CHAPTER VII

LINEAR CONNECTIONS

7.1. SCOPE OF THE CHAPTER

Derivatives of components of vector and tensor fields on differentiable manifolds with respect to local coordinates are not generally components of a tensor. However, it becomes possible to define a sort of derivative of a tensor field which proves to be also a tensor through a geometrical structure imposed on a manifold called a ***linear*** or ***affine connection***. In this way, we can properly accomplish the task of extending the classical differential geometry to higher dimensions. From another standpoint, the affine connection can be interpreted as a suitable structure interconnecting neighbouring tangent spaces of a smooth manifold and thus enabling us to differentiate tensor fields. Although the inception of the concept of the linear connection goes back to the developments in 19. century in the geometry and tensor calculus, its formal structure is merely established in early 1920s by Élie Cartan and Herman Weyl. The term *connection* was first used by Cartan. In Sec. 7.2, we define a third order linear connection that is not a tensor but whose coefficients transform obeying certain rules under change of coordinates. Except for this restriction this geometrical object on a manifold can be chosen arbitrarily. We then discuss the covariant derivative of a tensor preserving the tensorial properties by means of a linear connection and its characteristics. We further scrutinise the torsion and curvature tensors of a manifold introduced through the linear connection. Sec. 7.3 is concerned with the Cartan connection engendered by choosing an arbitrary basis and its dual in the tangent and cotangent bundles, respectively, instead of the natural basis and its dual. The torsion and curvature tensors are then defined via that connection. Cartan connection enables us to study the differential geometry of a manifold by employing a ***moving frame*** (***repère mobile***). We define the Levi-Civita connection on a Riemannian manifold in Sec. 7.4 as a connection that causes the covariant derivative of the metric tensor and the torsion tensor to vanish. It is shown that such a connection is determined uniquely. Finally, Sec. 7.5 is devoted to study the special structures of the

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operators d, δ, Δ introduced in Secs. 5.8 and 5.9 acquired by means of covariant derivatives within the context of the Levi-Civita connection.

7.2. CONNECTIONS ON MANIFOLDS

We know that vector and tensor fields on smooth manifolds are specified by their components that are differentiable functions depending on local coordinates in charts of an atlas. But derivatives of those components do not usually constitute components of a new tensor. Notwithstanding, we can manage to create new tensor fields by some kind of differentiation of vector and tensor fields on tangent and cotangent bundles by endowing the manifold with a new structure. Let us start by considering a rather simple example. A vector field $V \in T(M)$ on the manifold M is designated by

$$V = v^i(\mathbf{x})\frac{\partial}{\partial x^i}$$

in natural coordinates. We know that a coordinate transformation in the local chart like $y^i = y^i(x^j)$ gives rise to a transformation between *contravariant components* of the vector at the point $\mathbf{x}$ as follows

$$v'^i(\mathbf{y}) = \frac{\partial y^i}{\partial x^k}v^k(\mathbf{x}).$$

If we calculate the gradient of the components v'^i by using the chain rule of differentiation, we obtain

$$\frac{\partial v'^i}{\partial y^j} = \frac{\partial y^i}{\partial x^k}\frac{\partial x^l}{\partial y^j}\frac{\partial v^k}{\partial x^l} + \frac{\partial^2 y^i}{\partial x^k \partial x^l}\frac{\partial x^l}{\partial y^j}v^k. \qquad (7.2.1)$$

This transformation rule under a general change of coordinates shows clearly that the quantities $\partial v^k/\partial x^l$ cannot be covariant and contravariant components of a second order tensor due to the existence of the second part in (7.2.1). If only the coordinate transformation satisfy the relations

$$\frac{\partial^2 y^i}{\partial x^k \partial x^l} = 0,$$

that is, if it is an *affine transformation* given by $y^i = a^i_j x^j + b^i$ with constant coefficients, only then the gradient behaves like a tensor. We shall now try to modify m^2 quantities $\partial v^k/\partial x^l$ in such a way that it will acquire the properties of the components of a $\binom{1}{1}$-tensor. To this end, we introduce a new operation of differentiation that will be called the ***covariant derivative***. This derivative of the component v^i with respect to the variable x^j is identified by means of presently arbitrarily chosen m^3 functions $\Gamma^i_{jk}(\mathbf{x})$ as follows

$$\nabla_j v^i = v^i{}_{;j} = v^i{}_{,j} + \Gamma^i_{jk} v^k \tag{7.2.2}$$

where the functions $\Gamma^i_{jk}(\mathbf{x})$ will be called the ***coefficients of linear*** or ***affine connection***. Next, we shall attempt to determine these coefficients in such a way that $v^i{}_{;j} = \nabla_j v^i$ become components of a second order $\binom{1}{1}$-tensor. We thus wish that the following relation must be satisfied in a coordinate transformation $y^i = y^i(x^j)$:

$$\nabla'_j v'^i = \frac{\partial y^i}{\partial x^k}\frac{\partial x^l}{\partial y^j}\nabla_l v^k.$$

On utilising (7.2.1), we easily obtain the transformation rule

$$\frac{\partial y^i}{\partial x^k}\frac{\partial x^l}{\partial y^j}\frac{\partial v^k}{\partial x^l} + \frac{\partial^2 y^i}{\partial x^k \partial x^l}\frac{\partial x^l}{\partial y^j}v^k + \Gamma'^i_{jk}\frac{\partial y^k}{\partial x^l}v^l = \frac{\partial y^i}{\partial x^k}\frac{\partial x^l}{\partial y^j}\Big(\frac{\partial v^k}{\partial x^l} + \Gamma^k_{ln}v^n\Big)$$

where $\Gamma'^i_{jk}(\mathbf{y})$ denote coefficients of the linear connection in the new coordinate system. After having cancelled similar terms in both sides and modified the dummy indices appropriately, we are led to the conclusion

$$v^n\Big(\Gamma'^i_{jk}\frac{\partial y^k}{\partial x^n} - \Gamma^k_{ln}\frac{\partial y^i}{\partial x^k}\frac{\partial x^l}{\partial y^j} + \frac{\partial^2 y^i}{\partial x^n \partial x^l}\frac{\partial x^l}{\partial y^j}\Big) = 0.$$

In order that this expression holds for every component function v^n we have to choose the coefficients Γ^i_{jk} and Γ'^i_{jk} in such a way that they must obey the transformation rule

$$\Gamma'^i_{jk}(\mathbf{y}) = \frac{\partial y^i}{\partial x^m}\frac{\partial x^l}{\partial y^j}\frac{\partial x^n}{\partial y^k}\Gamma^m_{ln}(\mathbf{x}) - \frac{\partial^2 y^i}{\partial x^n \partial x^l}\frac{\partial x^l}{\partial y^j}\frac{\partial x^n}{\partial y^k}.$$

On the other hand, differentiating the expression

$$\frac{\partial y^i}{\partial x^l}\frac{\partial x^l}{\partial y^j} = \delta^i_j$$

resulting from the chain rule, with respect to the variable x^n we find that

$$\frac{\partial^2 y^i}{\partial x^n \partial x^l}\frac{\partial x^l}{\partial y^j} = -\frac{\partial y^i}{\partial x^l}\frac{\partial^2 x^l}{\partial y^j \partial x^n} = -\frac{\partial y^i}{\partial x^l}\frac{\partial^2 x^l}{\partial y^j \partial y^k}\frac{\partial y^k}{\partial x^n}.$$

Hence, we can write

$$-\frac{\partial^2 y^i}{\partial x^n \partial x^l}\frac{\partial x^l}{\partial y^j}\frac{\partial x^n}{\partial y^k} = \frac{\partial y^i}{\partial x^l}\frac{\partial^2 x^l}{\partial y^j \partial y^l}\frac{\partial y^l}{\partial x^n}\frac{\partial x^n}{\partial y^k} = \frac{\partial^2 x^l}{\partial y^j \partial y^k}\frac{\partial y^i}{\partial x^l}$$

and, consequently, we obtain

$$\Gamma'^i_{jk}(\mathbf{y}) = \frac{\partial y^i}{\partial x^m}\frac{\partial x^l}{\partial y^j}\frac{\partial x^n}{\partial y^k}\Gamma^m_{ln}(\mathbf{x}) + \frac{\partial^2 x^l}{\partial y^j \partial y^k}\frac{\partial y^i}{\partial x^l}. \tag{7.2.3}$$

Every choice of coefficients Γ^i_{jk} verifying the transformation rule (7.2.3) gives rise to a covariant derivative and conduces to a *linear connection.* Thus, it appears that it is possible to have many, probably infinitely many, choices for linear connections. The coefficients Γ^i_{jk} are usually named as the ***Christoffel symbols of the second kind***, because they were employed for the first time by German mathematician Elwin Bruno Christoffel (1829-1900) in tensor analysis within the context of the Riemannian geometry. The rule (7.2.3) indicates clearly that *the Christoffel symbols cannot be the components of a third order tensor*. Nevertheless, the symmetry of mixed partial derivatives leads us to the result

$$\begin{aligned}\Gamma'^i_{jk} - \Gamma'^i_{kj} &= \Big(\frac{\partial y^i}{\partial x^m}\frac{\partial x^l}{\partial y^j}\frac{\partial x^n}{\partial y^k} - \frac{\partial y^i}{\partial x^m}\frac{\partial x^l}{\partial y^k}\frac{\partial x^n}{\partial y^j}\Big)\Gamma^m_{ln} \\ &= \frac{\partial y^i}{\partial x^m}\frac{\partial x^l}{\partial y^j}\frac{\partial x^n}{\partial y^k}(\Gamma^m_{ln} - \Gamma^m_{nl}).\end{aligned}$$

This clearly shows that $\Gamma^i_{[jk]} = \frac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj})$ which are the antisymmetric parts of Christoffel symbols with respect to their subscripts, behave like the components of a third order $\binom{1}{2}$-tensor antisymmetric with respect to covariant indices.

When $f \in \Lambda^0(M)$, then we already know that $f_{,i}$ are components of a covariant vector since $df \in \Lambda^1(M)$. Hence, we can take $f_{,i} = f_{;i} = \nabla_i f$. Let us now represent the vectors $\nabla_j V, j = 1, \dots, m$ denoting covariant derivatives of a vector field V with respect to variables x^j by the vector field

$$\nabla_j V = (\nabla_j v^i)\frac{\partial}{\partial x^i} = v^i{}_{;j}\,\frac{\partial}{\partial x^i}$$

We can then introduce a second order $\binom{1}{1}$- tensor by

$$\boldsymbol{\nabla} V = dx^j \otimes \nabla_j V = (\nabla_j v^i)\, dx^j \otimes \frac{\partial}{\partial x^i} = v^i{}_{;j}\, dx^j \otimes \frac{\partial}{\partial x^i}. \tag{7.2.4}$$

The second order tensor $\boldsymbol{\nabla} V$ is characterised as the ***gradient*** of the vector field V. We also call it the ***covariant derivative of the vector field*** V. The operator $\boldsymbol{\nabla} = dx^j \otimes \nabla_j$ is in the form $\boldsymbol{\nabla} : T(M) = \mathfrak{T}(M)^1_0 \to \mathfrak{T}(M)^1_1$ and it assigns a second order tensor field to a vector field. It follows from the definition (7.2.4) that if $V_1, V_2 \in T(M)$, we get

$$\boldsymbol{\nabla}(V_1 + V_2) = \boldsymbol{\nabla} V_1 + \boldsymbol{\nabla} V_2.$$

On the other hand, if $f \in \Lambda^0(M)$, then we find that

$$\begin{aligned}\boldsymbol{\nabla}(fV) &= \nabla_j(fv^i)\, dx^j \otimes \frac{\partial}{\partial x^i} = (f_{,j}v^i + f\nabla_j v^i)\, dx^j \otimes \frac{\partial}{\partial x^i} \\ &= df \otimes V + f\boldsymbol{\nabla} V.\end{aligned}$$

Therefore, the operator $\boldsymbol{\nabla}$ *is linear only on real numbers*. Let us now especially choose $V = \partial/\partial x^k$ whose components are $v^i = \delta^i_k$. Then (7.2.2) gives $\nabla_j v^i = \Gamma^i_{jl}\delta^l_k = \Gamma^i_{jk}$ and it follows from (7.2.4) that

$$\nabla_j \frac{\partial}{\partial x^k} = \Gamma^i_{jk} \frac{\partial}{\partial x^i}, \quad \boldsymbol{\nabla} \frac{\partial}{\partial x^k} = \Gamma^i_{jk}\, dx^j \otimes \frac{\partial}{\partial x^i}. \tag{7.2.5}$$

These relations geometrically expresses the fact that the coefficients of a linear connection measure the change between basis vectors of a tangent space and those of an adjacent tangent space on a manifold. It seems that the operator $\boldsymbol{\nabla}$ interconnects neighbouring tangent spaces by means of the coefficients of connection providing thus a tool for transporting vectors in one tangent space into another tangent space. In this way, it becomes possible to differentiate vectors and endow these derivatives with tensorial properties independent of coordinate transformations, and to find a counterpart of the concept of parallel transport on differentiable manifolds that is almost trivial in the Euclidean space. That are the reasons why the operator $\boldsymbol{\nabla}$ is sometimes called a ***linear*** or ***affine connection*** on a manifold.

We postulate that the covariant derivative specified by the operator ∇_j will still obey the classical Leibniz rule. With the help of this postulate we can easily evaluate covariant derivative of a covariant vector, or in other words of a 1-form. Let us consider $\omega \in \Lambda^1(M)$ and $V \in T(M)$ so that we write $\omega = \omega_i dx^i$ and $V = v^i \partial_i$. We thus obtain $\omega(V) = \omega_i v^i \in \Lambda^0(M)$. Therefore, the relation

$$\begin{aligned}(\omega_i v^i)_{,j} &= \omega_{i,j} v^i + \omega_i v^i_{,j} = (\omega_i v^i)_{;j} = \omega_{i;j} v^i + \omega_i v^i_{;j} \\ &= \omega_{i;j} v^i + \omega_i \big(v^i{}_{,j} + \Gamma^i_{jk} v^k\big) = \omega_{i;j} v^i + \omega_i v^i{}_{,j} + \omega_k \Gamma^k_{ji} v^i\end{aligned}$$

yields $\big[\omega_{i;j} - \big(\omega_{i,j} - \Gamma^k_{ji}\omega_k\big)\big] v^i = 0$. Because the vector V is arbitrary, we finally obtain

$$\nabla_j \omega_i = \omega_{i;j} = \omega_{i,j} - \Gamma^k_{ji}\omega_k. \tag{7.2.6}$$

We can easily verify that $\omega_{i;j}$ are components of a second order covariant $\binom{0}{2}$-tensor. If we recall the transformation

$$\omega'_i = \frac{\partial x^k}{\partial y^i}\omega_k$$

and employ the relation (7.2.3) and the chain rule, we get

$$\begin{aligned}\frac{\partial \omega'_i}{\partial y^j} - \Gamma'^k_{ji}\omega'_k &= \frac{\partial^2 x^k}{\partial y^i \partial y^j}\omega_k + \frac{\partial x^k}{\partial y^i}\frac{\partial \omega_k}{\partial x^l}\frac{\partial x^l}{\partial y^j} \\ &\quad - \frac{\partial y^k}{\partial x^m}\frac{\partial x^l}{\partial y^j}\frac{\partial x^n}{\partial y^i}\Gamma^m_{ln}\frac{\partial x^p}{\partial y^k}\omega_p - \frac{\partial^2 x^l}{\partial y^j \partial y^i}\frac{\partial y^k}{\partial x^l}\frac{\partial x^p}{\partial y^k}\omega_p \\ &= \frac{\partial^2 x^k}{\partial y^i \partial y^j}\omega_k - \frac{\partial^2 x^l}{\partial y^i \partial y^j}\omega_l + \frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}\frac{\partial \omega_k}{\partial x^l} \\ &\quad - \frac{\partial x^l}{\partial y^j}\frac{\partial x^n}{\partial y^i}\Gamma^m_{ln}\omega_m = \frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}\Big(\frac{\partial \omega_k}{\partial x^l} - \Gamma^m_{lk}\omega_m\Big)\end{aligned}$$

Hence, we obtain

$$\nabla'_j \omega'_i = \frac{\partial x^l}{\partial y^j}\frac{\partial x^k}{\partial y^i}\nabla_l \omega_k$$

as it should be. Let us now define 1-forms $\nabla_j \omega$, $j = 1, \dots, m$ by

$$\nabla_j \omega = \nabla_j \omega_i \, dx^i = \omega_{i;j}\, dx^i.$$

Then the covariant derivative of a 1-form ω can be written as

$$\boldsymbol{\nabla}\omega = dx^j \otimes \nabla_j \omega = \nabla_j \omega_i \, dx^j \otimes dx^i = \omega_{i;j}\, dx^j \otimes dx^i. \tag{7.2.7}$$

If we choose a form $\omega = dx^k$, we have $\omega_i = \delta^k_i$ and (7.2.6) yields $\nabla_j \omega_i = -\Gamma^l_{ji}\delta^k_l = -\Gamma^k_{ji}$ so that we find

$$\nabla_j \, dx^k = -\Gamma^k_{ji}\, dx^i, \;\; \boldsymbol{\nabla} dx^k = -\Gamma^k_{ji}\, dx^j \otimes dx^i. \tag{7.2.8}$$

Hence, the same coefficients of a connection measure also the changes in basis forms in the cotangent bundle between adjacent dual spaces.

