CHAPTER VIII

INTEGRATION OF EXTERIOR FORMS

8.1. SCOPE OF THE CHAPTER

In this chapter, the integral of an exterior differential form over a submanifold of a given manifold, whose dimension is equal to the degree of the exterior form, is treated as a linear operator assigning a real number to that form. As is well known, the form reduces to a simple form on such a submanifold and the integral is roughly defined as a multiple Riemann integral of the single scalar function characterising that form. However, in order that this definition acquires a formal content, we have to exert quite a great effort and to equip the manifold with adequate structures such as simplices and chains. We also deal in this chapter with the cohomology and homology groups that are inspired by these structure and prove to be very helpful in revealing some hidden properties of closed forms. Sec. 8.2 introduces the concept of orientability of a manifold by means of a volume form on a manifold. In Sec. 8.3, the integration of forms is discussed on a very simple manifold, the Euclidean space. We treat the simplices in the Euclidean space that can be used as building blocks to generalise this approach to any smooth manifold in Sec. 8.4. We then discuss chains and their boundaries, and cycles. We further define differentiable singular simplices and chains that are images of a standard simplex at the origin of the Euclidean space on a differentiable manifold by means of smooth functions. In Sec. 8.5, we propose two different courses to follow in order to evaluate the integrals of forms on smooth manifolds. If we can manage to cover the manifold with a differentiable singular chain, the form can be pulled piecewise back to the standard simplex on which the integrations can be performed relatively easily, then these integral is summed up to obtain the integral on the manifold. In another approach, we can utilise the partition of unity on the manifold if it exists of course. The Stokes theorem that is one of the corner stones of the theory of integration of exterior forms is proven in Sec. 8.6 on the chains and also on manifolds with boundaries. This theorem matches the integral of the exterior derivative of a form on a

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manifold with the integral of this form on the boundary of this manifold. Sec. 8.7 is concerned with the determination of conservation laws corresponding to exact forms in an ideal that are annihilated by a solution submanifolds. Sec. 8.8 deals with the cohomology groups that are the quotient spaces of closed forms with respect to exact forms and homology groups that are quotient spaces of linear spaces of cycles with respect to linear spaces of cycles which are boundaries of chains. It is then tried to reveal important relationships between these two groups. These relationships connect the structure of closed forms on a manifold to the topological structure of that manifold. In Sec. 8.9, we define the inner product of forms on a Riemannian manifold by using the Hodge dual so that the exterior algebra is transformed into an inner product space. On making use of the structure so established, the properties of the Laplace-de Rham operator and the harmonic forms occupying the null space of this operator are investigated, and then the Hodge-de Rham decomposition theorem is explored. Finally, Sec. 8.10 is devoted to the Poincaré duality unravelling quite an interesting relation between cohomology groups in some kind of manifolds.

8.2. ORIENTABLE MANIFOLDS

We have already defined an orientable manifold on $p.$ 275. Let us hence recall that an m-dimensional manifold M is called an ***orientable manifold*** if we can find a form $\mu \in \Lambda^m(M)$ such that $\mu(p) \neq 0$ at every point $p \in M$. Such a form μ will be called a ***volume form***. Since the module $\Lambda^m(M)$ is 1-dimensional, every non-zero m-form Ω, consequently every new volume form is expressible as a multiple of the chosen volume form μ, namely, as $\Omega = f(p)\mu$ where $f \in \Lambda^0(M)$ and $f(p) \neq 0$ at every point of the manifold .

Let us assume that two volume forms μ_1 and μ_2 are related by an expression $\mu_1(p) = f(p)\mu_2(p)$ where $f \in \Lambda^0(M)$ and $f(p) > 0$ for all $p \in M$. This constitutes an equivalence relation on the set of volume forms because it is readily verified that it is reflexive, symmetric and transitive. Thus the set of volume forms is partitioned into equivalence classes $[\mu]$. An ***orientation*** of the manifold M is defined as an equivalence class $[\mu]$. We call the pair $(M, [\mu])$ as an ***oriented manifold***.

An oriented connected differentiable manifold M can possess only two orientations.

Let Ω and μ be volume forms. Hence, we can write $\Omega = f\mu$ for a non-zero function f. However, because M is connected a function $f \neq 0$ will be either $f(p) > 0$ or $f(p) < 0$ at every point $p \in M$. Thus Ω can only be a member of either the orientation $[\mu]$ or $[-\mu]$.

*The **positive orientation** of a connected manifold M is given by the equivalence class $[\mu]$ while its **negative orientation** by the equivalence class $[-\mu]$.* □

Let $e_1(p), e_2(p), \ldots, e_m(p)$ be a basis of the tangent space $T_p(M)$ and μ be a volume form. If $\mu(e_1, e_2, \ldots, e_m) > 0$, it is so at every point p of a connected manifold and for all equivalent volume forms. Such kind of basis vectors constitutes a ***right frame***. Similarly, if $\mu(e_1, e_2, \ldots, e_m) < 0$, then the basis vectors forms a ***left frame***. Since the form μ vanishes at no points of the manifold, it is evident that *the function $\mu(e_1, e_2, \ldots, e_m)$ cannot change its sign in an oriented manifold*. Hence, when moving on an oriented manifold the chosen basis vectors cannot change their *orientation*, in other words, their right or left characters. We can change the basis $e_1, e_2, \ldots, e_m$ to a basis $e'_1, e'_2, \ldots, e'_m$ through a linear transformation $e'_j = a^i_j e_i$, $i, j = 1, \ldots, m$ where $a^i_j \in \Lambda^0(M)$ and the matrix $\mathbf{A}(p) = \left[a^i_j(p)\right]$ must hold the condition $\det \mathbf{A} \neq 0$. On the other hand, because of the relation

$$\mu(e'_1, e'_2, \ldots, e'_m) = (\det \mathbf{A})\, \mu(e_1, e_2, \ldots, e_m)$$

the change of basis does not affect the right or the left character of bases if $\det \mathbf{A} > 0$. If only $\det \mathbf{A} < 0$, then a change of basis alters the orientation by shifting a right frame to the left one and vice versa.

We can immediately deduct from above the following result: *if the left-right character of a frame of basis vectors of the tangent bundle of a manifold changes when this frame is translated along a closed curve of the manifold as to bring it back to the initial point again, then this manifold is non-orientable*.

A non-connected manifold is still called orientable if its connected components are orientable. However, in each component its orientation can be chosen arbitrarily.

Theorem 8.2.1. *An m-dimensional connected paracompact differentiable manifold M is orientable if and only if there exists an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in \Gamma\}$ on M such that the differentiable transition mapping $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$, induced by the overlapping charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) having local coordinates $\mathbf{x}$ and $\mathbf{y}$, respectively, on the set $U_\alpha \cap U_\beta \neq \emptyset$, has a local representation $y^i = \Phi^i(x^1, x^2, \ldots, x^m)$, $i = 1, 2, \ldots m$ where $\mathbf{x} \in \varphi_\alpha(U_\alpha \cap U_\beta)$ and $\mathbf{y} \in \varphi_\beta(U_\alpha \cap U_\beta)$ possessing a positive Jacobian $J = \det(\partial y^i / \partial x^j)$.*

Let M be an oriented manifold. Hence, there is a volume form μ on M. By taking simple changes in local coordinates that might involve reflections if need be into consideration we may suppose in a chart $(U_\alpha, \varphi_\alpha)$ that $\mu\Big(\dfrac{\partial}{\partial x^1}, \dfrac{\partial}{\partial x^2}, \ldots \dfrac{\partial}{\partial x^m}\Big) > 0$. Such a coordinate system is said to be *positive* *local coordinates*. If the same kind of changes are performed, if necessary, in the chart (U_β, φ_β) to choose positive coordinates there, then the familiar relation

$$0 < \mu\Big(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \dots \frac{\partial}{\partial y^m}\Big) = \det\Big(\frac{\partial y^i}{\partial x^j}\Big)\mu\Big(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots \frac{\partial}{\partial x^m}\Big)$$

requires that $J = \det(\partial y^i/\partial x^j) > 0$.

We now conversely assume that the manifold M has an atlas with the above mentioned properties. Let us assume that $\{(V_a, f_a) : a \in A\}$ be a partition of unity subordinate to that atlas where $\{V_a\}$ is a locally finite open cover of M. *The paracompactness of the manifold M assures solely that such a partition of unity can always be found. In fact, if such a partition of unity on M is contrived, then the theorem turns out to be still valid even if M is not paracompact*. Since every open set V_a belongs to an open set U_{α_a} of a chart, the atlas $\{(V_a, \varphi_a) : a \in A\}$ formed by defining the mapping $\varphi_a = \varphi_{\alpha_a}|_{V_a}$ will satisfy the condition of positive Jacobian. Let us denote the positive local coordinates in the chart (V_a, φ_a) by $x_a^1, x_a^2, \dots, x_a^m$ and introduce a form $\omega \in \Lambda^m(M)$ in the following manner

$$\omega = \sum_a f_a \, dx_a^1 \wedge dx_a^2 \wedge \cdots \wedge dx_a^m$$

where each term can be extended to the entire manifold M if we recall that each f_a vanishes outside its support. Any point $p \in M$ is now located in a chart (V, φ) with local coordinates $x^1, x^2, \dots, x^m$ and for all charts such that $V_a \cap V \neq \emptyset$ we will get $\det(\partial x_a^i/\partial x^j) > 0$. We can thus write

$$\begin{aligned}\omega(p) &= \sum_a f_a(p) \, dx_a^1 \wedge \cdots \wedge dx_a^m \\ &= \sum_a f_a(p) \det(\partial x_a^i/\partial x^j) \, dx^1 \wedge \cdots \wedge dx^m.\end{aligned}$$

On the other hand, we know that all functions in the partition must satisfy $f_a(p) \geq 0$ and at each point $p \in M$ at least one function among them should be positive. Since the factor $\det(\partial x_a^i/\partial x^j)$ is positive by assumption, we conclude that $\omega(p) \neq 0$ at each point $p \in M$. Hence, ω is a volume form and the manifold M is orientable. □

Example 8.2.1. A non-zero n-form, namely, a standard volume form on the manifold $\mathbb{R}^n$ can be defined as $\mu = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$. Different arrangements of the forms dx^i yield either $+\mu$ or $-\mu$. Therefore, $\mathbb{R}^n$ is an oriented manifold. ■

Example 8.2.2. Let us consider the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. This submanifold of $\mathbb{R}^3$ is prescribed by the equation $x^2+y^2+z^2=R^2$. We now define a form $\mu \in \Lambda^2(\mathbb{R}^3)$ as follows

$$\mu = \frac{z\,dx\wedge dy + x\,dy\wedge dz + y\,dz\wedge dx}{(x^2+y^2+z^2)^{1/2}}.$$

It is clear that the form μ vanishes nowhere on $\mathbb{S}^2$. Thus μ is a volume, or in the true sense of the term, an area form. The structure of this form is best illustrated in spherical coordinates. The change of coordinates

$$x = R\sin\theta\cos\phi,\ y = R\sin\theta\sin\phi,\ z = R\cos\theta$$

reduces the volume form μ to

$$\mu = R^2 \sin\theta\, d\theta\wedge d\phi.$$

If delete the poles and choose $\theta \in (0,\pi)$, that is, if we consider two charts as it should be, we observe $\mu \neq 0$ in both charts. We also easily notice that the orientation of basis vectors in $T(\mathbb{S}^2)$ does not change along a closed curve on $\mathbb{S}^2$. Hence, $\mathbb{S}^2$ is an orientable manifold. ■

Example 8.2.3. As an example to non-orientable manifolds, we take the Möbius band introduced in Example 2.8.1 into consideration. We know that the Möbius band is a 2-dimensional submanifold of $\mathbb{R}^3$ prescribed by the parametric equations

$$x = \big(R + v\cos(u/2)\big)\cos u, y = \big(R + v\cos(u/2)\big)\sin u, z = v\sin(u/2)$$

where $u \in [0,2\pi)$ and $v \in [-w,w]$. A basis of the tangent bundle of this manifold can be chosen as the following linearly independent vectors

$$\begin{aligned}
V_u(u,v) &= \frac{\partial}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial}{\partial z}\\
&= -\frac{1}{2}\Big[4R\cos\frac{u}{2} + v(2+3\cos u)\Big]\sin\frac{u}{2}\frac{\partial}{\partial x}\\
&\quad + \frac{1}{4}\Big[4R\cos u + v\Big(\cos\frac{u}{2} + 3\cos\frac{3u}{2}\Big)\Big]\frac{\partial}{\partial y} + \frac{1}{2}v\cos\frac{u}{2}\frac{\partial}{\partial z}\\
V_v(u,v) &= \frac{\partial}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial}{\partial z}\\
&= \cos\frac{u}{2}\cos u\frac{\partial}{\partial x} + \cos\frac{u}{2}\sin u\frac{\partial}{\partial y} + \sin\frac{u}{2}\frac{\partial}{\partial z} = V_v(u)
\end{aligned}$$

that are tangent to the curves $v = \textit{constant}$ and $u = \textit{constant}$, respectively. Since the scalar product can be defined on $\mathbb{R}^3$, we can readily verify that the relation $V_u \cdot V_v = 0$ is satisfied. Thus the basis vectors so chosen are orthogonal. In the particular case $v = 0$, the vector field V_u takes the form

$$V_u(u,0) = R\Big(- \sin u \frac{\partial}{\partial x} + \cos u \frac{\partial}{\partial y}\Big).$$

We thus obtain

$$\begin{aligned} V_u(0,0) &= R\frac{\partial}{\partial y}, \quad V_v(0,0) = \frac{\partial}{\partial x}; \\ V_u(2\pi,0) &= R\frac{\partial}{\partial y}, \quad V_v(2\pi,0) = -\frac{\partial}{\partial x}. \end{aligned}$$

2π in the arguments of the vectors V_u and V_v must be interpreted as the limiting value as $u \to 2\pi$. The above relation clearly show that when we translate the basis vectors at the origin $(0,0)$ along the circle $v = 0$, they change their orientation as we approach again to the origin. Therefore, the Möbius band is not oriented. ■

Example 8.2.4. Let us consider the projective space $\mathbb{RP}^n$. We define open sets U_α and U_β of two charts of the manifold by the rules $x^\alpha \neq 0$, $x^\beta \neq 0$, $\alpha, \beta \in \{1, \dots, n+1\}$ as on p. 87. The local coordinates in those charts are, respectively, given by

$$\xi^i_\alpha = \frac{x^i}{x^\alpha}, \quad \xi^i_\beta = \frac{x^i}{x^\beta}, \quad i = 1, 2, \dots, n$$

We of course take $\xi^\alpha_\alpha = \xi^\beta_\beta = 1$ and $\xi^\alpha_\beta = x^\alpha / x^\beta$. The coordinate transformation in the open set $U_\alpha \cap U_\beta$ is depicted by the relations $\xi^i_\alpha = \xi^i_\beta / \xi^\alpha_\beta$, $\xi^\beta_\alpha = 1/\xi^\alpha_\beta$. Thus, the entries of the Jacobian matrix become

$$\frac{\partial \xi^i_\alpha}{\partial \xi^j_\beta} = \frac{\delta^i_j}{\xi^\alpha_\beta}, \ i \neq \alpha, \beta, \ j \neq \alpha, \beta; \quad \frac{\partial \xi^\beta_\alpha}{\partial \xi^\alpha_\beta} = -\frac{1}{(\xi^\alpha_\beta)^2}.$$

Hence, the determinant is found to be

$$\det\Big(\frac{\partial \xi^i_\alpha}{\partial \xi^j_\beta}\Big) = -\frac{1}{(\xi^\alpha_\beta)^{n+1}}.$$

If n is odd, and consequently, $n+1$ is an even number, then the sign of the determinant remains the same regardless of the sign of ξ^α_β and it can be rendered positive by a suitable change of coordinates. But if n is even, hence $n+1$ is an odd number the determinant changes its sign depending on the sign of ξ^α_β. In conclusion, according to Theorem 8.2.1 we understand that *the projective space* $\mathbb{RP}^n$ *is orientable if* n *is odd and non-orientable if* n *is even.* ■

Let us now consider a k-dimensional submanifold S of an m-dimensional oriented manifold M. According to our assumption there is a volume form $\mu \in \Lambda^m(M)$ on M and an equivalence class $[\mu]$. A basis for the tangent bundle $T(S)$ is given by k locally linearly independent vector fields $V_1, V_2, \dots, V_k$. We can then construct a basis $V_1, \dots, V_m$ for the tangent bundle $T(M)$ by supplying $m-k$ locally linearly independent vector fields $V_{k+1}, V_{k+2}, \dots, V_m$ that do not belong to $T(S)$. That is the reason why we may call these supplementary vectors as the *normal vectors* to the tangent bundle $T(S)$. Since μ is a volume form, we get $\mu(V_1, \dots, V_k, V_{k+1}, \dots, V_m) \neq 0$. Next, we define a form $\mu' \in \Lambda^k(S)$ as

$$\mu' = \mathbf{i}_{V_m} \circ \cdots \circ \mathbf{i}_{V_{k+1}}(\mu).$$

It is clear that this form can only be different from zero on the vectors in $T(S)$. Hence, the restriction $\mu' \in \Lambda^k(S)$ becomes meaningful. On the other hand, we have

$$\mu'(V_1, \dots, V_k) = \mu(V_1, \dots, V_k, V_{k+1}, \dots, V_m) \neq 0$$

so that μ' plays the part of a volume form on the submanifold S induced by the volume form μ. We then say that the submanifold S is ***externally oriented*** by the manifold M. However, this generally may not mean that the manifold S is *oriented internally* in the usual way.

Example 8.2.5. Let us consider the sphere $\mathbb{S}^{n-1}$ as a submanifold of the oriented manifold $\mathbb{R}^n$ determined by the equation $\sum_{i=1}^{n}(x^i)^2 = R^2$. A basis of the tangent bundle $T(\mathbb{S}^{n-1})$ can be chosen as the set of the following vector fields

$$V_i = \frac{\partial}{\partial x^i} - \frac{x^i}{x^n}\frac{\partial}{\partial x^n}, \quad i = 1, \dots, n-1$$

on noting that we can write $\partial x^n/\partial x^i = -x^i/x^n, i = 1, \dots, n-1$ on $\mathbb{S}^{n-1}$. Let us define a vector $V \in T(\mathbb{R}^n)$ by the relation

$$V = \frac{1}{R}\sum_{k=1}^{n} x^k \frac{\partial}{\partial x^k}.$$

Because of the scalar product $V \cdot V_i = (x^i - x^i)/R = 0$, we realise that the non-zero vector V is orthogonal to all vectors V_i. Therefore, it does not belong to $T(\mathbb{S}^{n-1})$. Consequently, the manifold $\mathbb{R}^n$ induces a volume form

$$\begin{aligned}\mu' = \mathbf{i}_V(\mu_n) &= \mathbf{i}_V(dx^1 \wedge \cdots \wedge dx^n)\\ &= \frac{1}{R}\sum_{k=1}^{n}(-1)^{k-1}x^k dx^1 \wedge \cdots \wedge dx^{k-1} \wedge dx^{k+1} \cdots \wedge dx^n\end{aligned}$$

on the submanifold $\mathbb{S}^{n-1}$ by externally orienting it. The structure of this form can be understood better if we employ hyperspherical coordinates. The ***hyperspherical coordinates*** are defined by the relations

$$\begin{aligned}x^1 &= R\cos\phi_1\\ x^2 &= R\sin\phi_1\cos\phi_2\\ x^3 &= R\sin\phi_1\sin\phi_2\cos\phi_3\\ &\vdots\\ x^k &= R\prod_{i=1}^{k-1}\sin\phi_i\cos\phi_k, \quad 1 \le k \le n-1\\ &\vdots\\ x^{n-1} &= R\sin\phi_1 \cdots \sin\phi_{n-2}\cos\phi_{n-1}\\ x^n &= R\sin\phi_1 \cdots \sin\phi_{n-2}\sin\phi_{n-1}\end{aligned}$$

where the conditions $0 \le \phi_1, \ldots, \phi_{n-2} \le \pi$ and $0 \le \phi_{n-1} \le 2\pi$ are to be satisfied. It is then immediately observed that the induced volume form on $\mathbb{S}^{n-1}$ can be written as follows

$$\mu' = R^{n-1}\sin^{n-2}\phi_1\sin^{n-3}\phi_2\cdots\sin\phi_{n-2}\, d\phi_1 \wedge d\phi_2 \wedge \cdots \wedge d\phi_{n-1}.$$

We can further realise that the submanifold $\mathbb{S}^{n-1}$ is also internally oriented by the form μ' if we restrict the coordinates $\phi_1, \ldots, \phi_{n-2}$ into the open interval $(0, \pi)$.

The volume form of the circle $\mathbb{S}^1$ $(n = 2)$ is

$$\mu' = \frac{1}{R}(x^1 dx^2 - x^2 dx^1) = R d\phi$$

whereas the volume form (area form) of the sphere $\mathbb{S}^2$ $(n = 3)$ becomes

$$\begin{aligned}\mu' &= \frac{1}{R^2}(x^1\, dx^2 \wedge dx^3 + x^2\, dx^3 \wedge dx^1 + x^3\, dx^1 \wedge dx^2)\\ &= R^2 \sin\phi_1 d\phi_1 \wedge d\phi_2.\end{aligned}$$

After having obtained the induced volume form, *the area* of the hypersurface $\mathbb{S}^n$ can be found easily by integrating this form. By using the definition of the hyperspherical coordinates, we obtain

$$\begin{aligned}\int_{\mathbb{S}^n}\mu' &= R^n\int_0^\pi\cdots\int_0^\pi\int_0^{2\pi}\sin^{n-1}\phi_1\sin^{n-2}\phi_2\cdots\sin\phi_{n-1}\,d\phi_1\cdots d\phi_{n-1}d\phi_n\\ &= 2\pi R^n\int_0^\pi\cdots\int_0^\pi\sin^{n-1}\phi_1\sin^{n-2}\phi_2\cdots\sin\phi_{n-1}\,d\phi_1\cdots d\phi_{n-1}\neq 0.\end{aligned}$$

In terms of the Gamma function $\Gamma(z)$, the relation

$$\int_0^\pi \sin^k\phi\,d\phi = \frac{\pi^{1/2}\,\Gamma\big(\frac{k+1}{2}\big)}{\Gamma\big(1+\frac{k}{2}\big)},\quad 1\le k\le n-1$$

leads to the result

$$\begin{aligned}S(\mathbb{S}^n) &= \int_{\mathbb{S}^n}\mu'\\ &= 2\pi^{\frac{n+1}{2}}R^n\frac{\Gamma(1)}{\Gamma(\frac{3}{2})}\frac{\Gamma(\frac{3}{2})}{\Gamma(2)}\cdots\frac{\Gamma\big(\frac{k+1}{2}\big)}{\Gamma\big(1+\frac{k}{2}\big)}\frac{\Gamma\big(1+\frac{k}{2}\big)}{\Gamma\big(1+\frac{k+1}{2}\big)}\cdots\frac{\Gamma\big(\frac{n}{2}\big)}{\Gamma\big(1+\frac{n-1}{2}\big)}\\ &= \frac{2\pi^{\frac{n+1}{2}}R^n}{\Gamma\big(\frac{n+1}{2}\big)}.\end{aligned}$$

In fact, since $\Gamma(1)=1, \Gamma(3/2)=\sqrt{\pi}/2, \Gamma(2)=1, \Gamma(5/2)=3\sqrt{\pi}/4,\cdots$ we find that $S(\mathbb{S}^1)=2\pi R,\ S(\mathbb{S}^2)=4\pi R^2,\ S(\mathbb{S}^3)=2\pi^2R^3,\ S(\mathbb{S}^4)=8\pi^2R^4/3$. ■

Example 8.2.6. We have seen in Example 8.2.3 that the Möbius band is non-orientable. We shall now demonstrate that the Möbius band can be externally oriented by the manifold $\mathbb{R}^3$ and an induced volume form may be defined on it. We can now introduce a vector field $W\in T(\mathbb{R}^3)$ that is not situated in the tangent bundle of the Möbius band on resorting to the vectorial product in $\mathbb{R}^3$ as follows

$$\begin{aligned}W = V_u\times V_v &= \frac{1}{2}\Big[2R\cos u + v\Big(\cos\frac{3u}{2}-\cos\frac{u}{2}\Big)\Big]\sin\frac{u}{2}\frac{\partial}{\partial x}\\ &\quad+\frac{1}{2}\Big[2R\cos u\sin\frac{u}{2}+v(\cos u+\sin^2 u)\Big)\Big]\frac{\partial}{\partial y}\\ &\quad-\cos\frac{u}{2}\Big(R+v\cos\frac{u}{2}\Big)\frac{\partial}{\partial z}.\end{aligned}$$

The length of this vector is

$$\|W\|^2 = W\cdot W = \Big[\frac{1}{4}(3+2\cos u)\,v^2 + 2R\,v\cos\frac{u}{2}+R^2\Big].$$

Then by employing the unit vector $N = W/\|W\|$, we obtain the 2-dimensional volume form induced by the volume form in $\mathbb{R}^3$ as

$$\begin{aligned}\mu' &= \mathbf{i}_N(dx \wedge dy \wedge dz)\\ &= \sqrt{\frac{1}{4}(3+2\cos u)\,v^2 + 2R\,v\cos\frac{u}{2} + R^2}\,du \wedge dv\end{aligned}$$

