbf{B}H\big(\mathbf{B}^{-1}(d\boldsymbol{\gamma}\wedge d\boldsymbol{\gamma})\mathbf{B}\big).$$

Because $\mathbf{B}\in\boldsymbol{\mathcal{A}}^0(U)$, we know that we can write $\mathbf{B} - \mathbf{B}_0 = Hd\mathbf{B}$ where $\mathbf{B}_0 = \mathbf{B}(\mathbf{x}_0)$. $\mathbf{B}H\big(\mathbf{B}^{-1}(d\boldsymbol{\gamma}\wedge d\boldsymbol{\gamma})\mathbf{B}\big)\in\boldsymbol{\mathcal{A}}^1(U)$ implies that this form is in the null space of the operator H. This means that on applying the operator H on $d\mathbf{B}$, the matrix $\mathbf{B}$ will have to satisfy the equation

$$\mathbf{B} = \mathbf{B}_0 - H(d\boldsymbol{\gamma}\,\mathbf{B}) = \mathbf{B}_0 - H(\boldsymbol{\Gamma}\mathbf{B}) + H(\boldsymbol{\Gamma}_a\mathbf{B}).$$

But since $\boldsymbol{\Gamma}_a\in\boldsymbol{\mathcal{A}}^1(U)$, we get $H(\boldsymbol{\Gamma}_a\mathbf{B}) = \mathbf{0}$. If we write $\mathbf{B} = \mathbf{A}\mathbf{B}_0$, we see at once that $\mathbf{A}$ is a regular matrix and $\mathbf{A}(\mathbf{x}_0) = \mathbf{I}$. We are thus led to the conclusion that the matrix $\mathbf{A}$ has to satisfy the integral equation

$$\mathbf{A} + H(\mathbf{\Gamma A}) = \mathbf{A} + H(d\boldsymbol{\gamma}\,\mathbf{A}) = \mathbf{I},\ \mathbf{A}(\mathbf{x}_0) = \mathbf{I}. \tag{6.8.11}$$

Let us now denote the forms $\mathbf{\Gamma}$ by $\Gamma^i_k = \Gamma^i_{kl}dx^l$. Then (6.8.11) is explicitly expressed in indicial notation as

$$a^i_j(\mathbf{x}) + \int_0^1 (x^l - x^l_0)\Gamma^i_{kl}[\mathbf{x}_0 + t(\mathbf{x} - \mathbf{x}_0)]a^k_j[\mathbf{x}_0 + t(\mathbf{x} - \mathbf{x}_0)]dt = \delta^i_j.$$

$\mathbf{A} = [a^i_j]$ is called the ***attitude matrix***. With the present choice of the matrix $\mathbf{B}$, we immediately see that one is able to write

$$d\overline{\mathbf{\Gamma}} = \overline{\mathbf{\Gamma}} \wedge \boldsymbol{\mu} + \boldsymbol{\mu} \wedge \overline{\mathbf{\Gamma}} + d\boldsymbol{\mu} - \boldsymbol{\mu} \wedge \boldsymbol{\mu} - \overline{\mathbf{\Gamma}} \wedge \overline{\mathbf{\Gamma}} + \widetilde{\mathbf{\Theta}}.$$

Since $\overline{\mathbf{\Gamma}}, \boldsymbol{\mu} \in \boldsymbol{\mathcal{A}}^1(U)$, when we apply the homotopy operator H to the foregoing equation we find $\overline{\mathbf{\Gamma}} = \boldsymbol{\mu} + H\widetilde{\mathbf{\Theta}}$ and introducing all these results into (6.8.7) we arrive at the expression

$$\begin{aligned}\mathbf{\Gamma} &= d\boldsymbol{\gamma} + \mathbf{B}\boldsymbol{\mu}\mathbf{B}^{-1} + \mathbf{B}H(\widetilde{\mathbf{\Theta}})\mathbf{B}^{-1} \\ &= d\boldsymbol{\gamma} - d\boldsymbol{\gamma} - d\mathbf{B}\mathbf{B}^{-1} + \mathbf{B}H(\widetilde{\mathbf{\Theta}})\mathbf{B}^{-1} \\ &= (\mathbf{B}H\widetilde{\mathbf{\Theta}} - d\mathbf{B})\mathbf{B}^{-1} = \big(\mathbf{B}H(\mathbf{B}^{-1}\mathbf{\Theta}\mathbf{B}) - d\mathbf{B}\big)\mathbf{B}^{-1}.\end{aligned}$$