We can now proceed to calculate the covariant derivative of a tensor field $\mathcal{T} \in \mathfrak{T}(M)^k_l$ by taking into account the equalities $(7.2.5)_1$ and $(7.2.8)_1$ associated with basis vectors and the Leibniz rule. Let the tensor field be given by

$$\mathcal{T} = t^{i_1\cdots i_k}_{j_1\cdots j_l}\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_k}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_l}.$$

The covariant derivative of this tensor is found to be a tensor specified by the expression

$$\begin{aligned}\nabla_j\mathcal{T} &= t^{i_1\cdots i_k}_{j_1\cdots j_l,j}\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_k}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_l}\\ &+\sum_{r=1}^{k} t^{i_1\cdots i_k}_{j_1\cdots j_l}\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\nabla_j\frac{\partial}{\partial x^{i_r}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_k}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_l}\\ &+\sum_{r=1}^{l} t^{i_1\cdots i_k}_{j_1\cdots j_l}\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_k}}\otimes dx^{j_1}\otimes\cdots\otimes\nabla_j\,dx^{j_r}\otimes\cdots\otimes dx^{j_l}\\ &= \nabla_j t^{i_1\cdots i_k}_{j_1\cdots j_l}\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_k}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_l}.\end{aligned}$$

It is easily verified that the covariant derivatives of the components of that tensor with respect to the variable x^j are expressed by the relation

$$\begin{aligned}\nabla_j t^{i_1\cdots i_k}_{j_1\cdots j_l} = t^{i_1\cdots i_k}_{j_1\cdots j_l;j} = \frac{\partial t^{i_1\cdots i_k}_{j_1\cdots j_l}}{\partial x^j} + \sum_{r=1}^{k}\Gamma^{i_r}_{jn}t^{i_1\cdots i_{r-1}ni_{r+1}\cdots i_k}_{j_1\cdots j_l}\\ -\sum_{r=1}^{l}\Gamma^{n}_{jj_r}t^{i_1\cdots i_k}_{j_1\cdots j_{r-1}nj_{r+1}\cdots j_l}.\end{aligned}$$

Thus the covariant derivative $\boldsymbol{\nabla}\mathcal{T}\in\mathfrak{T}(M)^k_{l+1}$ of a tensor field $\mathcal{T}\in\mathfrak{T}(M)^k_l$ is defined as

$$\begin{aligned}\boldsymbol{\nabla}\mathcal{T} &= dx^j\otimes\nabla_j\mathcal{T} \qquad (7.2.9)\\ &= t^{i_1\cdots i_k}_{j_1\cdots j_l;j}\,dx^j\otimes\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_k}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_l}\end{aligned}$$

This time, the operator $\boldsymbol{\nabla}$ is in the form $\boldsymbol{\nabla}:\mathfrak{T}(M)^k_l\to\mathfrak{T}(M)^k_{l+1}$. Let $\mathcal{T}$ and $\mathcal{S}$ be two tensor fields. It is straightforward to check that

$$\boldsymbol{\nabla}(\mathcal{T}\otimes\mathcal{S})\neq\boldsymbol{\nabla}\mathcal{T}\otimes\mathcal{S}+\mathcal{T}\otimes\boldsymbol{\nabla}\mathcal{S}.$$

Indeed, the two tensor products in the right hand side are actually different tensors because the forms dx^j appearing in covariant derivatives do not occupy the same place in them. So, it is not possible to add those tensors. Hence, covariant derivative of tensor products cannot satisfy the Leibniz rule. But, we can readily verify that the Leibniz rule holds for covariant derivative of tensor components. Let us now consider $\mathcal{T} \in \mathfrak{T}(M)^k_l$ and $\mathcal{S} \in \mathfrak{T}(M)^p_q$. It follows from the definition of the covariant derivative ∇_j that we find the relation below for the components of the tensor product $\mathcal{T} \otimes \mathcal{S}$:

$$
\begin{aligned}
(t^{i_1\cdots i_k}_{j_1\cdots j_l} s^{k_1\cdots k_p}_{l_1\cdots l_q})_{;j} &= \frac{\partial t^{i_1\cdots i_k}_{j_1\cdots j_l}}{\partial x^j} s^{k_1\cdots k_p}_{l_1\cdots l_q} + t^{i_1\cdots i_k}_{j_1\cdots j_l} \frac{\partial s^{k_1\cdots k_p}_{l_1\cdots l_q}}{\partial x^j} \\
&\quad + \sum_{r=1}^{k} \Gamma^{i_r}_{jn} t^{i_1\cdots i_{r-1} n i_{r+1}\cdots i_k}_{j_1\cdots j_l} s^{k_1\cdots k_p}_{l_1\cdots l_q} \\
&\quad + \sum_{r=1}^{p} t^{i_1\cdots i_k}_{j_1\cdots j_l} \Gamma^{k_r}_{jn} s^{k_1\cdots k_{r-1} n k_{r+1}\cdots k_p}_{l_1\cdots l_q} \\
&\quad - \sum_{r=1}^{l} \Gamma^{n}_{j j_r} t^{i_1\cdots i_k}_{j_1\cdots j_{r-1} n j_{r+1}\cdots j_l} s^{k_1\cdots k_p}_{l_1\cdots l_q} \\
&\quad - \sum_{r=1}^{l} t^{i_1\cdots i_k}_{j_1\cdots j_l} \Gamma^{n}_{j l_r} s^{k_1\cdots k_p}_{l_1\cdots l_{r-1} n l_{r+1}\cdots l_q} \\
&= t^{i_1\cdots i_k}_{j_1\cdots j_l;j} s^{k_1\cdots k_p}_{l_1\cdots l_q} + t^{i_1\cdots i_k}_{j_1\cdots j_l} s^{k_1\cdots k_p}_{l_1\cdots l_q;j}.
\end{aligned}
$$

Hence, we are now allowed to write the following relation

$$\nabla_j(\mathcal{T} \otimes \mathcal{S}) = \nabla_j \mathcal{T} \otimes \mathcal{S} + \mathcal{T} \otimes \nabla_j \mathcal{S}. \tag{7.2.10}$$

We can now define on the tangent bundle of a manifold the ***covariant derivative of a vector field*** V ***in the direction of a vector field*** U by means of the affine connection $\boldsymbol{\nabla}$ as follows

$$
\begin{aligned}
\nabla_U V = \mathbf{i}_U(dx^j)\nabla_j V = u^j \nabla_j V = (v^i{}_{;j}\, u^j)\frac{\partial}{\partial x^i} \\
= \big[U(v^i) + \Gamma^i_{jk} u^j v^k\big]\frac{\partial}{\partial x^i} \in T(M).
\end{aligned} \tag{7.2.11}
$$

where we have obviously defined the operator $\nabla_U = u^i \nabla_i$. We immediately observe that one obtains $\nabla_{\partial_j} V = \nabla_j V$ and

$$\nabla_{\partial_j} \partial_k = \Gamma^i_{jk} \partial_i. \tag{7.2.12}$$

Thus the connection coefficient Γ^i_{jk} clearly denotes the ith component of the covariant derivative of the kth natural basis vector in $T(M)$ in the direction of the jth natural basis vector. Therefore, the affine connection can also be interpreted as an operator $\boldsymbol{\nabla} : T(M) \times T(M) \to T(M)$. For $f \in \Lambda^0(M)$, we simply get

$$\nabla_U f = u^i f_{,i} = U(f). \tag{7.2.13}$$

On resorting to the definition (7.2.11), we can easily demonstrate that the following relations are satisfied:

$$\nabla_{U_1+U_2} V = \nabla_{U_1} V + \nabla_{U_2} V, \;\; \nabla_U (V_1 + V_2) = \nabla_U V_1 + \nabla_U V_2.$$

Moreover, for all functions $f \in \Lambda^0(M)$ we obtain for all $U, V \in T(M)$

$$\nabla_{fU} V = f\nabla_U V, \; \nabla_U (fV) = f\nabla_U V + U(f)V. \tag{7.2.14}$$

Thus, the operator ∇_U proves to be linear with respect to the vector U on the module $\Lambda^0(M)$. On the other hand, ∇_U becomes linear with respect to the vector V if only $U(f) = 0$, consequently, only on $\mathbb{R}$.

Let us consider a curve $C = \gamma(t)$ on the manifold M described by the mapping $\gamma : \mathcal{I} \to M$. If $U(t)$ is the tangent vector to this curve, we know that we can write

$$U = u^i \frac{\partial}{\partial x^i}, \quad u^i(t) = \frac{dx^i}{dt} = \frac{d\gamma^i}{dt}$$

where $\gamma^i = \varphi^i \circ \gamma$. We say that a vector field V is ***parallel*** along the curve C if $\nabla_U V = \mathbf{0}$. In view of (7.2.11), the components of such a vector field V must satisfy

$$\frac{\partial v^i}{\partial x^j}\frac{dx^j}{dt} + \Gamma^i_{jk}\frac{dx^j}{dt}v^k = \frac{dv^i}{dt} + \Gamma^i_{jk}\big(\gamma(t)\big)\frac{dx^j}{dt}v^k = 0. \tag{7.2.15}$$

(7.2.15) comprises a system of m first order linear ordinary differential equations to determine m dependent variables v^i. With a prescribed initial condition $V(t_0) = V_0$, the solution $V(t)$ of (7.2.15) is called the ***parallel translation*** of V_0 along the curve C. If $\Gamma^i_{jk} = 0$ on the manifold, then equations (7.2.15) yields $v^i =$ *constant* on C. A curve C is called a ***geodesic*** of the manifold if its tangent vectors are parallel along C. Therefore, the condition $\nabla_U U = \mathbf{0}$ must be satisfied on a geodesic.. Hence, the family of geodesics on a manifold are integral curves of the following system of second order, generally non-linear ordinary differential equations

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}(\mathbf{x})\frac{dx^j}{dt}\frac{dx^k}{dt} = 0. \tag{7.2.16}$$

Obviously, they are heavily dependent on connection coefficients. We can transform these equations to a system of first order differential equations by introducing auxiliary variables as follows

$$\frac{dx^i}{dt} = u^i, \ \frac{du^i}{dt} = -\Gamma^i_{jk}(\mathbf{x})u^j u^k$$

from which we conclude that there is a unique geodesic on a smooth manifold M through a point $p \in M$ and tangent to a vector U at that point. If $\Gamma^i_{jk} = 0$ on a manifold, the solution of (7.2.16) reduces merely to the family of *straight lines* $x^i = a^i t + b^i$.

*If a vector field V satisfies the condition $\nabla_U V = \mathbf{0}$ for every vector field U, we say that it is a **parallel vector field** on the manifold.* In this case, (7.2.11) leads to $(v^i{}_{,j} + \Gamma^i_{jk}v^k)u^j = 0$ for all $u^j \in \Lambda^0(M)$ or

$$v^i{}_{,j} + \Gamma^i_{jk}v^k = 0; \quad i,j,k = 1,\ldots,m. \tag{7.2.17}$$

(7.2.17) is a system of m^2 first order, linear partial differential equations involving only m variables v^i. Thus, it is usually no avail to expect to find a parallel vector field on a manifold unless its linear connection has a particular structure. It is quite easy to establish the integrability conditions of these differential equations. It follows from (7.2.17) that

$$v^i{}_{,jl} = -\Gamma^i_{jk,l}v^k - \Gamma^i_{jk}v^k_{,l} = -(\Gamma^i_{jn,l} + \Gamma^i_{jk}\Gamma^k_{ln})v^n.$$

Therefore, the compatibility relation $v^i{}_{,jl} = v^i{}_{,lj}$ can only be satisfied if

$$\Gamma^i_{jn,l} - \Gamma^i_{ln,j} + \Gamma^i_{jk}\Gamma^k_{ln} - \Gamma^i_{lk}\Gamma^k_{jn} = 0.$$

We shall show a little later that a connection whose coefficients are satisfying the above relations is curvature-free.

For a tensor $\mathcal{T} \in \mathfrak{T}(M)^k_l$, we define in a similar way

$$\nabla_U \mathcal{T} = \mathbf{i}_U(dx^j)\nabla_j \mathcal{T} = u^j \nabla_j \mathcal{T} \in \mathfrak{T}(M)^k_l. \tag{7.2.18}$$

The tensor $\nabla_U \mathcal{T}$ is called the ***covariant derivative of a tensor field $\mathcal{T}$ in the direction of the vector field*** U. Due to (7.2.10), we observe at once that the following rules are obeyed

$$\begin{aligned}\nabla_U(\mathcal{T}_1 + \mathcal{T}_2) &= \nabla_U \mathcal{T}_1 + \nabla_U \mathcal{T}_2,\\ \nabla_U(\mathcal{T}_1 \otimes \mathcal{T}_2) &= \nabla_U \mathcal{T}_1 \otimes \mathcal{T}_2 + \mathcal{T}_1 \otimes \nabla_U \mathcal{T}_2.\end{aligned}$$

It is also clear that *the operator ∇_U commutes with any operation of contraction on a tensor $\mathcal{T}$.*

Torsion and Curvature Tensors. We know that for any function $f \in \Lambda^0(M)$ partial derivatives are order-independent so that the symmetry relation $f_{,ij} = f_{,ji}$ is met. Since $f_{,i}$ are components of a covariant vector, we may ask whether this property is also preserved for covariant derivatives, that is, we may question the validity of the equality $\nabla_j f_{,i} = \nabla_i f_{,j}$. The relation (7.2.6) results then in

$$\nabla_j f_{,i} - \nabla_i f_{,j} = f_{,ij} - \Gamma^k_{ji} f_{,k} - f_{,ji} + \Gamma^k_{ij} f_{,k} = (\Gamma^k_{ij} - \Gamma^k_{ji}) f_{,k} = T^k_{ij} f_{,k}.$$

We have seen on p. 368 that

$$T^k_{ij} = -T^k_{ji} = \Gamma^k_{ij} - \Gamma^k_{ji} \tag{7.2.19}$$

is a third order $\binom{1}{2}$-tensor which is antisymmetric with respect to covariant indices. It is called the ***torsion tensor*** of the manifold. If the connection is symmetric, that is, if $\Gamma^k_{ij} = \Gamma^k_{ji}$, then the torsion tensor vanishes. For a non-zero T^k_{ij}, we necessarily get $\nabla_j f_{,i} \neq \nabla_i f_{,j}$. Let us now try to repeat the operation above associated with a scalar function for a vector V this time. On utilising (7.2.2) we obtain

$$\begin{aligned}\nabla_k \nabla_j v^i &= (v^i{}_{,j} + \Gamma^i_{jl} v^l)_{,k} + \Gamma^i_{kn}(v^n{}_{,j} + \Gamma^n_{jl} v^l) - \Gamma^n_{kj}(v^i{}_{,n} + \Gamma^i_{nl} v^l)\\ &= v^i{}_{,jk} + (\Gamma^i_{jl,k} + \Gamma^i_{kn}\Gamma^n_{jl}) v^l - \Gamma^l_{kj} \nabla_l v^i + \Gamma^i_{jl} v^l_{,k} + \Gamma^i_{kl} v^l_{,j}.\end{aligned}$$