This form will enable us to calculate the area of the Möbius band. ■

8.3. INTEGRATION OF FORMS IN THE EUCLIDEAN SPACE

We want to begin the study of the theory of integration of exterior forms with some rather simple examples that do not differ much from the classical integration. We first consider a differentiable curve C in $\mathbb{R}^n$ and a form $\omega \in \Lambda^1(\mathbb{R}^n)$. We know that the curve C is described by a smooth mapping $\gamma : [a,b] \to \mathbb{R}^n$. The curve C is a 1-dimensional manifold which is a submanifold of $\mathbb{R}^n$ if certain conditions are met and it is prescribed by the equations $x^i = x^i(t), 0 \le t \le 1$. On the curve C, the form $\omega = \omega_i(\mathbf{x})\,dx^i \in \Lambda^1(\mathbb{R}^n)$ is given by the expression

$$\omega(t) = \omega_i\big(\mathbf{x}(t)\big)\,\frac{dx^i}{dt}\,dt.$$

The integral of the 1-form ω on the curve C is a linear operator in the form of $\int_C : \Lambda^1(\mathbb{R}^n) \to \mathbb{R}$ that assigns a real number to this 1-form defined as follows

$$\int_C \omega = \int_0^1 \omega_i\big(\mathbf{x}(t)\big)\,\frac{dx^i}{dt}\,dt.$$

The integral in the right hand side is the well known Riemann integral. Sometimes it is not possible to describe the curve by just one parameter. In such a case, the interval $[a,b]$ is partitioned into subintervals such as $a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b$ making it possible to use a different parametrisation in each interval. If we denote the part of the curve corresponding to the interval $[t_j, t_{j+1}]$ by C_j, the integral may be expressed in the form

$$\int_C \omega = \sum_{j=0}^{m-1}\int_{C_j}\omega = \sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} \omega_i\big(\mathbf{x}(t)\big)\,\frac{dx^i}{dt}\,dt.$$

By generalising this approach, we can define the integral of a form $\omega \in \Lambda^n(\mathbb{R}^n)$ given by $\omega(\mathbf{x}) = \omega_{12\cdots n}(\mathbf{x})\,dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$ as the linear operator $\Lambda^n(\mathbb{R}^n) \to \mathbb{R}$

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} \omega_{12\cdots n}(\mathbf{x})\, dx^1 dx^2 \cdots dx^n$$

assigning a real number to the n-form ω. The right hand side of the above expression is the multiple Riemann integral of the function $\omega_{12\cdots n}(\mathbf{x})$ with n variables. Naturally, in order that the form ω can be integrated, this integral must exist. When the support of the form ω is compact, that is, when the smooth function $\omega_{12\cdots n}(\mathbf{x})$ vanishes outside a closed and bounded subset of $\mathbb{R}^n$, then it is bounded on this set and the integral will definitely exist. *It is obvious that the integral changes sign if we change the orientation of the manifold.*

Let us now consider a k-dimensional submanifold S_k of $\mathbb{R}^n$ prescribed by the parametric equations $x^i = x^i(u^1, \ldots, u^k), i = 1, \ldots, n$. We further assume that the parameters u^α,$\alpha = 1, \ldots, k$ vary in the region $\prod_{\alpha=1}^{k} [a^\alpha, b^\alpha]$ of $\mathbb{R}^k$ where $[a^\alpha, b^\alpha] \subset \mathbb{R}$ is a closed interval and the symbol $\prod$ represent the Cartesian product of intervals. This set will be called a ***closed k-rectangle***. We know that the value of a k-form

$$\omega(\mathbf{x}) = \frac{1}{k!}\, \omega_{i_1\cdots i_k}(\mathbf{x})\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Lambda^k(\mathbb{R}^n)$$

on the submanifold S_k is given by the expression

$$\begin{aligned}\omega(\mathbf{u}) &= \frac{1}{k!}\, \omega_{i_1\cdots i_k}\big(\mathbf{x}(\mathbf{u})\big) \frac{\partial x^{i_1}}{\partial u^{\alpha_1}} \cdots \frac{\partial x^{i_k}}{\partial u^{\alpha_k}}\, du^{\alpha_1} \wedge \cdots \wedge du^{\alpha_k} \\ &= \frac{1}{k!}\, \widetilde{\omega}_{\alpha_1\cdots\alpha_k}(\mathbf{u})\, du^{\alpha_1} \wedge \cdots \wedge du^{\alpha_k} = \widetilde{\omega}_{1\cdots k}(\mathbf{u})\, du^1 \wedge \cdots \wedge du^k\end{aligned}$$

where $a^\alpha \le u^\alpha \le b^\alpha$ with an appropriate ordering of parameters. Consequently, the integral of the k-form ω on the submanifold S_k will be defined as the following multiple Riemann integral

$$\int_{S_k} \omega = \int_{a^1}^{b^1} \int_{a^2}^{b^2} \cdots \int_{a^k}^{b^k} \widetilde{\omega}_{12\cdots k}(\mathbf{u})\, du^1 du^2 \cdots du^k.$$

Generally, the submanifold S_k may not be described by a single parametrisation. In such a case, the domain of integration may be the union of some k-rectangles and the integral is expressed as the sum of integrals over those sets. Naturally, these integrals must be convergent. However, in order to define the integral of a form on a differentiable manifold we shall need to equip the manifold with a much more different formal structure from those introduced sketchily in this section.

Example 8.3.1. The integral of the area form associated with the Möbius band given in Example 8.2.5 can be written as

$$A = \alpha R^2 \int_{-1}^{1}\int_{0}^{2\pi} \sqrt{1 + 2\alpha\nu\cos\frac{u}{2} + \frac{\alpha^2}{4}(3 + 2\cos u)\,\nu^2}\, du d\nu$$

where we defined the variable $\nu = v/w$ and the coefficient $\alpha = w/R$. w is the half width of the band. It is not possible to find the exact value of this integral. So we have to resort to numerical integration. For instance, we find $A = 3.1499R^2$ for $\alpha = 1/4$, $A = 1.2572R^2$ for $\alpha = 1/10$. It is readily verified that $A \to 4\pi\alpha R^2$ when $\alpha \to 0$. ■

8.4. SIMPLICES AND CHAINS

The one of the main building blocks in integrating forms over differentiable manifolds are made up by simplices and chains generated by them in the Euclidean space. Let us consider $k+1$ points $P_0, P_1, \dots P_k$ in the Euclidean space $\mathbb{R}^k$. We suppose that two ordered points (P, Q) in $\mathbb{R}^k$ designate the vector $Q - P$ connecting the first point P to the second point Q. Next, we assume that k vectors $P_1 - P_0, \dots, P_k - P_0$ are linearly independent. Hence, for any point $P \in \mathbb{R}^k$, the vector $P - P_0$ can be represented by

$$P - P_0 = \sum_{i=1}^{k} \xi^i (P_i - P_0),\ \ \xi^i \in \mathbb{R}.$$

If we choose $0 \le \xi^i \le 1$ and $\sum_{i=1}^{k} \xi^i \le 1$ for all $i = 1, \dots, k$, then we observe that the vector $P - P_0$ stays within the *k-dimensional closed and convex polyhedral region* formed by vectors $P_i - P_0, 1 \le i \le k$ as edges. Thus for a point P in this region, we can formally write

$$P = \Big[1 - \sum_{i=1}^{k} \xi^i\Big] P_0 + \sum_{i=1}^{k} \xi^i P_i = \sum_{i=0}^{k} \xi^i P_i \qquad (8.4.1)$$

where we define $\xi^0 = 1 - \sum_{i=1}^{k} \xi^i \ge 0$. Therefore, the conditions $\sum_{i=0}^{k} \xi^i = 1$ and $\xi^i \ge 0$ for all $0 \le i \le k$ will be satisfied. We shall now symbolise the *closed and convex set* produced by the *ordered points* P_0, $P_1, \dots P_k$ as

$$s_k = [P_0, P_1, \dots, P_k] \subset \mathbb{R}^k. \qquad (8.4.2)$$

s_k will be called a ***k-simplex***. Since it is a closed and bounded subset of $\mathbb{R}^k$, s_k becomes clearly a compact subset. If $P \in s_k$, then this point may now be represented by the formal expression (8.4.1). The non-negative real numbers $(\xi^0, \xi^1, \ldots, \xi^k)$ are called the ***barycentric coordinates*** of a point P inside the simplex s_k. The orientation of s_k is specified by the definite order of the successive generating points. We choose the order in (8.4.2) as the positive orientation of the simplex. A different ordering of these points specifies actually the same set. However, the orientation of the simplex may then change. We immediately recognise that if the new ordering is obtained from (8.4.2) by an even permutation of the order of the points in (8.4.2), then the sense in which the points follow each other, thus the orientation of the simplex, remains unchanged whereas if it is an odd permutation the orientation of the simplex is reversed. Let us denote a permutation of the numbers $0, \ldots, k$ by π. Hence, we can obviously write

$$[P_{\pi(0)}, P_{\pi(1)}, \ldots, P_{\pi(k)}] = \operatorname{sgn}(\pi)\,[P_0, P_1, \ldots, P_k]$$

where $\operatorname{sgn}(\pi) = 1$ if π is an even permutation while $\operatorname{sgn}(\pi) = -1$ if it is an odd permutation. If we make use of the coordinates in $\mathbb{R}^k$ and write $P = \{x^\alpha\}, P_i = \{x_i^\alpha\}, \alpha = 1, \ldots k; i = 0, 1, \ldots, k$, then (8.4.1) can be expressed more concretely as

$$x^\alpha = \sum_{i=0}^{k} \xi^i x_i^\alpha,\ \alpha = 1, \ldots k;\ \xi^i \geq 0,\ \sum_{i=0}^{k} \xi^i = 1.$$

The ***face*** opposite to the point P_i in a simplex s_k is defined as the $(k-1)$-simplex obtained by deleting this point from the k-simplex s_k. But in order to render its orientation compatible with the principal simplex, we first put this point into the first position in the ordering so that we obtain

$$[P_i, P_0, \ldots, P_{i-1}, P_{i+1}, \ldots, P_k] = (-1)^i [P_0, \ldots, P_{i-1}, P_i, P_{i+1}, \ldots, P_k]$$

from which we deduce that the faces of a k-simplex are found to be

$$s_{k-1}^i = (-1)^i [P_0, P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_k] \tag{8.4.3}$$

where $i = 0, 1, \ldots, k$. We now define the ***oriented boundary*** of a simplex s_k as a formal sum of its faces:

$$\partial s_k = \sum_{i=0}^{k} s_{k-1}^i. \tag{8.4.4}$$

Let a family of k-simplices $\{s_k^a : a \in A\}$ where A is an index set be given. The *formal* linear combination

$$c_k = \sum_{a \in A} \lambda_a s_k^a \tag{8.4.5}$$

where $\lambda_a \in \mathbb{R}$ is called a ***k-chain*** in the space $\mathbb{R}^k$. Thus appending simplices in a repetitive way if necessary and playing with their orientations, it becomes possible to produce rather complicated geometrical structures. According to this definition, *the boundary of a k-simplex becomes a* $(k-1)$-*chain*. In view of the definition (8.4.5), we may say that *all k-chains on* $\mathbb{R}^k$ *constitutes a linear vector space denoted by* $C_k(\mathbb{R}^k)$.

Let us now take without loss of generality $0 \le j < i \le k$ and consider the faces s_{k-1}^i and s_{k-1}^j of a simplex s_k:

$$\begin{aligned} s_{k-1}^i &= (-1)^i[P_0, P_1, \dots, P_j, \dots, P_{i-1}, P_{i+1}, \dots, P_k], \\ s_{k-1}^j &= (-1)^j[P_0, P_1, \dots, P_{j-1}, P_{j+1}, \dots, P_i, \dots, P_k]. \end{aligned}$$

It then follows from above that the jth face of s_{k-1}^i and the ith face of s_{k-1}^j are expressible as

$$\begin{aligned} s_{k-2}^{ij} &= (-1)^i(-1)^j[P_0, P_1, \dots, P_{j-1}, P_{j+1}, \dots, P_{i-1}, P_{i+1}, \dots, P_k], \\ s_{k-2}^{ji} &= (-1)^j(-1)^{i-1}[P_0, P_1, \dots, P_{j-1}, P_{j+1}, \dots, P_{i-1}, P_{i+1}, \dots, P_k]. \end{aligned}$$

These two sets are identical except for their orientations so that we get

$$s_{k-2}^{ij} = -\, s_{k-2}^{ji}.$$

Consequently, we conclude that

$$\partial(\partial s_k) = \partial^2 s_k = \sum_{i=0}^{k} \sum_{j=0}^{k} s_{k-2}^{ij} = 0. \tag{8.4.6}$$

This means that *the boundary of the boundary of a simplex is zero*.

Some low dimensional simplices can easily be visualised. $s_0 = [P_0]$ is just a point whereas $s_1 = [P_0, P_1]$ corresponds to a vector, $s_2 = [P_0, P_1, P_2]$ to an oriented triangle and $s_3 = [P_0, P_1, P_2, P_3]$ to an oriented tetrahedron. These simplices are displayed in Fig. 8.4.1.

The boundaries of simplices s_1, s_2, s_3 shown in Fig. 8.4.1 are then given by

$$\begin{aligned} \partial s_1 &= [P_1] - [P_0] \\ \partial s_2 &= [P_1, P_2] - [P_0, P_2] + [P_0, P_1] \\ \partial s_3 &= [P_1, P_2, P_3] - [P_0, P_2, P_3] + [P_0, P_1, P_3] - [P_0, P_1, P_2]. \end{aligned}$$

whereas $\partial s_0 = 0$.

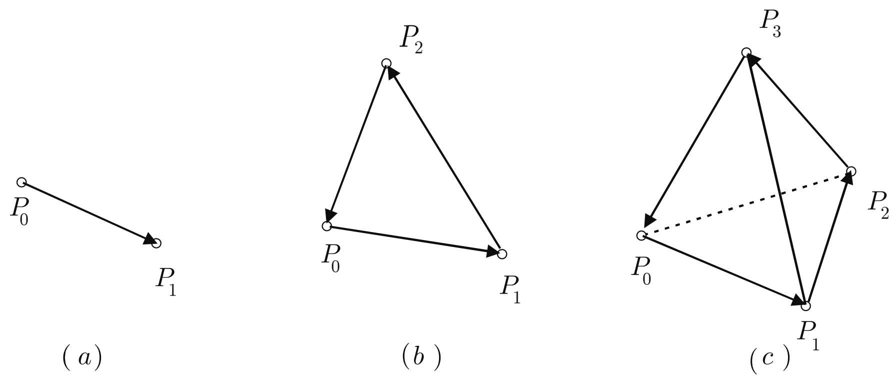

Fig. 8.4.1. Some simplices: (a) s_1-simplex, (b) s_2-simplex, (c) s_3-simplex.

The ***standard k-simplex*** in $\mathbb{R}^k$ is the k-simplex formed by the points $Q_0 = (0,0,\dots,0,0)$, $Q_1 = (1,0,\dots,0,0), Q_2 = (0,1,0,\dots,0),\dots,$ $Q_k = (0,0,\dots,0,1)$. Hence, the ***standard k-simplex*** is the set

$$\mathfrak{s}_k = \{(x^1,x^2,\dots,x^k) \in \mathbb{R}^k : 0 \le x^i \le 1, i = 1,\dots,k; \sum_{i=1}^k x^i \le 1\}.$$

It is straightforward to see that any k-simplex can be generated from the standard k-simplex via an *affine transformation*.

When we are treating in Sec. 8.3 the integration of exterior forms in the Euclidean space we encountered certain subset of $\mathbb{R}^k$ called k-rectangles. We observe at once that these subsets can be reduced to the Cartesian product $[0,1]^k$ called the ***k-cube*** by a very simple scaling transformation of coordinates. We can further show that a k-cube, or a ***box***, is diffeomorphic to the standard k-simplex. We define a mapping $\Phi : [0,1]^k \to \mathfrak{s}_k$ on the set

$$[0,1]^k = \{(y^1,y^2,\dots,y^k) \in \mathbb{R}^k : 0 \le y^i \le 1, i = 1,\dots,k\}$$

by the following relations

$$\begin{aligned} x^1 &= y^1, \\ x^2 &= y^2(1-y^1), \\ x^3 &= y^3(1-y^1)(1-y^2), \\ &\vdots \\ x^k &= y^k(1-y^1)(1-y^2)\cdots(1-y^{k-1}). \end{aligned}$$

We can easily verify that the inverse mapping $\Phi^{-1} : \mathfrak{s}_k \to [0,1]^k$ is given by

$$y^1 = x^1,\ y^2 = \frac{x^2}{1-x^1},\ y^3 = \frac{x^3}{1-x^1-x^2},\ \dots,\ y^k = \frac{x^k}{1-\sum\limits_{i=1}^{k-1} x^i}.$$

Thus the k-cube and the standard k-simplex can be diffeomorphically transformed to each other by means of the function Φ. Consequently, in developing a theory of integration on smooth manifolds, it does not cause a loss of generality to take only standard simplices into consideration. Usually, it proves to be more advantageous to utilise cubes in the numerical evaluation of integrals and simplices in revealing homological properties of the manifold which we will be dealing with later on.

Let us now consider a differentiable manifold M. A ***differentiable singular k-simplex*** σ_k on M is specified by a *smooth* function $f: V \to M$ mapping the standard k-simplex $\mathfrak{s}_k$ in $\mathbb{R}^k$ into the manifold M. In order to secure the differentiability of this function, its domain V is taken as an open neighbourhood of $\mathfrak{s}_k$. Since $\mathfrak{s}_k$ is compact, $\sigma_k = f(\mathfrak{s}_k)$ will necessarily be a compact subset of M. Thus a singular k-simplex on M is designated by the triple $\sigma_k = (\mathfrak{s}_k, V, f)$. The image points $\mathfrak{Q}_i = f(Q_i) \in M, i = 0, \dots, k$ correspond to the vertices of the singular k-simplex. A family of various singular k-simplices on the manifold M is naturally specified by the set $\{\sigma_k^a = (\mathfrak{s}_k, V_a, f_a : \mathfrak{s}_k \subset V_a), a \in A\}$ where A is an index set (Fig. 8.4.2).

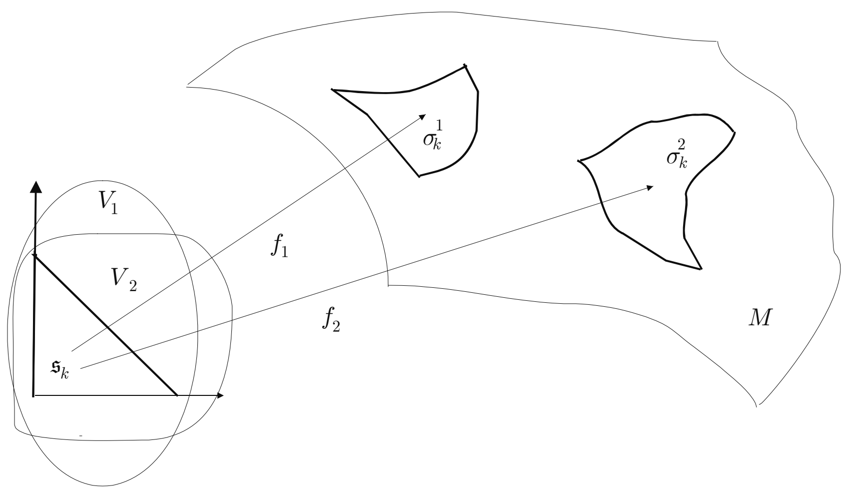

Fig. 8.4.2. Two singular simplices on a manifold M.

With $\lambda_a \in \mathbb{R}$ and $a \in A$, the *formal* linear combination

$$c_k = \sum_{a \in A} \lambda_a \, \sigma_k^a \tag{8.4.7}$$

is called a ***differentiable singular k-chain on M***. It is clear that a singular chain is the union of some singular simplices. If λ is a positive integer, this will imply that we pass over that simplex λ times. If λ is negative the orientation will be reversed. A single simplex σ_k can be regarded as a chain in the form $1 \cdot \sigma_k$. In accordance with this definition, we may say that the sum and multiplication with real numbers of chains is again a chain. Hence, we may think that all k-chains on a manifold M constitute a linear vector space $C_k(M)$.

Let us consider a singular k-simplex σ_k. Let the faces of the standard k-simplex $\mathfrak{s}_k$ be $\mathfrak{s}_{k-1}^i, i = 0, \dots, k$. The restriction of the function f to the set $\mathfrak{s}_{k-1}^i$ is expressible as $f|_{\mathfrak{s}_{k-1}^i} : V_i \to M$ where the subset $V_i \subset \mathbb{R}^{k-1}$ of V is an open neighbourhood of $\mathfrak{s}_{k-1}^i$. We characterise the following sets

$$\sigma_{k-1}^i = f(\mathfrak{s}_{k-1}^i) \;\text{ or }\; \sigma_{k-1}^i = (\mathfrak{s}_{k-1}^i, V_i, f), i = 0, \dots, k \tag{8.4.8}$$

as the ***faces*** of the singular k-simplex σ_k. The ***boundary*** of σ_k is the image of the boundary of $\mathfrak{s}_k$ under the function f. We thus get $\partial\sigma_k = f(\partial\mathfrak{s}_k)$ showing the validity of the commutation relation $\partial f(\mathfrak{s}_k) = f(\partial\mathfrak{s}_k)$. On the other hand, the boundary of σ_k may also be defined as

$$\partial\sigma_k = \sum_{i=0}^{k} \sigma_{k-1}^i. \tag{8.4.9}$$

Hence, it is a singular $(k-1)$-chain. Therefore, the function f must formally satisfy the relation

$$f\Big(\sum_{i=0}^{k} \mathfrak{s}_{k-1}^i\Big) = \sum_{i=0}^{k} f(\mathfrak{s}_{k-1}^i). \tag{8.4.10}$$

The boundary operator $\partial : C_k(M) \to C_{k-1}(M)$ introduced in (8.4.9) can be extended to an arbitrary chain by the following definition

$$\partial c_k = \partial\Big(\sum_{a \in A} \lambda_a \, \sigma_k^a\Big) = \sum_{a \in A} \lambda_a \, \partial\sigma_k^a. \tag{8.4.11}$$

This definition indicates clearly that ∂ is a linear operator. This operator can be applied for $k \geq 1$ without any problem. Since the boundary of a 0-simplex cannot be defined, we adopt the convention $\partial = 0$ on $C_0(M)$. We can state the theorem below concerning the boundary operator.

Theorem 8.4.1. *The boundary operator* ∂ *is linear and we have* $\partial\circ\partial = \partial^2 = 0$ *on* $C_k(M)$.

The linearity of the operator ∂ originates directly from the definition. On the other hand, the image of a zero simplex under f is obviously zero. Thus, we find that

$$\partial^2\sigma_k = \partial f(\partial\mathfrak{s}_k) = f(\partial^2\mathfrak{s}_k) = f(0) = 0.$$

Because of the linearity of the operator ∂, we immediately reach to the conclusion that

$$\partial(\partial c_k) = \partial^2 c_k = 0$$

for any chain. □

If the boundary of a chain c_k is zero, i.e., if we can write $\partial c_k = 0$, this chain is called a ***cycle***. Hence, the boundary of every chain is a cycle.

Let M and N be smooth manifolds and $\phi : M \to N$ be a smooth function. We consider a singular k-simplex $\sigma_k = (\mathfrak{s}_k, V, f)$ on the manifold M. The image of σ_k on the manifold N under the mapping ϕ is the set $\phi(\sigma_k) = \phi\big(f(\mathfrak{s}_k)\big) = (\phi\circ f)(\mathfrak{s}_k)$. But the set $\phi(\sigma_k)$ is a singular k-simplex on N because $f' = \phi\circ f : V \to N$ is a smooth function. In this case, we are led to the result

$$\partial\big(\phi(\sigma_k)\big) = f'(\partial\mathfrak{s}_k) = (\phi\circ f)(\partial\mathfrak{s}_k) = \phi\big(f(\partial\mathfrak{s}_k)\big) = \phi(\partial\sigma_k)$$

implying that the operators ∂ and ϕ commute. So we get the relation

$$\partial\circ\phi = \phi\circ\partial. \tag{8.4.12}$$

Because of the linearity of ∂ *this result will equally be valid for a chain* c_k:

$$\partial\big(\phi(c_k)\big) = \phi(\partial c_k).$$

Let S be an k-dimensional submanifold of an m-dimensional smooth manifold M. The usual coordinates in the standard k-simplex $\mathfrak{s}_k$ in $\mathbb{R}^k$ will be denoted by $u^\alpha, \alpha = 1,\dots,k$. *Let us assume that a singular k-chain* $c_k = \{\sigma_k^a = (\mathfrak{s}_k, V_a, f_a) : a\in A\}$ *can be found as satisfying the following conditions*:

(a). *Each singular k-simplex* σ_k^a *parametrizes a region* $S_a = \sigma_k^a = f_a(\mathfrak{s}_k)$ of S *by* $\mathbf{x} = (\varphi\circ f_a)(\mathbf{u})$ *where* φ *is the homeomorphism in a local chart.*

(b). *We have* $S = \bigcup\limits_{a\in A} S_a$.

(c). *Each* f_a is *injective and the rank of the differential* df_a *is* k. *Furthermore, for every* $a\neq b$ *we have* $f_a(\mathring{\mathfrak{s}}_k)\cap f_b(\mathring{\mathfrak{s}}_k) = \emptyset$. *Hence,* *the singular k-simplices can touch each other solely along their boundaries.*

Then we say that the chain c_k parametrizes the submanifold S by $u^1, \dots, u^k$.