On inserting now $\mathbf{B} = \mathbf{A}\mathbf{B}_0$ and $\mathbf{B}^{-1} = \mathbf{B}_0^{-1}\mathbf{A}^{-1}$ into the above relation, we finally conclude that

$$\mathbf{\Gamma} = \big[\mathbf{A}H(\mathbf{A}^{-1}\mathbf{\Theta}\mathbf{A}) - d\mathbf{A}\big]\mathbf{A}^{-1}. \tag{6.8.12}$$

Let us now take the equation $(6.8.6)_4$ into account. We then introduce a matrix form $\overline{\mathbf{\Theta}} = \mathbf{A}^{-1}\mathbf{\Theta}\mathbf{A}$ so that one can write $\mathbf{\Theta} = \mathbf{A}\overline{\mathbf{\Theta}}\mathbf{A}^{-1}$ whose exterior derivative has to satisfy

$$d\mathbf{A} \wedge \overline{\mathbf{\Theta}}\mathbf{A}^{-1} + \mathbf{A}\,d\overline{\mathbf{\Theta}}\,\mathbf{A}^{-1} + \mathbf{A}\overline{\mathbf{\Theta}} \wedge d\mathbf{A}^{-1} = \mathbf{A}\overline{\mathbf{\Theta}}\mathbf{A}^{-1} \wedge \mathbf{\Gamma} - \mathbf{\Gamma} \wedge \mathbf{A}\overline{\mathbf{\Theta}}\mathbf{A}^{-1}.$$

On recalling the equality $d\mathbf{A}^{-1}\mathbf{A} = -\mathbf{A}^{-1}d\mathbf{A}$, this equation leads to

$$\begin{aligned}d\overline{\mathbf{\Theta}} &= -\mathbf{A}^{-1}\mathbf{\Gamma}\mathbf{A} \wedge \overline{\mathbf{\Theta}} + \overline{\mathbf{\Theta}} \wedge \mathbf{A}^{-1}\mathbf{\Gamma}\mathbf{A} - \mathbf{A}^{-1}d\mathbf{A} \wedge \overline{\mathbf{\Theta}} - \overline{\mathbf{\Theta}} \wedge d\mathbf{A}^{-1}\mathbf{A} \\ &= -(\mathbf{A}^{-1}\mathbf{\Gamma}\mathbf{A} + \mathbf{A}^{-1}d\mathbf{A}) \wedge \overline{\mathbf{\Theta}} + \overline{\mathbf{\Theta}} \wedge (\mathbf{A}^{-1}\mathbf{\Gamma}\mathbf{A} + \mathbf{A}^{-1}d\mathbf{A}).\end{aligned}$$

On the other hand, it follows from (6.8.12) that

$$\mathbf{A}^{-1}\mathbf{\Gamma}\mathbf{A} + \mathbf{A}^{-1}d\mathbf{A} = H(\overline{\mathbf{\Theta}}).$$

Therefore, we obtain

$$d\overline{\mathbf{\Theta}} = -H(\overline{\mathbf{\Theta}}) \wedge \overline{\mathbf{\Theta}} + \overline{\mathbf{\Theta}} \wedge H(\overline{\mathbf{\Theta}}).$$

In view of Theorem 6.4.3, we can use the representation $\bar{\boldsymbol{\Theta}} = d\boldsymbol{\theta} + \bar{\boldsymbol{\Theta}}_a$ in which $\boldsymbol{\theta} = H\bar{\boldsymbol{\Theta}} \in \boldsymbol{\mathcal{A}}^1(U)$ and $\bar{\boldsymbol{\Theta}}_a = Hd\bar{\boldsymbol{\Theta}} \in \boldsymbol{\mathcal{A}}^2(U)$ are, respectively, 1- and 2-antiexact forms. We thus find $H(\bar{\boldsymbol{\Theta}}) = Hd\boldsymbol{\theta} + H\bar{\boldsymbol{\Theta}}_a = \boldsymbol{\theta}$. Therefore, we obtain

$$\begin{aligned} d\bar{\boldsymbol{\Theta}}_a = dHd\bar{\boldsymbol{\Theta}} = d\bar{\boldsymbol{\Theta}} &= -\boldsymbol{\theta}\wedge d\boldsymbol{\theta} + d\boldsymbol{\theta}\wedge\boldsymbol{\theta} - \boldsymbol{\theta}\wedge\bar{\boldsymbol{\Theta}}_a + \bar{\boldsymbol{\Theta}}_a\wedge\boldsymbol{\theta} \\ &= d(\boldsymbol{\theta}\wedge\boldsymbol{\theta}) - \boldsymbol{\theta}\wedge\bar{\boldsymbol{\Theta}}_a + \bar{\boldsymbol{\Theta}}_a\wedge\boldsymbol{\theta}. \end{aligned}$$