Hence, we are easily led to the conclusion

$$\nabla_k \nabla_j v^i - \nabla_j \nabla_k v^i = R^i_{lkj} v^l - T^l_{kj} \nabla_l v^i. \tag{7.2.20}$$

where we have defined

$$R^i_{jkl} = \Gamma^i_{lj,k} - \Gamma^i_{kj,l} + \Gamma^i_{kn}\Gamma^n_{lj} - \Gamma^i_{ln}\Gamma^n_{kj}. \tag{7.2.21}$$

Since the left hand side of (7.2.20) involves the components of a third order tensor, (7.2.21) are components of a fourth order tensor according to the quotient rule. This tensor is called the ***curvature tensor*** of the manifold. Hence, the second covariant derivatives of a vector commute if only the torsion and curvature tensors of a manifold vanish. It is evident that the curvature tensor is antisymmetric with respect to its last two covariant indices:

$$R^i_{jkl} = -R^i_{jlk} \tag{7.2.22}$$

We consider two vector fields $U, V \in T(M)$. We then obtain

$$\begin{aligned}\nabla_U V - \nabla_V U &= \big[(v^i{}_{,j}\, u^j + \Gamma^i_{jk} u^j v^k)\big]\frac{\partial}{\partial x^i} - \big[(u^i{}_{,j}\, v^j + \Gamma^i_{jk} v^j u^k)\big]\frac{\partial}{\partial x^i}\\ &= (\Gamma^i_{jk} - \Gamma^i_{kj}) u^j v^k \frac{\partial}{\partial x^i} + (v^i{}_{,j}\, u^j - u^i{}_{,j}\, v^j)\frac{\partial}{\partial x^i}\end{aligned}$$

$$= \left[T^i_{jk}u^j v^k + [U,V]^i\right]\frac{\partial}{\partial x^i}.$$

Thus the ***torsion operator***

$$\boldsymbol{\tau}(U,V) = \nabla_U V - \nabla_V U - [U,V] = T^i_{jk}u^j v^k \frac{\partial}{\partial x^i} \qquad (7.2.23)$$

assigns obviously a vector field to two vector fields through (7.2.23). As such it is of the form $\boldsymbol{\tau} : T(M)\times T(M) \to T(M)$. Let us now consider three vector fields $U,V,W \in T(M)$. We just obtain

$$\begin{aligned}\nabla_U\nabla_V W = \Big[&u^l v^j_{,l} w^i_{;j} + u^l v^j w^i_{,jl} + (\Gamma^i_{jk,l} + \Gamma^i_{ln}\Gamma^n_{jk})u^l v^j w^k \\ &+ \Gamma^i_{jk}(u^l v^j + u^j v^l) w^k_{;l}\Big]\frac{\partial}{\partial x^i}.\end{aligned}$$

If we recall the relation (7.2.20) and pay attention to symmetric terms with respect to vectors U and V, we then arrive at the result

$$\nabla_U\nabla_V W - \nabla_V\nabla_U W = \left([U,V]^j w^i_{;j} + R^i_{klj}u^l v^j w^k\right)\frac{\partial}{\partial x^i}.$$

Thus the ***curvature operator***

$$\boldsymbol{\rho}(U,V) = \nabla_U\nabla_V - \nabla_V\nabla_U - \nabla_{[U,V]} = [\nabla_U,\nabla_V] - \nabla_{[U,V]} \qquad (7.2.24)$$

is instrumental in assigning a vector field to three vector fields U,V,W by the relation

$$\boldsymbol{\rho}(U,V)W = (R^i_{klj}u^l v^j)w^k\frac{\partial}{\partial x^i} = \rho^i_k(U,V)\,w^k\frac{\partial}{\partial x^i}. \qquad (7.2.25)$$

It is of the form $\boldsymbol{\rho} : T(M)\times T(M) \to \mathfrak{T}(M)^1_1$. It follows from (7.2.23) and (7.2.24) that

$$\boldsymbol{\tau}(U,V) = -\boldsymbol{\tau}(V,U), \quad \boldsymbol{\rho}(U,V) = -\boldsymbol{\rho}(V,U).$$

Making use of (2.10.19) and (7.2.14), we readily observe that we can write

$$\boldsymbol{\tau}(fU,gV) = fg\,\boldsymbol{\tau}(U,V), \quad \boldsymbol{\rho}(fU,gV)hW = fgh\,\boldsymbol{\rho}(U,V)W$$

for all $f,g,h \in \Lambda^0(M)$.

Some results obtained in this section by employing Christoffel symbols can be reversed in order to define the connection in the sense of Koszul [French mathematician Jean-Louis Koszul (1921)]. Koszul has abstractly defined the affine connection ∇ as a mapping that assigns a vector $\nabla_U V$ to every pair of vectors (U, V). This mapping is desired to satisfy the rules mentioned on p. 373. With this definition Christoffel symbols are prescribed by the relation (7.2.12) and the covariant derivative is introduced in a similar way. The expressions (7.2.23) and (7.2.24) are employed to define the torsion and curvature tensors, respectively.

7.3. CARTAN CONNECTION

In the previous section, we have employed natural basis vectors in $T(M)$ and $T^*(M)$ determined by local charts. We now prefer a more general approach in introducing the linear connection. Let us construct a basis for the tangent bundle $T(M)$ by collecting arbitrarily chosen m linearly independent vectors $e_1, e_2, \dots, e_m \in T_p(M)$ associated with every point $p \in M$. Since $T(M)$ is closed under the Lie product, there exist some functions $c_{ij}^k \in \Lambda^0(M)$ satisfying the relations

$$[e_i, e_j] = c_{ij}^k e_k, \quad i, j, k = 1, \dots, m. \tag{7.3.1}$$

We know that the conditions (2.11.3) and (2.11.4) must be imposed on these functions. They may now be written as

$$\begin{aligned} &c_{ij}^k = -c_{ji}^k, \\ &c_{ij}^q c_{pq}^k + c_{jp}^q c_{iq}^k + c_{pi}^q c_{jq}^k + e_p(c_{ij}^k) + e_i(c_{jp}^k) + e_j(c_{pi}^k) = 0. \end{aligned}$$

Cartan has called the basis vectors $\{e_i\}$ attached to every point of the manifold as the ***moving frame***. Let us now denote the reciprocal basis vectors, namely, 1-forms in $T^*(M)$ by $\theta^1, \dots, \theta^m$. They of course satisfy the relations $\theta^i(e_j) = \mathbf{i}_{e_j}(\theta^i) = \delta_j^i$. Resorting to the path followed in Sec. 5.14, we immediately discern that the reciprocal basis vector must obey the rules

$$d\theta^k = -\frac{1}{2} c_{ij}^k \theta^i \wedge \theta^j. \tag{7.3.2}$$

We shall now introduce an affine connection on a manifold M through a mapping $\boldsymbol{\nabla} : T(M) = \mathfrak{T}(M)_0^1 \to \mathfrak{T}(M)_1^1$ satisfying the following rules:

$$\begin{aligned} &\boldsymbol{\nabla}(U + V) = \boldsymbol{\nabla}U + \boldsymbol{\nabla}V, \\ &\boldsymbol{\nabla}(fV) = df \otimes V + f\boldsymbol{\nabla}V, \ \boldsymbol{\nabla}f = df, \forall f \in \Lambda^0(M), \end{aligned} \tag{7.3.3}$$

The last rule specifies the action of the mapping $\boldsymbol{\nabla}$ on a scalar function f. We call $\boldsymbol{\nabla}$ the ***Cartan connection*** and the tensor $\boldsymbol{\nabla}V$ the *covariant derivative* of a vector V with respect to that connection. If we express 1-form df as $\boldsymbol{\nabla}f = df = \alpha_i \theta^i$, then the familiar properties imply that

$\mathbf{i}_{e_j}(df) = e_j(f) = \alpha_j$ so that $\boldsymbol{\nabla} f$ is expressible as

$$\boldsymbol{\nabla} f = df = e_i(f)\theta^i \in \Lambda^1(M).$$

Next, let us employ the description $\boldsymbol{\nabla} f = (\nabla_i f)\theta^i$ that is tantamount to say that the covariant derivative of a function $f \in \Lambda^0(M)$ in the direction of the vector e_i with respect to Cartan connection is given by

$$\nabla_i f = e_i(f). \tag{7.3.4}$$

Covariant derivatives of basis vector can be expressed in the form

$$\boldsymbol{\nabla} e_i = \gamma^j_{ki}\theta^k \otimes e_j, \quad \gamma^j_{ki} \in \Lambda^0(M) \tag{7.3.5}$$

where the functions γ^j_{ki} play now the part of *connection coefficients*. Then, the covariant derivative of a vector V is evaluated from (7.3.3)$_2$, (7.3.4) and (7.3.5) as the following expression

$$\boldsymbol{\nabla} V = \boldsymbol{\nabla}(v^i e_i) = dv^i \otimes e_i + v^i \boldsymbol{\nabla} e_i = \big[e_j(v^i) + \gamma^i_{jk} v^k\big]\theta^j \otimes e_i.$$

If we denote the components of this tensor by $\nabla_j v^i$, we can write

$$\boldsymbol{\nabla} V = \nabla_j v^i \, \theta^j \otimes e_i, \quad \nabla_j v^i = e_j(v^i) + \gamma^i_{jk} v^k. \tag{7.3.6}$$

Let us further define the vectors $\nabla_j V = (\nabla_j v^i) e_i$ that allows us to write $\boldsymbol{\nabla} V = \theta^j \otimes \nabla_j V$. Then, we obviously draw the conclusion

$$\nabla_j e_i = \gamma^k_{ji} e_k. \tag{7.3.7}$$

Let us now choose a new basis in $T(M)$ via a regular matrix $\mathbf{B}(\mathbf{x})$ as follows: $e'_i = b^j_i e_j$. If we suppose that reciprocal basis vectors transform in the form $\theta'^i = a^i_j \theta^j$, then we find that

$$\delta^i_j = \theta'^i(e'_j) = a^i_k b^l_j \, \theta^k(e_l) = a^i_k b^l_j \delta^k_l = a^i_k b^k_j$$

implying that $\mathbf{A} = \mathbf{B}^{-1}$. Components of a vector with respect to the new basis will become $v'^i = a^i_j v^j$. In this case, we can write

$$dv^i = d(b^i_j v'^j) = v'^j db^i_j + b^i_j \, dv'^j = b^i_j \, dv'^j + v'^j e_k(b^i_j)\theta'^k.$$

Hence, we obtain

$$\begin{aligned} \boldsymbol{\nabla} V &= dv^i \otimes e_i + \gamma^i_{jk} v^k \, \theta^j \otimes e_i \\ &= a^l_i \big[b^i_j \, dv'^j + v'^j e_k(b^i_j)\theta'^k\big] \otimes e'_l + \gamma^i_{jk} b^k_l b^j_m a^n_i v'^l \theta'^m \otimes e'_n \end{aligned}$$

$$= \left[dv'^j + v'^k a^j_i e'_l(b^i_k)\theta'^l\right] \otimes e'_j + \gamma^i_{jk} b^k_l b^j_m a^n_i v'^l \theta'^m \otimes e'_n$$
$$= dv'^i \otimes e'_i + \gamma'^i_{jk} v'^k \theta'^j \otimes e'_i$$

from which we deduce after properly changing the dummy indices that the connection coefficients in the new basis must satisfy the relation

$$\gamma'^i_{jk} = a^i_l b^m_j b^n_k \gamma^l_{mn} + a^i_l e'_j(b^l_k) = a^i_l b^m_j b^n_k \gamma^l_{mn} + a^i_l b^m_j e_m(b^l_k).$$

Because of the last terms, we understand that the coefficients γ^i_{jk} cannot be components of a third order tensor.

Let us now consider a vector $V = v^i e_i$ and a 1-form $\omega = \omega_i \theta^i$. Since we have assumed that the operator ∇_j satisfies the Leibniz rule, the expression $\omega(V) = \omega_i v^i \in \Lambda^0(M)$ yields

$$\nabla_j(\omega_i v^i) = e_j(\omega_i v^i) = e_j(\omega_i)v^i + \omega_i e_j(v^i) = v^i \nabla_j \omega_i + \omega_i \nabla_j v^i$$
$$= v^i \nabla_j \omega_i + \omega_i \left[e_j(v^i) + \gamma^i_{jk} v^k\right]$$

whence we deduce that $v^i\left[\nabla_j \omega_i - \left(e_j(\omega_i) - \gamma^k_{ji}\omega_k\right)\right] = 0$ and since V is an arbitrary vector, we are led to the conclusion

$$\nabla_j \omega_i = e_j(\omega_i) - \gamma^k_{ji}\omega_k. \tag{7.3.8}$$

We can thus write $\nabla_j \omega = (\nabla_j \omega_i)\,\theta^i$ and $\boldsymbol{\nabla}\omega = \theta^j \otimes \nabla_j \omega = \nabla_j \omega_i\, \theta^j \otimes \theta^i$. Hence, the operator $\boldsymbol{\nabla}$ is now a mapping $\boldsymbol{\nabla} : T^*(M) = \mathfrak{T}(M)^0_1 \to \mathfrak{T}(M)^0_2$. On the other hand, we can readily reach to the following relations

$$\nabla_j\, \theta^k = -\gamma^k_{ji}\, \theta^i, \;\; \boldsymbol{\nabla}\theta^k = -\gamma^k_{ji}\, \theta^j \otimes \theta^i. \tag{7.3.9}$$

Covariant derivative with respect to Cartan connection that we have dealt with so far can easily be extended to any tensor $\mathcal{T} \in \mathfrak{T}(M)^k_l$. As an example, let us take the tensor

$$\mathcal{T} = t^{i_1\cdots i_k}_{j_1\cdots j_l} e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes \theta^{j_1} \otimes \cdots \otimes \theta^{j_l}$$

into consideration. We then find $\boldsymbol{\nabla}\mathcal{T} = \theta^j \otimes \nabla_j \mathcal{T} \in \mathfrak{T}(M)^k_{l+1}$ and

$$\nabla_j \mathcal{T} = \nabla_j t^{i_1\cdots i_k}_{j_1\cdots j_l} e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes \theta^{j_1} \otimes \cdots \otimes \theta^{j_l}$$

where the components $\nabla_j t^{i_1\cdots i_k}_{j_1\cdots j_l}$ are given by

$$\nabla_j t^{i_1\cdots i_k}_{j_1\cdots j_l} = e_j\left(t^{i_1\cdots i_k}_{j_1\cdots j_l}\right) + \sum_{r=1}^{k} \gamma^{i_r}_{jn} t^{i_1\cdots i_{r-1} n i_{r+1}\cdots i_k} - \sum_{r=1}^{l} \gamma^n_{jj_r} t^{i_1\cdots i_k}_{j_1\cdots j_{r-1} n j_{r+1}\cdots j_l}$$

We observe at once that the relation (7.2.10) is also valid in Cartan connection as well.