All singular chains under the operator $\partial : C_{k+1}(M) \to C_k(M)$ constitute a ***chain complex*** specified by the following *decreasing* sequence

$$\cdots \longrightarrow C_{k+1}(M) \overset{\partial}{\longrightarrow} C_k(M) \overset{\partial}{\longrightarrow} C_{k-1}(M) \longrightarrow \cdots \quad (8.4.13)$$

because of the fact that $\partial \circ \partial = \partial^2 = 0$. This implies that $\mathcal{R}_k(\partial) \subseteq \mathcal{N}_k(\partial)$. $\mathcal{N}_k(\partial) = \text{Ker}\,(\partial) \subseteq C_k(M)$ is called the space of ***k-cycles***, and $\mathcal{R}_k(\partial) = \text{Im}\,(\partial) \subseteq C_k(M)$ is called the space of ***k-boundaries***.

Let us now consider the dual space $C_k^*(M)$ of the vector space $C_k(M)$. If $f_k \in C_k^*(M)$, then it is a *linear functional* $f_k : C_k(M) \to \mathbb{R}$. Such an f_k will be instumental in creating a ***singular k-cochain on M***. In order to justify this terminology, we shall introduce the ***coboundary operator*** $\mathfrak{d}$ acting on the dual space $C_k^*(M)$ by the relation

$$f_{k+1}(c_{k+1}) = (\mathfrak{d} f_k)(c_{k+1}) = f_k(\partial c_{k+1}) \quad (8.4.14)$$

for all $c_{k+1} \in C_{k+1}(M)$. Obviously $\mathfrak{d} : C_k^*(M) \to C_{k+1}^*(M)$ is a homomorphism and one writes $f_{k+1} = \mathfrak{d} f_k$. Moreover, it is straightforward to see that for any $f_k \in C_k^*(M)$ we obtain

$$(\mathfrak{d} \circ \mathfrak{d}) f_k(c_{k+2}) = \mathfrak{d}^2 f_k(c_{k+2}) = \mathfrak{d} f_k(\partial c_{k+2}) = f_k(\partial^2 c_{k+2}) = 0$$

implying that $\mathfrak{d} \circ \mathfrak{d} = \mathfrak{d}^2 = 0$. Hence, we find that $\mathcal{R}_k(\mathfrak{d}) \subseteq \mathcal{N}_{k+1}(\mathfrak{d})$. This means that all singular cochains under the coboundary operator $\mathfrak{d}$ constitute a ***cochain complex*** given by

$$\cdots \longrightarrow C_{k-1}^*(M) \overset{\mathfrak{d}}{\longrightarrow} C_k^*(M) \overset{\mathfrak{d}}{\longrightarrow} C_{k+1}^*(M) \longrightarrow \cdots. \quad (8.4.15)$$

$\mathcal{N}_k(\mathfrak{d}) = \text{Ker}\,(\mathfrak{d}) \subseteq C_k^*(M)$ is called the space of ***k-cocycles***, and $\mathcal{R}_k(\mathfrak{d}) = \text{Im}\,(\mathfrak{d}) \subseteq C_k^*(M)$ is called the space of ***k-coboundaries***.

8.5. INTEGRATION OF FORMS ON MANIFOLDS

We assume that M is an m-dimensional differentiable manifold. Let us consider a form $\omega \in \Lambda^k(M)$. It is known that this form is expressed in local coordinates as

$$\omega(\mathbf{x}) = \frac{1}{k!}\, \omega_{i_1 \cdots i_k}(\mathbf{x})\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

We shall now try to define the integral of this form on a $k \le m$ dimensional submanifold S of M. To this end, we first assume that there exists a singular k-chain c_k that parametrizes S just as we have depicted at the end of Sec. 8.4. If we recall that a chain is a linear combination of singular simplices, we realise at once that it would be entirely sufficient to define the integral on a single singular k-simplex $\sigma_k = (\mathfrak{s}_k, V, f)$. The smooth function $f : V \to M$ enables us to establish a smooth relationship between the natural coordinates $\mathbf{x} \in \mathbb{R}^m$ and the parameters $\mathbf{u} \in \mathfrak{s}_k \subset \mathbb{R}^k$ in the form $\mathbf{x} = \mathbf{x}(\mathbf{u})$. Let $f^* : \Lambda(M) \to \Lambda(V)$ be the pull-back operator induced by the mapping f. This operator pulls the k-form ω defined on the simplex σ_k back to the form $\omega^* = f^*\omega$ on the standard simplex $\mathfrak{s}_k$ as follows

$$\begin{aligned}\omega^*(\mathbf{u}) &= \frac{1}{k!}\,\omega_{i_1\cdots i_k}\big(\mathbf{x}(\mathbf{u})\big)\frac{\partial x^{i_1}}{\partial u^{\alpha_1}}\cdots\frac{\partial x^{i_k}}{\partial u^{\alpha_k}}\,du^{\alpha_1}\wedge\cdots\wedge du^{\alpha_k}\\ &= \frac{1}{k!}\,\widetilde{\omega}_{\alpha_1\cdots\alpha_k}(\mathbf{u})\,du^{\alpha_1}\wedge\cdots\wedge du^{\alpha_k} = \widetilde{\omega}_{1\cdots k}(\mathbf{u})\,du^1\wedge\cdots\wedge du^k.\end{aligned}$$

The integral on σ_k is now defined by the relation

$$\int_{\sigma_k}\omega = \int_{\mathfrak{s}_k} f^*\omega = \int_{\mathfrak{s}_k}\widetilde{\omega}_{1\cdots k}(\mathbf{u})\,du^1\cdots du^k \in \mathbb{R} \tag{8.5.1}$$

reducing this integral to a multiple Riemann integral on the standard k-simplex $\mathfrak{s}_k$ in the Euclidean space. Since we have assumed that the k-chain $c_k = \sum_{a\in A}\lambda_a\,\sigma_k^a$ parametrizes the submanifold S, the integral of the k-form ω on S is eventually given by the sum

$$\int_S\omega = \int_{c_k}\omega = \sum_{a\in A}\lambda_a\int_{\sigma_k^a}\omega = \sum_{a\in A}\lambda_a\int_{\mathfrak{s}_k^a} f_a^*\omega. \tag{8.5.2}$$

In order this definition to be consistent, we have to show that this integral is independent of the choice of parametrisation of S. Without loss of generality, we may suppose that S is subject to two different parametrisations by two chains $c_k = \{(\mathfrak{s}_k, V_a, f_a) : a \in A\}$ and $c'_k = \{(\mathfrak{s}_k, U_\mathfrak{a}, g_\mathfrak{a}) : \mathfrak{a} \in \mathfrak{A}\}$ with all real coefficients are $\lambda_a = \lambda_\mathfrak{a} = 1$. Because we can write

$$S = \underset{a\in A}{\cup} f_a(V_a) = \underset{\mathfrak{a}\in\mathfrak{A}}{\cup} g_\mathfrak{a}(U_\mathfrak{a})$$

we evidently obtain

$$S = \big(\underset{a\in A}{\cup} f_a(V_a)\big)\cap\big(\underset{\mathfrak{a}\in\mathfrak{A}}{\cup} g_\mathfrak{a}(U_\mathfrak{a})\big) = \underset{a\in A,\mathfrak{a}\in\mathfrak{A}}{\cup} f_a(V_a)\cap g_\mathfrak{a}(U_\mathfrak{a}).$$

Since the mappings f_a and $g_{\mathfrak{a}}$ are injective, they are bijective mappings over their ranges. Consequently, their inverses exist so that one is able to write

$$f_a^{-1} \circ g_{\mathfrak{a}} : (g_{\mathfrak{a}}^{-1} \circ f_a)(V_a) \cap U_{\mathfrak{a}} \to V_a \cap (f_a^{-1} \circ g_{\mathfrak{a}})(U_{\mathfrak{a}})$$

We thus reach to the desired result as follows

$$\begin{aligned}\int_{c_k} \omega &= \sum_{a \in A} \int_{V_a} f_a^* \omega = \sum_{a \in A, \mathfrak{a} \in \mathfrak{A}} \int_{V_a \cap (f_a^{-1} \circ g_{\mathfrak{a}})(U_{\mathfrak{a}})} f_a^* \omega \\ &= \sum_{a \in A, \mathfrak{a} \in \mathfrak{A}} \int_{(g_{\mathfrak{a}}^{-1} \circ f_a)(V_a) \cap U_{\mathfrak{a}}} (f_a^{-1} \circ g_{\mathfrak{a}})^* f_a^* \omega \\ &= \sum_{a \in A, \mathfrak{a} \in \mathfrak{A}} \int_{(g_{\mathfrak{a}}^{-1} \circ f_a)(V_a) \cap U_{\mathfrak{a}}} g_{\mathfrak{a}}^* \circ (f_a^{-1})^* f_a^* \omega = \sum_{\mathfrak{a} \in \mathfrak{A}} \int_{U_{\mathfrak{a}}} g_{\mathfrak{a}}^* \omega = \int_{c_k'} \omega.\end{aligned}$$

We thus realise that the integral of a k-form on a k-dimensional submanifold can be evaluated as the sum of some multiple integrals over a simple standard k-simplex once we manage to parametrize this submanifold by a suitable chain. If the chain is finite, then this procedure does not cause undue difficulties. But if the chain is infinite, we may then have to face up with a serious problem of convergence. In such a case, if the support of the form ω, i.e., the set *supp* $(\omega) = \overline{\{p \in M : \omega(p) \neq 0\}}$ is compact so that it can be covered with finitely many open sets, then surely no problems occur.

Let M, N be smooth manifolds and $\phi : M \to N$ be a smooth mapping. If c_k is a k-chain on M, we know that $c_k' = \phi(c_k)$ is a k-chain on N. Hence, if $\omega \in \Lambda^k(N)$ we immediately observe that

$$\int_{c_k'} \omega = \int_{\phi(c_k)} \omega = \int_{c_k} \phi^* \omega. \tag{8.5.3}$$

Example 8.5.1. We want to calculate the integral of the form $\omega = xyz^2 dx \wedge dy \wedge dz \in \Lambda^3(\mathbb{R}^3)$ on the standard 3-simplex $\mathfrak{s}_3$. On using the familiar method of calculation of multiple integrals, we obtain

$$\begin{aligned}\int_{\mathfrak{s}_3} \omega &= \int_{\mathfrak{s}_3} xyz^2 dx dy dz = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} xyz^2 dz \\ &= \int_0^1 dx \int_0^{1-x} \frac{1}{3} xy(1-x-y)^3 dy \\ &= \frac{1}{60} \int_0^1 x(1-x)^5 dx = \frac{1}{2520}\end{aligned}$$

■

Example 8.5.2. We shall calculate the integral of the 2-form

$$\omega = (x^2 + z^2)dx \wedge dy + (x^2 + y^2)dy \wedge dz + (y^2 + z^2)dz \wedge dx \in \Lambda^2(\mathbb{R}^3)$$

on the 2-chain $c_2 = \sum_{i=0}^{3} \sigma_2^i$ made up of the *faces* of the tetrahedron formed by the points $Q_0 = (0,0,0)$, $Q_1 = (a,0,0)$, $Q_2 = (0,b,0)$, $Q_3 = (0,0,c)$ in $\mathbb{R}$. The simplices of the chain are given by

$$\sigma_2^0 = [Q_1, Q_2, Q_3], \quad \sigma_2^1 = -[Q_0, Q_2, Q_3],$$
$$\sigma_2^2 = [Q_0, Q_1, Q_3], \quad \sigma_2^3 = -[Q_0, Q_1, Q_2].$$

We define the standard 2-simplex by

$$\mathfrak{s}_2 = \{(u,v) \in \mathbb{R}^2 : 0 \le u, v \le 1,\ u + v \le 1\}.$$

Then the functions $f_i(u,v) = (x,y,z), i = 0,1,2,3$ identifying singular 2-simplices σ_2^i become

$$\begin{aligned} f_1(u,v) &= (0, bu, cv), \\ f_2(u,v) &= (au, 0, cv), \\ f_3(u,v) &= (au, bv, 0) \\ f_0(u,v) &= (au, bv, c(1-u-v)). \end{aligned}$$

Indeed, we can readily verify that these functions provide the following mappings

$$\begin{aligned} f_1 &: \mathfrak{s}_2 \to [Q_0, Q_2, Q_3], \\ f_2 &: \mathfrak{s}_2 \to [Q_0, Q_1, Q_3], \\ f_3 &: \mathfrak{s}_2 \to [Q_0, Q_1, Q_2], \\ f_0 &: \mathfrak{s}_2 \to [Q_1, Q_2, Q_3]. \end{aligned}$$

When we pull the form ω from those faces back to $\mathfrak{s}_2$, we obtain the forms

$$\begin{aligned} f_1^*\omega &= b^3c\, u^2 du \wedge dv, \\ f_2^*\omega &= -ac^3v^2 du \wedge dv, \\ f_3^*\omega &= a^3b\, u^2 du \wedge dv, \\ f_0^*\omega &= \big[a^2b(a+c)u^2 + b^2c(a+b)v^2 + ac^2(c+b)(1-u-v)^2\big] du \wedge dv. \end{aligned}$$

We thus find

$$\int_{\sigma_2^1} \omega = -b^3c \int_{u=0}^{1} \int_{v=0}^{1-u} u^2 du dv = -\frac{b^3c}{12},$$

$$\int_{\sigma_2^2}\omega = -ac^3\int_{u=0}^{1}\int_{v=0}^{1-u} v^2dudv = -\frac{ac^3}{12},$$

$$\int_{\sigma_2^3}\omega = -a^3b\int_{u=0}^{1}\int_{v=0}^{1-u} u^2dudv = -\frac{a^3b}{12},$$

$$\int_{\sigma_2^0}\omega =$$

$$\int_{u=0}^{1}\int_{v=0}^{1-u}\big[a^2b(a+c)u^2+b^2c(a+b)v^2+c^2a(c+b)(1-u-v)^2\big]dudv$$

$$=\frac{1}{12}\big[a^3b+a^2bc+b^3c+ac(b^2+bc+c^2)\big]$$

whence we arrive at the result

$$\int_{c_2}\omega = \frac{abc}{12}(a+b+c). \qquad \blacksquare$$

The approach we have followed above to evaluate the integral of a k-form on a k-dimensional manifold consists of decomposing a complicated region to much simpler regions by means of k-chains and summing all integrals calculated relatively easily on those regions. We shall now discuss a second approach that may prove to be more effective in certain cases. In that approach, we decompose the form into some forms that vanish outside of some simple regions covering the manifold and we add the integrals of these forms together to obtain the final result.

We consider a k-dimensional smooth submanifold S of an m-dimensional smooth manifold M. Let $\mathcal{A}_M = \{(U'_\lambda, \varphi'_\lambda) : \lambda \in \Lambda\}$ be an atlas of M. We know that this atlas induces an atlas $\mathcal{A}_S = \{(U_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$ on S [*see p.* 105] where $U_\lambda = U'_\lambda \cap S, \varphi_\lambda = \varphi'_\lambda \circ i : U_\lambda \to \mathbb{R}^k$ and $\mathcal{I} : S \to M$ is the inclusion mapping $\mathcal{I}(p) = p$ for all $p \in S$. Let us now assume that there exists a partition of unity $\{V_a, f_a : a \in A\}$ on the submanifold S subordinate to the atlas $\mathcal{A}_S$ [*see p.* 62]. Each set V_a belongs to an open set U_{λ_a} of a chart of this atlas. We now consider a form $\omega \in \Lambda^k(M)$ and try to evaluate its integral over S. Since the partition of unity implies that $\sum_{a\in A} f_a(p) = 1$ for all $p \in S$, we can write

$$\omega|_S = \omega(p) = \sum_{a\in A}\omega_a(p), \quad \omega_a(p) = f_a(p)\,\omega(p) \in \Lambda^k(U_{\lambda_a}),$$
$$\textit{supp}\,(\omega_a) \subseteq \textit{supp}\,(f_a) \subset V_a \subseteq U_{\lambda_a}.$$

We thus obtain

$$\int_S \omega = \sum_{a\in A} \int_{supp\,(f_a)} f_a\,\omega = \sum_{a\in A} \int f_a\,\omega. \tag{8.5.4}$$

If the sum at the right hand side is convergent, the integral of the form ω on S is expressed as the sum integrals of forms that vanish outside of certain regions. When S is a paracompact manifold, we had mentioned before [*see p.* 95] that a partition of unity can be found subordinate to every atlas. We know that there exist merely finitely many functions f_a in a neighbourhood of each point $p \in S$. However, if $\mathcal{A}_S$ does not contain a finite number of open sets, infinitely many terms may nevertheless be involved in the sum and we naturally have to face up with a problem of convergence. When the support of the form ω on the submanifold S is compact, it can always be covered by a finitely many open sets, so the expression (8.5.4) becomes a finite sum in this case. Therefore, the problem of convergence disappears naturally. If the submanifold S itself is compact, this situation will always occur.

In order that the integral of a form given by (8.5.4) has a meaning, it should not be dependent on the chosen atlas and the partition of unity. To show this, let us consider two atlases and their two charts $\{U_\lambda, \varphi_\lambda : \lambda \in \Lambda\}$ and $\{W_\gamma, \psi_\gamma : \gamma \in \Gamma\}$ on S and two partitions of unity $\{V_a, f_a : a \in A\}$ and $\{Z_b, g_b : b \in B\}$ on S subordinate to those atlases, respectively. Since $S = \bigcup_{\lambda\in\Lambda} U_\lambda = \bigcup_{\gamma\in\Gamma} W_\gamma$, we can obviously write $S = \bigcup_{\lambda\in\Lambda,\gamma\in\Gamma} (U_\lambda \cap W_\gamma)$. Thus, the family $\{U_\lambda \cap W_\gamma : \lambda \in \Lambda, \gamma \in \Gamma\}$ is likewise an open cover of S. We then realise that $\{V_a \cap Z_b, f_a g_b : a \in A, b \in B\}$ is the partition of unity subordinate to the open cover $\{U_\lambda \cap W_\gamma\}$. Accordingly, the integral of the form ω can be written in two different ways as follows

$$\begin{aligned}\int_S \omega &= \sum_{a\in A} \int f_a\omega = \sum_{a\in A}\sum_{b\in B} \int f_a g_b\,\omega \\ &= \sum_{b\in B} \int g_b\omega = \sum_{b\in B}\sum_{a\in A} \int g_b f_a\omega.\end{aligned}$$

since we can write $\omega(p) = \sum_{a\in A} f_a(p)\omega(p) = \sum_{b\in B} g_b(p)\omega(p)$.

As a matter of fact, if the above sums converge absolutely, then we are allowed to interchange freely the order of summations in the above expressions. Furthermore, if the support of the form ω is compact, then this will happen naturally. Hence, if we consider two partitions of unity we obtain

$$\int_S \omega = \sum_{a\in A} \int f_a\,\omega = \sum_{b\in B} \int g_b\,\omega.$$

Hence, the integral is independent of the chosen charts and partitions of unity subordinate to them.

8.6. THE STOKES THEOREM

We had defined a manifold with boundary in *pp*. 90-93. We had seen there that the boundary ∂S of such a k-dimensional differentiable manifold S is a $(k-1)$-dimensional differentiable manifold and the local coordinates $(x^1, x^2, \dots, x^{k-1}, x^k) \in \mathbb{R}^k$ can be so chosen that the boundary in $\mathbb{R}^{k-1}$ is represented by $(x^1, x^2, \dots, x^{k-1}, x^k = 0)$. The Stokes theorem that is rather simple looking at a first glance but having a great potential in provoking very important developments [it is commemorated by the name of English mathematician Sir George Gabriel Stokes (1819-1903) who utilised a similar theorem in the context of classical vector analysis[1]] states that the following relation

$$\int_S d\omega = \int_{\partial S} \omega \tag{8.6.1}$$

is valid for every form $\omega \in \Lambda^{k-1}(S)$. This theorem is very important because it helps derive the classical theorems of Green-Gauss and Kelvin-Stokes as well as the fundamental theorem of calculus. It also links topology and analysis because the boundary operator ∂ on the right hand side is purely geometric whereas the integral and the exterior derivative on the left hand side are purely analytic. We shall first prove this theorem for a manifold with boundary prescribed by a singular chain c_k whose boundary is given by ∂c_k.

Theorem 8.6.1 (The Stokes Theorem on Chains). *Let M be a differentiable manifold. We assume that there exists a k-chain $c_k \in C_k(M)$ and consider an exterior differential form $\omega \in \Lambda^{k-1}(M)$. We then have the equality*

[1]It was actually Sir William Thomson (Lord Kelvin) (1824-1907) who discovered this relation within the context of classical vector analysis and communicated it to Stokes in July 1850. However, Stokes is identified with this theorem because he asked its proof on 1854 Smith's Prize examination in Cambridge University. It is not known whether the students were able to answer that question. That is the reason why some authors call this theorem as the Kelvin-Stokes theorem.

$$\int_{c_k} d\omega = \int_{\partial c_k} \omega$$

provided that the integrals converge.

For a chain given by $c_k = \sum_a \lambda_a \sigma_k^a$, its boundary is expressed as $\partial c_k = \sum_a \lambda_a \, \partial\sigma_k^a$. Hence, it would suffice to show that the above relation is valid for a single singular k-simplex σ_k. Since $\sigma_k = (\mathfrak{s}_k, V, f)$ or, in short, $\sigma_k = f(\mathfrak{s}_k)$, we can write

$$\int_{\sigma_k} d\omega = \int_{\mathfrak{s}_k} f^* d\omega = \int_{\mathfrak{s}_k} d(f^*\omega)$$

on resorting to the pull-back operation where we make use of the property $f^* \circ d = d \circ f^*$ in accordance with Theorem 5.8.2. The form $\theta = f^*\omega \in \Lambda^{k-1}(\mathbb{R}^k)$ will be defined on an open neighbourhood V of the standard k-simplex $\mathfrak{s}_k$ in $\mathbb{R}^k$. Let us now denote the local coordinates in $\mathbb{R}^k$ by $u^1, \dots, u^k$. Hence, the form θ becomes expressible *in terms of its essential components* as

$$\begin{aligned}\theta &= \sum_{i=1}^{k} (-1)^{i-1} \theta_i(\mathbf{u}) \, du^1 \wedge \cdots \wedge du^{i-1} \wedge du^{i+1} \wedge \cdots \wedge du^k \\ &= \sum_{i=1}^{k} \vartheta_i\end{aligned}$$

where we have obviously introduced the forms $\vartheta_i \in \Lambda^{k-1}(\mathbb{R}^k), i = 1, 2, \dots, k$ as follows

$$\vartheta_i = (-1)^{i-1} \theta_i(\mathbf{u}) \, du^1 \wedge \cdots \wedge du^{i-1} \wedge du^{i+1} \wedge \cdots \wedge du^k.$$

The factor $(-1)^{i-1}$ is inserted for convenience. Thus the exterior derivative $d\theta$ may be expressed in the following manner

$$\begin{aligned}d\theta &= \sum_{i=1}^{k} (-1)^{i-1} \frac{\partial \theta_i}{\partial u^j} \, du^j \wedge du^1 \wedge \cdots \wedge du^{i-1} \wedge du^{i+1} \wedge \cdots \wedge du^k \\ &= \frac{\partial \theta_i}{\partial u^i} \, du^1 \wedge \cdots \wedge du^k = \sum_{r=1}^{k} \frac{\partial \theta_{\underline{r}}}{\partial u^{\underline{r}}} du^1 \wedge \cdots \wedge du^r \wedge \cdots \wedge du^k\end{aligned}$$

where the summation convention is suspended as usual on underscored indices. Therefore, we can write

$$\int_{\mathfrak{s}_k} d\theta = \sum_{r=1}^{k} \int_{\mathfrak{s}_k} \frac{\partial \theta_{\underline{r}}}{\partial u^{\underline{r}}}\, du^1 \cdots du^r \cdots du^k. \tag{8.6.2}$$

These integrals can now easily be evaluated by consulting to the standard simplex represented in Fig. 8.6.1.

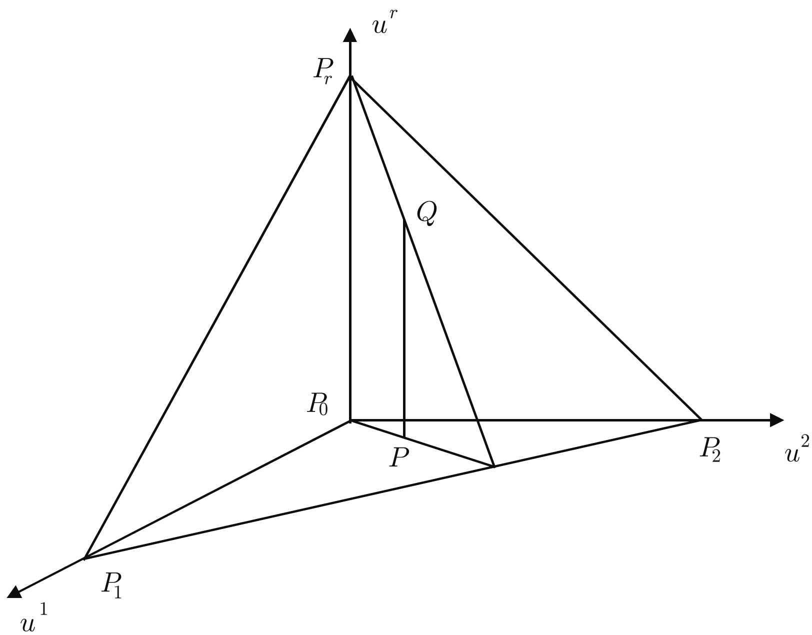

Fig. 8.6.1. Standard simplex $\mathfrak{s}_k$ positioned with respect to the preferred variable u^r.