When we apply the operator H to that expression, we get $Hd\bar{\boldsymbol{\Theta}}_a = \bar{\boldsymbol{\Theta}}_a = \boldsymbol{\theta}\wedge\boldsymbol{\theta}$ and we obtain the following representation

$$\bar{\boldsymbol{\Theta}} = d\boldsymbol{\theta} + \boldsymbol{\theta}\wedge\boldsymbol{\theta}, \quad \boldsymbol{\Theta} = \mathbf{A}(d\boldsymbol{\theta} + \boldsymbol{\theta}\wedge\boldsymbol{\theta})\mathbf{A}^{-1} \tag{6.8.13}$$

whereas (6.8.12) takes the shape

$$\boldsymbol{\Gamma} = (\mathbf{A}\boldsymbol{\theta} - d\mathbf{A})\mathbf{A}^{-1} \;\text{ and }\; \boldsymbol{\theta} = \mathbf{A}^{-1}(\boldsymbol{\Gamma}\mathbf{A} + d\mathbf{A}). \tag{6.8.14}$$

Let us now define the matrix forms $\boldsymbol{\omega} \in \boldsymbol{\Lambda}^k(U)$ and $\boldsymbol{\sigma} \in \boldsymbol{\Lambda}^{k+1}(U)$ through the relations

$$\boldsymbol{\Omega} = \mathbf{A}\boldsymbol{\omega}, \quad \boldsymbol{\Sigma} = \mathbf{A}\boldsymbol{\sigma}.$$

So the equation $(6.8.6)_1$ is transformed into

$$d\boldsymbol{\Omega} = d\mathbf{A}\wedge\boldsymbol{\omega} + \mathbf{A}d\boldsymbol{\omega} = -\boldsymbol{\Gamma}\mathbf{A}\wedge\boldsymbol{\omega} + \mathbf{A}\boldsymbol{\sigma}$$

from which we extract the expression

$$d\boldsymbol{\omega} = -\mathbf{A}^{-1}(\boldsymbol{\Gamma}\mathbf{A} + d\mathbf{A})\wedge\boldsymbol{\omega} + \boldsymbol{\sigma} = -\boldsymbol{\theta}\wedge\boldsymbol{\omega} + \boldsymbol{\sigma}. \tag{6.8.15}$$

Similarly, the equation $(6.8.6)_2$ becomes

$$d\boldsymbol{\Sigma} = d\mathbf{A}\wedge\boldsymbol{\sigma} + \mathbf{A}d\boldsymbol{\sigma} = \mathbf{A}(d\boldsymbol{\theta} + \boldsymbol{\theta}\wedge\boldsymbol{\theta})\mathbf{A}^{-1}\wedge\mathbf{A}\boldsymbol{\omega} - \boldsymbol{\Gamma}\mathbf{A}\wedge\boldsymbol{\sigma}$$

and one obtains

$$d\boldsymbol{\sigma} = -\boldsymbol{\theta}\wedge\boldsymbol{\sigma} + (d\boldsymbol{\theta} + \boldsymbol{\theta}\wedge\boldsymbol{\theta})\wedge\boldsymbol{\omega}. \tag{6.8.16}$$

After having resorted to Theorem 6.4.3, we can write

$$\boldsymbol{\omega} = d\boldsymbol{\phi} + \boldsymbol{\omega}_a, \quad \boldsymbol{\sigma} = d\boldsymbol{\eta} + \boldsymbol{\sigma}_a, \quad \boldsymbol{\beta} = \boldsymbol{\theta}\wedge\boldsymbol{\omega} \tag{6.8.17}$$

where we introduced the matrix forms $\boldsymbol{\phi} = H\boldsymbol{\omega} = H(\mathbf{A}^{-1}\boldsymbol{\Omega}) \in \boldsymbol{\mathcal{A}}^{k-1}(U)$, $\boldsymbol{\omega}_a \in \boldsymbol{\mathcal{A}}^k(U)$ and $\boldsymbol{\eta} = H\boldsymbol{\sigma} = H(\mathbf{A}^{-1}\boldsymbol{\Sigma}) \in \boldsymbol{\mathcal{A}}^k(U)$, $\boldsymbol{\sigma}_a \in \boldsymbol{\mathcal{A}}^{k+1}(U)$. It then follows from (6.8.17), (6.8.15) and (6.8.16) that

$$\begin{aligned} &d\boldsymbol{\omega} = d\boldsymbol{\omega}_a = \boldsymbol{\sigma} - \boldsymbol{\beta} = d\boldsymbol{\eta} + \boldsymbol{\sigma}_a - \boldsymbol{\beta}, \\ &d\boldsymbol{\sigma} = d\boldsymbol{\sigma}_a = d\boldsymbol{\theta}\wedge\boldsymbol{\omega} - \boldsymbol{\theta}\wedge(\boldsymbol{\sigma} - \boldsymbol{\beta}) = d\boldsymbol{\theta}\wedge\boldsymbol{\omega} - \boldsymbol{\theta}\wedge d\boldsymbol{\omega} = d\boldsymbol{\beta}. \end{aligned}$$