Covariant derivative of a vector field $V = v^i e_i \in T(M)$ in the direction of a vector field $U = u^i e_i \in T(M)$ can be defined as in Sec. 7.2 by the vector

$$\nabla_U V = \mathbf{i}_U(\theta^j)\nabla_j V = u^j \nabla_j V = \big[U(v^i) + \gamma^i_{jk} u^j v^k\big] e_i. \quad (7.3.10)$$

Similarly, the covariant derivative of a tensor field $\mathcal{T} \in \mathfrak{T}(M)^k_l$ in the direction of a vector field U is designated by

$$\nabla_U \mathcal{T} = u^j \nabla_j \mathcal{T} \in \mathfrak{T}(M)^k_l.$$

It is also evident that one is able to write

$$\nabla_{e_k} e_j = \nabla_k e_j = \gamma^i_{kj} e_i, \quad \nabla_{e_k} \theta^j = \nabla_k\, \theta^j = -\gamma^j_{ki} \theta^i. \quad (7.3.11)$$

In order to define the torsion tensor $\boldsymbol{\mathcal{T}}$ and the curvature tensor $\boldsymbol{\mathcal{R}}$ in Cartan connection, we can make use of the relations (7.2.23) and (7.2.24). Consider a form field $\omega \in \Lambda^1(M)$ and vector fields $U, V, W \in T(M)$ and write

$$\boldsymbol{\mathcal{T}}(\omega, U, V) = \omega\big(\boldsymbol{\tau}(U, V)\big) = T^i_{jk} \omega_i u^j v^k,$$
$$\boldsymbol{\mathcal{R}}(\omega, U, V, W) = \omega\big(\boldsymbol{\rho}(U, V) W\big) = R^i_{klj} \omega_i u^l v^j w^k.$$

It follows from (7.2.23) that

$$\nabla_{u^j e_j} v^k e_k - \nabla_{v^k e_k} u^j e_j - [u^j e_j, v^k e_k] = \big(\nabla_{e_j} e_k - \nabla_{e_k} e_j - [e_j, e_k]\big) u^j v^k$$
$$= (\gamma^i_{jk} - \gamma^i_{kj} - c^i_{jk}) u^j v^k e_i.$$

Therefore, the third order torsion tensor is found as

$$\boldsymbol{\mathcal{T}} = T^i_{jk} e_i \otimes \theta^j \otimes \theta^k, \;\; T^i_{jk} = -T^i_{kj} = \gamma^i_{jk} - \gamma^i_{kj} - c^i_{jk}. \quad (7.3.12)$$

In the like fashion, the relation

$$\boldsymbol{\rho}(U, V) W = u^j v^k w^l \boldsymbol{\rho}(e_j, e_k) e_l$$

leads us to

$$\boldsymbol{\rho}(e_j, e_k) e_l = \big([\nabla_{e_j}, \nabla_{e_k}] - \nabla_{[e_j, e_k]}\big) e_l = \big([\nabla_{e_j}, \nabla_{e_k}] - c^m_{jk} \nabla_{e_m}\big) e_l$$
$$= \nabla_{e_j} \nabla_{e_k} e_l - \nabla_{e_k} \nabla_{e_j} e_l - c^m_{jk} \nabla_{e_m} e_l$$
$$= \nabla_{e_j} \gamma^i_{kl} e_i - \nabla_{e_k} \gamma^i_{jl} e_i - c^m_{jk} \gamma^i_{ml} e_i$$

$$= \gamma^i_{kl}\nabla_{e_j}e_i + e_j(\gamma^i_{kl})\,e_i - \gamma^i_{jl}\nabla_{e_k}e_i - e_k(\gamma^i_{jl})\,e_i - c^m_{jk}\gamma^i_{ml}e_i$$
$$= \big[e_j(\gamma^i_{kl}) - e_k(\gamma^i_{jl}) + \gamma^i_{jm}\gamma^m_{kl} - \gamma^i_{km}\gamma^m_{jl} - c^m_{jk}\gamma^i_{ml}\big]e_i = R^i_{ljk}e_i$$

from which we can manage to extract the components of the fourth order curvature tensor $\boldsymbol{\mathcal{R}} = R^i_{ljk}e_i \otimes \theta^l \otimes \theta^j \otimes \theta^k$ as follows

$$R^i_{ljk} = e_j(\gamma^i_{kl}) - e_k(\gamma^i_{jl}) + \gamma^i_{jm}\gamma^m_{kl} - \gamma^i_{km}\gamma^m_{jl} - c^m_{jk}\gamma^i_{ml}. \tag{7.3.13}$$

It is easily seen from the definition that this tensor possesses the antisymmetry property $R^i_{ljk} = -R^i_{lkj}$.

When we choose a connection determined by the coefficients γ^i_{jk}, this makes it possible to generate a new connection without a torsion. Let us define the new connection coefficients by

$$\gamma'^i_{jk} = \gamma^i_{jk} - \frac{1}{2}T^i_{jk},$$

we then find that

$$T'^i_{jk} = \gamma'^i_{jk} - \gamma'^i_{kj} - c^i_{jk} = \gamma^i_{jk} - \gamma^i_{kj} - c^i_{jk} - T^i_{jk} = 0.$$

We shall now try to discuss the action of the commutator $[\nabla_j, \nabla_k]$ on diverse tensor fields. Let us first consider the scalar function $f \in \Lambda^0(M)$. We obtain from (7.3.4) and (7.3.8) that

$$\nabla_k\nabla_j f = \nabla_k e_j(f)$$
$$= e_k\big(e_j(f)\big) - \gamma^m_{kj}e_m(f).$$

Because of the commutation relation $e_ke_j - e_je_k = c^m_{kj}\,e_m$, we get

$$(\nabla_k\nabla_j - \nabla_j\nabla_k)f = -\,(\,\gamma^m_{kj} - \gamma^m_{jk} - c^m_{kj})\,e_m(f) \tag{7.3.14}$$
$$= -\,T^m_{kj}\,e_m(f).$$

Next, we consider a vector field $V \in T(M)$. Since we have

$$\nabla_j v^i = e_j(v^i) + \gamma^i_{jm}v^m$$

we find that

$$\nabla_k\nabla_j v^i = e_k\big(e_j(v^i)\big) + \big[e_k(\gamma^i_{jm}) + \gamma^i_{kn}\gamma^n_{jm} - \gamma^n_{kj}\gamma^i_{nm}\big]v^m$$
$$+\,\gamma^i_{jm}e_k(v^m) + \gamma^i_{km}e_j(v^m) - \gamma^m_{kj}e_m(v^i).$$

Hence, we arrive at

$$(\nabla_k\nabla_j - \nabla_j\nabla_k)\,v^i = \big[e_k(\gamma^i_{jm}) - e_j(\gamma^i_{km}) + \gamma^i_{kn}\gamma^n_{jm} - \gamma^i_{jn}\gamma^n_{km} - c^n_{kj}\gamma^i_{nm}\big]v^m - T^m_{kj}\big(e_m(v^i) + \gamma^i_{mn}v^n\big).$$

Consequently, we obtain

$$[\nabla_k, \nabla_j]v^i = (\nabla_k\nabla_j - \nabla_j\nabla_k)v^i = R^i_{mkj}v^m - T^m_{kj}\nabla_m v^i. \quad (7.3.15)$$

Similarly, for a form $\omega \in \Lambda^1(M)$, we have $\nabla_j\omega_i = e_j(\omega_i) - \gamma^m_{ji}\omega_m$ and we thus find

$$\nabla_k\nabla_j\omega_i = e_k\big(e_j(\omega_i)\big) + \big[-e_k(\gamma^m_{ji}) + \gamma^n_{kj}\gamma^m_{ni} + \gamma^n_{ki}\gamma^m_{jn}\big]\omega_m - \gamma^m_{ji}e_k(\omega_m) - \gamma^m_{ki}e_j(\omega_m) - \gamma^m_{kj}e_m(\omega_i).$$

This relation gives rise to

$$(\nabla_k\nabla_j - \nabla_j\nabla_k)\,\omega_i = \big[e_j(\gamma^m_{ki}) - e_k(\gamma^m_{ji}) + \gamma^n_{ki}\gamma^m_{jn} - \gamma^n_{ji}\gamma^m_{kn} - c^n_{jk}\gamma^m_{ni}\big]\omega_m + T^m_{jk}\big(e_m(\omega_i) - \gamma^n_{mi}\omega_n\big)$$

from which we obtain at once

$$[\nabla_k, \nabla_j]\omega_i = (\nabla_k\nabla_j - \nabla_j\nabla_k)\,\omega_i = R^m_{ijk}\omega_m + T^m_{jk}\nabla_m\omega_i = -R^m_{ikj}\omega_m - T^m_{kj}\nabla_m\omega_i. \quad (7.3.16)$$

With these information at hand, we can easily evaluate the action of the operator $[\nabla_k, \nabla_j]$ on any tensor. But, let us first verify that the operator $[\nabla_k, \nabla_j]$ obeys the Leibniz rule on tensor products. If $\mathcal{T}$ and $\mathcal{S}$ are two tensor fields, we can obviously write

$$\nabla_j(\mathcal{T}\otimes\mathcal{S}) = \nabla_j\mathcal{T}\otimes\mathcal{S} + \mathcal{T}\otimes\nabla_j\mathcal{S}$$
$$\nabla_k\nabla_j(\mathcal{T}\otimes\mathcal{S}) = \nabla_k\nabla_j\mathcal{T}\otimes\mathcal{S} + \nabla_j\mathcal{T}\otimes\nabla_k\mathcal{S} + \nabla_k\mathcal{T}\otimes\nabla_j\mathcal{S} + \mathcal{T}\otimes\nabla_k\nabla_j\mathcal{S}$$

whence we draw at once the conclusion

$$[\nabla_k, \nabla_j](\mathcal{T}\otimes\mathcal{S}) = [\nabla_k, \nabla_j](\mathcal{T})\otimes\mathcal{S} + \mathcal{T}\otimes[\nabla_k, \nabla_j](\mathcal{S}). \quad (7.3.17)$$

Let us consider a tensor $\mathcal{T}$ of order $k+l$. Then we can produce a scalar function $f = t^{i_1\cdots i_k}_{j_1\cdots j_l}v^{j_1}_{(1)}\cdots v^{j_l}_{(l)}\omega^{(1)}_{i_1}\cdots\omega^{(k)}_{i_k}$ by considering its action on l vectors $V_{(1)},\dots V_{(l)}$ and k 1-forms $\omega^{(1)},\dots,\omega^{(k)}$. On utilising (7.3.14), we have

$$[\nabla_k, \nabla_j](f) = -T^m_{kj}\,e_m(f) = -T^m_{kj}\,e_m\big(t^{i_1\cdots i_k}_{j_1\cdots j_l}\big)v^{j_1}_{(1)}\cdots v^{j_l}_{(l)}\omega^{(1)}_{i_1}\cdots\omega^{(k)}_{i_k} - T^m_{kj}\,t^{i_1\cdots i_k}_{j_1\cdots j_l}e_m\big(v^{j_1}_{(1)}\cdots v^{j_l}_{(l)}\omega^{(1)}_{i_1}\cdots\omega^{(k)}_{i_k}\big).$$

On the other hand, because of the Leibniz rule (7.3.17), on applying the commutator $[\nabla_k, \nabla_j]$ on the function f defined above, we get

$$\begin{aligned}[\nabla_k, \nabla_j](f) &= [\nabla_k, \nabla_j]\big(t^{i_1\cdots i_k}_{j_1\cdots j_l} v^{j_1}_{(1)}\cdots v^{j_l}_{(l)}\omega^{(1)}_{i_1}\cdots\omega^{(k)}_{i_k}\big)\\ &= [\nabla_k, \nabla_j]\big(t^{i_1\cdots i_k}_{j_1\cdots j_l}\big) v^{j_1}_{(1)}\cdots v^{j_l}_{(l)}\omega^{(1)}_{i_1}\cdots\omega^{(k)}_{i_k}\\ &\quad + t^{i_1\cdots i_k}_{j_1\cdots j_l}\Big[\sum_{r=1}^{l} v^{j_1}_{(1)}\cdots[\nabla_k, \nabla_j]\, v^{j_r}_{(r)}\cdots v^{j_l}_{(l)}\omega^{(1)}_{i_1}\cdots\omega^{(k)}_{i_k}\\ &\quad + \sum_{r=1}^{k} v^{j_1}_{(1)}\cdots v^{j_l}_{(l)}\omega^{(1)}_{i_1}\cdots[\nabla_k, \nabla_j]\,\omega^{(r)}_{i_r}\cdots\omega^{(k)}_{i_k}\Big].\end{aligned}$$

Let us now employ (7.3.15) and (7.3.16) to transform the terms within brackets in the third and fourth lines of the above expression into the following form

$$\begin{aligned}&\sum_{r=1}^{l} v^{j_1}_{(1)}\cdots(R^{j_r}_{mkj}v^m_{(r)} - T^m_{kj}\nabla_m v^{j_r}_{(r)})\cdots v^{j_l}_{(l)}\omega^{(1)}_{i_1}\cdots\omega^{(k)}_{i_k}\\ &\qquad + \sum_{r=1}^{k} v^{j_1}_{(1)}\cdots v^{j_l}_{(l)}\omega^{(1)}_{i_1}\cdots(-R^m_{i_r kj}\omega^{(r)}_m - T^m_{kj}\nabla_m\omega^{(r)}_{i_r})\cdots\omega^{(k)}_{i_k}.\end{aligned}$$

Next, we write

$$\begin{aligned}\nabla_m v^{j_r}_{(r)} &= e_m(v^{j_r}_{(r)}) + \gamma^{j_r}_{mn}v^n_{(r)},\\ \nabla_m\omega^{(r)}_{i_r} &= e_m(\omega^{(r)}_{i_r}) - \gamma^n_{mi_r}\omega^{(r)}_n\end{aligned}$$

and change the dummy indices properly to obtain

$$\begin{aligned}[\nabla_k, \nabla_j](f) &= \Big\{[\nabla_k, \nabla_j]\big(t^{i_1\cdots i_k}_{j_1\cdots j_l}\big) + \sum_{r=1}^{l} R^m_{j_r kj} t^{i_1\cdots i_k}_{j_1\cdots j_{r-1} m j_{r+1}\cdots j_l} -\\ &\quad \sum_{r=1}^{k} R^{i_r}_{mkj} t^{i_1\cdots i_{r-1} m i_{r+1}\cdots i_k}_{j_1\cdots j_l} - T^m_{kj}\Big[\sum_{r=1}^{l}\gamma^n_{mj_r} t^{i_1\cdots i_k}_{j_1\cdots j_{r-1} n j_{r+1}\cdots j_l}\\ &\quad - \sum_{r=1}^{l}\gamma^{i_r}_{mn} t^{i_1\cdots i_{r-1} n i_{r+1}\cdots i_k}_{j_1\cdots j_l}\Big]\Big\} v^{j_1}_{(1)}\cdots v^{j_l}_{(l)}\omega^{(1)}_{i_1}\cdots\omega^{(k)}_{i_k}\\ &= -T^m_{kj}\, e_m(t^{i_1\cdots i_k}_{j_1\cdots j_l}) v^{j_1}_{(1)}\cdots v^{j_l}_{(l)}\omega^{(1)}_{i_1}\cdots\omega^{(k)}_{i_k}\\ &\quad - T^m_{kj}\, t^{i_1\cdots i_k}_{j_1\cdots j_l} e_m(v^{j_1}_{(1)}\cdots v^{j_l}_{(l)}\omega^{(1)}_{i_1}\cdots\omega^{(k)}_{i_k}).\end{aligned}$$

Consequently, on collecting terms suitably, we end up with the relations

$$\Big\{[\nabla_k,\nabla_j](t^{i_1\cdots i_k}_{j_1\cdots j_l})+\sum_{r=1}^{l}R^m_{j_rkj}t^{i_1\cdots i_k}_{j_1\cdots j_{r-1}mj_{r+1}\cdots j_l}-\sum_{r=1}^{k}R^{i_r}_{mkj}t^{i_1\cdots i_{r-1}mi_{r+1}\cdots i_k}_{j_1\cdots j_l} + T^m_{kj}\nabla_m t^{i_1\cdots i_k}_{j_1\cdots j_l}\Big\}v^{j_1}_{(1)}\cdots v^{j_l}_{(l)}\omega^{(1)}_{i_1}\cdots\omega^{(k)}_{i_k}=0.$$

Since this expression must be satisfied for arbitrary vectors and 1-forms, we finally reach to the desired relation

$$(\nabla_k\nabla_j-\nabla_j\nabla_k)\big(t^{i_1\cdots i_k}_{j_1\cdots j_l}\big)=\sum_{r=1}^{k}R^{i_r}_{mkj}\,t^{i_1\cdots i_{r-1}mi_{r+1}\cdots i_k}_{j_1\cdots j_l} -\sum_{r=1}^{l}R^m_{j_rkj}\,t^{i_1\cdots i_k}_{j_1\cdots j_{r-1}mj_{r+1}\cdots j_l}-T^m_{kj}\nabla_m\big(t^{i_1\cdots i_k}_{j_1\cdots j_l}\big). \tag{7.3.18}$$

Next, we define m^2 ***connection*** $\mathbf{1}$***-forms*** by

$$\Gamma^i_j=\gamma^i_{kj}\,\theta^k, \tag{7.3.19}$$

m ***torsion*** $\mathbf{2}$***-forms*** by

$$\Sigma^i=\frac{1}{2}T^i_{jk}\theta^j\wedge\theta^k, \tag{7.3.20}$$

and m^2 ***curvature*** $\mathbf{2}$***-forms*** by

$$\Theta^i_j=\frac{1}{2}R^i_{jkl}\theta^k\wedge\theta^l. \tag{7.3.21}$$

Theorem 7.13.1. *Torsion and curvature forms satisfy the Cartan structural equations*

$$d\theta^i=-\Gamma^i_j\wedge\theta^j+\Sigma^i,\;\;\Theta^i_j=d\Gamma^i_j+\Gamma^i_k\wedge\Gamma^k_j \tag{7.3.22}$$

in the moving frame.