In fact, on recalling the relation

$$[P_0, P_1, \dots, P_{r-1}, P_{r+1}, \dots, P_k] = (-1)^r \mathfrak{s}_{k-1}^r$$

[*see* (8.4.3)], we obtain

$$\begin{aligned} &\int_{\mathfrak{s}_k} \frac{\partial \theta_{\underline{r}}}{\partial u^{\underline{r}}}\, du^1 \cdots du^{r-1} du^r du^{r+1} \cdots du^k \\ &\qquad = (-1)^r \int_{\mathfrak{s}_{k-1}^r} \big[\theta_r(Q) - \theta_r(P)\big]\, du^1 \cdots du^{r-1} du^{r+1} \cdots du^k \end{aligned} \tag{8.6.3}$$

where we define on relevant faces

$$\theta_r(P) = \theta_r(u^1, \cdots, u^{r-1}, 0, u^{r+1}, \cdots, u^k),$$

$$\theta_r(Q) = \theta_r\Big(u^1, \cdots, u^{r-1}, 1 - \sum_{i=1, i\neq r}^{k} u^i, u^{r+1}, \cdots, u^k\Big).$$

Since $u^r = 0$ on the face $\mathfrak{s}^r_{k-1}$, we get $du^r = 0$ there and it follows from the definition of the form θ that only one term in the expression for θ survives on $\mathfrak{s}^r_{k-1}$:

$$\theta|_{\mathfrak{s}^r_{k-1}} = (-1)^{r-1}\theta_r(\mathbf{u}|_{u^r=0})\, du^1 \wedge \cdots \wedge du^{r-1} \wedge du^{r+1} \wedge \cdots \wedge du^k.$$

We can thus write

$$(-1)^{r-1}\int_{\mathfrak{s}^r_{k-1}} \theta_r(P)\, du^1 \cdots du^{r-1} du^{r+1} \cdots du^k = \int_{\mathfrak{s}^r_{k-1}} \theta. \qquad (8.6.4)$$

On the other hand, on projecting the integral of the form ϑ_r on the face $\mathfrak{s}^0_{k-1} = [P_1, \ldots, P_{r-1}, P_r, P_{r+1}, \ldots, P_k]$ in the direction of the preferred u^r-axis, we find that

$$\begin{aligned}\int_{\mathfrak{s}^0_{k-1}} \vartheta_r &\\ &= \int_{[P_1,\ldots,P_{r-1},P_0,P_{r+1},\ldots,P_k]} (-1)^{r-1}\theta_r(Q)\, du^1 \cdots du^{r-1} du^{r+1} \cdots du^k \\ &= \int_{[P_0,P_1,\ldots,P_{r-1},P_{r+1},\ldots,P_k]} \theta_r(Q)\, du^1 \cdots du^{r-1} du^{r+1} \cdots du^k \\ &= (-1)^r \int_{\mathfrak{s}^r_{k-1}} \theta_r(Q)\, du^1 \cdots du^{r-1} du^{r+1} \cdots du^k. \qquad (8.6.5)\end{aligned}$$

In order to facilitate the computation of the integral in the fourth line above, let us introduce the change of variables $(u^1, u^2, \ldots, u^{r-1}, u^{r+1}, \ldots, u^k) \to (v^1, \ldots, v^{k-1})$ through the relations

$$\begin{aligned} &v^1 = u^2, \ldots, v^{r-2} = u^{r-1}, \;\; v^{r-1} = 1 - \sum_{i=1, i\neq r}^{k} u^i, \\ &v^r = u^{r+1}, \ldots, v^{k-1} = u^k. \end{aligned}$$

We readily observe that the inverse relations are then given by

$$u^1 = 1 - \sum_{i=1}^{k-1} v^i, u^2 = v^1, \ldots u^{r-1} = v^{r-2}, u^{r+1} = v^r, \ldots, u^k = v^{k-1}.$$

The relation

$$du^1 \wedge \cdots \wedge du^{r-1} \wedge du^{r+1} \wedge \cdots \wedge du^k = (-1)^{r-1} dv^1 \wedge dv^2 \wedge \cdots \wedge dv^{k-1}$$

implies that the Jacobian of the transformation is $J = (-1)^{r-1}$. We thus get $|J| = 1$ and

$$du^1 \cdots du^{r-1} du^{r+1} \cdots du^k = dv^1 dv^2 \cdots dv^{k-1}.$$

Since $0 \leq u^i \leq 1$, $\sum_{i=1}^{k} u^i \leq 1$ in the standard k-simplex $\mathfrak{s}_k$, the new variables $v^1, \ldots, v^{k-1}$ will evidently satisfy the conditions $\sum_{i=1}^{k-1} v^i \leq 1$ and $0 \leq v^i \leq 1$, $i = 1, \ldots, k-1$. Therefore, they depict the standard $(k-1)$-simplex $\mathfrak{s}_{k-1}$ in $\mathbb{R}^{k-1}$. Hence, we can write

$$\begin{aligned} \int_{\mathfrak{s}_{k-1}^r} \theta_r(Q)\, du^1 \cdots du^{r-1} du^{r+1} \cdots du^k & \\ &= \int_{\mathfrak{s}_{k-1}} \overline{\theta}_r(v^1, \ldots, v^{k-1})\, dv^1 \cdots dv^{k-1} \\ &= \int_{\mathfrak{s}_{k-1}} \theta_r\Big(1 - \sum_{i=1}^{k-1} v^i, v^1, \ldots, v^{k-1}\Big)\, dv^1 \cdots dv^{k-1}. \end{aligned}$$

On inserting the expressions (8.6.4) and (8.6.5) into the relations (8.6.3) and (8.6.2), we reach to the conclusion

$$\int_{\mathfrak{s}_k} d\theta = \sum_{r=1}^{k} \int_{\mathfrak{s}_{k-1}^r} \theta + \sum_{r=1}^{k} \int_{\mathfrak{s}_{k-1}^0} \vartheta_r = \sum_{r=0}^{k} \int_{\mathfrak{s}_{k-1}^r} \theta = \int_{\partial \mathfrak{s}_k} \theta.$$

If we take the relations (8.4.12) and (8.5.3) into consideration, the above equality leads to the following expression

$$\int_{\sigma_k} d\omega = \int_{\partial \sigma_k} \omega$$

whence we deduce the Stokes theorem on k-chains in the following form

$$\int_{c_k} d\omega = \int_{\partial c_k} \omega \tag{8.6.6}$$

where $\omega \in \Lambda^{k-1}(M)$. □

If the chain c_k is the boundary of a chain b_{k+1}, i.e., if ∂b_{k+1} then $\partial c_k = \partial^2 b_{k+1} = 0$ and consequently, we obtain

$$\int_{c_k} d\omega = \int_{\partial b_{k+1}} d\omega = \int_{\partial^2 b_{k+1}} \omega = 0.$$

Similarly, if a chain c_k is a cycle, we then have $\partial c_k = 0$ and we clearly find this time

$$\int_{c_k} d\omega = 0.$$

On the other hand, if ω is a *closed form*, namely, if $d\omega = 0$, then on the boundary of every chain, we find

$$\int_{\partial c_k} \omega = 0.$$

But, we have to warn that satisfaction of this condition on the boundary of every k-chain does not generally mean that the $(k-1)$-form ω is closed.

If the difference of two k-chains is the boundary of a $(k+1)$-chain, that is, if $c_k - c'_k = \partial b_{k+1}$, we then get $\partial c_k - \partial c'_k = \partial(c_k - c'_k) = \partial^2 b_{k+1} = 0$. Hence, the difference of such kind of chains is a cycle and the relation (8.6.6) yields

$$\int_{c_k} d\omega = \int_{c'_k} d\omega.$$

On the other hand, if $\omega \in \Lambda^k(M)$, we then find

$$\int_{c_k} \omega - \int_{c'_k} \omega = \int_{c_k - c'_k} \omega = \int_{\partial b_{k+1}} \omega = \int_{b_{k+1}} d\omega$$

Thus, if ω is a closed form, i.e., if $d\omega = 0$, we also observe the following equality for this sort of chains

$$\int_{c_k} \omega = \int_{c'_k} \omega.$$

If $\omega \in \Lambda^{k-1}(M)$ is an *exact form*, we have to write $\omega = d\theta$ and (8.6.6) leads to the identity

$$0 = \int_{c_k} d^2\theta = \int_{\partial c_k} d\theta = \int_{\partial^2 c_k} \theta = 0.$$

Example 8.6.1. We now want to evaluate the integral in Example 8.5.2 by means of the relation

$$\sum_{i=0}^{3}\int_{\sigma_2^i}\omega = \int_{\partial\mathfrak{s}_3}\omega = \int_{\mathfrak{s}_3} d\omega.$$

The integral of the form

$$d\omega = 2(x+y+z)\,dx\wedge dy\wedge dz$$

on $\mathfrak{s}_3$ can be calculated as

$$\begin{aligned}\int_{\mathfrak{s}_3} d\omega &= 2\int_{x=0}^{a}\int_{y=0}^{-(b/a)x+b}\int_{z=0}^{c[1-(x/a)-(y/b)]}(x+y+z)\,dxdydz\\ &= \frac{abc}{12}(a+b+c).\end{aligned}$$

We thus arrive at the same result. ■

Let us now consider a k-dimensional differentiable manifold S with a boundary ∂S. We know that we are able to choose the local coordinates $(x^1, x^2, \dots, x^k)$ in a chart in such a way that $x^k = 0$ defines the boundary ∂S. Thus local coordinates of any point at the boundary are then given by $(x^1, x^2, \dots, x^{k-1})$. Let the vectors $e_1, e_2, \dots, e_{k-1}$ be a local basis for the tangent bundle $T(\partial S)$. Two vectors that do not belong to $T(\partial S)$ are $\partial/\partial x^k$ and $-\partial/\partial x^k$. The former vector is called as the ***interior normal*** of the boundary ∂S while the latter as the ***exterior normal***. We assume that S is positively oriented by the volume form $\mu_k = dx^1\wedge\cdots\wedge dx^k$ so that we have $\mu_k(\partial/\partial x^1, \dots, \partial/\partial x^k) > 0$. We shall now adopt the convention that the $(k-1)$-dimensional boundary manifold ∂S will be ***positively oriented*** with respect to its exterior normal if $\mu_k(-\partial/\partial x^k, e_1, e_2, \dots, e_{k-1}) > 0$. We now propose the following form of the Stokes theorem for smooth manifolds with boundary.

Theorem 8.6.2 (Stokes' Theorem on Manifolds with Boundary). *Let S be a k-dimensional smooth manifold with boundary and $\omega\in\Lambda^{k-1}(S)$ be an exterior form with a compact support. If $\mathcal{I}:\partial S\to S$ is the inclusion mapping identifying boundary points as points of the manifold, then the form $\mathcal{I}^*\omega\in\Lambda^{k-1}(\partial S)$ will satisfy the relation*

$$\int_{\partial S}\mathcal{I}^*\omega = \int_S d\omega \quad \textit{or in short} \quad \int_{\partial S}\omega = \int_S d\omega. \tag{8.6.7}$$

The manifold ∂S is supposed to be positively oriented.

Let us first assume that the support D_ω of the form ω lies within a chart (U, φ) whose local coordinates are $(x^1, x^2, \dots, x^k)$. Thus the form ω can be represented as

$$\omega = \sum_{i=1}^{k} (-1)^{i-1} \omega_i(\mathbf{x})\, dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^k$$

if $\mathbf{x} \in \varphi(D_\omega) \subset \mathbb{R}^k$ and $\omega = 0$ if $\mathbf{x} \notin \varphi(D_\omega)$. We shall further assume that $x^k = 0$ on the boundary ∂S. In this case, the exterior derivative of the form ω can be written as

$$d\omega = \sum_{r=1}^{k} \frac{\partial \omega_{\underline{r}}}{\partial x^{\underline{r}}} dx^1 \wedge \cdots \wedge dx^r \wedge \cdots \wedge dx^k \in \Lambda^k(S)$$

as we have attested previously. Next, we have to distinguish two different situations.

(i). Let $\partial S \cap U = \emptyset$ so that we obviously obtain

$$\int_{\partial S} \omega = 0.$$

On the other hand, the set $\varphi(D_\omega) \subset \varphi(U) = V$ in $\mathbb{R}^k$ is closed and bounded because it is a compact set being the image of a compact set under a homeomorphism. Consequently, we can assume that $\varphi(D_\omega) \subset K_k$ where the k-dimensional ***box*** K_k is actually a k-rectangle defined by $K_k = \prod_{r=1}^{k} [a^r, b^r]$, $0 < a^r < b^r < \infty$. At all of the end points of the box the following conditions will evidently be satisfied since the support of ω is supposed to be compact:

$$\omega_r(x^1, \ldots, a^r, \ldots, x^k) = \omega_r(x^1, \ldots, b^r, \ldots, x^k) = 0, \quad 1 \le r \le k.$$

We thus find that

$$\int_S d\omega = \int_{\varphi(D_\omega)} d\omega = \sum_{r=1}^{k} \int_{K_{k-1}} \Big[\int_{a^r}^{b^r} \frac{\partial \omega_{\underline{r}}}{\partial x^{\underline{r}}}\, dx^r dx^1 \cdots dx^{r-1} dx^{r+1} \cdots dx^k$$

$$= \sum_{r=1}^{k} \int_{K_{k-1}} \omega_r(x^1, \ldots, x^r, \ldots, x^k) \Big|_{x^r = a^r}^{x^r = b^r} dx^1 \cdots dx^{r-1} dx^{r+1} \cdots dx^k = 0$$

which proves that the relation (8.6.7) will hold in this case.

(ii). Let $\partial S \cap U \neq \emptyset$. In this case, with $x^k = 0$ on ∂S we obtain

$$\mathcal{I}^* \omega = (-1)^{k-1} \omega_k(x^1, x^2, \ldots, x^{k-1}, 0)\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{k-1}.$$

But, we have to take now $0 = a^k < b^k < \infty$ in the box K_k containing the image $\varphi(D_\omega)$ of the support of the ω. Hence, this time we get

$$\int_S d\omega = \int_{\varphi(D_\omega)} d\omega = \sum_{r=1}^{k} \int_{K_{k-1}} \Big[\int_{a^r}^{b^r} \frac{\partial \omega_{\underline{r}}}{\partial x^{\underline{r}}}\, dx^r dx^1 \cdots dx^{r-1} dx^{r+1} \cdots dx^k$$

$$= \sum_{r=1}^{k} \int_{K_{k-1}} \omega_r(x^1, \dots, x^r, \dots, x^k)\big|_{x^r=a^r}^{x^r=b^r} dx^1 \cdots dx^{r-1} dx^{r+1} \cdots dx^k$$

$$= -\int_{K_{k-1}} \omega_k(x^1, x^2, \dots, x^{k-1}, 0)\, dx^1 dx^2 \cdots dx^{k-1}.$$

On the other hand, since $\mu_k(-\partial/\partial x^k, \partial/\partial x^1, \dots, \partial/\partial x^{k-1}) = (-1)^k$, we cannot say that the basis $(\partial/\partial x^1, \dots, \partial/\partial x^{k-1})$ is positively oriented in $T(\partial S)$. Accordingly, we find

$$\int_{\partial S} \mathcal{I}^* \omega = (-1)^{2k-1} \int_{K_{k-1}} \omega_k(x^1, x^2, \dots, x^{k-1}, 0)\, dx^1 dx^2 \cdots dx^{k-1}$$

$$= -\int_{K_{k-1}} \omega_k(x^1, x^2, \dots, x^{k-1}, 0)\, dx^1 dx^2 \cdots dx^{k-1}$$

that results in

$$\int_{\partial S} \mathcal{I}^* \omega = \int_S d\omega.$$

We now wish to relax the condition that the support of the form ω is compact as it appears in the statement of the theorem. However, we shall instead suppose that there is an atlas on S subordinate to which there exists a partition of unity $\{V_\alpha, f_\alpha\}$ where $V_\alpha \subseteq U$ and (U, φ) is a chart of the atlas. We now impose the restriction that $\varphi(\textit{supp}\, f_\alpha) \subseteq \mathbb{R}^k$ is bounded for each member of the family. Since $\varphi(\textit{supp}\, f_\alpha)$ is also closed, the image of *supp* f_α is a compact set in $\mathbb{R}^k$. Hence, due to the homeomorphism *supp* f_α becomes a compact subset in S. Let us now define the forms $\omega_\alpha = f_\alpha \omega \in \Lambda^{k-1}(S)$ associated with the form $\omega \in \Lambda^{k-1}(S)$. The support of ω_α is the same as that of f_α, i.e., it is a compact subset. Thus, Stokes' theorem can be applied to such forms. Because of the relation $\sum_\alpha f_\alpha = 1$ we get $\sum_\alpha df_\alpha = 0$. Therefore, we can write

$$\omega = \sum_\alpha f_\alpha \omega = \sum_\alpha \omega_\alpha, \;\; d\omega = \sum_\alpha (df_\alpha\, \omega + f_\alpha d\omega) = \sum_\alpha d\omega_\alpha$$

and, consequently, obtain

$$\int_S d\omega = \sum_\alpha \int_S d\omega_\alpha = \sum_\alpha \int_{\partial S} \mathcal{I}^* \omega_\alpha = \int_{\partial S} \sum_\alpha \mathcal{I}^* \omega_\alpha$$
$$= \int_{\partial S} \mathcal{I}^* \sum_\alpha \omega_\alpha = \int_{\partial S} \mathcal{I}^* \omega$$

provided that the above sum is convergent and we are allowed to interchange summation and integration operations. When S is a paracompact manifold and the support of the form ω is compact, the number of the functions f_α involved is finite so these operations can always be performed. □

If ω is a closed form, i.e., if $d\omega = 0$, then we get $\int_{\partial S} \omega = 0$ on a manifold S with boundary ∂S. However, this condition does not imply in general that the form ω is exact, namely, there exists a form σ such that $\omega = d\sigma$ and $\int_{\partial S} d\sigma = 0$.

Example 8.6.2. We consider the form

$$\omega = \frac{-\,y\,dx + x\,dy}{x^2 + y^2} \in \Lambda^1(M)$$

defined on the manifold $M = \mathbb{R}^2 - (0,0)$. Let $D \subset M$ be a region bounded by a closed curve C containing the point $(0,0)$. We can immediately see that $d\omega = 0$. Furthermore, we can easily verify that one is able to write

$$\omega = d\theta, \;\; \theta = \arctan\frac{y}{x}.$$

Hence, we find that

$$\int_C \omega = \int_C d\theta = 2\pi \neq 0.$$

It is clear that this result is not in essence in contradiction with Stokes' theorem because it is originated from the fact that the real boundary of the region is described by $C \cup \{(0,0)\}$. ■

We shall now attempt to obtain Stokes' theorem by following a completely different path. This approach will also prove to be rather advantageous from the standpoint of giving rise to new interpretations. We consider a k-dimensional submanifold S of an m-dimensional differentiable manifold M where $k \leq m$. Let us assume that this submanifold is specified by a smooth mapping $\phi : S \to M$. So in local coordinates, the submanifold S is defined by a parametrisation $x^i = \phi^i(u^\alpha), i = 1, \ldots, m; \alpha = 1, \ldots, k$. Let $U \subseteq S$ be a region with boundary. Its $(k-1)$-dimensional boundary ∂U

may be determined by a mapping $\psi : \partial U \to S$ or through functions $u^\alpha = \psi^\alpha(v^a),\ a = 2, \dots, k$. Since the dimension of the boundary ∂U is $k-1$, the rank of the matrix $[\partial u^\alpha / \partial v^a]$ must be $k-1$. This amounts to say that we can take $\det [\partial u^b / \partial v^a] \neq 0$ by changing the ordering of coordinates if necessary. We can thus write $v^a = \xi^a(u^b)$ and the manifold ∂U may be described by the equation $u^1 = \xi^1(u^2, \dots, u^k)$. Next, we select the new local coordinates (w^1, w^a) for the manifold S by the expressions $w^a = u^a$, $w^1 = u^1 - \xi^1(u^2, \dots, u^k)$. Hence, the boundary ∂U of the region U is determined by the condition $w^1 = 0$. In this case, the parameters $w^a, a = 2, \dots, k$ constitute the local coordinates of ∂U. All the vectors $\partial / \partial w^a$ belong to $T(\partial U)$. Only the vector $\partial / \partial w^1$ is not in the tangent bundle of ∂U and lies in $T(S)$. We now introduce a vector field $V = \partial / \partial w^1$. This vector field creates a flow, that is, a one-parameter mapping $e^{tV} : S \to S$ on the submanifold S dragging the region U onto a region $U(t) \subset S$ (Fig. 8.6.2).

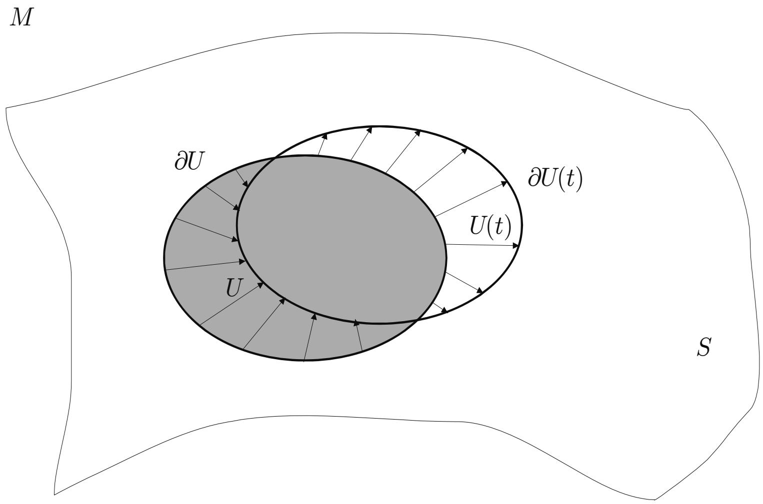

Fig. 8.6.2. The region $U \subset S$ dragged along the flow e^{tV}.

A form $\theta \in \Lambda^k(M)$ on S can be written as

$$\theta = \Theta(w^1, w^2, \dots, w^k)\, dw^1 \wedge dw^2 \wedge \cdots \wedge dw^k.$$

Let us now consider the set difference $\delta U(t) = U(t) - U$ so that the integration of the form θ over which can be expressed as

$$\int_{\delta U(t)} \theta = \int_{U(t)} \theta - \int_U \theta$$

For a small parameter t, we can choose local coordinates in the vicinity of ∂U as $(t, w^2, \dots, w^k)$. We can then expand the function Θ into a Maclaurin series about $t = 0$ and write

$$\Theta(t, w^2, \dots, w^k) = \Theta|_{t=0} + \left.\frac{\partial \Theta}{\partial t}\right|_{t=0} t + \cdots.$$

We can obviously write the following expression for small values of the parameter t

$$\begin{aligned}\int_{\delta U(t)} \theta &= \int_{\delta U(t)} \Theta(w^1, w^2, \dots, w^k)\, dw^1 dw^2 \cdots dw^k \\ &= \int_{\partial U} \Big(\int_0^t \Theta(w^1, w^2, \dots, w^k)\, dw^1 \Big) dw^2 \cdots dw^k.\end{aligned}$$

Inserting the relation

$$\int_0^t \Theta(w^1, w^2, \dots, w^k)\, dw^1 = t\Theta|_{t=0} + o(t)$$

into the foregoing integral, we find

$$\begin{aligned}\int_{\delta U(t)} \theta &= t \int_{\partial U} \Theta(0, w^2, \dots, w^k)\, dw^2 \cdots dw^k + o(t) \\ &= t \int_{\partial U} \mathbf{i}_V(\theta) + o(t).\end{aligned}$$

On the other hand, we can write

$$\left.\frac{d}{dt} \int_{U(t)} \theta \right|_{t=0} = \lim_{t \to 0} \frac{1}{t} \Big[\int_{U(t)} \theta - \int_U \theta \Big] = \int_{\partial U} \mathbf{i}_V(\theta).$$

But the above expression can also be calculated in a rather different way:

$$\begin{aligned}\left.\frac{d}{dt} \int_{U(t)} \theta \right|_{t=0} &= \lim_{t \to 0} \frac{1}{t} \Big[\int_{U(t)} \theta - \int_U \theta \Big] = \lim_{t \to 0} \frac{1}{t} \Big[\int_U (e^{tV})^* \theta - \int_U \theta \Big] \\ &= \int_U \lim_{t \to 0} \frac{(e^{tV})^* \theta - \theta}{t} \\ &= \int_U \pounds_V \theta.\end{aligned}$$

We are thus led to quite an interesting result given below

$$\int_U \pounds_V \theta = \int_{\partial U} \mathbf{i}_V(\theta).$$

However, we know that we can write $\pounds_V \theta = \mathbf{i}_V(d\theta) + d\mathbf{i}_V(\theta)$ in view of (5.11.5). $d\theta \in \Lambda^{k+1}(M)$ vanishes identically on the k-dimensional manifold S so that we get $\pounds_V \theta = d\mathbf{i}_V(\theta)$ on S and arrive at the result

$$\int_U d\mathbf{i}_V(\theta) = \int_{\partial U} \mathbf{i}_V(\theta). \tag{8.6.8}$$

Let us now write $\omega = \mathbf{i}_V(\theta) \in \Lambda^{k-1}(M)$. Since θ and to some extent V are arbitrary, we can take ω as an arbitrary $(k-1)$-form. Therefore, we derive again the Stokes theorem in its familiar form:

$$\int_U d\omega = \int_{\partial U} \omega.$$

Let us now take an m-dimensional *complete* Riemannian manifold M into account. If we denote the local coordinates by $(x^1, \dots, x^m)$, the elementary arc length on the manifold is given by

$$ds^2 = g_{ij}(\mathbf{x})\, dx^i dx^j$$

where we know that g_{ij} is a positive definite, covariant symmetric tensor. The volume form is prescribed by

$$\mu_m = \sqrt{g}\, dx^1 \wedge \cdots \wedge dx^m$$

where $g = \det[g_{ij}] > 0$. Let S be a k-dimensional submanifold of M. This submanifold is parametrically determined by relations $x^i = x^i(u^\alpha), \alpha = 1, \dots, k$. In this circumstance, the elementary arc length on the submanifold S can be introduced as

$$ds^2 = g_{ij}\big(\mathbf{x}(\mathbf{u})\big) \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta}\, du^\alpha du^\beta = a_{\alpha\beta}(\mathbf{u})\, du^\alpha du^\beta > 0 \tag{8.6.9}$$

where the second order, covariant symmetric tensor

$$a_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} \tag{8.6.10}$$

denotes the metric tensor $\boldsymbol{\mathcal{A}}$ on S induced by the metric tensor $\boldsymbol{\mathcal{G}}$ on M. It follows at once from (8.6.9) that $\boldsymbol{\mathcal{A}}$ is also positive definite. The volume form on S can now be defined as

$$\mu_k = \sqrt{a}\, du^1 \wedge \cdots \wedge du^k \tag{8.6.11}$$

where $a = \det[a_{\alpha\beta}] > 0$ due to the positive definiteness of $\mathcal{A}$.