On applying the operator H to these equations, we get

$$\begin{aligned}\boldsymbol{\omega}_a &= -H\boldsymbol{\beta}+\boldsymbol{\eta},\\ \boldsymbol{\sigma}_a &= Hd\boldsymbol{\beta}=\boldsymbol{\beta}-dH\boldsymbol{\beta}\end{aligned}$$

and, consequently, we arrive at the result

$$\boldsymbol{\omega}=d\boldsymbol{\phi}-H\boldsymbol{\beta}+\boldsymbol{\eta},\quad \boldsymbol{\sigma}=d\boldsymbol{\eta}-dH\boldsymbol{\beta}+\boldsymbol{\beta}. \tag{6.8.18}$$

In the relation

$$\boldsymbol{\beta}=\boldsymbol{\theta}\wedge d\boldsymbol{\phi}-\boldsymbol{\theta}\wedge H\boldsymbol{\beta}+\boldsymbol{\theta}\wedge\boldsymbol{\eta}$$

$\boldsymbol{\theta}\wedge H\boldsymbol{\beta}$ and $\boldsymbol{\theta}\wedge\boldsymbol{\eta}$ are antiexact forms. Therefore, we can write

$$H\boldsymbol{\beta}=H(\boldsymbol{\theta}\wedge d\boldsymbol{\phi})\quad\text{and}\quad \boldsymbol{\beta}=\boldsymbol{\theta}\wedge d\boldsymbol{\phi}-\boldsymbol{\theta}\wedge H(\boldsymbol{\theta}\wedge d\boldsymbol{\phi})+\boldsymbol{\theta}\wedge\boldsymbol{\eta}.$$

On the other hand, if we take notice of the relation $dH\boldsymbol{\beta}=dH(\boldsymbol{\theta}\wedge d\boldsymbol{\phi})=\boldsymbol{\theta}\wedge d\boldsymbol{\phi}-Hd(\boldsymbol{\theta}\wedge d\boldsymbol{\phi})=\boldsymbol{\theta}\wedge d\boldsymbol{\phi}-H(d\boldsymbol{\theta}\wedge d\boldsymbol{\phi})$, then (6.8.18) leads to

$$\begin{aligned}\boldsymbol{\omega}&=d\boldsymbol{\phi}+\boldsymbol{\eta}-H(\boldsymbol{\theta}\wedge d\boldsymbol{\phi}),\\ \boldsymbol{\sigma}&=d\boldsymbol{\eta}+\boldsymbol{\theta}\wedge\boldsymbol{\eta}+H(d\boldsymbol{\theta}\wedge d\boldsymbol{\phi})-\boldsymbol{\theta}\wedge H(\boldsymbol{\theta}\wedge d\boldsymbol{\phi}).\end{aligned}$$

Thus, the solution of the system of exterior differential equations (6.8.6) is provided by

$$\begin{aligned}\boldsymbol{\Omega}&=\mathbf{A}\big[d\boldsymbol{\phi}+\boldsymbol{\eta}-H(\boldsymbol{\theta}\wedge d\boldsymbol{\phi})\big],\\ \boldsymbol{\Sigma}&=\mathbf{A}\big[d\boldsymbol{\eta}+\boldsymbol{\theta}\wedge\boldsymbol{\eta}+H(d\boldsymbol{\theta}\wedge d\boldsymbol{\phi})-\boldsymbol{\theta}\wedge H(\boldsymbol{\theta}\wedge d\boldsymbol{\phi})\big],\\ \boldsymbol{\Gamma}&=(\mathbf{A}\boldsymbol{\theta}-d\mathbf{A})\mathbf{A}^{-1},\\ \boldsymbol{\theta}&=\mathbf{A}^{-1}(\boldsymbol{\Gamma}\mathbf{A}+d\mathbf{A}),\\ \boldsymbol{\Theta}&=\mathbf{A}(d\boldsymbol{\theta}+\boldsymbol{\theta}\wedge\boldsymbol{\theta})\mathbf{A}^{-1}\end{aligned} \tag{6.8.19}$$