Indeed, it follows from (7.3.2) and (7.3.12) that

$$\begin{aligned}d\theta^i&=-\frac{1}{2}\,c^i_{jk}\,\theta^j\wedge\theta^k\\&=\frac{1}{2}\,T^i_{jk}\,\theta^j\wedge\theta^k-\frac{1}{2}(\gamma^i_{jk}-\gamma^i_{kj})\theta^j\wedge\theta^k\\&=\Sigma^i-\gamma^i_{kj}\theta^k\wedge\theta^j\\&=-\Gamma^i_j\wedge\theta^j+\Sigma^i.\end{aligned}$$

On the other hand, the relation (7.13.14) leads to

$$\begin{aligned}d\Gamma^i_j &= d\gamma^i_{lj} \wedge \theta^l + \gamma^i_{mj}\, d\theta^m \\ &= \big[e_k(\gamma^i_{lj}) - \frac{1}{2}\gamma^i_{mj}c^m_{kl}\big]\theta^k \wedge \theta^l \\ \Gamma^i_m \wedge \Gamma^m_j &= \gamma^i_{km}\gamma^m_{lj}\, \theta^k \wedge \theta^l.\end{aligned}$$

We thus conclude that

$$\begin{aligned}d\Gamma^i_j + \Gamma^i_m \wedge \Gamma^m_j &= \big[e_k(\gamma^i_{lj}) + \gamma^i_{km}\gamma^m_{lj} - \frac{1}{2}\gamma^i_{mj}c^m_{kl}\big]\theta^k \wedge \theta^l \\ &= \frac{1}{2}\big[e_k(\gamma^i_{lj}) - e_l(\gamma^i_{kj}) + \gamma^i_{km}\gamma^m_{lj} - \gamma^i_{lm}\gamma^m_{kj} - c^m_{kl}\gamma^i_{mj}\big]\theta^k \wedge \theta^l \\ &= \frac{1}{2}R^i_{jkl}\theta^k \wedge \theta^l = \Theta^i_j. \qquad \square\end{aligned}$$

The expressions (7.3.22) provide us with a quite an effective tool to discover relatively easily some interesting relations between the curvature and torsion tensors. From $(7.3.22)_1$, we can write

$$\begin{aligned}d\Sigma^i &= d\Gamma^i_j \wedge \theta^j - \Gamma^i_j \wedge d\theta^j \\ &= (d\Gamma^i_j + \Gamma^i_k \wedge \Gamma^k_j) \wedge \theta^j - \Gamma^i_j \wedge \Sigma^j \\ &= \Theta^i_j \wedge \theta^j - \Gamma^i_j \wedge \Sigma^j.\end{aligned}$$

Employing the definitions (7.3.20) and (7.3.21), we get

$$\Theta^i_j \wedge \theta^j - \Gamma^i_j \wedge \Sigma^j = \frac{1}{2}(R^i_{jkl} - \gamma^i_{jm}T^m_{kl})\, \theta^j \wedge \theta^k \wedge \theta^l.$$

By appropriately renaming the dummy indices, we obtain from (7.3.20) and (7.3.2) that

$$\begin{aligned}d\Sigma^i &= \frac{1}{2}(dT^i_{jk} \wedge \theta^j \wedge \theta^k + T^i_{jk}d\theta^j \wedge \theta^k - T^i_{jk}\theta^j \wedge d\theta^k) \\ &= \frac{1}{2}\big[e_j(T^i_{kl}) - \frac{1}{2}\, c^m_{jk}T^i_{ml} + \frac{1}{2}\, c^m_{kl}T^i_{jm}\big]\theta^j \wedge \theta^k \wedge \theta^l.\end{aligned}$$

Hence, we get

$$\Big[R^i_{jkl} - e_j(T^i_{kl}) - \gamma^i_{jm}T^m_{kl} + \frac{1}{2}\, c^m_{jk}T^i_{ml} - \frac{1}{2}\, c^m_{kl}T^i_{jm}\Big]\theta^j \wedge \theta^k \wedge \theta^l = 0.$$

Since the covariant derivative of the torsion tensor is

$$\nabla_j T^i_{kl} = e_j(T^i_{kl}) + \gamma^i_{jm}T^m_{kl} - \gamma^m_{jk}T^i_{ml} - \gamma^m_{jl}T^i_{km},$$

the foregoing equality can be transformed into

$$\big[R^i_{jkl} - \nabla_j T^i_{kl} + (\gamma^m_{jk} - \tfrac{1}{2}c^m_{jk})T^i_{lm} - \gamma^m_{jl}T^i_{km} - \tfrac{1}{2}\,c^m_{kl}T^i_{jm}\big]\theta^j \wedge \theta^k \wedge \theta^l = 0.$$

If we add and subtract the terms $\frac{1}{2}c^m_{jl}T^i_{km}$ into the brackets above and note that $\frac{1}{2}(c^m_{jl}T^i_{km} + c^m_{kl}T^i_{jm})\,\theta^j \wedge \theta^k = 0$, then we get

$$\big[R^i_{jkl} - \nabla_j T^i_{kl} + (\gamma^m_{jk} - \tfrac{1}{2}c^m_{jk})T^i_{lm} - (\gamma^m_{jl} - \tfrac{1}{2}\,c^m_{jl})T^i_{km}\big]\theta^j \wedge \theta^k \wedge \theta^l = 0.$$

On utilising the antisymmetry with respect to indices k and l, we obtain

$$\big[R^i_{jkl} - \nabla_j T^i_{kl} + (2\gamma^m_{[jk]} - c^m_{jk})T^i_{lm}\big]\theta^j \wedge \theta^k \wedge \theta^l = 0.$$

Let us now insert (7.3.12) into the above expression to cast it into the form

$$(R^i_{jkl} - \nabla_j T^i_{kl} + T^m_{jk}T^i_{lm})\big]\theta^j \wedge \theta^k \wedge \theta^l = 0.$$

Then we finally reach to the following identity

$$R^i_{[jkl]} = \nabla_{[j}T^i_{kl]} - T^m_{[jk}T^i_{l]m}. \qquad (7.3.23)$$

The explicit form of the expression (7.3.23) becomes

$$\begin{aligned} &R^i_{jkl} + R^i_{klj} + R^i_{ljk} = \\ &\qquad \nabla_j T^i_{kl} + \nabla_k T^i_{lj} + \nabla_l T^i_{jk} - T^m_{jk}T^i_{lm} - T^m_{kl}T^i_{jm} - T^m_{lj}T^i_{km}. \end{aligned}$$

Next, we evaluate the exterior derivative of $(7.3.22)_2$ to obtain

$$\begin{aligned} d\Theta^i_j &= d\Gamma^i_k \wedge \Gamma^k_j - \Gamma^i_k \wedge d\Gamma^k_j = \Theta^i_k \wedge \Gamma^k_j - \Gamma^i_m \wedge \Gamma^m_k \wedge \Gamma^k_j \\ &\quad - \Gamma^i_k \wedge \Theta^k_j + \Gamma^i_k \wedge \Gamma^k_m \wedge \Gamma^m_j = \Theta^i_k \wedge \Gamma^k_j - \Gamma^i_k \wedge \Theta^k_j. \end{aligned}$$

As is easily seen, we can write

$$\begin{aligned} \Theta^i_k \wedge \Gamma^k_j - \Gamma^i_k \wedge \Theta^k_j &= \frac{1}{2}(R^i_{nlm}\gamma^n_{kj} - R^n_{jlm}\gamma^i_{kn})\,\theta^k \wedge \theta^l \wedge \theta^m, \\ d\Theta^i_j &= \frac{1}{2}\big[e_k(R^i_{jlm}) - \frac{1}{2}\,c^n_{kl}R^i_{jnm} + \frac{1}{2}\,c^n_{mk}R^i_{jln}\big]\theta^k \wedge \theta^l \wedge \theta^m. \end{aligned}$$

We thus obtain

$$\big[e_k(R^i_{jlm}) + \gamma^i_{kn}R^n_{jlm} - \gamma^n_{kj}R^i_{nlm} - \frac{1}{2}\,c^n_{kl}R^i_{jnm} + \frac{1}{2}\,c^n_{mk}R^i_{jln}\big]\theta^k \wedge \theta^l \wedge \theta^m = 0.$$

On account of the expression

$$\nabla_k R^i_{jlm} = e_k(R^i_{jlm}) + \gamma^i_{kn}R^n_{jlm} - \gamma^n_{kj}R^i_{nlm} - \gamma^n_{kl}R^i_{jnm} - \gamma^n_{km}R^i_{jln}$$

the above equality can be transformed into

$$\big[\nabla_k R^i_{jlm} - (\gamma^n_{kl} - \tfrac{1}{2}c^n_{kl})R^i_{jmn} + (\gamma^n_{km} - \tfrac{1}{2}c^n_{km})R^i_{jln}\big]\theta^k \wedge \theta^l \wedge \theta^m = 0$$

If we take notice the antisymmetry of the exterior products of 1-forms in this expression and properly rename the dummy indices, we get

$$\big[\nabla_k R^i_{jlm} - (2\gamma^n_{[kl]} - c^n_{kl})R^i_{jmn}\big]\theta^k \wedge \theta^l \wedge \theta^m = 0$$

and

$$(\nabla_k R^i_{jlm} - T^n_{kl}R^i_{jmn})\,\theta^k \wedge \theta^l \wedge \theta^m = 0$$

after having inserted (7.3.12). We then finally obtain the following identity

$$\nabla_{[k} R^i_{|j|lm]} = T^n_{[kl} R^i_{|j|m]n} \tag{7.3.24}$$

where the operation of alternation will be suspended on the index j occupying the space inside two vertical bars. The expressions (7.3.23) and (7.3.24) are called the ***1st and 2nd Bianchi identities***, respectively, because they were first discovered by Italian mathematician Luigi Bianchi (1856-1928) albeit in a different framework. The explicit form of the relations (7.3.24) becomes

$$\nabla_k R^i_{jlm} + \nabla_l R^i_{jmk} + \nabla_m R^i_{jkl} = T^n_{kl}R^i_{jmn} + T^n_{lm}R^i_{jkn} + T^n_{mk}R^i_{jln}.$$

When the torsion tensor vanishes, then we would necessarily get $R^i_{[jkl]} = 0$ and $\nabla_{[k} R^i_{|j|lm]} = 0$.

Let us now introduce the matrix forms $\mathbf{\Omega} = [\theta^i] \in \mathbf{\Lambda}^1(M)$ where all 1-forms θ^i are linearly independent, $\mathbf{\Gamma} = [\Gamma^i_j] \in \mathbf{\Lambda}^1(M)$, $\mathbf{\Sigma} = [\Sigma^i] \in \mathbf{\Lambda}^2(M)$ and $\mathbf{\Theta} = [\Theta^i_j] \in \mathbf{\Lambda}^2(M)$. Hence, the Cartan structural equations (7.3.22) can now be expressed as follows in the matrix form

$$d\mathbf{\Omega} = -\mathbf{\Gamma} \wedge \mathbf{\Omega} + \mathbf{\Sigma}, \quad \mathbf{\Theta} = d\mathbf{\Gamma} + \mathbf{\Gamma} \wedge \mathbf{\Gamma}.$$

The exterior derivatives of these forms satisfy the relations

$$d\mathbf{\Sigma} = \mathbf{\Theta} \wedge \mathbf{\Omega} - \mathbf{\Gamma} \wedge \mathbf{\Sigma}, \quad d\mathbf{\Theta} = \mathbf{\Theta} \wedge \mathbf{\Gamma} - \mathbf{\Gamma} \wedge \mathbf{\Theta}.$$

These equations coincide with the system of exterior differential equations given by (6.8.6). Thus the local solutions of these differential equations on an open set $U \subseteq M$ contractible to one of its interior points are provided by the expressions (6.8.19). By employing these expressions for various purposes we can determine the basis forms and connection coefficients generating certain torsion and curvature tensors with desired properties.

7.4. LEVI-CIVITA CONNECTION

As we have cited in Sec. 5.9, a Riemannian manifold is equipped with a symmetric, second order covariant metric tensor

$$\boldsymbol{\mathcal{G}} = g_{ij}(\mathbf{x})\,\theta^i \otimes \theta^j, \;\; g_{ij} = g_{ji}.$$

We assume that the basis in the cotangent bundle $T^*(M)$ can be so chosen that the matrix $\mathbf{G} = [g_{ij}]$ is regular at every point of the manifold, that is, the inverse matrix denoted by $\mathbf{G}^{-1} = [(g^{-1})^{ij}] = [g^{ij}]$ exists. The elementary arc length on this manifold will be measured by the relation

$$ds^2 = g_{ij}\,\theta^i\theta^j$$

similar to that given on p. 274.