Let us now write a form $\omega \in \Lambda^{m-k}(M)$ as

$$\omega = \frac{1}{k!}\, \omega^{i_1 \cdots i_k} \mu_{i_k \cdots i_1}$$

where top down generated basis form $\mu_{i_k \cdots i_1}$ are given [*see* (5.9.17)] by

$$\mu_{i_k \cdots i_1} = \frac{1}{(m-k)!}\, \epsilon_{i_1 \cdots i_k i_{k+1} \cdots i_m} dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_m}$$

Let us now evaluate these forms on an $(m-k)$-dimensional submanifold S of M. On supposing that the submanifold S is specified by the parameters $(u^1, \ldots, u^{m-k})$, we get

$$\begin{aligned} &\mu_{i_k \cdots i_1} \\ &\quad = \frac{1}{(m-k)!}\, \epsilon_{i_1 \cdots i_k i_{k+1} \cdots i_m} \frac{\partial x^{i_{k+1}}}{\partial u^{\alpha_1}} \cdots \frac{\partial x^{i_m}}{\partial u^{\alpha_{m-k}}} du^{\alpha_1} \wedge \cdots \wedge du^{\alpha_{m-k}} \\ &\quad = \frac{1}{(m-k)!}\, \epsilon_{i_1 \cdots i_k i_{k+1} \cdots i_m} e^{\alpha_1 \cdots \alpha_{m-k}} \frac{\partial x^{i_{k+1}}}{\partial u^{\alpha_1}} \cdots \frac{\partial x^{i_m}}{\partial u^{\alpha_{m-k}}} du^1 \wedge \cdots \wedge du^{m-k} \end{aligned}$$

where Greek indices take the values $1, 2, \ldots, m-k$. On employing the relation (8.6.11) for a volume form μ_k in the form μ_{m-k}, we can write $\mu_{m-k} = \sqrt{a}\, du^1 \wedge \cdots \wedge du^{m-k}$. If we introduce the Levi-Civita tensor

$$\epsilon^{\alpha_1 \cdots \alpha_{m-k}} = \frac{e^{\alpha_1 \cdots \alpha_{m-k}}}{\sqrt{a}},$$

we end up with the result

$$\mu_{i_k \cdots i_1} = \frac{1}{(m-k)!}\, \epsilon_{i_1 \cdots i_k i_{k+1} \cdots i_m} \epsilon^{\alpha_1 \cdots \alpha_{m-k}} \frac{\partial x^{i_{k+1}}}{\partial u^{\alpha_1}} \cdots \frac{\partial x^{i_m}}{\partial u^{\alpha_{m-k}}} \mu_{m-k}.$$

We now define a completely antisymmetric covariant tensor on S through the following components

$$n_{i_1 \cdots i_k} = \frac{1}{(m-k)!}\, \epsilon_{i_1 \cdots i_k i_{k+1} \cdots i_m} \epsilon^{\alpha_1 \cdots \alpha_{m-k}} \frac{\partial x^{i_{k+1}}}{\partial u^{\alpha_1}} \cdots \frac{\partial x^{i_m}}{\partial u^{\alpha_{m-k}}}. \tag{8.6.12}$$

We then see that we can write

$$\mu_{i_k \cdots i_1} = n_{i_1 \cdots i_k} \mu_{m-k}. \tag{8.6.13}$$

It follows from (8.6.12) that

$$n_{i_1\cdots i_k}\frac{\partial x^{i_1}}{\partial u^{\alpha_{m-k+1}}}\cdots\frac{\partial x^{i_k}}{\partial u^{\alpha_m}}=0.$$

This is true because the expression

$$\epsilon_{i_1\cdots i_k i_{k+1}\cdots i_m}\frac{\partial x^{i_1}}{\partial u^{\alpha_{m-k+1}}}\cdots\frac{\partial x^{i_k}}{\partial u^{\alpha_m}}\frac{\partial x^{i_{k+1}}}{\partial u^{\alpha_1}}\cdots\frac{\partial x^{i_m}}{\partial u^{\alpha_{m-k}}}$$

is completely antisymmetric with respect to indices $i_1,\dots,i_m$ and this entails that it becomes also completely antisymmetric with respect to indices $\alpha_1,\ \dots,\alpha_{m-k}$. However, this latter indices take on only $m-k$ different values. Therefore, it is not possible to avoid getting repeated indices in the set $i_1,\dots,i_m$ taking m different values.

The exterior derivative of the form ω is [*see* (5.9.19)]

$$\begin{aligned}d\omega&=\frac{1}{(k-1)!}\,\omega^{i_1\cdots i_{k-1}i}{}_{;i}\,\mu_{i_{k-1}\cdots i_1}\\&=\frac{1}{(k-1)!}\,\omega^{i_1\cdots i_{k-1}i}{}_{;i}\,n_{i_1\cdots i_{k-1}}\mu_{m-k+1}.\end{aligned}$$

Similarly, we can find

$$\omega=\frac{1}{k!}\,\omega^{i_1\cdots i_k}n_{i_1\cdots i_k}\mu_{m-k}.$$

In view of (8.6.11), we can introduce the volume element on S as

$$dV_k=\sqrt{a}\,du^1\cdots du^k$$

Let us now consider a form $\omega\in\Lambda^{m-k}(M)$ defined on a region U_{m-k+1} on an $(m-k+1)$-dimensional submanifold S whose boundary is given by the manifold ∂U_{m-k}. Application of the Stokes theorem by using parameters peculiar to those submanifolds yields

$$\begin{aligned}\int_{U_{m-k+1}}\omega^{i_1\cdots i_{k-1}i}{}_{;i}\,n_{i_1\cdots i_{k-1}}dV_{m-k+1}=&\\ \frac{1}{k}\int_{\partial U_{m-k}}\omega^{i_1\cdots i_k}n_{i_1\cdots i_k}dV_{m-k.}&\end{aligned}\tag{8.6.14}$$

An important special case that may be deduced from the above relation corresponds to $k=1$. In this case, one has

$$\omega=\omega^i\mu_i\quad\text{and}\quad d\omega=\omega^i{}_{;i}\,\mu$$

so we obtain

$$\int_{U_m} \omega^i{}_{;i}\, dV_m = \int_{\partial U_{m-1}} \omega^i n_i\, dV_{m-1} \tag{8.6.15}$$

where the components $n_i, i = 1, \ldots, m$ are defined by

$$n_i = \frac{1}{(m-1)!}\, \epsilon_{ii_1\cdots i_{m-1}} \epsilon^{\alpha_1\cdots\alpha_{m-1}} \frac{\partial x^{i_1}}{\partial u^{\alpha_1}} \cdots \frac{\partial x^{i_{m-1}}}{\partial u^{\alpha_{m-1}}}. \tag{8.6.16}$$

It is clearly seen that the relations

$$n_i \frac{\partial x^i}{\partial u^\alpha} = 0,\ \ \alpha = 1, \ldots, m-1$$

will be satisfied. The quantities

$$\frac{\partial x^i}{\partial u^\alpha},\quad \alpha = 1, \ldots, m-1$$

are contravariant components of $m-1$ vectors in $T(\partial U_{m-1})$. The vector whose covariant components are n_i in $T(M)$ is orthogonal to all those vectors. Hence, it is called the ***exterior normal*** **n** to the boundary ∂U_{m-1} (if U_m is positively oriented). We shall now show that **n** is a unit vector. We first evaluate the expression

$$\begin{aligned} n_i n^i &= \frac{1}{\big[(m-1)!\big]^2}\, \epsilon_{ii_1\cdots i_{m-1}} \epsilon^{i}_{j_1\cdots j_{m-1}} \epsilon^{\alpha_1\cdots\alpha_{m-1}} \epsilon^{\beta_1\cdots\beta_{m-1}} \\ &\quad \times \frac{\partial x^{i_1}}{\partial u^{\alpha_1}} \cdots \frac{\partial x^{i_{m-1}}}{\partial u^{\alpha_{m-1}}} \frac{\partial x^{j_1}}{\partial u^{\beta_1}} \cdots \frac{\partial x^{j_{m-1}}}{\partial u^{\beta_{m-1}}} \\ &= \frac{1}{\big[(m-1)!\big]^2}\, g_{j_1k_1} \cdots g_{j_{m-1}k_{m-1}}\, \epsilon_{ii_1\cdots i_{m-1}} \epsilon^{ik_1\cdots k_{m-1}} \\ &\quad \times \epsilon^{\alpha_1\cdots\alpha_{m-1}} \epsilon^{\beta_1\cdots\beta_{m-1}} \frac{\partial x^{i_1}}{\partial u^{\alpha_1}} \cdots \frac{\partial x^{i_{m-1}}}{\partial u^{\alpha_{m-1}}} \frac{\partial x^{j_1}}{\partial u^{\beta_1}} \cdots \frac{\partial x^{j_{m-1}}}{\partial u^{\beta_{m-1}}} \\ &= \frac{1}{\big[(m-1)!\big]^2}\, g_{j_1k_1} \cdots g_{j_{m-1}k_{m-1}}\, \delta^{k_1\cdots k_{m-1}}_{i_1\cdots i_{m-1}} \epsilon^{\alpha_1\cdots\alpha_{m-1}} \epsilon^{\beta_1\cdots\beta_{m-1}} \\ &\quad \times \frac{\partial x^{i_1}}{\partial u^{\alpha_1}} \cdots \frac{\partial x^{i_{m-1}}}{\partial u^{\alpha_{m-1}}} \frac{\partial x^{j_1}}{\partial u^{\beta_1}} \cdots \frac{\partial x^{j_{m-1}}}{\partial u^{\beta_{m-1}}} \end{aligned}$$

where we have utilised the relations (5.5.7) and (5.5.5). The last line above is completely antisymmetric with respect to indices $\alpha_1, \cdots, \alpha_{m-1}$. Thus it also becomes completely antisymmetric with respect to indices $i_1, \cdots, i_{m-1}$. Hence, according to (1.4.8) we conclude that

$$
\begin{aligned}
n_i n^i &= \frac{1}{(m-1)!}\,\epsilon^{\alpha_1\cdots\alpha_{m-1}}\epsilon^{\beta_1\cdots\beta_{m-1}} g_{j_1k_1}\cdots g_{j_{m-1}k_{m-1}} \frac{\partial x^{j_1}}{\partial u^{\beta_1}}\frac{\partial x^{k_1}}{\partial u^{\alpha_1}} \\
&\quad \cdots \frac{\partial x^{j_{m-1}}}{\partial u^{\beta_{m-1}}}\frac{\partial x^{k_{m-1}}}{\partial u^{\alpha_{m-1}}} \\
&= \frac{1}{(m-1)!}\,\epsilon^{\alpha_1\cdots\alpha_{m-1}}\epsilon^{\beta_1\cdots\beta_{m-1}} a_{\beta_1\alpha_1}\cdots a_{\beta_{m-1}\alpha_{m-1}}.
\end{aligned}
$$

On the other hand, the relation

$$a = \det[a_{\alpha\beta}] = \frac{1}{(m-1)!}\,e^{\alpha_1\cdots\alpha_{m-1}}e^{\beta_1\cdots\beta_{m-1}} a_{\beta_1\alpha_1}\cdots a_{\beta_{m-1}\alpha_{m-1}}$$

yields $n_i n^i = 1$. In a similar fashion, it is a simple exercise to demonstrate the validity of the relation

$$n_{i_1\cdots i_k} n^{i_1\cdots i_k} = k!.$$

The relation (8.6.15) is called the ***Green-Gauss-Ostrogradski*** or ***divergence formula*** generalised to an m-dimensional manifold [after English, German and Russian mathematicians, respectively, George Green (1793-1841), Johann Carl Friedrich Gauss (1777-1855) and Mikhail Vasilevich Ostrogradski (1801-1862)].

Example 8.6.3. We consider a bounded region $U \subset \mathbb{R}^3$ and the 2-form $\omega = X dy \wedge dz + Y dz \wedge dx + Z dx \wedge dy$. In view of (8.6.13), we can write

$$\mu_x = dy \wedge dz = n_x \mu_2,\;\; \mu_y = dz \wedge dx = n_y \mu_2,\;\; \mu_z = dx \wedge dy = n_z \mu_2$$

on ∂U. The components of the unit exterior normal vector ∂U to the closed surface ∂U are given by

$$n_i = e_{ijk} e^{\alpha\beta} \frac{\partial x^j}{\partial u^\alpha}\frac{\partial x^k}{\partial u^\beta},$$

where we denote $x^1 = x, x^2 = y, x^3 = z$. On the region U, we get

$$d\omega = \Big(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\Big) dx \wedge dy \wedge dz.$$

Hence, the Stokes theorem takes the form

$$\int_U \Big(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\Big) dv = \int_{\partial U} (X n_x + Y n_y + Z n_z) da.$$

Let us introduce vectors $\mathbf{F} = (X, Y, Z)$ and $\mathbf{n} = (n_x, n_y, n_z)$. Then the Stokes theorem leads to the quite familiar formula

$$\int_U \operatorname{div}\mathbf{F}\,dv = \int_U \boldsymbol{\nabla}\cdot\mathbf{F}\,dv = \int_{\partial U}\mathbf{F}\cdot\mathbf{n}\,da. \qquad \blacksquare$$

Example 8.6.4. Let U be a 2-dimensional submanifold, in other words a surface, in $\mathbb{R}^3$ and $\partial U = C$ be the closed curve that supports this surface. We consider a 1-form $\omega = Xdx + Ydy + Zdz$. We denote by s the arc length of the curve C. Then the form ω is expressible on C as

$$\omega = \Big(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\Big)\,ds = \mathbf{F}\cdot\mathbf{t}\,ds$$

on C where $\mathbf{t}$ is the unit tangent vector of C. On the other hand, one has

$$d\omega = \Big(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\Big)\,dy\wedge dz + \Big(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\Big)\,dz\wedge dx + \Big(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\Big)\,dx\wedge dy.$$

Hence, the exterior derivative of the form ω can be written as follows

$$d\omega = \mathbf{n}\cdot\operatorname{curl}\mathbf{F}\;\mu_2.$$

Therefore, the Stokes theorem associated with exterior forms leads to the familiar expression

$$\int_U \mathbf{n}\cdot\operatorname{curl}\mathbf{F}\,da = \int_U \mathbf{n}\cdot(\boldsymbol{\nabla}\times\mathbf{F})\,da = \int_C \mathbf{F}\cdot\mathbf{t}\,ds$$

known as the ***Kelvin-Stokes formula*** in the classical vector analysis. $\blacksquare$

Through the Stokes theorem, we can generalise a relation known as the integration by parts in the classical analysis. We take two forms $\omega \in \Lambda^k(M)$ and $\sigma \in \Lambda^l(M)$ into consideration. Let U be a region on a submanifold of M with dimension $k + l + 1 \le m$. It follows from the exterior derivative of the form $\omega\wedge\sigma$ that

$$\int_U d\omega\wedge\sigma = \int_{\partial U}\omega\wedge\sigma - (-1)^k\int_U \omega\wedge d\sigma. \qquad (8.6.17)$$

8.7. CONSERVATION LAWS

Let $\mathcal{I}(\omega^a), a = 1,\dots,A$ be an ideal of the exterior algebra $\Lambda(M)$. We know that if the mapping $\phi : S \to M$ satisfy the condition $\phi^*\omega = 0$ for all $\omega \in \mathcal{I}$, then it is a solution of this ideal. Here the solution hypersurface S is a submanifold with dimension, say, $k \le m$. We shall now try to determine *non-zero exact k-forms* in the ideal $\mathcal{I}(\omega^a)$ annihilated by the solution submanifold. To this end, we consider a form $\omega \in \Lambda^k(M)$ in the ideal and look for a form $\Omega \in \Lambda^{k-1}(M)$ such that $\omega = d\Omega$. Since $\omega \in \mathcal{I}$, we can write

$$0 = \phi^*\omega = \phi^* d\Omega = d\phi^*\Omega.$$

Let $U_k \subseteq S$ be a smooth k-dimensional region and ∂U_k be its boundary. It follows from the Stokes theorem that

$$\int_{\partial U_k} \phi^*\Omega = \int_{U_k} d\phi^*\Omega = 0.$$

Consequently, the form Ω must satisfy the relation

$$\int_{\partial U_k} \phi^*\Omega = 0 \tag{8.7.1}$$

on every $U_k \subseteq S$ with boundary. (8.7.1) is called a ***conservation law*** in the integral form. Let us now suppose that the mapping ϕ is parametrically prescribed by the relations $x^i = \phi^i(u^1, \dots, u^k), i = 1, \dots, m$. We take the volume form on S as $\mu = du^1 \wedge \cdots \wedge du^k$ and define basis $(k-1)$-forms $\mu_\alpha = \mathbf{i}_{\partial_\alpha}\mu, \alpha = 1, \dots, k$ in $\Lambda^{k-1}(S)$. Since the form Ω will eventually be pulled back on the submanifold S we can choose

$$\Omega = \Omega^\alpha(\mathbf{x})\mu_\alpha(\mathbf{u}) \text{ and } \phi^*\Omega = \Omega^\alpha\big(\mathbf{x}(\mathbf{u})\big)\mu_\alpha(\mathbf{u})$$

without loss of generality. Accordingly, in order that a form $\omega \in \mathcal{I}$ is to be exact, we have to find suitable forms $\gamma_a \in \Lambda(M)$ so that Ω satisfies

$$\omega = \gamma_a \wedge \omega^a = d\Omega = d\Omega^\alpha(\mathbf{x}) \wedge \mu_\alpha(\mathbf{u}). \tag{8.7.2}$$

On the other hand, the relation

$$d\phi^*\Omega = d\Omega^\alpha\big(\mathbf{x}(\mathbf{u})\big) \wedge \mu_\alpha = \frac{\partial\Omega^\alpha}{\partial x^i}\frac{\partial x^i}{\partial u^\beta}du^\beta \wedge \mu_\alpha = \frac{\partial\Omega^\alpha}{\partial x^i}\frac{\partial x^i}{\partial u^\alpha}\mu = 0$$

implies that the functions Ω^α ought to satisfy the divergence equation

$$\frac{\partial\Omega^\alpha}{\partial x^i}\frac{\partial x^i}{\partial u^\alpha} = \frac{\partial\phi^*\Omega^\alpha(\mathbf{u})}{\partial u^\alpha} = 0. \tag{8.7.3}$$

Example 8.7.1. The coordinate cover in the manifold $M = \mathbb{R}^3$ is given by $\mathbf{x} = (x, v, t)$. We consider the ideal $\mathcal{I}$ generated by the following forms:

$$\omega^1 = dv + n^2 x\, dt,$$
$$\omega^2 = dx - v\, dt.$$

On a 1-dimensional solution submanifold prescribed by the mapping $x = x(t), v = v(t)$, these forms will have to satisfy

$$\phi^*\omega^1 = \Big(\frac{dv}{dt} + n^2 x\Big)dt = 0,\ \phi^*\omega^2 = \Big(\frac{dx}{dt} - v\Big)dt = 0.$$

Hence, the solution submanifold is determined through the differential equation below associated with 1-dimensional oscillating systems

$$\frac{d^2x}{dt^2} + n^2 x = 0.$$

Since the solution submanifold is 1-dimensional, we have to look naturally for exact 1-forms. Let $\Omega \in \Lambda^0(M)$. Due to (8.7.3), we find that

$$\frac{d\phi^*\Omega}{dt} = \frac{\partial\Omega}{\partial x}\frac{dx}{dt} + \frac{\partial\Omega}{\partial v}\frac{dv}{dt} + \frac{\partial\Omega}{\partial t} = 0.$$

Thus, we must have $\phi^*\Omega = constant$.

On the other hand, the condition (8.7.2) takes the form

$$\gamma_1(dv + n^2 x\, dt) + \gamma_2(dx - v\, dt) = d\Omega = \Omega_x\, dx + \Omega_v\, dv + \Omega_t\, dt$$

where $\gamma_1, \gamma_2 \in \Lambda^0(M)$. The subscripts indicate the variables with respect to which partial derivatives will be evaluated. We thereby obtain

$$\gamma_1 = \Omega_v,\ \gamma_2 = \Omega_x,\ n^2 x\gamma_1 - v\gamma_2 = \Omega_t$$

or

$$-\,n^2 x\Omega_v + v\Omega_x + \Omega_t = 0.$$

It is obvious that this equation corresponds to the relation $d\phi^*\Omega/dt = 0$ on the solution submanifold. In order to solve the foregoing partial differential equation, we can employ the method of characteristics. To this end, we have to solve the following system of ordinary differential equations

$$\frac{dv}{n^2 x} = -\,\frac{dx}{v} = -\,dt.$$

It is a simple exercise to see that $\Omega = \Omega(\xi, \eta)$ where

$$\xi = \frac{1}{2}(v^2 + n^2 x^2), \quad \eta = t - \int \frac{dx}{v}.$$

Thus, independent conservation laws become $\xi = constant$, $\eta = constant$. In this case, every function $\Omega(\xi, \eta)$ remains constant on the solution submanifold. If we take $U_1 = [t_1, t_2]$, then we find on ∂U_1 the known result

$$\frac{1}{2}(v^2 + n^2 x^2)\Big|_{t_1}^{t_2} = 0, \quad t_2 - t_1 - \int_{x_1}^{x_2} \frac{dx}{v} = 0. \qquad \blacksquare$$

Example 8.7.2. Let $\mathbf{x} = (\theta, u, v, x, t)$ be a coordinate cover on the manifold $M = \mathbb{R}^5$. We consider the ideal $\mathcal{I}$ generated by the following forms

$$\omega^1 = du \wedge dt - v\, dx \wedge dt, \quad \omega^2 = d\theta - u\, dx - v\, dt.$$

On a 2-dimensional solution submanifold prescribed by the mapping $\theta = \theta(x,t), u = u(x,t)$ and $v = v(x,t)$, we have to satisfy the conditions below

$$\begin{aligned} \phi^*\omega^1 &= (u_x - v)\, dx \wedge dt = 0, \\ \phi^*\omega^2 &= (\theta_x - u)\, dx + (\theta_t - v)\, dt = 0 \end{aligned}$$

whence we deduce that

$$v = u_x,\ u = \theta_x,\ v = \theta_t \ \text{ or } \ \theta_t = \theta_{xx}.$$

The last equation describes a one-dimensional heat conduction,. More generally it models a diffusion process. Let us take the volume form in a 2-dimensional solution submanifold as $\mu = dx \wedge dt$. We then get $\mu_1 = dt$, $\mu_2 = \ -dx$. Hence, we have to look for a form in the following shape

$$\Omega = \Phi\, dt - \Psi\, dx \in \Lambda^1(M)$$

where we have defined $\Omega^1 = \Phi \in \Lambda^0(M)$, $\Omega^2 = \Psi \in \Lambda^0(M)$. The condition (8.7.2) then yields

$$\gamma_1 \omega^1 + \gamma_2 \wedge \omega^2 = d\Omega = d\Phi \wedge dt - d\Psi \wedge dx$$

where $\gamma_1 \in \Lambda^0(M), \gamma_2 \in \Lambda^1(M)$. If we express γ_2 as

$$\gamma_2 = a\, d\theta + b\, du + c\, dv + \ e\, dx \ + f dt$$

where $a, b, c, e, f \in \Lambda^0(M)$, then the above relation is transformed into

$$\begin{aligned} (\gamma_1 - bv - \Phi_u)\, du \wedge dt - (\gamma_1 v - fu + ev + \Phi_x + \Psi_t)\, dx \wedge dt + b\, du \wedge d\theta \\ + c\, dv \wedge d\theta - (e + au - \Psi_\theta)\, d\theta \wedge dx - (f + av + \Phi_\theta)\, d\theta \wedge dt \\ - (bu - \Psi_u)\, du \wedge dx - (cu - \Psi_v)\, dv \wedge dx - (cv + \Phi_v)\, dv \wedge dt = 0 \end{aligned}$$

whence we extract the relations

$$\begin{aligned} &\gamma_1 - bv = \Phi_u, \quad -\gamma_1 v + fu - ev = \Phi_x + \Psi_t, \quad b = 0, \quad c = 0, \\ &e + au = \Psi_\theta,\ f + av = \ -\Phi_\theta,\ bu = \Psi_u,\ cu = \Psi_v,\ cv = \ -\Phi_v. \end{aligned}$$

by equating the coefficients of the linearly independent 2-forms to zero. Hence, we end up with the relations

$$\gamma_1 = \Phi_u,\ e = \Psi_\theta - au,\ f = -\Phi_\theta - av,\ \Psi_u = 0,\ \Psi_v = 0,\ \Phi_v = 0$$
$$\Phi_x + \Psi_t + v(\Phi_u + \Psi_\theta) + u\Phi_\theta = 0$$

implying first that we must have $\Psi = \Psi(\theta, x, t)$, $\Phi = \Phi(\theta, u, x, t)$. Since Φ and Ψ are independent of v, it is required that the coefficient of v in the last equation above must vanish yielding $\Phi_u = -\Psi_\theta$. On noting that Ψ does not depend on u, this expression is easily integrated to give

$$\Phi = -\Psi_\theta u + \phi(\theta, x, t)$$

where ϕ is an arbitrary function. Thus, the expression $\Phi_x + \Psi_t + u\Phi_\theta = 0$ yields the equation

$$\phi_x + \Psi_t + u(\phi_\theta - \Psi_{\theta x}) - u^2\Psi_{\theta\theta} = 0.$$