where $\boldsymbol{\phi}\in\boldsymbol{\mathcal{A}}^{k-1}(U)$ and $\boldsymbol{\eta}\in\boldsymbol{\mathcal{A}}^{k}(U)$ are arbitrary matrix forms and the matrix $\mathbf{A}$ is determined by solving the integral equation (6.8.11) once the matrix form $\boldsymbol{\Gamma}$ is prescribed. The matrix form $\boldsymbol{\theta}\in\boldsymbol{\mathcal{A}}^{1}(U)$ is then found from $(6.8.19)_4$. On the contrary, if we choose a matrix form $\boldsymbol{\theta}$, then the admissible matrix form $\boldsymbol{\Gamma}$ is deduced from $(6.8.19)_3$. The matrix $\mathbf{A}$ has to be the solution of the integrodifferential equation

$$\mathbf{A}+H(\mathbf{A}\boldsymbol{\theta}-d\mathbf{A})=\mathbf{I}$$

obtained from (6.8.11) by replacing $\boldsymbol{\Gamma}$ by $(6.8.19)_3$. Let us now define a matrix form $\boldsymbol{\psi}\in\boldsymbol{\mathcal{A}}^{k}(U)$ by $\boldsymbol{\psi}=\boldsymbol{\eta}-H(\boldsymbol{\theta}\wedge d\boldsymbol{\phi})$ whose exterior derivative is expressible as $d\boldsymbol{\psi}=d\boldsymbol{\eta}-\boldsymbol{\theta}\wedge d\boldsymbol{\phi}+H(d\boldsymbol{\theta}\wedge d\boldsymbol{\phi})$. Then we easily verify that the relations $(6.8.19)_{1-2}$ are reduced to simpler forms given below

$$\mathbf{\Omega} = \mathbf{A}(d\boldsymbol{\phi} + \boldsymbol{\psi}), \ \ \mathbf{\Sigma} = \mathbf{A}\big[d\boldsymbol{\psi} + \boldsymbol{\theta} \wedge (d\boldsymbol{\phi} + \boldsymbol{\psi})\big]. \tag{6.8.20}$$

But, in order that these equations are to be legitimate, we have to demonstrate that the matrix $\mathbf{A}$ determined through (6.8.11) is regular, that is, $A = \det \mathbf{A} \neq 0$. We rewrite (6.8.10) as $d\boldsymbol{\gamma}\mathbf{A} + d\mathbf{A} = -\mathbf{A}\mathbf{B}_0\boldsymbol{\mu}\mathbf{B}_0^{-1}$. We can then easily find that dA is expressible as

$$\begin{aligned} dA &= (\partial A/\partial a^i_j)da^i_j = (\mathit{Cofactor}\, a^i_j)\, da^i_j \\ &= A\overline{a}^{1j}_{\ i}\, da^i_j = A \operatorname{tr}(d\mathbf{A}\mathbf{A}^{-1}). \end{aligned}$$

Hence, we obtain $dA = -A \operatorname{tr}(d\boldsymbol{\gamma} + \mathbf{A}\mathbf{B}_0\boldsymbol{\mu}\mathbf{B}_0^{-1}\mathbf{A}^{-1}) = -A \operatorname{tr}(d\boldsymbol{\gamma} + \boldsymbol{\mu})$ and consequently

$$d\log A = -d\operatorname{tr}\boldsymbol{\gamma} - \operatorname{tr}\boldsymbol{\mu}, \ \ Hd\log A = -\operatorname{tr} Hd\boldsymbol{\gamma} - \operatorname{tr} H\boldsymbol{\mu}.$$

Since $H\boldsymbol{\mu} = \mathbf{0}$ and $Hd\boldsymbol{\gamma} = \boldsymbol{\gamma}$, we get $\log A(\mathbf{x}) - \log 1 = -\operatorname{tr}\boldsymbol{\gamma}(\mathbf{x})$. We thus conclude that

$$A(\mathbf{x}) = e^{-\operatorname{tr}\gamma} = e^{-\operatorname{tr} H\mathbf{\Gamma}} \neq 0$$

proving that $\mathbf{A}$ is a regular matrix.

Next, we define two systems of exterior differential equations on an open set $U \subseteq M$:

$$\begin{aligned} d\mathbf{\Omega} &= -\mathbf{\Gamma}\wedge\mathbf{\Omega} + \mathbf{\Sigma}, & d\mathbf{\Omega}' &= -\mathbf{\Gamma}'\wedge\mathbf{\Omega}' + \mathbf{\Sigma}', \\ d\mathbf{\Sigma} &= \mathbf{\Theta}\wedge\mathbf{\Omega} - \mathbf{\Gamma}\wedge\mathbf{\Sigma}, & d\mathbf{\Sigma}' &= \mathbf{\Theta}'\wedge\mathbf{\Omega}' - \mathbf{\Gamma}'\wedge\mathbf{\Sigma}', \\ d\mathbf{\Gamma} &= -\mathbf{\Gamma}\wedge\mathbf{\Gamma} + \mathbf{\Theta}, & d\mathbf{\Gamma}' &= -\mathbf{\Gamma}'\wedge\mathbf{\Gamma}' + \mathbf{\Theta}', \\ d\mathbf{\Theta} &= \mathbf{\Theta}\wedge\mathbf{\Gamma} - \mathbf{\Gamma}\wedge\mathbf{\Theta}, & d\mathbf{\Theta}' &= \mathbf{\Theta}'\wedge\mathbf{\Gamma}' - \mathbf{\Gamma}'\wedge\mathbf{\Theta}'. \end{aligned}$$