Theorem 7.4.1. *There is a unique torsion-free affine connection on a Riemannian manifold with respect to which the covariant derivative of the metric tensor vanishes.*

The conditions $\boldsymbol{\nabla}\boldsymbol{\mathcal{G}} = \mathbf{0}$ and $\boldsymbol{\mathcal{T}} = \mathbf{0}$ necessitate, respectively

$$\nabla_k g_{ij} = e_k(g_{ij}) - \gamma^l_{ki} g_{lj} - \gamma^l_{kj} g_{il} = 0,$$
$$T^i_{jk} = \gamma^i_{jk} - \gamma^i_{kj} - c^i_{jk} = 0.$$

Let us now write $\gamma^i_{jk} = \gamma^i_{[jk]} + \gamma^i_{(jk)}$ where the symmetric and antisymmetric parts with respect to the subscripts of the connection coefficients are denoted, respectively, by

$$\gamma^i_{(jk)} = \frac{1}{2}(\gamma^i_{jk} + \gamma^i_{kj}), \quad \gamma^i_{[jk]} = \frac{1}{2}(\gamma^i_{jk} - \gamma^i_{kj}).$$

Then the condition $\nabla_k g_{ij} = 0$ yields

$$\gamma^i_{[jk]} = \frac{1}{2}\,c^i_{jk} \;\;\text{ and }\;\; \gamma^l_{(ki)} g_{lj} + \gamma^l_{(kj)} g_{il} = e_k(g_{ij}) - \frac{1}{2}(c^l_{ki} g_{lj} + c^l_{kj} g_{il}).$$

On introducing the definitions

$$\gamma_{ijk} = g_{il}\gamma^l_{jk}, \quad c_{ijk} = g_{il} c^l_{jk} = \, - \, c_{ikj} \tag{7.4.1}$$

we can cast the above expressions into the forms

$$\gamma_{i[jk]} = \frac{1}{2}\,c_{ijk}, \;\; \gamma_{j(ki)} + \gamma_{i(kj)} = e_k(g_{ij}) - \frac{1}{2}(c_{jki} + c_{ikj}). \tag{7.4.2}$$

Since the relations $(7.4.2)_2$ must be valid for all values of indices, we can employ cyclic permutations to write

$$\gamma_{j(ki)} + \gamma_{i(kj)} = e_k(g_{ij}) - \tfrac{1}{2}(c_{jki} + c_{ikj}),$$
$$\gamma_{k(ij)} + \gamma_{j(ik)} = e_i(g_{jk}) - \tfrac{1}{2}(c_{kij} + c_{jik}),$$
$$\gamma_{i(jk)} + \gamma_{k(ji)} = e_j(g_{ki}) - \tfrac{1}{2}(c_{ijk} + c_{kji}).$$

If we add the first and the third lines and subtract from the resulting expression the second line, consider the symmetries in indices and recall that we must write $\gamma_{ijk} = \gamma_{i[jk]} + \gamma_{i(jk)}$, we obtain

$$\gamma_{ijk} = \frac{1}{2}\big[e_j(g_{ki}) + e_k(g_{ij}) - e_i(g_{jk})\big] + \frac{1}{2}(c_{ijk} + c_{jik} + c_{kij}). \quad (7.4.3)$$

As to the connection coefficients, they are found from (7.4.1) as

$$\gamma^i_{jk} = g^{il}\gamma_{ljk}. \quad (7.4.4)$$

It is clear that when the metric tensor is specified the unique connection satisfying the conditions $\boldsymbol{\mathcal{T}} = \mathbf{0}$ and $\nabla\boldsymbol{\mathcal{G}} = \mathbf{0}$ is given by (7.4.4). □

Although this connection is known as the ***Levi-Civita connection***, some authors prefer to use term the ***Riemannian connection***. (7.4.4) is then explicitly written as follows

$$\gamma^i_{jk} = \frac{1}{2}g^{il}\big[e_j(g_{kl}) + e_k(g_{lj}) - e_l(g_{jk})\big] + \frac{1}{2}(c^i_{jk} + g_{jm}g^{il}c^m_{lk} + g_{km}g^{il}c^m_{lj}).$$

In natural coordinates we should take $e_i = \partial_i = \partial/\partial x^i$. Let us denote the connection coefficients corresponding to this case by $\Gamma^i_{jk} = \gamma^i_{jk}$. Since, in this case we have $c^i_{jk} = 0$, we find that $\Gamma^i_{jk} = \Gamma^i_{kj}$ and the *symmetric connection* is determined by the coefficients

$$\Gamma^i_{jk} = g^{il}\Gamma_{ljk}, \quad (7.4.5)$$
$$\Gamma_{ljk} = \frac{1}{2}\Big(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\Big).$$

the quantities Γ_{ijk} are called the ***Christoffel symbols of the first kind*** while we know that Γ^i_{jk} are the *Christoffel symbols of the second kind*. We thus conclude that the natural coordinates and the metric tensor on a Riemannian manifold create in a concrete way the linear connection whose existence was anticipated in Sec. 7.2.

If we choose vectors $\{e_i\}$ in the tangent spaces describing the moving frame as an orthonormal basis and forms $\{\theta^i\}$ as their reciprocal basis in the dual spaces, we know that the metric tensor reduces to $g_{ij} = \mp\,\delta_{ij}$. Then (7.4.3) gives in view of $(7.4.1)_2$

$$\gamma_{ijk} = \frac{1}{2}(c_{ijk} + c_{jik} + c_{kij}) = -\gamma_{kji}.$$

Because of the fact that we have selected the Levi-Civita connection in such a way that $\nabla_k g_{ij} = 0$, the relation $g^{il}g_{lj} = \delta^i_j$ leads immediately to $\nabla_k g^{ij} = 0$. Since the torsion tensor is zero, the curvature tensor, which will be called henceforth in this section as the ***Riemann curvature tensor*** or the ***Riemann-Christoffel tensor***, will satisfy the relations

$$R^i_{[jkl]} = 0 \text{ or } R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0 \tag{7.4.6}$$
$$\nabla_{[k}R^i_{|j|lm]} = 0 \text{ or } \nabla_k R^i_{jlm} + \nabla_l R^i_{jmk} + \nabla_m R^i_{jkl} = 0$$

in accordance with (7.3.23) and (7.3.24). We can now define the *covariant curvature tensor* by

$$R_{ijkl} = g_{im}R^m_{jkl.}$$

Some properties of this tensor can be revealed most easily in natural coordinates. In these coordinates, it follows from the relation (7.2.21) that

$$\begin{aligned} R_{ijkl} &= g_{im}(g^{mn}\Gamma_{nlj})_{,k} - g_{im}(g^{mn}\Gamma_{nkj})_{,l} + \Gamma_{ikn}\Gamma^n_{lj} - \Gamma_{iln}\Gamma^n_{kj} \\ &= g_{im}g^{mn}{}_{,k}\Gamma_{nlj} + \Gamma_{ilj,k} - g_{im}g^{mn}{}_{,l}\Gamma_{nkj} - \Gamma_{ikj,l} + \Gamma_{ikn}\Gamma^n_{lj} - \Gamma_{iln}\Gamma^n_{kj}. \end{aligned}$$

On the other hand, if we insert the relation

$$g^{mn}{}_{,k} = -\Gamma^m_{kr}g^{rn} - \Gamma^n_{kr}g^{mr}$$

obtained from the condition $\nabla_k g^{mn} = 0$ into the above expression, we find

$$\begin{aligned} R_{ijkl} = &-\Gamma_{ikn}\Gamma^n_{lj} - \Gamma^n_{ki}\Gamma_{nlj} + \Gamma_{iln}\Gamma^n_{kj} + \Gamma^n_{li}\Gamma_{nkj} + \Gamma_{ilj,k} - \Gamma_{ikj,l} \\ &+ \Gamma_{ikn}\Gamma^n_{lj} - \Gamma_{iln}\Gamma^n_{kj} \\ = &\ \Gamma_{ilj,k} - \Gamma_{ikj,l} + \Gamma^n_{li}\Gamma_{nkj} - \Gamma^n_{ki}\Gamma_{nlj} \end{aligned}$$

and on making use of (7.4.5) we finally arrive at the relation

$$R_{ijkl} = \frac{1}{2}(g_{il,jk} - g_{ik,jl} + g_{jk,il} - g_{jl,ik}) + \Gamma^n_{li}\Gamma_{nkj} - \Gamma^n_{ki}\Gamma_{nlj}. \tag{7.4.7}$$

Because of the symmetry properties $g_{ij} = g_{ji}$ and $\Gamma_{ijk} = \Gamma_{ikj}$, we readily observe the existence of the block symmetries

$$R_{ijkl} = R_{klij} = -R_{lkij}. \tag{7.4.8}$$

Since R_{ijkl} is a tensor, this symmetry properties will be valid in every coordinate systems.

The ***Ricci tensor*** R_{ij} [after Italian mathematician Gregorio Ricci-Curbastro who is rightly considered one of the principal founders of the absolute differential calculus connected with covariant differentiation, or tensor analysis as we call it today] is defined as a contraction of the curvature tensor R^i_{jkl}:

$$R_{ij} = R^k_{ikj} = \Gamma^k_{ji,k} - \Gamma^k_{ki,j} + \Gamma^k_{km}\Gamma^m_{ji} - \Gamma^k_{jm}\Gamma^m_{ki}. \qquad (7.4.9)$$

If we note that in view of (7.4.8) we can write $R_{kilj} = R_{ljki}$, and consequently $R^k_{ilj} = R_{lj}{}^k{}_i$ we find $R^k_{ikj} = R_{kj}{}^k{}_i = R^k_{jki}$ and finally we arrive at the relation

$$R_{ij} = R_{ji}.$$

Hence the *Ricci tensor is symmetric*. Moreover, we can easily deduce from (7.4.5) that a contraction on the Christoffel symbols of the second kind is found to be

$$\Gamma^j_{ji} = \frac{1}{2} g^{kl} \frac{\partial g_{kl}}{\partial x^i}.$$

Let $g = \det[g_{kl}]$. We then obtain

$$\frac{\partial g}{\partial x^i} = \frac{\partial g}{\partial g_{kl}} \frac{\partial g_{kl}}{\partial x^i} = g g^{kl} \frac{\partial g_{kl}}{\partial x^i}.$$

Therefore, we get

$$\Gamma^j_{ji} = \frac{1}{2g}\frac{\partial g}{\partial x^i} = \frac{1}{\sqrt{|g|}}\frac{\partial \sqrt{|g|}}{\partial x^i} = \frac{\partial \log |g|^{1/2}}{\partial x^i}. \qquad (7.4.10)$$

Because of the symmetry of the Ricci tensor, $(7.4.6)_1$ is satisfied identically. As to $(7.4.6)_2$, the contraction on indices i and l yields

$$\nabla_k R^i_{jim} + \nabla_i R^i_{jmk} + \nabla_m R^i_{jki} = \nabla_k R_{jm} - \nabla_m R_{jk} + \nabla_i R^i_{jmk} = 0.$$

We now raise the index j by recalling that the covariant derivative of the tensor g^{ij} vanishes to obtain

$$\nabla_k R^j_m - \nabla_m R^j_k + \nabla_i R^{ij}{}_{mk} = 0.$$

By taking notice of $R^{ij}{}_{mk} = R^{ji}{}_{km}$ and contracting indices j and k, we get

$$\nabla_j R^j_m - \nabla_m R^j_j + \nabla_i R^i_m = 0 \quad \text{or} \quad 2\nabla_i R^i_m - \nabla_m R = 0$$

where $R = R^i_i$ is a scalar quantity. Accordingly, the relation

$$\nabla_i(R^i_j - \tfrac{1}{2}\delta^i_j R) = 0 \tag{7.4.11}$$

must be satisfied. (7.4.11) is called the ***contracted Bianchi identity***. The second order tensor

$$G^i_j = R^i_j - \tfrac{1}{2}\delta^i_j R \tag{7.4.12}$$

is sometimes named as the ***Einstein tensor*** because of its association with the theory of general relativity. If we recall (7.4.10), we see that this tensor satisfies the following relation

$$\nabla_i G^i_j = G^i_{j,i} + \Gamma^k_{ki} G^i_j - \Gamma^k_{ij} G^i_k = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i}(\sqrt{|g|}\, G^i_j) - \Gamma^k_{ij} G^i_k = 0.$$

Since a Riemannian manifold equipped with the Levi-Civita connection is torsion-free, then $(7.3.22)_1$ takes the shape $d\theta^i = -\Gamma^i_j \wedge \theta^j$. On the other hand, we know that we can write

$$e_k(g_{ij})\, \theta^k = \gamma_{ikj}\theta^k + \gamma_{jki}\theta^k$$

since $\nabla_k g_{ij} = 0$. If we define the forms $\Gamma_{ij} = g_{il}\Gamma^l_j \in \Lambda^1(M)$, the above relation may be cast into the form

$$e_k(g_{ij})\, \theta^k = \Gamma_{ij} + \Gamma_{ji}.$$

If the metric tensor is *constant*, we find that $e_k(g_{ij}) = 0$ and $\Gamma_{ji} = -\Gamma_{ij}$, namely, the matrix 1-form $\mathbf{\Gamma} = [\Gamma_{ij}]$ is antisymmetric. This property will always exist in an orthonormal frame since one then has $g_{ij} = \delta_{ij}$. Moreover the inverse matrix g^{ij} is also the identity matrix. Hence, we can write $\Gamma_{ij} = \Gamma^i_j = -\Gamma_{ji} = -\Gamma^j_i$.

Example 7.4.1. We define a spherically symmetric *indefinite* metric in the 4-dimensional space-time manifold by the relation

$$ds^2 = e^{2\lambda}dt^2 - e^{2\mu}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2$$

where (r, θ, ϕ) are spherical space coordinates and t is the time coordinate. This metric was proposed by German astronomer and mathematician Karl Schwarzschild (1873-1916) in order to obtain the first exact analytical solution of Einstein's field equations of the general relativity in the vacuum. The exponents functions in this expression are taken as $\lambda = \lambda(r, t)$ and $\mu = \mu(r, t)$. The field equations of the general relativity connect the Einstein's tensor to the energy-momentum tensor reflecting physical properties of the medium involved through the relation

$$G^i_j = -\kappa T^i_j, \;\; i,j = 0,1,2,3$$

where κ is a constant. Due to (7.4.11), the energy-momentum tensor must satisfy $\nabla_i T^i_j = 0$, that is, it must be divergence-free. Since $T^i_j = 0$ in the vacuum, we ought to find $G^i_j = 0$. The relation $G^i_i = -R = 0$ then implies that the following equations must also hold:

$$R_{ij} = 0.$$

Let us now define four linearly independent 1-forms by the expressions given below

$$\theta^0 = e^{\lambda} dt, \;\; \theta^1 = e^{\mu} dr, \;\; \theta^2 = r\, d\theta, \;\; \theta^3 = r \sin\theta\, d\phi.$$

With this choice of basis, we can write

$$\boldsymbol{\mathcal{G}} = g_{ij} \theta^i \otimes \theta^j = \theta^0 \otimes \theta^0 - \theta^1 \otimes \theta^1 - \theta^2 \otimes \theta^2 - \theta^3 \otimes \theta^3$$

where $0 \leq i,j \leq 3$. Hence, all components g_{ij} of the metric tensor become constant being equal to 0, 1 or -1. Thus, the connection forms Γ_{ij} have to be antisymmetric. If we denote the partial derivatives with respect to the variables r and t by subscripts r and t, we then obtain

$$d\theta^0 = -\lambda_r\, e^{-\mu}\, \theta^0 \wedge \theta^1, \; d\theta^1 = \mu_t\, e^{-\lambda}\, \theta^0 \wedge \theta^1, \; d\theta^2 = \frac{e^{-\mu}}{r}\, \theta^1 \wedge \theta^2,$$
$$d\theta^3 = \frac{e^{-\mu}}{r}\, \theta^1 \wedge \theta^3 + \frac{\cot\theta}{r}\, \theta^2 \wedge \theta^3.$$

In this case, the coefficients $c^i_{jk} = -c^i_{kj}$ are found to be

$$c^0_{01} = \lambda_r\, e^{-\mu}, \;\; c^1_{01} = -\mu_t\, e^{-\lambda}, \;\; c^2_{12} = -\frac{e^{-\mu}}{r},$$
$$c^3_{13} = -\frac{e^{-\mu}}{r}, \; c^3_{23} = -\frac{\cot\theta}{r}.$$

All other coefficients are zero. When we carefully scrutinise the Cartan structural equations $d\theta^i = -\Gamma^i_j \wedge \theta^j$, we realise that connection forms must be designated as follows

$$\Gamma^0_1 = \lambda_r\, e^{-\mu}\, \theta^0 + \mu_t\, e^{-\lambda}\, \theta^1 = -\Gamma^1_0, \;\; \Gamma^2_1 = \frac{e^{-\mu}}{r}\, \theta^2 = -\Gamma^1_2,$$
$$\Gamma^3_1 = \frac{e^{-\mu}}{r}\, \theta^3 = -\Gamma^1_3, \; \Gamma^3_2 = \frac{\cot\theta}{r}\, \theta^3 = -\Gamma^2_3, \; \Gamma^2_0 = -\Gamma^0_2 = 0,$$
$$\Gamma^3_0 = -\Gamma^0_3 = 0.$$