However, this equation is satisfied if only

$$\Psi_{\theta\theta} = 0,\ \ \phi_\theta = \Psi_{\theta x},\ \ \Psi_t + \phi_x = 0.$$

The first two equations give

$$\Psi = \alpha(x,t)\,\theta + \beta(x,t), \phi = \Psi_x + \varphi(x,t) \text{ and } \phi = \alpha_x\theta + \beta_x + \varphi.$$

As to the last equation, it yields

$$(\alpha_{xx} + \alpha_t)\theta + \beta_{xx} + \varphi_x + \beta_t = 0.$$

Therefore, the functions Φ and Ψ are finally given by

$$\Phi = -\alpha u + \alpha_x\theta + \beta_x + \varphi,\ \ \Psi = \alpha\theta + \beta$$

provided that the functions α, β and φ are to satisfy the equations

$$\alpha_{xx} + \alpha_t = 0,\ \ \beta_{xx} + \beta_t + \varphi_x = 0.$$

Thus the conservation law takes the form

$$\frac{\partial}{\partial x}(-\alpha\theta_x + \alpha_x\theta + \beta_x + \varphi) + \frac{\partial}{\partial t}(\alpha\theta + \beta) = 0$$

and we arrive at the integral relation

$$\int_C (\Phi\,dt - \Psi\,dx) = 0$$

on every closed curve C in the (x,t)-plane. ∎

Example 8.7.3. The coordinate cover of the manifold $M = \mathbb{R}^4$ is given by $\mathbf{x} = (x, t, u, c)$. Let us consider the ideal $\mathcal{I}$ generated by the following 2-forms

$$\begin{aligned} \omega^1 &= -\,du \wedge dx + u\,du \wedge dt + \alpha c\,dc \wedge dt, \\ \omega^2 &= -\,dc \wedge dx + u\,dc \wedge dt + \frac{1}{\alpha}\,c\,du \wedge dt. \end{aligned}$$

On a 2-dimensional solution submanifold prescribed by the mapping $u = u(x,t)$, $c = c(x,t)$, the following relations must hold

$$\begin{aligned} \phi^*\omega^1 &= (u_t + uu_x + \alpha cc_x)\,dx \wedge dt = 0, \\ \phi^*\omega^2 &= (c_t + uc_x + \frac{1}{\alpha}\,cu_x)\,dx \wedge dt = 0. \end{aligned}$$

Subscripts denote partial derivatives with respect to relevant variables. They of course give rise to partial differential equations

$$\begin{aligned} u_t + uu_x + \alpha cc_x &= 0, \\ c_t + uc_x + \frac{1}{\alpha}\,cu_x &= 0 \end{aligned}$$

to determine the functions $u(x,t)$ and $c(x,t)$ prescribing the solution manifold. These equations are modelling the *one-dimensional isentropic gas flow* for the choice

$$\alpha = \frac{2}{\gamma - 1}$$

and the *shallow water theory* in hydrodynamics describing the propagation gravity waves on the free surface of an incompressible fluid of infinite extent in x-direction on a horizontal flat bottom for the choice $\alpha = 2$. In isentropic gas flow, γ denotes the ratio of specific heats of the gas under constant pressure and constant volume. u is the velocity of the gas while c denotes the local sound speed. In the shallow water theory, u is the velocity of the fluid and $c = \sqrt{gh}$ where h is the elevation of the water surface during the propagation of the gravity wave from the horizontal bottom. g denotes the well known gravitational acceleration.

We shall now attempt to find conservation laws by taking into account a form $\Omega = \Phi(\mathbf{x})dt - \Psi(\mathbf{x})dx \in \Lambda^1(M)$. In order that $d\Omega$ is to be in the ideal $\mathcal{I}(\omega^1, \omega^2)$, we have to write

$$\gamma_1\omega^1 + \gamma_2\omega^2 = d\Omega = d\Phi \wedge dt - d\Psi \wedge dx$$

where $\gamma_1, \gamma_2 \in \Lambda^0(M)$. Therefore, the relation

$$
\begin{aligned}
&- \gamma_1 du \wedge dx - \gamma_2\, dc \wedge dx + \Big(u\gamma_1 + \frac{c}{\alpha}\gamma_2\Big) du \wedge dt + (\alpha c \gamma_1 + u\gamma_2) dc \wedge dt \\
&= (\Phi_x + \Psi_t) dx \wedge dt + \Phi_u du \wedge dt + \Phi_c dc \wedge dt - \Psi_u du \wedge dx - \Psi_c dc \wedge dx
\end{aligned}
$$

leads to the partial differential equations

$$
\begin{aligned}
&\Phi_x + \Psi_t = 0, \qquad \gamma_1 = \Psi_u, \qquad \gamma_2 = \Psi_c \\
&u\Psi_u + \frac{c}{\alpha}\Psi_c = \Phi_u, \quad \alpha c \Psi_u + u\Psi_c = \Phi_c.
\end{aligned}
$$

to determine the functions Φ and Ψ. On the other hand, the symmetry relation $\Phi_{uc} = \Phi_{cu}$ leads from the last two equations above to the second order linear partial differential equation for the function Ψ

$$\alpha^2 \Psi_{uu} - \Psi_{cc} + \frac{\alpha - 1}{c}\Psi_c = 0.$$

On the other hand, we can find from the above relations

$$\Psi_u = \frac{\alpha u \Phi_u - c\Phi_c}{\alpha(u^2 - c^2)}, \quad \Psi_c = \frac{u\Phi_c - \alpha c \Phi_u}{u^2 - c^2}$$

from which we obtain

$$\alpha^2 \Phi_{uu} - \Phi_{cc} + \frac{(\alpha - 1)(u^2 + c^2)}{c(u^2 - c^2)}\Phi_c - \frac{2\alpha(\alpha - 1)}{c(u^2 - c^2)}\Phi_u = 0.$$

Of course the solution functions $\Phi(x,t,u,c)$ and $\Psi(x,t,u,c)$ are interrelated through the relations above. We anticipate that our field equations may possess infinitely many conservation laws since they are originated from solutions of partial differential equations. Indeed, certain particular solutions of those equations justify this expectation[1]. It can be shown that a polynomial type of conservation laws that are independent of x, t can be found as

$$\Psi_n = c^n C_n^{(\frac{\alpha}{2} - n)}\Big(\frac{u}{\alpha c}\Big), \; \Phi_n = c^{n+1}\Big[\frac{u}{c} C_n^{(\frac{\alpha}{2} - n)}\Big(\frac{u}{\alpha c}\Big) - C_{n+1}^{(\frac{\alpha}{2} - n)}\Big(\frac{u}{\alpha c}\Big)\Big]$$

where $n = 1, 2, \ldots$ and $n \neq \alpha/2$. $C_n^{(\lambda)}$ denotes a Gegenbauer polynomial [after German mathematician Leopold Bernhard Gegenbauer (1849-1903)]. These sequence of orthogonal polynomials is found from a generating function through the expansion

[1] For a detailed analysis one may consult to Şuhubi, E. S., Conservation laws for one-dimensional isentropic gas flows, *International Journal of Engineering Science*, **22**, 119-126, 1984.

$$\frac{1}{(1-2xt+t^2)^\lambda}=\sum_{n=0}^{\infty}C_n^{(\lambda)}(x)t^n.$$

They can also be obtained by the following recurrence relation

$$\begin{aligned}&C_0^{(\lambda)}(x)=1,\ C_1^{(\lambda)}(x)=2\lambda x\\&C_n^{(\lambda)}(x)=\frac{1}{n}\big[2x(n+\lambda-1)C_{n-1}^{(\lambda)}(x)-(n+2\lambda-2)C_{n-2}^{(\lambda)}(x)\big].\end{aligned}$$

We confine ourselves here in giving only a few samples of this infinite set:

$$\begin{aligned}\Psi_1&=\frac{\alpha-2}{\alpha}u,\Phi_1=\frac{\alpha-2}{2}\Big(\frac{1}{\alpha}u^2+c^2\Big),\\\Psi_2&=-\frac{\alpha-4}{2}\Big(c^2-\frac{\alpha-2}{\alpha^2}u^2\Big),\Phi_2=-\frac{\alpha-4}{\alpha}u\Big(c^2-\frac{\alpha-2}{3\alpha}u^2\Big),\\\Psi_3&=\frac{(\alpha-4)(\alpha-6)}{6}u\Big(\frac{\alpha-2}{\alpha^3}u^2-\frac{3}{\alpha}c^2\Big),\\\Phi_3&=-\frac{(\alpha-4)(\alpha-6)}{4}\Big(\frac{1}{2}c^4+\frac{\alpha+2}{\alpha^2}u^2c^2-\frac{\alpha-2}{2\alpha^3}u^4\Big),\\\Psi_4&=\frac{(\alpha-6)(\alpha-8)}{24}u\Big(\frac{(\alpha-2)(\alpha-4)}{\alpha^4}u^4-6\frac{\alpha-4}{\alpha^2}u^2c^2+3c^4\Big),\\\Phi_4&=\frac{(\alpha-6)(\alpha-8)}{2\alpha}u\Big(c^4-\frac{(\alpha+1)(\alpha-4)}{3\alpha^2}u^2c^2+\frac{(\alpha-2)(\alpha-4)}{15\alpha^3}u^4\Big),\\&\vdots\end{aligned}$$

■

Example 8.7.4. As a final example, we shall try to establish conservation laws associated with the field equations of a hyperelastic body in motion occupying an open region $\Omega\subseteq\mathbb{R}^3$ initially[2]. To facilitate our investigation we employ Cartesian coordinates. The position of a material particle before deformation will be determined by *material coordinates* $X_K, K=1,2,3$ whereas the place of the same particle in $\mathbb{R}^3$ at time t will be denoted by the *spatial coordinates* $x_k, k=1,2,3$. The motion of this continuous medium is determined by the diffeomorphism $x_k=x_k(\mathbf{X},t)$ with parameter t. A homogeneous hyperelastic material is characterised by the *stress potential* $\Sigma(\mathbf{C})$ in which $\mathbf{C}=\mathbf{F}^{\mathrm{T}}\mathbf{F}$ is the deformation tensor where $\mathbf{F}=[x_{k,K}]$ is the deformation gradient tensor, or matrix, whose components are denoted

[2] Şuhubi, E. S., Conservation laws in nonlinear elastodynamics, *International Journal of Engineering Science*, **27**, 441-453, 1989.

by $F_{kK} = \dfrac{\partial x_k}{\partial X_K} = x_{k,K}$. The equations of motion of the body are

$$\frac{\partial T_{Kk}}{\partial X_K} + \rho_0 f_k = \rho_0 \frac{\partial v_k}{\partial t} \tag{8.7.4}$$

where T_{Kk} is the Piola-Kirchhoff stress tensor of the first kind [after Italian mathematician and physicist Gabrio Piola (1794-1850) and German mathematician and physicist Gustav Robert Kirchhoff (1824-1887)] and $v_k = \dfrac{\partial x_k}{\partial t}$ are the components of the velocity vector of a particle. ρ_0 is the constant density of the undeformed medium and $f_k = f_k(\mathbf{X}, t)$ represents the components of the given body force density. Constitutive equations characterising the elastic behaviour of the medium are of the form

$$\begin{aligned} T_{Kk} &= \frac{\partial \Sigma}{\partial F_{kK}}, \\ T_{Kk} F_{lK} &= T_{Kl} F_{kK}. \end{aligned} \tag{8.7.5}$$

The relations $(8.7.5)_2$ arise from the symmetry of the Cauchy stress tensor. Therefore, equations of motion may be reduced to the following system of first order partial differential equations

$$\begin{aligned} C_{kKlL} \frac{\partial F_{lL}}{\partial X_K} - \rho_0 \frac{\partial v_k}{\partial t} + \rho_0 f_k &= 0, \\ \frac{\partial F_{kK}}{\partial X_L} - \frac{\partial F_{kL}}{\partial X_K} &= 0, \\ \frac{\partial F_{kK}}{\partial t} - \frac{\partial v_k}{\partial X_K} &= 0 \end{aligned}$$

where the coefficients

$$C_{kKlL}(\mathbf{F}) = \frac{\partial T_{Kk}}{\partial F_{lL}} = \frac{\partial^2 \Sigma}{\partial F_{kK} \partial F_{lL}} = C_{lLkK}(\mathbf{F}) \tag{8.7.6}$$

are called the *elasticities* of the medium. Let us now consider the 19-dimensional manifold K with a coordinate cover $(X_K, t, x_k, v_k, F_{kK})$. We first introduce the 3- and 2-forms below

$$\begin{aligned} \mu &= dX_1 \wedge dX_2 \wedge dX_3 = \frac{1}{3!} e_{KLM}\, dX_K \wedge dX_L \wedge dX_M, \\ \mu_K &= \mathbf{i}_{\partial_K} \mu = \frac{1}{2} e_{KLM}\, dX_L \wedge dX_M. \end{aligned}$$

We then define the following 4-forms:

$$
\begin{aligned}
\omega_k &= \rho_0\, dv_k \wedge \mu + C_{kKlL} dF_{lL} \wedge \mu_K \wedge dt + \rho_0 f_k\, \mu \wedge dt,\\
\omega_{kK} &= dF_{kK} \wedge \mu + dv_k \wedge \mu_K \wedge dt,\\
\pi_{kK} &= e_{KLM}\, dF_{kL} \wedge \mu_M \wedge dt,\\
\sigma_k &= dx_k \wedge \mu + v_k\, \mu \wedge dt,\\
\sigma_{kK} &= dx_k \wedge \mu_K \wedge dt - F_{kK}\, \mu \wedge dt.
\end{aligned}
$$

We can readily verify that

$$
\begin{aligned}
&d\omega_k = 0, \qquad d\omega_{kK} = 0, \qquad d\pi_{kK} = 0,\\
&d\sigma_k = \rho_0^{-1} dt\ \wedge \omega_k,\ \ d\sigma_{kK} = \ - dt\ \wedge \omega_{kK}
\end{aligned}
$$

These relations mean that the ideal $\mathcal{I}$ generated by these 4-forms is closed. Let the submanifold with the coordinate cover (X_K, t) be M. We can easily check that the mapping $\phi : M \to K$ annihilating these forms, and consequently the ideal $\mathcal{I}$, provides the solution of the differential field equations. In fact, we find that

$$
\begin{aligned}
\phi^*\omega_k &= \rho_0\, \frac{\partial v_k}{\partial t} dt \wedge \mu + C_{kKlL} F_{lL,M} dX_M \wedge \mu_K \wedge dt + \rho_0 f_k\, \mu \wedge dt\\
&= \Big(- \rho_0\, \frac{\partial v_k}{\partial t} + C_{kKlL} F_{lL,K} + \rho_0 f_k \Big) \mu \wedge dt = 0,\\
\phi^*\omega_{kK} &= \Big(- \frac{\partial F_{kK}}{\partial t} + \frac{\partial v_k}{\partial X_K} \Big) \mu \wedge dt = 0,\\
\phi^*\pi_{kK} &= e_{KLM}\, \frac{\partial F_{kL}}{\partial X_M} \mu \wedge dt = 0,\\
\phi^*\sigma_k &= \Big(- \frac{\partial x_k}{\partial t} + v_k \Big) \mu \wedge dt = 0,\\
\phi^*\sigma_{kK} &= \Big(\frac{\partial x_k}{\partial X_K} - F_{kK} \Big) \mu \wedge dt = 0.
\end{aligned}
$$

We shall now look for the *exact* 4-forms in the ideal $\mathcal{I}$. If $\omega \in \mathcal{I}$, then we can write

$$
\omega = \phi_k \omega_k + \phi_{kK} \omega_{kK} + \nu_{kK} \pi_{kK} + \psi_k \sigma_k + \psi_{kK} \sigma_{kK}
$$

where $\phi_k, \phi_{kK}, \nu_{kK}, \psi_k, \psi_{kK} \in \Lambda^0(K)$. Let us now introduce a 3-form

$$
\Omega = \Phi \mu - \Phi_K\, \mu_K \wedge dt \in \Lambda^3(K)
$$

where $\Phi, \Phi_K \in \Lambda^0(K)$. Next, we try to determine the functions ϕ_k, ϕ_{kK}, $\nu_{kK}, \psi_k, \psi_{kK}$ as to satisfy the relation

$$
\omega = d\Omega = d\Phi \wedge \mu - d\Phi_K\ \wedge \mu_K \wedge dt.
$$

Under the solution mapping, we have

$$0 = \phi^* \omega = \phi^* d\Omega = d\phi^* \Omega$$

so that we obtain the conservation equations

$$\frac{\partial(\phi^* \Phi)}{\partial t} + \frac{\partial(\phi^* \Phi_K)}{\partial X_K} = 0.$$

The relation $\omega = d\Omega$ now yields

$$\begin{aligned}
&\Big(\rho_0 \phi_k - \frac{\partial \Phi}{\partial v_k}\Big) dv_k \wedge \mu + \Big(C_{kKlL}\phi_k - e_{KLM}\nu_{lM} + \frac{\partial \Phi_K}{\partial F_{lL}}\Big) dF_{lL} \wedge \mu_K \wedge dt \\
&\quad + \Big(\rho_0 f_k \phi_k + \psi_k v_k - \psi_{kK} F_{kK} + \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi_K}{\partial X_K}\Big) \mu \wedge dt \\
&\quad\quad + \Big(\phi_{kK} - \frac{\partial \Phi}{\partial F_{kK}}\Big) dF_{kK} \wedge \mu + \Big(\phi_{kK} + \frac{\partial \Phi_K}{\partial v_k}\Big) dv_k \wedge \mu_K \wedge dt \\
&\quad\quad\quad + \Big(\psi_k - \frac{\partial \Phi}{\partial x_k}\Big) dx_k \wedge \mu + \Big(\psi_{kK} + \frac{\partial \Phi_K}{\partial x_k}\Big) dx_k \wedge \mu_K \wedge dt = 0
\end{aligned}$$

from which we extract the following expressions

$$\begin{aligned}
\rho_0 \phi_k &= \frac{\partial \Phi}{\partial v_k}, \\
\phi_{kK} = \frac{\partial \Phi}{\partial F_{kK}} &= -\frac{\partial \Phi_K}{\partial v_k}, \\
\psi_k = \frac{\partial \Phi}{\partial x_k}, \ \ \psi_{kK} &= -\frac{\partial \Phi_K}{\partial x_k} \\
C_{kKlL}\phi_k - e_{KLM}\nu_{lM} + \frac{\partial \Phi_K}{\partial F_{lL}} &= 0, \\
\rho_0 f_k \phi_k + \psi_k v_k - \psi_{kK} F_{kK} + \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi_K}{\partial X_K} &= 0.
\end{aligned}$$

It follows from the fifth expression above by employing the first one, recalling the relation $e_{KLM} e_{KLN} = 2\delta_{MN}$ and evaluating its symmetric part with respect to indices K and L that

$$\begin{aligned}
&\nu_{lM} = \frac{1}{2} e_{KLM} \Big(\frac{1}{\rho_0} C_{kKlL} \frac{\partial \Phi}{\partial v_k} + \frac{\partial \Phi_K}{\partial F_{lL}}\Big) \\
&\frac{1}{\rho_0}(C_{kKlL} + C_{kLlK}) \frac{\partial \Phi}{\partial v_k} + \frac{\partial \Phi_K}{\partial F_{lL}} + \frac{\partial \Phi_L}{\partial F_{lK}} = 0.
\end{aligned}$$

Hence, the equations to be satisfied by the functions Φ and Φ_K depending on 19 variables are reduced to

$$\frac{\partial \Phi}{\partial F_{kK}} + \frac{\partial \Phi_K}{\partial v_k} = 0, \tag{8.7.7}$$
$$\frac{\partial \Phi_K}{\partial F_{lL}} + \frac{\partial \Phi_L}{\partial F_{lK}} + B_{KLkl}\frac{\partial \Phi}{\partial v_k} = 0,$$
$$\frac{\partial \Phi}{\partial t} + \frac{\partial \Phi_K}{\partial X_K} + \frac{\partial \Phi}{\partial x_k}v_k + \frac{\partial \Phi_K}{\partial x_k}F_{kK} + \frac{\partial \Phi}{\partial v_k}f_k = 0.$$

where the functions $B_{KLkl}(\mathbf{F})$ are defined by

$$B_{KLkl} = B_{LKkl} = B_{KLlk} = \frac{1}{\rho_0}(C_{kKlL} + C_{kLlK}). \tag{8.7.8}$$

enjoy several symmetries in subscripts depicted above that can be verified just by inspection. The system (8.7.7) contains 28 equations to determine only four functions Φ and Φ_K for an arbitrary stress potential Σ. In order to find the solution of this system, let us start by differentiating $(8.7.7)_2$ and employing $(8.7.7)_1$ to obtain

$$\frac{\partial^2 \Phi}{\partial F_{lL}\partial F_{mK}} + \frac{\partial^2 \Phi}{\partial F_{lK}\partial F_{mL}} = B_{KLkl}\frac{\partial^2 \Phi}{\partial v_k \partial v_m}$$

The left hand side of this expression is symmetric in indices l and m imposing the following restriction on the right hand side:

$$B_{KLkl}\frac{\partial^2 \Phi}{\partial v_k \partial v_m} = B_{KLkm}\frac{\partial^2 \Phi}{\partial v_k \partial v_l}.$$

For fixed K and L, this implies that the symmetric matrix $\partial^2\Phi/\partial v_k \partial v_l$ commutes with arbitrary symmetric matrices B_{KLkl}. According to the well known *Schur lemma* of the group theory [Russian born German mathematician Issai Schur (1875-1941)] this matrix can only be a multiple of the unit matrix. Therefore, we ought to write that

$$\frac{\partial^2 \Phi}{\partial v_k \partial v_l} = \rho_0 \phi(\mathbf{X}, t, \mathbf{x}, \mathbf{v}, \mathbf{F})\delta_{kl}$$

When $k \neq l$, we are evidently led to $\partial^2\Phi/\partial v_k \partial v_l = 0$.Hence we readily observe that

$$\frac{\partial^3 \Phi}{\partial v_k \partial v_l \partial v_l} = \rho_0 \frac{\partial \phi}{\partial v_k} = 0$$

so that the function ϕ becomes independent of the variables $\mathbf{v} = \{v_k\}$. We thus obtain

$$\Phi = \frac{1}{2}\rho_0\phi(\mathbf{X},t,\mathbf{x},\mathbf{F})v_k v_k + \lambda_k(\mathbf{X},t,\mathbf{x},\mathbf{F})v_k + \mu(\mathbf{X},t,\mathbf{x},\mathbf{F})$$

Let us now insert this expression into $(8.7.7)_1$ to obtain

$$\frac{\partial\Phi_K}{\partial v_k} = -\frac{1}{2}\rho_0\frac{\partial\phi}{\partial F_{kK}}v_m v_m - \frac{\partial\lambda_m}{\partial F_{kK}}v_m - \frac{\partial\mu}{\partial F_{kK}}$$

and

$$\frac{\partial^2\Phi_K}{\partial v_k\partial v_l} = -\rho_0\frac{\partial\phi}{\partial F_{kK}}v_l - \frac{\partial\lambda_l}{\partial F_{kK}}.$$

The symmetry on the left hand side with respect to indices k and l now requires that

$$\rho_0\Big(\frac{\partial\phi}{\partial F_{kK}}v_l - \frac{\partial\phi}{\partial F_{lK}}v_k\Big) + \frac{\partial\lambda_l}{\partial F_{kK}} - \frac{\partial\lambda_k}{\partial F_{lK}} = 0.$$

Since ϕ and $\boldsymbol{\lambda}$ do not depend on $\mathbf{v}$, we immediately obtain

$$\frac{\partial\phi}{\partial F_{kK}} = 0,$$
$$\frac{\partial\lambda_l}{\partial F_{kK}} = \frac{\partial\lambda_k}{\partial F_{lK}}.$$

Hence, we see that $\phi = \phi(\mathbf{X},t,\mathbf{x})$ We thus conclude that

$$\Phi_K = -\frac{1}{2}\frac{\partial\lambda_l}{\partial F_{kK}}v_k v_l - \frac{\partial\mu}{\partial F_{kK}}v_k + \Psi_K(\mathbf{X},t,\mathbf{x},\mathbf{F})$$