We say that these two systems are *equivalent* if we have $\mathbf{\Omega} = \mathbf{\Omega}'$ on U, namely, if they lead to the same solution. In such a case we first observe that the relation

$$\mathbf{\Sigma}' = \mathbf{\Sigma} + (\mathbf{\Gamma}' - \mathbf{\Gamma})\wedge\mathbf{\Omega}$$

must be satisfied. Actually both systems involve same kind of solutions as (6.8.19). But these solutions should be interrelated in order to obtain the same $\mathbf{\Omega}$ from those two systems. These relations turns out to be quite complicated. That is the reason why they are not included here. Notwithstanding, a specific situation bears a particular importance. In the second system, let us take

$$\mathbf{\Gamma}' = -d\mathbf{A}\,\mathbf{A}^{-1}.$$

Then $(6.8.19)_3$ gives rise to $\mathbf{\Gamma} - \mathbf{\Gamma}' = \mathbf{A}\boldsymbol{\theta}\mathbf{A}^{-1}$ and $\mathbf{\Sigma}' = \mathbf{\Sigma} - \mathbf{A}\boldsymbol{\theta}\mathbf{A}^{-1} \wedge \mathbf{\Omega}$. We know that $\boldsymbol{\theta} \in \boldsymbol{\mathcal{A}}^1(U)$. We thus get $\mathbf{A}\boldsymbol{\theta}\mathbf{A}^{-1} \in \boldsymbol{\mathcal{A}}^1(U)$ so we find that

$$\boldsymbol{\gamma}' = H(\mathbf{\Gamma}') = -H(d\mathbf{A}\,\mathbf{A}^{-1}) = H(\mathbf{A}\boldsymbol{\theta}\mathbf{A}^{-1} - d\mathbf{A}\,\mathbf{A}^{-1}) = H(\mathbf{\Gamma}) = \boldsymbol{\gamma}.$$

This implies that $\mathbf{A}$ and $\mathbf{A}'$ satisfy the same matrix integral equation (6.8.11) so that we can take $\mathbf{A}' = \mathbf{A}$. On the other hand, it follows from the relation $d\mathbf{A}^{-1} = -\mathbf{A}^{-1}d\mathbf{A}\mathbf{A}^{-1}$ that

$$d\mathbf{\Gamma}' = d\mathbf{A} \wedge d\mathbf{A}^{-1} = -d\mathbf{A}\mathbf{A}^{-1} \wedge d\mathbf{A}\mathbf{A}^{-1} = -\mathbf{\Gamma}' \wedge \mathbf{\Gamma}'.$$

We thus obtain $\mathbf{\Theta}' = \mathbf{0}$ and $\boldsymbol{\theta}' = H(\mathbf{A}^{-1}\mathbf{\Theta}'\mathbf{A}) = \mathbf{0}$. Furthermore, we find

$$\begin{aligned}\boldsymbol{\phi}' &= H(\mathbf{A}^{-1}\mathbf{\Omega}) = \boldsymbol{\phi},\\ \boldsymbol{\eta}' &= H(\mathbf{A}^{-1}\mathbf{\Sigma}') = H(\mathbf{A}^{-1}\mathbf{\Sigma} - \boldsymbol{\theta}\mathbf{A}^{-1} \wedge \mathbf{\Omega}) = \boldsymbol{\eta} - H(\boldsymbol{\theta} \wedge \boldsymbol{\omega})\\ &= \boldsymbol{\eta} - H(\boldsymbol{\theta} \wedge d\boldsymbol{\phi}) - H(\boldsymbol{\theta} \wedge \boldsymbol{\omega}_a) = \boldsymbol{\eta} - H(\boldsymbol{\theta} \wedge d\boldsymbol{\phi}).\end{aligned}$$

We thus understand that any system of exterior differential equations is equivalent to the system

$$d\mathbf{\Omega}' = -\mathbf{\Gamma}' \wedge \mathbf{\Omega}' + \mathbf{\Sigma}',\ d\mathbf{\Sigma}' = -\mathbf{\Gamma}' \wedge \mathbf{\Sigma}', d\mathbf{\Gamma}' = -\mathbf{\Gamma}' \wedge \mathbf{\Gamma}',\ \mathbf{\Theta}' = \mathbf{0}.$$