The curvature forms $\Theta^i_j = d\Gamma^i_j + \Gamma^i_k \wedge \Gamma^k_j$ will then possess the property $\Theta^i_j = -\Theta^j_i$. These relations between the forms $\mathbf{\Theta}$ and $\mathbf{\Gamma}$ result readily in the curvature forms below

$$\begin{aligned}
\Theta^0_1 &= d\Gamma^0_1 = \big[e^{-2\lambda}(\mu_{tt} - \mu_t\lambda_t + \mu_t^2) - e^{-2\mu}(\lambda_{rr} - \mu_r\lambda_r + \lambda_r^2)\big]\theta^0 \wedge \theta^1,\\
\Theta^0_2 &= \Gamma^0_1 \wedge \Gamma^1_2 = -\frac{e^{-2\mu}\lambda_r}{r}\,\theta^0 \wedge \theta^2 - \frac{e^{-(\lambda+\mu)}\mu_t}{r}\,\theta^1 \wedge \theta^2,\\
\Theta^0_3 &= \Gamma^0_1 \wedge \Gamma^1_3 = -\frac{e^{-2\mu}\lambda_r}{r}\,\theta^0 \wedge \theta^3 - \frac{e^{-(\lambda+\mu)}\mu_t}{r}\,\theta^1 \wedge \theta^3,\\
\Theta^1_2 &= d\Gamma^1_2 + \Gamma^1_3 \wedge \Gamma^3_2 = \frac{e^{-(\lambda+\mu)}\mu_t}{r}\,\theta^0 \wedge \theta^2 + \frac{e^{-2\mu}\mu_r}{r}\,\theta^1 \wedge \theta^2,\\
\Theta^1_3 &= d\Gamma^1_3 + \Gamma^1_2 \wedge \Gamma^2_3 = \frac{e^{-(\lambda+\mu)}\mu_t}{r}\,\theta^0 \wedge \theta^3 + \frac{e^{-2\mu}\mu_r}{r}\,\theta^1 \wedge \theta^3,\\
\Theta^2_3 &= d\Gamma^2_3 + \Gamma^2_1 \wedge \Gamma^1_3 = \frac{1 - e^{-2\mu}}{r^2}\,\theta^2 \wedge \theta^3,\\
\Theta^1_0 &= -\Theta^0_1, \Theta^2_0 = -\Theta^0_2, \Theta^3_0 = -\Theta^0_3,\\
\Theta^2_1 &= -\Theta^1_2, \Theta^3_1 = -\Theta^1_3, \Theta^3_2 = -\Theta^2_3.
\end{aligned}$$

By making use of these forms, the components of the curvature tensor are found as

$$\begin{aligned}
R^0_{101} &= -R^1_{001} = -R^0_{110} = R^1_{010}\\
&= e^{-2\lambda}(\mu_{tt} - \mu_t\lambda_t + \mu_t^2) - e^{-2\mu}(\lambda_{rr} - \mu_r\lambda_r + \lambda_r^2)\\
R^0_{202} &= -R^2_{002} = -R^0_{220} = R^2_{020} = R^0_{303} = -R^3_{003} = -R^0_{330}\\
&= R^3_{030} = -\frac{e^{-2\mu}\lambda_r}{r}\\
R^0_{212} &= -R^2_{012} = -R^0_{221} = R^2_{021} = R^0_{313} = -R^3_{013} = -R^0_{331}\\
&= R^3_{031} = R^1_{202} = -R^2_{102} = -R^2_{220} = R^2_{120} = -R^1_{303}\\
&= R^3_{103} = -R^3_{130} = R^1_{330} = -\frac{e^{-(\lambda+\mu)}\mu_t}{r}\\
R^1_{212} &= -R^2_{112} = R^2_{121} = -R^1_{221} = R^1_{313} = -R^3_{113} = R^3_{131}\\
&= -R^1_{331} = \frac{e^{-2\mu}\mu_r}{r}\\
R^2_{323} &= -R^3_{223} = R^3_{232} = -R^2_{332} = \frac{1 - e^{-2\mu}}{r^2}
\end{aligned}$$

The components of the Ricci tensor are easily obtained as

$$R_{02} = R^1_{012} + R^3_{032} = 0, \qquad R_{03} = R^1_{013} + R^2_{023} = 0,$$

$$
\begin{aligned}
&R_{13} = R^0_{103} + R^2_{123} = 0, \qquad R_{12} = R^0_{102} + R^3_{132} = 0,\\
&R_{23} = R^0_{203} + R^1_{213} = 0,\\
&R_{00} = R^1_{010} + R^2_{020} + R^3_{030} = e^{-2\lambda}(\mu_{tt} - \mu_t\lambda_t + \mu_t^2)\\
&\qquad - e^{-2\mu}(\lambda_{rr} - \mu_r\lambda_r + \lambda_r^2) - 2\frac{e^{-2\mu}\lambda_r}{r},\\
&R_{11} = R^0_{101} + R^2_{121} + R^3_{131} = e^{-2\lambda}(\mu_{tt} - \mu_t\lambda_t + \mu_t^2)\\
&\qquad - e^{-2\mu}(\lambda_{rr} - \mu_r\lambda_r + \lambda_r^2) + 2\frac{e^{-2\mu}\mu_r}{r},\\
&R_{22} = R^0_{202} + R^1_{212} + R^3_{232} = \frac{e^{-2\mu}(\mu_r - \lambda_r)}{r} + \frac{1 - e^{-2\mu}}{r^2},\\
&R_{33} = R^0_{303} + R^1_{313} + R^2_{323} = \frac{e^{-2\mu}(\mu_r - \lambda_r)}{r} + \frac{1 - e^{-2\mu}}{r^2},\\
&R_{01} = R^1_{011} + R^2_{021} + R^3_{031} = -\frac{e^{-(\lambda+\mu)}\mu_t}{r}.
\end{aligned}
$$

In this situation, the Einstein equations $R_{ij} = 0$ yield $\mu_t = 0$ and $\mu = \mu(r)$ for the component $R_{01} = 0$. When we employ this property in the other equations, we conclude that

$$
\begin{aligned}
\lambda_{rr} - \mu_r\lambda_r + \lambda_r^2 - \frac{2\lambda_r}{r} &= 0,\\
\lambda_{rr} - \mu_r\lambda_r + \lambda_r^2 + \frac{2\mu_r}{r} &= 0,\\
\frac{e^{-2\mu}(\mu_r - \lambda_r)}{r} + \frac{1 - e^{-2\mu}}{r^2} &= 0.
\end{aligned}
$$

Let us subtract the first equation in the above list from the second one to find $\mu_r + \lambda_r = 0$, and consequently $\lambda(r,t) = -\mu(r) + f(t)$. As to the last equation, it yields

$$
2e^{-2\mu}\mu_r = -\frac{de^{-2\mu}}{dr} = \frac{e^{-2\mu} - 1}{r}.
$$

By integrating this equation, we obtain

$$
e^{-2\mu} = 1 - \frac{K}{r}.
$$

Hence, we can deduce the expression

$$
e^{2\lambda} = e^{2f(t)}\Big(1 - \frac{K}{r}\Big) = F(t)\Big(1 - \frac{K}{r}\Big), \ \ F(t) > 0.
$$

We can immediately observe that the first and the second equations will be satisfied as well with these expressions for λ and μ. Let us introduce a new variable τ by the relation $d\tau/dt = \sqrt{F(t)}$. We then see that we can take $F(t) = 1$ without loss of generality. Therefore, the metric satisfying the Einstein equations takes the form

$$ds^2 = \Big(1 - \frac{K}{r}\Big)dt^2 - \Big(1 - \frac{K}{r}\Big)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

For physical reasons, we choose $K > 0$. The characteristic components of the curvature tensor become

$${R^0}_{101} = {R^2}_{323} = \frac{K}{r^3},\ {R^0}_{202} = {R^1}_{212} = -\frac{K}{2r^3},\ {R^0}_{212} = 0.$$

This metric that was obtained by Schwarzschild in 1915 constitutes the first and simplest exact solution of the Einstein equations. It determines the curvature of space-time, in other words the gravitational field, created by spherical symmetric static body. Let us choose

$$\sqrt{F(t)} = c = \text{speed of light},$$
$$K = r_S = 2Gm/c^2 = \text{Schwarzschild radius}$$

where m is the mass of the body, G is the universal gravitation constant. With these physical parameters, the Schwarzschild metric takes the form

$$ds^2 = c^2\Big(1 - \frac{2Gm}{c^2 r}\Big)dt^2 - \Big(1 - \frac{2Gm}{c^2 r}\Big)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

This metric involving singularities at $r = 0$ is the first ever solution that predicts the existence of black holes. ■

Since the torsion tensor vanishes in Levi-Civita connection, (7.3.18) takes the form

$$[\nabla_k, \nabla_j]\, t^{i_1\cdots i_k}_{j_1\cdots j_l} = \sum_{r=1}^{k} {R^{i_r}}_{mkj} t^{i_1\cdots i_{r-1} m i_{r+1}\cdots i_k}_{j_1\cdots j_l} - \sum_{r=1}^{l} {R^m}_{j_r kj} t^{i_1\cdots i_k}_{j_1\cdots j_{r-1} m j_{r+1}\cdots j_l}.$$

In addition, because of the symmetries (7.4.8), the effect of the curvature tensor on a vector $\boldsymbol{a}$ can be written as

$${R^m}_{ikj}\, a_m = R_{mikj}\, a^m = -R_{imkj}\, a^m = -{R_i}^m{}_{kj}\, a_m.$$

Hence, the effect of the commutator $[\nabla_k, \nabla_j]$ is expressible by the relation

$$[\nabla_k, \nabla_j]\, t^{i_1\cdots i_k}_{j_1\cdots j_l} = (\nabla_k \nabla_j - \nabla_j \nabla_k)\, t^{i_1\cdots i_k}_{j_1\cdots j_l} = \qquad (7.4.13)$$

$$\sum_{r=1}^{k} R^{i_r}{}_{mkj}\, t^{i_1\cdots i_{r-1}mi_{r+1}\cdots i_k}_{j_1\cdots j_l} + \sum_{r=1}^{l} R_{j_r}{}^{m}{}_{kj}\, t^{i_1\cdots i_k}_{j_1\cdots j_{r-1}mj_{r+1}\cdots j_l}$$

In natural coordinates, the covariant derivative of a tensor $\mathcal{T}$ in the direction of a vector V is given by

$$\begin{aligned} v^i\nabla_i t^{i_1\cdots i_k}_{j_1\cdots j_l} = & \\ & t^{i_1\cdots i_k}_{j_1\cdots j_l,i}v^i + \sum_{r=1}^{k} v^i\Gamma^{i_r}_{in} t^{i_1\cdots i_{r-1}ni_{r+1}\cdots i_k}_{j_1\cdots j_l} - \sum_{r=1}^{l} v^i\Gamma^{n}_{ij_r} t^{i_1\cdots i_k}_{j_1\cdots j_{r-1}nj_{r+1}\cdots j_l}. \end{aligned}$$

Since the connection is symmetric, we can obviously write

$$v^i\Gamma^{i_r}_{in} = \Gamma^{i_r}_{ni}v^i = v^{i_r}_{;n} - v^{i_r}_{,n}.$$

We are thus led to the relation

$$\begin{aligned} v^i\nabla_i t^{i_1\cdots i_k}_{j_1\cdots j_l} = & \\ & t^{i_1\cdots i_k}_{j_1\cdots j_l,i}v^i - \sum_{r=1}^{k} t^{i_1\cdots i_{r-1}ni_{r+1}\cdots i_k}_{j_1\cdots j_l} v^{i_r}_{,n} + \sum_{r=1}^{l} t^{i_1\cdots i_k}_{j_1\cdots j_{r-1}nj_{r+1}\cdots j_l} v^{n_r}_{,j_r} \\ & + \sum_{r=1}^{k} t^{i_1\cdots i_{r-1}ni_{r+1}\cdots i_k}_{j_1\cdots j_l} v^{i_r}_{;n} - \sum_{r=1}^{l} t^{i_1\cdots i_k}_{j_1\cdots j_{r-1}nj_{r+1}\cdots j_l} v^{n}_{;j_r} \end{aligned}$$

where the first line is none other than the Lie derivative of the tensor $\mathcal{T}$ with respect to the vector field V [*see* (5.11.16)]. We thereby obtain the relation

$$(\nabla_V\mathcal{T} - \pounds_V\mathcal{T})^{i_1\cdots i_k}_{j_1\cdots j_l} = \sum_{r=1}^{k} t^{i_1\cdots i_{r-1}ni_{r+1}\cdots i_k}_{j_1\cdots j_l} v^{i_r}_{;n} - \sum_{r=1}^{l} t^{i_1\cdots i_k}_{j_1\cdots j_{r-1}nj_{r+1}\cdots j_l} v^{n}_{;j_r}.$$

If we apply this relation to the metric tensor $\boldsymbol{\mathcal{G}}$, we then find

$$(\pounds_V\boldsymbol{\mathcal{G}})_{ij} = g_{kj}v^k_{;i} + g_{ik}v^k_{;j} = v_{i;j} + v_{j;i}$$

because $\boldsymbol{\nabla}\boldsymbol{\mathcal{G}} = \mathbf{0}$. This means that a vector field V that leaves the metric tensor invariant under the flow created by this vector field, namely, satisfying the relation $\pounds_V\boldsymbol{\mathcal{G}} = \mathbf{0}$ has to verify the partial differential equations

$$v_{i;j} + v_{j;i} = v_{i,j} + v_{j,i} - 2\Gamma^k_{ij}v_k = 0. \qquad (7.4.14)$$

(7.4.14) are known as the ***Killing equations*** and a vector field satisfying these equations is called a ***Killing vector field***. If V is a Killing vector field, we then obtain by cyclic permutation of indices

$$\begin{aligned} v_{i;jk} - v_{i;kj} &= v_{i;jk} + v_{k;ij} = R_{ilkj} v^l, \\ v_{j;ki} + v_{i;jk} &= R_{jlik} v^l, \\ v_{k;ij} + v_{j;ki} &= R_{klji} v^l. \end{aligned}$$

If we subtract the third equation from the sum of the first two equations in the above list, we get

$$2v_{i;jk} = (R_{ilkj} + R_{jlik} - R_{klji}) v^l.$$

However, the symmetries of the curvature tensor yield $R_{ilkj} + R_{jlik} - R_{klji} = 2R_{ijkl}$ [*see* (7.4.6)$_1$ with the lowered index i]. Thus a Killing vector field must satisfy the relation

$$v_{i;jk} = R_{ijkl} v^l$$

from which we easily deduct the expressions

$$v^j_{;ij} = R_{ij} v^j, \quad v^{i\ j}_{;j} = -R^i_j v^j.$$

7.5. DIFFERENTIAL OPERATORS

Let us assume that the exterior differential form $\omega \in \Lambda^k(M)$ is defined on a Riemannian manifold M. We have introduced the exterior derivative operator d in Sec. 5.8 while the operators of co-differential δ and Laplace-de Rham Δ in Sec. 5.9. In this section, we shall try to present a more detailed discussion of the structure of these operators when such a manifold is endowed with the Levi-Civita connection. We first consider the exterior derivative operator $d : \Lambda^k(M) \to \Lambda^{k+1}(M)$. We know that the exterior derivative of a form $\omega \in \Lambda^k(M)$ is

$$d\omega = \frac{1}{k!}\, \omega_{[i_1 \cdots i_k, i]}\, dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

If we now insert the expression

$$\omega_{i_1 \cdots i_k, i} = \nabla_i \omega_{i_1 \cdots i_k} + \sum_{r=1}^{k} \Gamma^j_{i i_r} \omega_{i_1 \cdots i_{r-1} j i_{r+1} \cdots i_k}$$

into the above relation, pay attention to the complete antisymmetry of the exterior product $dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ and the symmetry $\Gamma^j_{i i_r} = \Gamma^j_{i_r i}$ of the connection coefficients, we reach to the conclusion that can be written as

$$d\omega = \frac{k+1}{(k+1)!}\,\nabla_{[i}\omega_{i_1\cdots i_k]}\,dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \qquad (7.5.1)$$

The co-differential operator $\delta : \Lambda^k(M) \to \Lambda^{k-1}(M)$ was defined by the expression (5.9.30):

$$\delta\omega = \frac{(-1)^k}{(k-1)!}\,\omega_{[i_1\cdots i_{k-1}]i;}{}^{i}\,dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}.$$