In order to determine the arbitrary functions appearing in Φ and Φ_K, we have to introduce these expressions into $(8.7.7)_2$ and $(8.7.7)_3$. After tedious, but not overly complicated manipulations, which we abstain from repeating them here, we arrive at the following result when $f_k = 0$

$$\begin{aligned}\Phi &= a\Big(\Sigma + \frac{1}{2}\rho_0 v_k v_k\Big) + b_k\rho_0 v_k + c_k e_{klm}\rho_0 x_l v_m + d_L\rho_0 x_{k,L}v_k\\ \Phi_K &= -aT_{Kk}v_k - b_kT_{Kk} - c_k e_{klm}x_l T_{Km}\\ &\qquad + d_L\Big[\Big(\Sigma - \frac{1}{2}\rho_0 v_k v_k\Big)\delta_{KL} - T_{Kk}x_{k,L}\Big]\end{aligned}$$

where a, b_k, c_k and d_L are arbitrary constants. A reader interested with details may be referred to the work mentioned above. Therefore, independent conservation laws will be, respectively

$$\frac{\partial}{\partial t}\Big(\Sigma+\frac{1}{2}\rho_0 v_k v_k\Big)=\frac{\partial}{\partial X_K}(T_{Kk}v_k), \qquad \textit{balance of energy} \tag{8.7.9}$$

$$\rho_0\frac{\partial v_k}{\partial t}=\frac{\partial T_{Kk}}{\partial X_K}, \qquad \textit{balance of linear momentum}$$

$$\rho_0\frac{\partial}{\partial t}(e_{klm}x_l v_m)=\frac{\partial}{\partial X_K}(e_{klm}x_l T_{Km}), \; \textit{balance of angular momentum}$$

$$\rho_0\frac{\partial}{\partial t}(x_{k,L}v_k)+\frac{\partial}{\partial X_K}\Big[\Big(\Sigma-\frac{1}{2}\rho_0 v_k v_k\Big)\delta_{KL}-T_{Kk}x_{k,L}\Big]=0$$

The first three expressions corresponds in the framework of the classical mechanics to conservation laws to which every correctly formulated conservative system must obey. However, the last conservation law is of different character and it is peculiar only to the field equations of elasticity. If we integrate the conservation laws in the differential form on the region Ω and employ the divergence theorem we obtain

$$\frac{\partial}{\partial t}\int_\Omega \Phi\,dV+\int_{\partial\Omega}\Phi_K N_K\,dA=0$$

where the vector $\mathbf{N}$ is the unit exterior normal to the boundary $\partial\Omega$ of the region Ω. Hence, conservation laws in integral form are given by

$$\frac{\partial}{\partial t}\int_\Omega\Big(\Sigma+\frac{1}{2}\rho_0|\mathbf{v}|^2\Big)\,dV-\int_{\partial\Omega}T_{Kk}v_k N_K\,dA=0, \tag{8.7.10}$$

$$\frac{\partial}{\partial t}\int_\Omega \rho_0 v_k\,dV-\int_{\partial\Omega}T_{Kk}N_K\,dA=0,$$

$$\frac{\partial}{\partial t}\int_\Omega e_{klm}\rho_0 x_l v_m\,dV-\int_{\partial\Omega}e_{klm}x_l T_{Km}N_K\,dA=0,$$

$$\frac{\partial}{\partial t}\int_\Omega \rho_0 x_{k,L}v_k\,dV+\int_{\partial\Omega}\Big[\Big(\Sigma-\frac{1}{2}\rho_0|\mathbf{v}|^2\Big)N_L-T_{Kk}x_{k,L}N_K\Big]dA=0$$

The last integral is the non-linear dynamical counterpart of the J-integral that is frequently utilised in fracture mechanics ■

8.8. THE COHOMOLOGY OF DE RHAM

In Chapter VI, we had shown through the homotopy operator that all closed forms on a differentiable manifold M are *locally* exact. This property may not valid, however, *globally*, in other words, over the entire manifold. That the character of the connection between closed and exact forms depends only on the topology of the manifold, particularly on the *holes* within, but not on its differentiable structure has been demonstrated by de Rham through the investigation of cohomology groups on the module of exterior forms and homology groups on the topology of the manifold.

We had already seen that all closed and exact forms defined on a differentiable manifold M^m constitute subalgebras $\mathcal{C}(M)$ and $\mathcal{E}(M)$ of the exterior algebra $\Lambda(M)$, respectively, on $\mathbb{R}$ whereas $\mathcal{C}^k(M)$ and $\mathcal{E}^k(M)$ are vector subspaces of the module $\Lambda^k(M)$ on $\mathbb{R}$ [*see* Theorem 5.8.3]. We obviously have $\mathcal{E}^k(M) \subseteq \mathcal{C}^k(M)$, namely, $\mathcal{E}^k(M)$ is a subspace of $\mathcal{C}^k(M)$. *We shall now define a relation* $\sim$ *on the vector space* $\mathcal{C}^k(M)$ *as follows*: *two closed forms are related if their difference is an exact form*. Hence, for two forms $\omega_1, \omega_2 \in \mathcal{C}^k(M)$ the relation $\omega_1 \sim \omega_2$ implies that $\omega_1 - \omega_2 = d\theta$ where $\theta \in \Lambda^{k-1}(M)$. $\sim$ is an equivalence relation. Indeed, $\omega \sim \omega$ since $0 = \omega - \omega = d0$ *so the relation is reflexive*. If $\omega_1 \sim \omega_2$, then one has $\omega_2 - \omega_1 = d(-\theta)$ and $\omega_2 \sim \omega_1$ *so the relation is symmetric*. If $\omega_1 \sim \omega_2$ and $\omega_2 \sim \omega_3$, then we get $\omega_1 - \omega_2 = d\theta_1$, $\omega_2 - \omega_3 = d\theta_2$ and, consequently, $\omega_1 - \omega_3 = d\theta_1 + d\theta_2 = d(\theta_1 + \theta_2)$ and $\omega_1 \sim \omega_3$ *so the relation is transitive.* Therefore, the vector space $\mathcal{C}^k(M)$ is partitioned into disjoint equivalence classes. An equivalence class associated with a form $\omega \in \mathcal{C}^k(M)$ will be the set

$$[\omega] = \{\omega + \sigma : \sigma \in \mathcal{E}^k(M)\} = \{\omega + d\theta : \theta \in \Lambda^{k-1}(M)\}. \quad (8.8.1)$$

This set is called a ***cohomology class***. All forms belong to the cohomology class of the form ω are called as ***cohomologous forms*** to ω. We have seen on p. 5 that the quotient set of these equivalence classes may be equipped with a structure of a linear vector space on $\mathbb{R}$. We shall denote the *quotient space* of $\mathcal{C}^k(M)$ with respect to its subspace $\mathcal{E}^k(M)$ by the *vector space*

$$H^k(M) = \mathcal{C}^k(M)/\mathcal{E}^k(M). \quad (8.8.2)$$

If we consider the cochain complex (5.8.6) given by

$$\Lambda^0(M) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^k(M) \xrightarrow{d} \Lambda^{k+1}(M) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^m(M) \to 0$$

we see that this quotient space is also expressible in the equivalent form

$$H^k(M) = \mathcal{N}_k(d)/\mathcal{R}_k(d).$$

The zero element of this vector space is given by $[\mathbf{0}] = \mathcal{E}^k(M)$. Since the linear vector space $H^k(M)$ is known to be an Abelian group, it will thus be named as the ***kth de Rham cohomology group*** of the manifold M. The dimension $b_k(M)$ of the linear vector space $H^k(M)$ which is the number of the linearly independent equivalence classes is called by Poincaré as the ***kth***

Betti number [after Italian mathematician Enrico Betti (1823-1892)] of the manifold M. Evidently b_k is a positive integer that might be infinite. As we shall observe later that Betti numbers are dependent on the topology of the manifold M, particularly on its connectedness and number of holes within M. The sum

$$\chi(M) = \sum_{k=0}^{m}(-1)^k b_k(M) \tag{8.8.3}$$

formed by Betti numbers is called the ***Euler-Poincaré characteristic*** of the manifold M [Swiss mathematician Leonhard Euler (1707-1783)]. In order that all closed k-forms on a manifold are to be exact k-forms, we must clearly have $\mathcal{C}^k(M) = \mathcal{E}^k(M)$. This is of course tantamount to say that $H^k(M) = \mathbf{0}$. Hence, $b_k(M) = 0$ in such a case.

Since $\mathcal{E}^0(M) = \{0\}$, we naturally get $H^0(M) = \mathcal{C}^0(M)$. On the other hand, if a function $f \in \Lambda^0(M)$ is closed, that is, if $df = 0$, we find that f is constant. When M is a connected manifold, the function f takes of course a unique constant value on M. Hence, we get $H^0(M) = \mathbb{R}$ and consequently $b_0(M) = 1$. But, if the manifold M is a disconnected union of r connected components, the function will be allowed to take a different constant value on each component. So we find $H^0(M) = \mathbb{R}^r$ and $b_0(M) = r$. If $k > m$, then all k-forms on M vanish leading to the result $H^k(M) = \mathbf{0}$. Since all closed forms on $\mathbb{R}^m$ with $m > 0$ are exact [*see p.* 334], we deduce that $\mathcal{C}^k(\mathbb{R}^m) = \mathcal{E}^k(\mathbb{R}^m)$. Accordingly, we obtain $H^k(\mathbb{R}^m) = \mathbf{0}$ for $1 \le k \le m$. Thus Betti numbers become $b_0(\mathbb{R}^m) = 1$, $b_k(\mathbb{R}^m) = 0$, $1 \le k \le m$. When M is a contractible manifold, we similarly have $H^k(M) = \mathbf{0}, 1 \le k \le m$ and Betti numbers are found to be $b_0(M) = 1$, $b_k(M) = 0$, $1 \le k \le m$.

The direct sum $H(M) = \overset{m}{\underset{k=0}{\oplus}} H^k(M)$ is a linear vector space on $\mathbb{R}$. Let us take the cohomology classes $[\omega] \in H^k(M)$, $[\sigma] \in H^l(M)$ into consideration where the representatives of classes are $\omega \in \mathcal{C}^k(M)$ and $\sigma \in \mathcal{C}^l(M)$. An operation of multiplication $\sqcup$ on $H(M)$ will now be defined by

$$[\omega] \sqcup [\sigma] = [\omega \wedge \sigma]. \tag{8.8.4}$$

Endowed with this operation, $H(M)$ is named as the ***de Rham algebra***.

Let us now consider smooth manifolds M and N and a smooth mapping $\phi : M \to N$. We know that the mapping ϕ generates the pull-back operator $\phi^* : \Lambda(N) \to \Lambda(M)$. Since the operator ϕ^*, that is linear on $\mathbb{R}$, and d are commutative, ϕ^* transforms closed forms into closed forms and also exact forms into exact forms. In fact, owing to Theorem 5.8.2 for a form $\omega \in \mathcal{C}^k(N)$ we immediately obtain $\phi^*\omega \in \mathcal{C}^k(M)$ because of the relation

$0 = \phi^*(d\omega) = d(\phi^*\omega)$. In the same manner, for a form $\omega \in \mathcal{E}^k(N)$ we have $\omega = d\sigma$ and we obtain $\phi^*\omega = \phi^*(d\sigma) = d(\phi^*\sigma)$, or $\phi^*\omega \in \mathcal{E}^k(M)$. For a closed form $\omega \in \mathcal{C}^k(N)$ the vector $[\omega] \in H^k(N)$ is the set of forms $\omega + d\theta$ for all forms $\theta \in \Lambda^{k-1}(N)$. In this case, we get

$$\phi^*(\omega + d\theta) = \phi^*\omega + d(\phi^*\theta) \in [\phi^*\omega]$$

so that we obtain $\phi^*[\omega] = [\phi^*\omega] \in H^k(M)$ for every $[\omega] \in H^k(N)$. This means that a *linear transformation* $\phi^* : H(N) \to H(M)$ between de Rham algebras arises from the mapping ϕ. ϕ^* *is actually a homomorphism*. Indeed if $[\omega], [\sigma] \in H(N)$, we can easily obtain

$$\begin{aligned}\phi^*([\omega] \sqcup [\sigma]) &= \phi^*[\omega \wedge \sigma] = [\phi^*(\omega \wedge \sigma)] = [\phi^*\omega \wedge \phi^*\sigma] \\ &= [\phi^*\omega] \sqcup [\phi^*\sigma].\end{aligned}$$

If ϕ is a diffeomorphism, then $\phi^* : H(N) \to H(M)$ becomes naturally an isomorphism.

Example 8.8.1. We consider a submanifold in $\mathbb{R}^{n+1}$ given by the unit sphere $\mathbb{S}^n$. We suppose that that the poles are the points defined by $x^{n+1} = \pm 1$. If we employ the hyperspherical coordinates introduced on *p.* 412 satisfying the conditions $0 \leq \phi_1, \ldots, \phi_{n-1} \leq \pi$ and $0 \leq \phi_n \leq 2\pi$, we know that the volume form on $\mathbb{S}^n$ can be chosen as

$$\mu = \sin^{n-1}\phi_1 \sin^{n-2}\phi_2 \cdots \sin\phi_{n-1}\, d\phi_1 \wedge d\phi_2 \wedge \cdots \wedge d\phi_n \in \Lambda^n(\mathbb{S}^n)$$

Since $d\mu = 0$, μ is a closed form. But it is not an exact form. Indeed, because one has $\partial\mathbb{S}^n = 0$, an exact form $\omega = d\sigma \in \Lambda^n(\mathbb{S}^n)$ must satisfy the condition

$$\int_{\mathbb{S}^n} \omega = \int_{\mathbb{S}^n} d\sigma = \int_{\partial\mathbb{S}^n} \sigma = 0$$

in accordance with the Stokes theorem. However, we had already seen that [*see p.* 413]

$$\int_{\mathbb{S}^n} \mu = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\big(\frac{n+1}{2}\big)} \neq 0.$$

We shall now try to demonstrate the following proposition: *A closed form $\omega \in \Lambda^n(\mathbb{S}^n) = \mathcal{C}^n(\mathbb{S}^n)$ is an exact form if and only if $\int_{\mathbb{S}^n} \omega = 0$*. If the form ω is exact, this condition is satisfied straightforwardly as is seen above. We shall use the method of mathematical induction to show that it is also the necessary condition. To this end, we shall first prove this proposition for

$n = 1$. The embedding $\phi : \mathbb{R} \to \mathbb{S}^1 \subset \mathbb{R}^2$ prescribed by $\phi(\theta) = e^{i\theta}$ determines the 1-dimensional manifold $\mathbb{S}^1$. If $\omega \in \Lambda^1(\mathbb{S}^1)$, then we have $\phi^*\omega = f(\theta)\, d\theta$. In order that this form is to be uniquely defined, the function must be 2π-periodic. We then introduce a function $F(\theta) = \int_0^\theta f(\tau)\, d\tau$. If $\int_{\mathbb{S}^1} \omega = 0$, then we get

$$0 = \int_{\mathbb{S}^1} \omega = \int_\theta^{\theta+2\pi} f(\tau)\, d\tau = F(\theta + 2\pi) - F(\theta), \;\; \forall \theta \in \mathbb{R}.$$

This means that $F(\theta)$ ought to be a 2π-periodic function. Thus, a *unique* function $G \in \Lambda^0(\mathbb{S}^1)$ may be defined through the relation $\phi^* G = F$. Hence, we can write $\phi^*\omega = f(\theta)\, d\theta = dF = d(\phi^* G) = \phi^* dG$ from which it follows that $\omega = dG$, i.e., $\omega \in \mathcal{E}^1(\mathbb{S}^1)$. In order to apply the mathematical induction, we shall now suppose that *the proposition in question is true in the manifold* $\mathbb{S}^{n-1}$ and then try to prove that it will also be true in the manifold $\mathbb{S}^n$. We know that the manifold $\mathbb{S}^n$ can be prescribed by an atlas with two charts. The following open sets of these charts

$$U_1 = \{\mathbf{x} \in \mathbb{S}^n : x^{n+1} < 1\} \quad \text{and} \quad U_2 = \{\mathbf{x} \in \mathbb{S}^n : x^{n+1} > -1\}$$

yield $U_1 \cup U_2 = \mathbb{S}^n$ and by a stereographic projection [*see p.* 81] these sets become homeomorphic to $\mathbb{R}^n$. We define the *north* and *south hemispheres* of $\mathbb{S}^n$ as the closed sets

$$N = \{\mathbf{x} \in \mathbb{S}^n : x^{n+1} \geq 0\} \subset U_2 \;\text{ and }\; S = \{\mathbf{x} \in \mathbb{S}^n : x^{n+1} \leq 0\} \subset U_1,$$

respectively. We see at once $N \cup S = \mathbb{S}^n$ and we observe that $N \cap S = \{\mathbf{x} \in \mathbb{S}^n : x^{n+1} = 0\} = \mathbb{S}^{n-1}$. The latter set can of course be taken as the common boundary of N and S and it should be oriented in reverse directions whether it is considered as the boundary ∂N or ∂S. Since the sets U_1 and U_2 are homeomorphic to $\mathbb{R}^n$, they are contractible sets. Let $\omega \in \mathcal{C}^n(\mathbb{S}^n)$ be a closed form satisfying the condition $\int_{\mathbb{S}^n} \omega = 0$. According to the Poincaré lemma, restrictions of the form ω to regions U_1 and U_2 are exact forms, namely, there exist forms $\sigma_1 \in \Lambda^{n-1}(U_1)$ and $\sigma_2 \in \Lambda^{n-1}(U_2)$ such that the relations $\omega|_{U_1} = d\sigma_1$ and $\omega|_{U_2} = d\sigma_2$ are held. Therefore, if we choose that ∂S is positively oriented, then the Stokes theorem leads to

$$\begin{aligned} 0 &= \int_{\mathbb{S}^n} \omega = \int_S \omega + \int_N \omega = \int_S d\sigma_1 + \int_N d\sigma_2 = \int_{\partial S} \sigma_1 + \int_{\partial N} \sigma_2 \\ &= \int_{\mathbb{S}^{n-1}} \sigma_1 - \int_{\mathbb{S}^{n-1}} \sigma_2 = \int_{\mathbb{S}^{n-1}} (\sigma_1 - \sigma_2). \end{aligned}$$

Hence, the integral of the form $(\sigma_1 - \sigma_2)|_{\mathbb{S}^{n-1}} \in \Lambda^{n-1}(\mathbb{S}^{n-1})$ on $\mathbb{S}^{n-1}$ vanishes. According to our assumption the form $(\sigma_1 - \sigma_2)|_{\mathbb{S}^{n-1}}$ is *exact*. Let us now define a smooth mapping $\psi : U \to \mathbb{S}^{n-1}$ on the open set $U = U_1 \cap U_2$ by assigning to each point $\mathbf{x} \in U$ the point of intersection of the meridian through the point $\mathbf{x}$ with the equator $\mathbb{S}^{n-1}$. In this case, the form $(\sigma_1 - \sigma_2)|_U = \psi^*(\sigma_1 - \sigma_2)|_{\mathbb{S}^{n-1}}$ on U will also be exact. Thus, there exists a form $\alpha \in \Lambda^{n-2}(U)$ such that $(\sigma_1 - \sigma_2)|_U = d\alpha$. Let us choose a form $\beta \in \Lambda^{n-2}(\mathbb{S}^n)$ so that one gets $d\beta|_U = d\alpha$ on U and introduce a form $\sigma \in \Lambda^{n-1}(\mathbb{S}^n)$ as follows

$$\sigma = \begin{cases} \sigma_1, & \text{on } U_1, \\ \sigma_2 + d\beta, & \text{on } U_2. \end{cases}$$

On $U = U_1 \cap U_2$, we find $\sigma_1 = \sigma_2 + d\beta$ and $\sigma_1 - \sigma_2 = d\beta = d\alpha$ whence we conclude that $\omega = d\sigma$ and $\omega \in \mathcal{E}^n(\mathbb{S}^n)$. We have thus shown that if we assume that the proposition is true for $n-1$, then it becomes also true for n. Since we have already seen that the proposition is true for $n = 1$, we are led to the conclusion that it is true for every n.

We shall now demonstrate that if $\omega \in \Lambda^n(\mathbb{S}^n)$ is a closed form, we can always find a number $c \in \mathbb{R}$ such that $\omega - c\mu$ is rendered as an exact form. On employing hyperspherical coordinates, we can generally express this form as follows

$$\omega = f(\phi_1, \ldots, \phi_{n-1}, \phi_n)\, d\phi_1 \wedge \cdots \wedge d\phi_{n-1} \wedge d\phi_n$$

where the function f is π-periodic in variables $\phi_1, \ldots, \phi_{n-1}$ and 2π-periodic in the variable ϕ_n. We thus get

$$\int_{\mathbb{S}^n} \omega = \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} f(\phi_1, \ldots, \phi_{n-1}, \phi_n)\, d\phi_1 \cdots d\phi_{n-1} d\phi_n.$$

Let us choose a real number c as follows

$$c = \frac{\displaystyle\int_{\mathbb{S}^n} \omega}{\displaystyle\int_{\mathbb{S}^n} \mu} = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2\pi^{\frac{n+1}{2}}} \int_{\mathbb{S}^n} \omega.$$

With this choice the closed form $\alpha = \omega - c\mu \in \Lambda^n(\mathbb{S}^n)$ will clearly satisfy the condition

$$\int_{\mathbb{S}^n} \alpha = 0.$$

Hence α is an exact form, i.e., $\alpha \in \mathcal{E}^n(\mathbb{S}^n)$. If $c = 0$, then the closed form $\omega \in \Lambda^n(\mathbb{S}^n)$ will obviously be an exact form. According to this result, all closed n-forms on $\mathbb{S}^n$ are to be cohomologous to a real constant multiple of the volume form μ. Hence, we can write $H^n(\mathbb{S}^n) = \mathbb{R}$. On the other hand, because $\mathbb{S}^n$ is a connected manifold we know that $H^0(\mathbb{S}^n) = \mathbb{R}$. Therefore, the corresponding Betti numbers are $b_0(\mathbb{S}^n) = b_n(\mathbb{S}^n) = 1$.

We shall now try to determine the cohomology groups $H^k(\mathbb{S}^n)$ for $1 \le k \le n-1$ of the n-sphere. To this end, we shall resort once more to mathematical induction. Let us first take a closed form $\omega \in \mathcal{C}^1(\mathbb{S}^n)$ into account. Since $d\omega = 0$ on U_1 and U_2, the Poincaré lemma implies that there are functions $f \in \Lambda^0(U_1)$ and $g \in \Lambda^0(U_2)$ such that one writes $\omega|_{U_1} = df$ and $\omega|_{U_2} = dg$. On the open set $U = U_1 \cap U_2$, we thus get $d(f-g)|_U = (df - dg)|_U = (\omega - \omega)|_U = 0$ so that we find $f - g = c =$ *constant*. Let us now define a function $\varphi \in \Lambda^0(\mathbb{S}^n)$ as follows

$$\varphi = \begin{cases} f & \text{on } U_1, \\ g + c & \text{on } U_2. \end{cases}$$

Then we obtain $\omega = d\varphi$, i.e., $\omega \in \mathcal{E}^1(\mathbb{S}^n)$. This means that every closed 1-form on $\mathbb{S}^n$ is exact. In consequence, we find $H^1(\mathbb{S}^n) = 0$. *Let us now assume that every closed $(k-1)$-form on $\mathbb{S}^n$ is exact.* We consider a closed form $\omega \in \mathcal{C}^k(\mathbb{S}^n)$. Since $d\omega = 0$ again on open sets U_1 and U_2, the Poincaré lemma indicates that there are the forms $\sigma_1 \in \Lambda^{k-1}(U_1)$ and $\sigma_2 \in \Lambda^{k-1}(U_2)$ so that one has $\omega|_{U_1} = d\sigma_1$ and $\omega|_{U_2} = d\sigma_2$. Thereby, we obtain

$$d(\sigma_1 - \sigma_2)|_U = (d\sigma_1 - d\sigma_2)|_U = (\omega - \omega)|_U = 0$$

on $U = U_1 \cap U_2$. We thus conclude that $\sigma_1 - \sigma_2 \in \mathcal{C}^{k-1}(U)$ and our assumption assures us that there exists a form $\alpha \in \Lambda^{k-2}(U)$ such that we have $(\sigma_1 - \sigma_2)|_U = d\alpha$. Let us now choose a form $\beta \in \Lambda^{k-2}(\mathbb{S}^n)$ as to satisfy the relation $d\beta|_U = d\alpha$ on U and define a form $\sigma \in \Lambda^{k-1}(\mathbb{S}^n)$ in the following manner

$$\sigma = \begin{cases} \sigma_1 & \text{on } U_1, \\ \sigma_2 + d\beta & \text{on } U_2. \end{cases}$$

We thus conclude that $\omega = d\sigma$, that is, $\omega \in \mathcal{E}^k(\mathbb{S}^n)$. Hence, the mathematical induction prove the proposition that *every closed k-form on $\mathbb{S}^n$ satisfying the condition $1 \le k \le n-1$ is exact* so that one obtains $H^k(\mathbb{S}^n) = 0$ for $1 \le k \le n-1$. Betti numbers thus become $b_k(\mathbb{S}^n) = 0, 1 \le k \le n-1$. The relation (8.8.3) then yields the Euler-Poincaré characteristic of the n-sphere as $\chi(\mathbb{S}^n) = b_0 + (-1)^n b_n = 1 + (-1)^n$. Hence $\chi(\mathbb{S}^n) = 0$ if n is an odd number and $\chi(\mathbb{S}^n) = 2$ if n is an even number. ■

The salient property of the sphere $\mathbb{S}^n$ is that it is a connected, oriented and compact manifold.