As to the solution of the equivalent system, it is easily found that

$$\mathbf{\Omega} = \mathbf{A}(d\boldsymbol{\phi} + \boldsymbol{\eta}'), \quad \mathbf{\Sigma}' = \mathbf{A}d\boldsymbol{\eta}', \quad \mathbf{\Gamma}' = -d\mathbf{A}\,\mathbf{A}^{-1}$$

from which we reach to a sort of generalisation of the Frobenius theorem: a representation $\mathbf{\Omega} = \mathbf{A}d\boldsymbol{\phi}$ for a matrix k-form $\mathbf{\Omega}$ becomes possible if and only if $\boldsymbol{\eta}' = \mathbf{0}$. This entails of course the condition $\mathbf{\Sigma}' = \mathbf{0}$.

Example 6.8.1. The matrix forms $\mathbf{\Omega} \in \mathbf{\Lambda}^k(U)$ and $\mathbf{\Gamma} \in \mathbf{\Lambda}^1(U)$ are given by

$$\mathbf{\Omega} = \begin{bmatrix}\Omega_1\\ \Omega_2\end{bmatrix},\ \mathbf{\Gamma} = -\mathbf{G}\,df,\ \mathbf{G} = \begin{bmatrix}a & a\\ 2a & 0\end{bmatrix},\ f \in \Lambda^0(U),\ f(\mathbf{x}_0) = 0$$

where a is a constant. We search for the solution of the system of exterior differential equations $d\mathbf{\Omega} = -\mathbf{\Gamma} \wedge \mathbf{\Omega}$. In terms of the component forms, this system is expressed as follows

$$d\Omega_1 = a\,df \wedge (\Omega_1 + \Omega_2),\ \ d\Omega_2 = 2a\,df \wedge \Omega_1.$$

We immediately observe that

$$d\mathbf{\Gamma} = \mathbf{0},\ \ \mathbf{\Gamma} \wedge \mathbf{\Gamma} = \mathbf{G}^2 df \wedge df = \mathbf{0}.$$

In this case (6.8.6) yields $\mathbf{\Theta} = \mathbf{0}$ and, consequently, $\boldsymbol{\theta} = \mathbf{0}$. It then follows from (6.8.19) that $\mathbf{\Gamma} = -d\mathbf{A}\,\mathbf{A}^{-1}$. The integral equation (6.8.11) now takes the form $\mathbf{A} - H(d\mathbf{A}) = \mathbf{I}$. This in turn gives $\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}) + \mathbf{A}(\mathbf{x}_0) = \mathbf{I}$, that is, the integral equation is satisfied identically. Hence, the matrix $\mathbf{A}$ is determined from the solution of the differential equation $d\mathbf{A} = -\boldsymbol{\Gamma}\mathbf{A} = \mathbf{G}\mathbf{A}\,df$. We know that the solution is expressible as

$$\mathbf{A}(\mathbf{x}) = e^{f(\mathbf{x})\mathbf{G}},\ \ \mathbf{A}(\mathbf{x}_0) = e^{f(\mathbf{x}_0)\mathbf{G}} = \mathbf{I}.$$

Since $f\mathbf{G}$ is a 2×2 matrix, we can write $e^{f(\mathbf{x})\mathbf{G}} = \alpha_0(\mathbf{x})\mathbf{I} + \alpha_1(\mathbf{x})f(\mathbf{x})\mathbf{G}$ according to the celebrated Hamilton-Cayley theorem which states that every square matrix satisfies its characteristic equation. The eigenvalues of the matrix $\mathbf{G}$ are $2a$ and $-a$ so that the coefficient functions $\alpha_0(\mathbf{x})$ and $\alpha_1(\mathbf{x})$ are found from the equations

$$e^{2af} = \alpha_0 + 2af\alpha_1,\ \ e^{-af} = \alpha_0 - af\alpha_1$$

as the following expressions

$$\alpha_0(\mathbf{x}) = \frac{e^{2af(\mathbf{x})} + 2e^{-af(\mathbf{x})}}{3},\ \ \alpha_1(\mathbf{x}) = \frac{e^{2af(\mathbf{x})} - e^{-af(\mathbf{x})}}{3af(\mathbf{x})}.$$

Therefore, the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} \dfrac{2e^{2af(\mathbf{x})} + e^{-af(\mathbf{x})}}{3} & \dfrac{e^{2af(\mathbf{x})} - e^{-af(\mathbf{x})}}{3} \\ \dfrac{2(e^{2af(\mathbf{x})} - e^{-af(\mathbf{x})})}{3} & \dfrac{e^{2af(\mathbf{x})} + 2e^{-af(\mathbf{x})}}{3} \end{bmatrix}.$$