The components of this form is determined by

$$\omega^{i_1\cdots i}{}_{;i} = \frac{(\sqrt{g}\,\omega^{i_1\cdots i})_{,i}}{\sqrt{g}}$$

as given in (5.9.19) and through $\omega_{i_1\cdots i_{k-1}i;}{}^{i} = g_{i_1j_1}\cdots g_{i_{k-1}j_{k-1}}\omega^{j_1\cdots j_{k-1}i}{}_{;i}$. We shall now demonstrate that these relations are actually associated with the covariant derivative generated by the Levi-Civita connection. The relation

$$\begin{aligned}\omega^{i_1\cdots i_k}{}_{;i} &= \nabla_i\omega^{i_1\cdots i_k}\\ &= \omega^{i_1\cdots i_k}{}_{,i} + \sum_{r=1}^{k}\Gamma^{i_r}_{ij}\omega^{i_1\cdots i_{r-1}ji_{r+1}\cdots i_k}\end{aligned}$$

leads to

$$\begin{aligned}\nabla_i\omega^{i_1\cdots i_{k-1}i} &= \omega^{i_1\cdots i_{k-1}i}{}_{,i} + \sum_{r=1}^{k-1}\Gamma^{i_r}_{ij}\omega^{i_1\cdots i_{r-1}ji_{r+1}\cdots i_{k-1}i} + \Gamma^{i}_{ij}\omega^{i_1\cdots i_{k-1}j}\\ &= \omega^{i_1\cdots i_{k-1}i}{}_{,i} + \Gamma^{i}_{ij}\omega^{i_1\cdots i_{k-1}j} = \frac{1}{\sqrt{g}}(\sqrt{g}\,\omega^{i_1\cdots i_{k-1}i})_{,i}\end{aligned}$$

where we have employed the complete antisymmetry of the components $\omega^{i_1\cdots i_k}$, the symmetry of the connection coefficients Γ^k_{ij} and the relation (7.4.10). Hence, by raising and lowering indices by means of the metric tensor, the co-differential operator becomes expressible as follows

$$\begin{aligned}\delta\omega &= \frac{(-1)^k}{(k-1)!}\,\nabla^i\omega_{i_1\cdots i_{k-1}i}\,dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}} \qquad (7.5.2)\\ &= -\,\frac{1}{(k-1)!}\,\nabla^i\omega_{ii_1\cdots i_{k-1}}\,dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}\end{aligned}$$

where we have evidently defined $\nabla^i = g^{ij}\nabla_j$.

The Laplace-de Rham operator

$$\varDelta = \delta d + d\delta : \Lambda^k(M) \to \Lambda^k(M)$$

can now be evaluated by using (7.5.1) and (7.5.2). We can thus write

$$d\delta\omega = -\frac{k}{k!}\,\nabla_{[i_1}(\nabla^i\omega_{|i|i_2\cdots i_k]})\,dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_k},$$

and

$$\delta d\omega = -\frac{k+1}{k!}\,\nabla^i(\nabla_{[i}\omega_{i_1 i_2\cdots i_k]})\,dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_k}$$

where the antisymmetries in the coefficients are explicitly described. If we express the form $\varDelta\omega$ as

$$\varDelta\omega = \frac{1}{k!}(\varDelta\omega)_{i_1\cdots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k}\in\Lambda^k(M)$$

then its components are determined by the following expression

$$(\varDelta\omega)_{i_1 i_2\cdots i_k} = -(k+1)\,\nabla^i\nabla_{[i}\omega_{i_1 i_2\cdots i_k]} - k\,\nabla_{[i_1}\nabla^i\omega_{|i|i_2\cdots i_k]}.$$

If we utilise the relation (5.5.2), we obtain at once

$$(k+1)\nabla^i\nabla_{[i}\omega_{i_1 i_2\cdots i_k]} = \nabla^i\nabla_i\,\omega_{i_1 i_2\cdots i_k} - \sum_{r=1}^{k}\nabla^i\nabla_{i_r}\,\omega_{i_1\cdots i_{r-1}i i_{r+1}\cdots i_k}$$

and

$$\begin{aligned} k\nabla_{[i_1}\nabla^i\omega_{|i|i_2\cdots i_k]} &= \nabla_{i_1}\nabla^i\,\omega_{i i_2\cdots i_k} - \sum_{r=2}^{k}\nabla_{i_r}\nabla^i\,\omega_{i i_2\cdots i_{r-1}i_1 i_{r+1}\cdots i_k}\\ &= \sum_{r=1}^{k}\nabla_{i_r}\nabla^i\,\omega_{i_1\cdots i_{r-1}i i_{r+1}\cdots i_k}. \end{aligned}$$

Hence, we draw the conclusion

$$(\varDelta\omega)_{i_1\cdots i_k} = -\nabla^i\nabla_i\omega_{i_1 i_2\cdots i_k} - \sum_{r=1}^{k}(\nabla_{i_r}\nabla^i - \nabla^i\nabla_{i_r})\,\omega_{i_1\cdots i_{r-1}i i_{r+1}\cdots i_k}.$$

On the other hand, let us note that one is able to write

$$[\nabla_{i_r},\nabla^i]\,\omega_{i_1\cdots i_{r-1}i i_{r+1}\cdots i_k} = (-1)^{r-1}[\nabla_{i_r},\nabla_i]\,\omega^i_{i_1\cdots i_{r-1}i_{r+1}\cdots i_k}$$

Employing then the relation (7.4.13), we find that

$$\begin{aligned}
[\nabla_{i_r}, \nabla_i]\, \omega^i_{i_1\cdots i_{r-1}i_{r+1}\cdots i_k} &= R^i_{mi_ri}\, \omega^m_{i_1\cdots i_{r-1}i_{r+1}\cdots i_k} \\
&\quad + \sum_{s=1, s\neq r}^{k} R_{i_s}{}^{m}{}_{i_ri}\, \omega^i_{i_1\cdots i_{s-1}mi_{s+1}\cdots i_{r-1}i_{r+1}\cdots i_k} \\
= -\, R_{mi_r}\, \omega^m_{i_1\cdots i_{r-1}i_{r+1}\cdots i_k} &+ \sum_{s=1, s\neq r}^{k} (-1)^{s-1} R_{i_s}{}^{m}{}_{i_ri}\, \omega^i_{mi_1\cdots i_{s-1}i_{s+1}\cdots i_{r-1}i_{r+1}\cdots i_k} \\
= -\, R^m_{i_r}\, \omega_{mi_1\cdots i_{r-1}i_{r+1}\cdots i_k} &+ \sum_{s=1, s\neq r}^{k} (-1)^{s-1} R_{i_s}{}^{m}{}_{i_r}{}^{i}\, \omega_{imi_1\cdots i_{s-1}i_{s+1}\cdots i_{r-1}i_{r+1}\cdots i_k}
\end{aligned}$$

Therefore, we see that the Laplace-de Rham operator is completely determined by the components

$$\begin{aligned}
(\Delta\omega)_{i_1\cdots i_k} = -\, g^{ij}\nabla_j\nabla_i\, \omega_{i_1i_2\cdots i_k} - \sum_{r=1}^{k}(-1)^r R^m_{i_r}\, \omega_{mi_1\cdots i_{r-1}i_{r+1}\cdots i_k} \qquad (7.5.3) \\
- \sum_{r=1}^{k} \sum_{s=1, s\neq r}^{k} (-1)^{r+s} R_{i_s}{}^{m}{}_{i_r}{}^{i}\, \omega_{imi_1\cdots i_{s-1}i_{s+1}\cdots i_{r-1}i_{r+1}\cdots i_k}.
\end{aligned}$$

VII. EXERCISES

7.1. A manifold M is equipped with two connections defined by the Christoffel symbols Γ^i_{jk} and Γ'^i_{jk}. Show that the quantities $\Upsilon^i_{jk} = \Gamma^i_{jk} - \Gamma'^i_{jk}$ are components of a $\binom{1}{2}$-tensor.

7.2. Let $\boldsymbol{\nabla}$ be a connection on a manifold M. Show that the operator $\boldsymbol{\nabla}^*$ defined by the relation $\nabla^*_U V = \nabla_U V + \boldsymbol{\tau}(U, V)$ is also a connection on M whose torsion tensor is determined by $-\,\boldsymbol{\tau}$. $\boldsymbol{\nabla}^*$ is called the *conjugate connection*.

7.3. $\boldsymbol{\nabla}$ is a connection on a manifold M. Show that the connection defined by the relation $\boldsymbol{\nabla}^s = \frac{1}{2}(\boldsymbol{\nabla} + \boldsymbol{\nabla}^*)$ is symmetric. Find the connection coefficients.

7.4. Show that the connections $\boldsymbol{\nabla}$, $\boldsymbol{\nabla}^*$ and $\boldsymbol{\nabla}^s$ have the same geodesics on the manifold M.

7.5. A connection on the manifold $\mathbb{R}^2$ whose coordinate cover is (x^1, x^2) is prescribed by Christoffel symbols $\Gamma^1_{12} = \Gamma^1_{21} = 1$ and all other coefficients $\Gamma^i_{jk} = 0$. Determine the geodesics.

7.6. A connection $\boldsymbol{\nabla}$ and a tensor S^i_{jk} that is antisymmetric with respect to its covariant indices are given on a manifold M. Show that there is a unique connection on M with the same geodesics as those of $\boldsymbol{\nabla}$ and its torsion tensor being equal to S^i_{jk}.

7.7. If the parameter of a geodesic curve determined by the equation (7.2.16) is t, then show that a change of the parameter in the form $\tau = \alpha t + \beta$ where α and β are constants still satisfies that equation. Thus, a parameter of a geodesic curve may be named as an *affine parameter*.

7.8. The *indefinite Lorentz metric* on the manifold $\mathbb{R}^{n+1}$ [Dutch physicist Hendrik Antoon Lorentz (1853-1928)] is introduced by the relation

$$\boldsymbol{\mathcal{G}}' = -\,dx_0 \otimes dx_0 + \sum_{i=1}^{n} dx_i \otimes dx_i.$$

Hence, the Lorentz inner product and the *length* of a vector is determined, respectively, by

$$\boldsymbol{\mathcal{G}}'(U,V) = -\,u_0 v_0 + \sum_{i=1}^{n} u_i v_i,$$
$$\boldsymbol{\mathcal{G}}'(U,U) = -\,u_0^2 + \sum_{i=1}^{n} u_i^2.$$

In this case, $(\mathbb{R}^{n+1}, \boldsymbol{\mathcal{G}}')$ becomes obviously a pseudo-Riemannian manifold. Let us now define an n-dimensional submanifold of the manifold $\mathbb{R}^{n+1}$ as follows

$$H^n = \{\mathbf{x} \in \mathbb{R}^{n+1} : x_0^2 - \sum_{i=1}^{n} x_i^2 = 1, x_0 > 0\} \subset \mathbb{R}^{n+1}.$$

H^n is called a *hyperbolic space*. Show that the metric $\boldsymbol{\mathcal{G}}'$ generates a definite metric $\boldsymbol{\mathcal{G}}$ on H^n whose components are given by

$$\boldsymbol{\mathcal{G}} = g_{ij}\, dx_i \otimes dx_j, \quad g_{ij} = \delta_{ij} - \frac{x_i x_j}{x_0^2}$$

and $(H^n, \boldsymbol{\mathcal{G}})$ becomes a complete Riemannian manifold.

7.9. *Hyperbolic plane* H^2 is defined as the submanifold

$$H^2 = \{\mathbf{x} \in \mathbb{R}^3 : x_0^2 - x_1^2 - x_2^2 = 1, x_0 > 0\} \subset \mathbb{R}^3$$

where $\mathbb{R}^3$ is equipped with the Lorentz metric. By using a coordinate transformation

$$x_1 = r\cos\theta, \;\; x_2 = r\sin\theta$$

show that the metric tensor of this Riemannian manifold is given by the relation

$$\boldsymbol{\mathcal{G}} = \frac{dr \otimes dr}{1 + r^2} + r^2 d\theta \otimes d\theta$$

Find further the form of this metric tensor under the coordinate transformation $r = \sinh s$.

7.10. Compute the metric tensor of the 3-dimensional hyperbolic space H^3. Use first the spherical coordinates (r, θ, ϕ) given by

$$x = r\sin\theta\cos\phi,$$
$$y = r\sin\theta\sin\phi,$$
$$z = r\cos\theta$$

then the transformation $r = \sinh s$.

7.11. Show that the number of the independent components of the curvature tensor R_{ijkl} satisfying the symmetry relations (7.4.8) in an n-dimensional manifold is given by

$$\frac{1}{12}n^2(n^2-1).$$

7.12. The Weyl tensor is defined by the relation

$$W^{ij}{}_{kl} = R^{ij}{}_{kl} - 2\delta^{[i}{}_{[k}R^{j]}{}_{l]} + \frac{1}{3}\delta^{[i}{}_{[k}\,\delta^{j]}{}_{l]}R.$$

Show that all contractions of this tensor yield zero tensors.

7.13. The metric tensor in a 2-dimensional Riemannian manifold is given in the following form

$$\boldsymbol{\mathcal{G}} = dr \otimes dr + [f(r,\theta)]^2 d\theta \otimes d\theta.$$

Let us choose the basis forms in $T^*(M)$ as

$$\theta^1 = dr, \quad \theta^2 = f(r,\theta)\,d\theta.$$

Find the reciprocal basis vectors e_1, e_2 in $T(M)$ and the coefficients c^k_{ij}. Determine the Christoffel symbols and the curvature tensor.

7.14. Show that the operator of covariant differentiation satisfies the Jacobi identity

$$[\nabla_i, [\nabla_j, \nabla_k]] + [\nabla_j, [\nabla_k, \nabla_i]] + [\nabla_k, [\nabla_i, \nabla_j]] = 0.$$

7.15. Let M be a Riemannian manifold and $\boldsymbol{\mathcal{R}}$ be its curvature tensor. Show that the relation $\boldsymbol{\mathcal{R}}(V_1, V_2, V_3, V_4) = \boldsymbol{\mathcal{G}}\big(\boldsymbol{\mathcal{R}}(V_2, V_3, V_4), V_1\big)$ is satisfied for all vector fields $V_i \in T(M), i = 1,2,3,4$. Verify further the following identities:

(a) $\boldsymbol{\mathcal{R}}(V_1, V_2, V_3, V_4) = -\boldsymbol{\mathcal{R}}(V_2, V_1, V_3, V_4)$,
(b) $\boldsymbol{\mathcal{R}}(V_1, V_2, V_3, V_4) = -\boldsymbol{\mathcal{R}}(V_1, V_2, V_4, V_3)$,
(c) $\boldsymbol{\mathcal{R}}(V_1, V_2, V_3, V_4) = \boldsymbol{\mathcal{R}}(V_3, V_4, V_1, V_2)$,
(d) $\boldsymbol{\mathcal{R}}(V_1, V_2, V_3, V_4) + \boldsymbol{\mathcal{R}}(V_1, V_3, V_4, V_2) + \boldsymbol{\mathcal{R}}(V_1, V_4, V_2, V_3) = 0$.

7.16. Calculate the function Δf in cylindrical coordinates where $f \in \Lambda^0(\mathbb{R}^3)$.

7.17. Calculate the form $\Delta\omega$ in cylindrical and spherical coordinates where $\omega \in \Lambda^1(\mathbb{R}^3)$.

7.18. Let $(M, \boldsymbol{\mathcal{G}})$ and $(N, \boldsymbol{\Gamma})$ be two Riemannian manifolds. If a diffeomorphism $\phi : M \to N$ fulfil the condition $\phi^*\boldsymbol{\Gamma} = \boldsymbol{\mathcal{G}}$, then it is called an ***isometry***. If such an isometry is established, then we say that the manifolds M and N are *isometric*. Show that a diffeomorphism $\phi : M \to M$ is an isometry if and only if the condition

$$\phi^*(\Delta f) = \Delta(\phi^* f)$$

holds for all functions $f \in \Lambda^0(M)$.