Since, in the present example we have $\boldsymbol{\Sigma} = \mathbf{0}$, the solution will be in the form $\boldsymbol{\Omega} = \mathbf{A}d\boldsymbol{\phi}$. We thus obtain the solution

$$\Omega_1 = \frac{1}{3}(2e^{2af(\mathbf{x})} + e^{-af(\mathbf{x})})\,d\phi_1 + \frac{1}{3}(e^{2af(\mathbf{x})} - e^{-af(\mathbf{x})})\,d\phi_2$$
$$\Omega_2 = \frac{2}{3}(e^{2af(\mathbf{x})} - e^{-af(\mathbf{x})})\,d\phi_1 + \frac{1}{3}(e^{2af(\mathbf{x})} + 2e^{-af(\mathbf{x})})\,d\phi_2$$

where $\boldsymbol{\phi}(\mathbf{x}) = [\phi_1(\mathbf{x})\ \ \phi_2(\mathbf{x})]^{\mathrm{T}}$ is an arbitrary vector function.

VI. EXERCISES

6.1. For forms $\omega \in \Lambda(\mathbb{R}^4)$ given below evaluate the forms $H\omega$ and their ω_e exact and ω_a antiexact parts. H is the homotopy operator with the centre $(x,y,z,t) = (0,0,0,0)$:

$(a)\ \omega = (1+t^2)\,dx + z\,dy + x^3 dz + xyz\,dt.$

$(b)\ \omega = t^2 dx \wedge dy + y\ dx \wedge dz + z^3 dx \wedge dt + x^2\,dy \wedge dz + xy\ dz \wedge dt.$

$(c)\ \omega = x^2 t\,dx \wedge dy \wedge dz + (x^2+z^2)\,dx \wedge dy \wedge dt + yt\,dy \wedge dt \wedge dz.$

$(d)\ \omega = (x^2 + y^2 + z^2 + t^2)\, dx \wedge dy \wedge dz \wedge dt.$

6.2. Repeat the above operations by shifting the centre of the homotopy operator to the point $(1, 0, 1, 0)$.

6.3. For the form $\omega = \cos(3t)\, dx + \sin(2z)\, dy \in \Lambda^1(\mathbb{R}^4)$ evaluate the forms $H\omega, \omega_e, \omega_a$. The centre of the homotopy operator H is the point $(0, 0, 0, 0)$. Determine the same forms when the centre is changed to the point $(1, 1, 1, 1)$.

6.4. Determine the Darboux classes, ranks and indices of the forms $\omega \in \Lambda^1(\mathbb{R}^4)$ given below:

$(a)\ \omega = y^2 dx + x^2 dy$

$(b)\ \omega = yz\, dx + x^2 yt\, dt$

$(c)\ \omega = (1 - t^2)\, dx + (x^2 + y^2 - z^3)\, dy + xyzt\, dt$

$(d)\ \omega = (t + z^3)\, dx + (x^2 - y^2 + 1)\, dz + (y^2 - z)\, dt$

$(e)\ \omega = yt\, dx + (x^2 + t^2 - z)\, dy + (1 + y)\, dz + (z^2 - x)\, dt$

6.5. Let $\omega \in \Lambda(M)$. Show that the following relations

$$(e^{tdH})\omega = Hd\omega + e^t dH\omega, \ \ (e^{tHd})\omega = dH\omega + e^t Hd\omega$$

can locally be validated.

6.6. The function $u : M \to \mathbb{R}$ vanishes at the centre of the homotopy operator, that is, it satisfies the condition $u(\mathbf{x}_0) = 0$. Consider the following integral equation for the function $f : M \to \mathbb{R}$:

$$f = 1 + H(f\, du).$$

Show that the solution of this integral equation is given by $f = e^u$.

6.7. Investigate the same problem for the integral equation

$$f = k + H(f\, du)$$

where $k \neq 0$ is a given constant. Discuss the case $u(\mathbf{x}_0) \neq 0$.

6.8. Assume that $\Omega \in \Lambda^1(\mathbb{R}^3), \Sigma \in \Lambda^2(\mathbb{R}^3)$ and

$$\Gamma = (2x + z)\, dx + (2y + z)\, dy - (x + y)\, dz \in \Lambda^1(\mathbb{R}^3).$$

Find the solution of the exterior differential equation $d\Omega = \Gamma \wedge \Omega + \Sigma$. The centre of the homotopy operator will be taken as the point $\mathbf{0} \in \mathbb{R}^3$.