We had seen that all singular k-chains on a manifold M constitute a linear vector space $C_k(M)$ [*see* p. 421]. We know that the set $\{C_k(M)\}$ constitutes a chain complex under the boundary operator ∂. Let us then consider the subset of $C_k(M)$ formed by k-cycles $\overset{\circ}{C}_k(M) = \{c_k \in C_k(M) : \partial c_k = 0\}$. Because the sum of two k-cycles and a real multiple of k-cycle is also a k-cycle, $\overset{\circ}{C}_k(M)$ is a subspace of the linear vector space $C_k(M$, hence it is a linear vector space by itself. Let us denote the set of k-cycles that are boundaries of $(k+1)$-chains by

$$B_k(M) = \{c_k = \partial b_{k+1} : \ b_{k+1} \in C_{k+1}(M)\}.$$

Evidently, the set $B_k(M)$ is also a linear vector space and it is clear that $B_k(M) \subseteq \overset{\circ}{C}_k(M)$ since $\partial c_k = 0$ if $c_k \in B_k(M)$. We now define a relation $\sim$ on $\overset{\circ}{C}_k(M)$ as follows: $c'_k, c''_k \in \overset{\circ}{C}_k(M)$ are related if their difference is a boundary of a $(k+1)$-chain, namely $c'_k \sim c''_k$ if only $c'_k - c''_k = \partial b_{k+1} \in B_k(M)$. Two cycles whose difference is a boundary will be called ***homologous cycles***. It can readily be verified that this relation defining the ***homology*** on the manifold M is an equivalence relation. Equivalence classes, in other words ***homology classes***, are defined by

$$[c_k] = \{c_k + \partial b_{k+1} : c_k \in \overset{\circ}{C}_k(M), b_{k+1} \in C_{k+1}(M)\}$$

Let us denote the quotient space generated by those classes by

$$H_k(M) = \overset{\circ}{C}_k(M)/B_k(M). \tag{8.8.5}$$

$H_k(M)$ is a linear vector space on real numbers $\mathbb{R}$. As such it is an Abelian group and is named as the ***kth differentiable singular homology group*** of the manifold M. If we consider the chain complex (8.4.13), this quotient space can also be expressed equivalently as

$$H_k(M) = \mathcal{N}_k(\partial)/\mathcal{R}_k(\partial).$$

We can roughly say that homology groups illustrate the existence and the distribution of holes in topological spaces. The zero element of the vector space $H_k(M)$ is naturally given as $[\mathbf{0}] = B_k(M)$.

Sometimes, it does not prove to be very convenient to work with a chain complex with decreasing indices. Especially, quite a difficult problem arises if we wish to establish a relationship between the de Rham cohomology groups and the homology groups. To circumvent this obstacle we may employ the cochain complex (8.4.15) with the coboundary operator $\mathfrak{d}$. Then, the ***kth singular cohomology group*** M is defined as the following quotient space of the vector space $\mathcal{N}_k(\mathfrak{d})$ with respect to its subspace $\mathcal{R}_k(\mathfrak{d})$

$$\mathcal{H}_k(M) = \mathcal{N}_k(\mathfrak{d})/\mathcal{R}_k(\mathfrak{d}).$$

Hence, *the vector space $\mathcal{H}_k(M)$ is the quotient space of k-cocycles with respect to k-boundaries*. The equivalence class $[f_k] \in \mathcal{H}_k(M)$ related to an element $f_k \in C_k^*(M)$ is the set of all linear functionals on $C_k(M)$ given by $[f_k] = f_k + \mathfrak{d}g_{k-1}$ for all $g_{k-1} \in C_{k-1}^*(M)$ and $\mathfrak{d}f_k = 0$. $[f_k]$ is known to be a ***singular cohomology class***. We shall now try to demonstrate the following proposition:

The kth singular cohomology group is isomorphic to the dual space $H_k^*(M)$ of the kth singular homology group $H_k(M)$, namely, there is a natural isomorphism $\mathfrak{C}_k:\mathcal{H}_k(M) \to H_k^*(M)$ such that $\mathfrak{C}_k([f_k]) \in H_k^*(M)$.

Let us consider an arbitrary equivalence class $[f_k] \in \mathcal{H}_k(M)$ where the linear functional $f_k \in C_k^*(M)$ satisfying $\mathfrak{d}f_k = 0$ is a representative of this class. Consequently, f_k is also a linear functional on the subspace $\overset{\circ}{C}_k(M)$ of k-cycles. On the other hand, the relation $0 = \mathfrak{d}f_k(c_{k+1}) = f_k(\partial c_{k+1})$ implies that f_k vanishes on all k-boundaries in the form $c_k = \partial b_{k+1}$ in $C_k(M)$ forming the subspace $B_k(M)$. Hence, f_k becomes a linear functional defined on the homology group $H_k(M)$ because if $[c_k] \in H_k(M)$ where $c_k \in C_k(M)$ is arbitrary, we get $f_k(c_k + \partial b_{k+1}) = f_k(c_k)$. Thus the value of the functional f_k on an equivalence class $[c_k]$ is independent of the representative of this class. However, in order to say that the linear functional $f_k \in H_k^*(M)$ is well defined, we have to show that its value is also independent of the representative of the equivalence class $[f_k]$. This is, however, easily deduced from

$$\begin{aligned} f_k + \mathfrak{d}g_{k-1})(c_k + \partial b_{k+1}) &= f_k(c_k) + \mathfrak{d}g_{k-1}(c_k) + \mathfrak{d}g_{k-1}(\partial b_{k+1}) \\ &= f_k(c_k) \end{aligned}$$

since $\mathfrak{d}g_{k-1}(c_k) = g_{k-1}(\partial c_k) = 0$ because $\partial c_k = 0$ and $\mathfrak{d}g_{k-1}(\partial b_{k+1}) = g_{k-1}(\partial^2 b_{k+1}) = 0$ because $\partial^2 = 0$. We have thus found that to each element of $\mathcal{H}_k(M)$ we can assign a unique element of $H_k^*(M)$. Obviously, this mapping is linear and in order to prove that it is an isomorphism, we must show that it is both injective and surjective. Since equivalence classes are disjoint, injectivity of the mapping is evident. To show surjectivity let us consider a functional $f_k \in H_k^*(M)$. Since $\mathfrak{d}f_k \in C_{k+1}^*(M)$, we get $0 = \mathfrak{d}f_k(c_{k+1}) = f_k(\partial c_{k+1})$ for all $c_{k+1} \in \overset{\circ}{C}_{k+1}(M)$. We thus find $\mathfrak{d}f_k = 0$. We can then define the set of functionals $[f] = f_k + \mathfrak{d}g_{k-1}$ with $g_{k-1} \in C_{k-1}^*(M)$. Clearly, $[f] \in \mathcal{H}_k(M)$. On the other hand, for $c_k \in \overset{\circ}{C}_k(M)$ we get

$$(f_k + \mathfrak{d} g_{k-1})(c_k) = f_k(c_k) + g_{k-1}(\partial c_k) = f_k(c_k).$$

Therefore, we have shown that a functional f_k in $H_k^*(M)$ is the image of an equivalence class $[f]$ in $\mathcal{H}_k(M)$, namely, the mapping is surjective. Thus, *the mapping* $\mathfrak{C}_k{:}\mathcal{H}_k(M) \to H_k^*(M)$ *is an isomorphism*. □

Let us now define a mapping $\mathcal{B}_k : \Lambda^k(M) \times C_k(M) \to \mathbb{R}$ as the integral of a k-form over a k-chain as follows

$$\mathcal{B}_k(\omega, c_k) = \int_{c_k} \omega \in \mathbb{R}. \tag{8.8.6}$$

Naturally, in order that this definition is justifiable, the integral (8.8.6) must exist. This mapping is obviously linear with respect to both the k-form ω and the k-chain c_k. In other words, $\mathcal{B}_k(\omega, c_k)$ is a bilinear, real valued functional. Whenever we consider a *fixed* form $\omega_0 \in \Lambda^k(M)$, then the real valued function

$$\mathcal{F}_{\omega_0}^{(k)}(c_k) = \mathcal{B}_k(\omega_0, c_k) = \int_{c_k} \omega_0 \tag{8.8.7}$$

turns out to be intrinsically a linear functional on the vector space $C_k(M)$. Thus, (8.8.6) is actually generating a mapping $\mathcal{F}_k : \Lambda^k(M) \to C_k(M)^*$ over real numbers from the *vector space* $\Lambda^k(M)$ into the dual space $C_k(M)^*$ designated by $\mathcal{F}_k(\omega) = \mathcal{F}_\omega^{(k)}$. The definition (8.8.7) signify at once that the mapping $\mathcal{F}_k$ is linear, in other words, it is a homomorphism.

Next, we introduce in similar fashion a real valued and bilinear functional $\overset{\circ}{\mathcal{B}}_k : H^k(M) \times H_k(M) \to \mathbb{R}$ through the relation

$$\overset{\circ}{\mathcal{B}}_k([\omega], [c_k]) = \int_{c_k} \omega \in \mathbb{R} \tag{8.8.8}$$

where the closed form $\omega \in C^k(M)$ and cycle $c_k \in \overset{\circ}{C}_k(M)$ are arbitrarily selected representatives of the equivalence classes $[\omega] \in H^k(M)$ and $[c_k] \in H_k(M)$. On the other hand, in order that the definition (8.8.8) bears a meaning the value of the functional must be independent of the chosen representatives of the equivalence classes. This can be proven quite easily, however, if we recall that

$$[\omega] = \{\omega + d\theta : \omega \in C^k(M), \theta \in \Lambda^{k-1}(M)\},$$
$$[c_k] = \{c_k + \partial b_{k+1} : c_k \in \overset{\circ}{C}_k(M), b_{k+1} \in C_{k+1}(M)\}$$

and then utilise the Stokes theorem to obtain

$$\int_{c_k+\partial b_{k+1}}(\omega+d\theta)=\int_{c_k}\omega+\int_{c_k}d\theta+\int_{\partial b_{k+1}}\omega+\int_{\partial b_{k+1}}d\theta$$
$$=\int_{c_k}\omega+\int_{\partial c_k}\theta+\int_{b_{k+1}}d\omega+\int_{\partial^2 b_{k+1}}\theta=\int_{c_k}\omega$$

for all forms $\theta \in \Lambda^{k-1}(M)$ and all boundaries $b_{k+1} \in C_{k+1}(M)$ where we employed the relations $d\omega = 0, \partial c_k = 0, \partial^2 b_{k+1} = 0$. It now clear that the functional (8.8.8) determines a homomorphism $\overset{\circ}{\mathcal{F}}_k : H^k(M) \to H_k(M)^*$ defined by

$$\overset{\circ}{\mathcal{F}}_k([\omega])([c_k]) = \overset{\circ}{\mathcal{B}}_k([\omega],[c_k]) = \int_{c_k}\omega \in \mathbb{R}$$

from the cohomology group $H^k(M)$ into the vector space $H_k(M)^*$ that is the dual of the homology group $H_k(M)$. We have seen above that the vector spaces $H_k(M)^*$ and $\mathcal{H}_k(M)$ are isomorphic. Therefore, $\overset{\circ}{\mathcal{F}}_k$ may as well be regarded as a homomorphism between $H^k(M)$ and $\mathcal{H}_k(M)$. We can now show the simple lemma given below:

Lemma 8.8.1. *M and N are smooth manifolds and $\phi : M \to N$ is a smooth mapping. Then the following diagram commutes:*

$$\begin{array}{ccc} H^k(N) & \xrightarrow{\phi^*} & H^k(M) \\ \downarrow \overset{\circ}{\mathcal{F}}_k & & \downarrow \overset{\circ}{\mathcal{F}}_k \\ \mathcal{H}_k(N) & \xrightarrow{\phi^*} & \mathcal{H}_k(M). \end{array}$$

We know that if $\omega \in \Lambda^k(N)$, then $\phi^*\omega = \omega \circ \phi \in \Lambda^k(M)$. Similarly, if $f \in C_k^*(N)$, we then obtain $f(c_k^*) = f\big(\phi(c_k)\big) = (f\circ\phi)(c_k) = \phi^* f(c_k)$ where $c_k \in C_k(M)$. Thus, $f\circ\phi = \phi^* f \in C_k^*(M)$. The relation (8.5.3) then requires that

$$\overset{\circ}{\mathcal{F}}_k(\phi^*[\omega])([c_k]) = \overset{\circ}{\mathcal{F}}_k([\omega])(\phi[c_k])$$

for all $[\omega] \in H^k(N)$ and $[c_k] \in H_k(M)$. □

De Rham's theorem proven in 1931 states that *if M is a Hausdorff, locally compact, second countable and oriented smooth manifold, then this homomorphism $\overset{\circ}{\mathcal{F}}_k$, called **de Rham homomorphism**, is actually an isomorphism*. In order to prove this theorem, we need first to investigate certain properties of Mayer-Vietoris sequences [after Austrian mathematicians

Walther Mayer (1887-1948) and supercentenarian Leopold Vietoris (1891-2002)].

Theorem 8.8.1 (Mayer-Vietoris). *Let M be an m-dimensional smooth manifold supporting a partition of unity and $U, V \subset M$ open subsets such that $U \cup V = M$. We consider the cochain complex*

$$\Lambda^0(M) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^k(M) \xrightarrow{d} \Lambda^{k+1}(M) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^m(M) \to 0$$

and the cohomology groups $H^k(M) = \mathcal{N}_k(d)/\mathcal{R}_k(d)$. Then for all $0 \le k \le m$, there exists a homomorphism $\Gamma : H^k(U \cap V) \to H^{k+1}(M)$ such that the following ***Mayer-Vietoris sequence*** *is exact*:

$$\cdots \xrightarrow{\Gamma} H^k(M) \xrightarrow{\varphi} H^k(U) \oplus H^k(V) \xrightarrow{\psi} H^k(U \cap V) \xrightarrow{\Gamma} H^{k+1}(M) \xrightarrow{\varphi} \cdots$$

The homomorphisms φ and ψ are defined by $\varphi = \mathcal{I}_3^* \oplus \mathcal{I}_4^*$ and $\psi = \mathcal{I}_1^* - \mathcal{I}_2^*$ where $\mathcal{I}_1 : U \cap V \to U, \mathcal{I}_2 : U \cap V \to V, \mathcal{I}_3 : U \to M$ and $\mathcal{I}_4 : V \to M$ are inclusion mappings and $\mathcal{I}_1^* : \Lambda^k(U) \to \Lambda^k(U \cap V)$, $\mathcal{I}_2^* : \Lambda^k(V) \to \Lambda^k(U \cap V)$, $\mathcal{I}_3^* : \Lambda^k(M) \to \Lambda^k(U)$, $\mathcal{I}_4^* : \Lambda^k(M) \to \Lambda^k(V)$ are corresponding pull-back operators..

Although we have proven only for Hausdorff, locally compact and second countable manifolds, we had mentioned that if the manifold M is paracompact, then for each open cover $\{U_\lambda : \lambda \in \Lambda\}$ of M there exists a partition of unity subordinate to this cover. We shall see that only the existence of the partition of unity will be crucial for our proof of this theorem.

In view of Theorem 1.2.3, we merely need to show that the short sequence

$$0 \to \Lambda^k(M) \xrightarrow{\varphi} \Lambda^k(U) \oplus \Lambda^k(V) \xrightarrow{\psi} \Lambda^k(U \cap V) \to 0$$

is exact. For a form $\omega \in \Lambda^k(M)$, we have

$$\varphi(\omega) = \big(\mathcal{I}_3^*(\omega), \mathcal{I}_4^*(\omega)\big) = (\omega|_U, \omega|_V).$$

If $\alpha \in \Lambda^k(U)$, $\beta \in \Lambda^k(V)$, then we get

$$\psi(\alpha, \beta) = \mathcal{I}_1^*(\alpha) - \mathcal{I}_2^*(\beta) = \alpha|_{U \cap V} - \beta|_{U \cap V}$$

We first demonstrate that the sequence is exact at $\Lambda^k(M)$. To this end, we only need to show that φ is injective. Let us take $\varphi(\omega) = (\omega|_U, \omega|_V) = (0, 0)$ that leads to $\omega|_U = 0$ and $\omega|_V = 0$. Since $U \cup V = M$, this implies that $\omega = 0$ which proves the injectivity.

In order to prove the exactness at $\Lambda^k(U) \oplus \Lambda^k(V)$ let us apply the operator $\psi \circ \varphi$ on a form $\omega \in \Lambda^k(M)$ to obtain

$$(\psi\circ\varphi)(\omega)=\psi(\omega|_U,\omega|_V)=\omega|_{U\cap V}-\omega|_{U\cap V}=0.$$

Hence, $\psi\circ\varphi=0$ on the module $\Lambda^k(M)$ so that we get $\mathcal{R}(\varphi)\subseteq\mathcal{N}(\psi)$. Next, let us consider $(\alpha,\beta)\in\mathcal{N}(\psi)$ which means that $\alpha|_{U\cap V}=\beta|_{U\cap V}$. Thus, there exists a form $\omega\in\Lambda^k(M)$ such that $\omega|_U=\alpha$ and $\omega|_V=\beta$. We then clearly write $(\alpha,\beta)=\varphi(\omega)$ implying that $\mathcal{N}(\psi)\subseteq\mathcal{R}(\varphi)$. Hence, we get $\mathcal{R}(\varphi)=\mathcal{N}(\psi)$ which proves the exactness.

To prove the exactness at $\Lambda^k(U\cap V)$ we just have to show that ψ is surjective. Since (U,V) is an open cover of M, there exists a partition of unity (f_1,f_2) subordinate to (U,V) such that *supp* $(f_1)\subset U$, *supp* $(f_2)\subset V$. Let $\sigma\in\Lambda^k(U\cap V)$. We define the forms $\lambda\in\Lambda^k(U)$ and $\mu\in\Lambda^k(V)$ as follows

$$\lambda=\begin{cases} f_1\sigma & \text{on } U\cap V\\ 0 & \text{on } U-supp\,(f_1)\end{cases},\quad \mu=\begin{cases} -f_2\sigma & \text{on } U\cap V\\ 0 & \text{on } V-supp\,(f_1)\end{cases}.$$

We then obtain $\psi(\lambda,\mu)=\lambda|_{U\cap V}-\mu|_{U\cap V}=f_1\sigma-(-f_2\sigma)=(f_1+f_2)\sigma$ $=\sigma$ that amounts to say that $\mathcal{R}(\psi)=\Lambda^k(U\cap V)$. We thus conclude that Mayer-Vietoris sequence is exact. □

In exactly the same fashion we can show that Mayer-Vietoris sequence

$$\cdots\xrightarrow{\Gamma'}\mathcal{H}_k(M)\xrightarrow{\varphi}\mathcal{H}_k(U)\oplus\mathcal{H}_k(V)\xrightarrow{\psi}\mathcal{H}_k(U\cap V)\xrightarrow{\Gamma'}\mathcal{H}_{k+1}(M)\xrightarrow{\varphi^*}\cdots$$

based on the cochain complex [*see* (8.4.15)]

$$\cdots\longrightarrow C^*_{k-1}(M)\xrightarrow{\mathfrak{d}}C^*_k(M)\xrightarrow{\mathfrak{d}}C^*_{k+1}(M)\longrightarrow\cdots$$

is exact. $\mathfrak{d}$ is the coboundary operator defined in (8.4.14). To prove the existence, we only have to show that the short sequence

$$0\to C^*_k(M)\xrightarrow{\varphi}C^*_k(U)\oplus C^*_k(V)\xrightarrow{\psi}C^*_k(U\cap V)\to 0$$

is exact. The inclusion operators $\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3$ and $\mathcal{I}_4$ are the same as those given above in Theorem 8.8.1. Pull-back operators

$$\begin{aligned}&\mathcal{I}_1^*:C^*_k(U)\to C^*_k(U\cap V),\ \mathcal{I}_2^*:C^*_k(V)\to C^*_k(U\cap V),\\ &\mathcal{I}_3^*:C^*_k(M)\to C^*_k(U),\qquad \mathcal{I}_2^*:C^*_k(V)\to C^*_k(U\cap V)\end{aligned}$$

simply produce restrictions of functionals. For instance, $\mathcal{I}_1^*(f)=f|_{U\cap V}$ for a functional $f\in C^*_k(U)$. For a functional $f\in C^*_k(M)$, we get

$$\varphi(f)=\big(\mathcal{I}_3^*(f),\mathcal{I}_4^*(f)\big)=(f|_U,f|_V)$$

and for $g \in C_k^*(U)$, $h \in C_k^*(V)$

$$\psi(g,h) = \mathcal{I}_1^*(g) - \mathcal{I}_2^*(h) = g|_{U\cap V} - h|_{U\cap V}.$$

A smooth manifold M is called a ***de Rham manifold*** if the homomorphism on M is an isomorphism.

Lemma 8.8.2. *An open convex subset $U \subseteq \mathbb{R}^n$ is a de Rham manifold.*

Since a convex open set in $\mathbb{R}^n$ is star-shaped, the Poincaré lemma is applicable. Hence we find that $H^k(U) = 0$ for $k > 0$ and $H^0(U) = \mathbb{R}$. This automatically implies that $\mathcal{H}_k(U) = 0$ for $k > 0$ and $\mathcal{H}_0(U) = \mathbb{R}$ since the dual space of $\mathbb{R}$ is also $\mathbb{R}$. Therefore we only have to demonstrate that $\mathring{\mathcal{F}}_0 : H^0(U) \to \mathcal{H}_0(U)$ is an isomorphism. But elements of $H^0(U)$ are constant functions and a σ_0 singular simplex is just a single point. Thus, $\mathring{\mathcal{F}}_0$ assigns the same real number to a real number. □

Lemma 8.8.3. *Let $\{U_\lambda : \lambda \in \Lambda\}$ be a class of open, pairwise disjoint, de Rham subsets of a smooth manifold M. Then $U = \bigcup_{\lambda\in\Lambda} U_\lambda$ is also a de Rham manifold.*

In order to prove this lemma, we must show that the following diagram commutes isomorphically:

$$\begin{array}{ccc} H^k(U) & \xrightarrow{\mathcal{A}} & \bigcup_{\lambda\in\Lambda} H^k(U_\lambda) \\ \downarrow \mathfrak{I}_k & & \downarrow \bigoplus_{\lambda\in\Lambda} \mathring{\mathcal{F}}_{k,\lambda} \\ \mathcal{H}_k(U) & \xrightarrow{\mathcal{B}} & \bigcup_{\lambda\in\Lambda} \mathcal{H}_k(U_\lambda). \end{array}$$

In view of Lemma 8.8.1 the diagram commutes. To show that the homomorphism $\mathfrak{I}_k$ is an isomorphism, we only need to prove that $\mathcal{A}$ and $\mathcal{B}$ are isomorphisms because $\mathring{\mathcal{F}}_{k,\lambda} : H^k(U_\lambda) \to \mathcal{H}_k(U_\lambda), \lambda \in \Lambda$ are isomorphisms by definition on pairwise disjoint de Rham subsets U_λ. In order to determine the homomorphism $\mathcal{A} : H^k(U) \to \bigcup_{\lambda\in\Lambda} H^k(U_\lambda)$, we first define $\mathcal{I}_\lambda^*\omega$ on the set $\bigcup_{\lambda\in\Lambda} H^k(U_\lambda)$ by

$$\mathcal{I}_\lambda^*\omega = \begin{cases} \mathcal{I}_\lambda^*\omega & \text{on } H^k(U_\lambda) \\ 0 & \text{on } \bigcup_{\mu\in\Lambda,\mu\neq\lambda} H^k(U_\mu) \end{cases}$$

where $\mathcal{I}_\lambda = U_\lambda \to U$ are the inclusion mappings and, thus for $\omega \in \Lambda^k(U)$ we get $\mathcal{I}_\lambda^*\omega \in \Lambda^k(U_\lambda)$ that is none other than the restriction of ω on U_λ. We then take for $[\omega] \in H^k(U)$

$$\mathcal{A}[\omega] = \bigoplus_{\lambda\in\Lambda} [\mathcal{I}_\lambda^*\omega] \in \bigcup_{\lambda\in\Lambda} H^k(U_\lambda)$$

which clearly indicates that $\mathcal{A}$ is an isomorphism. $\mathcal{A}$ is injective because equivalence classes are disjoint. Next, let us choose $\omega_\lambda \in \Lambda^k(U_\lambda)$. Since the sets U_λ are pairwise disjoint, then $\omega = \bigoplus_{\lambda\in\Lambda} \omega_\lambda \in \Lambda^k(U)$ with $\mathcal{I}_\lambda^*\omega = \omega_\lambda$. We thus obtain $\mathcal{A}[\omega] = \bigoplus_{\lambda\in\Lambda} [\mathcal{I}_\lambda^*\omega]$ so that $\mathcal{A}$ is surjective. In exactly similar way, we can show that $\mathcal{B}$ is likewise an isomorphism. Since the diagram commutes, we deduce that $\mathfrak{I}_k : H^k(U) \to \mathcal{H}_k(U)$ is also an isomorphism. This means that $U = \bigcup_{\lambda\in\Lambda} U_\lambda$ is a de Rham manifold.

Lemma 8.8.4. *Let U and V be open subsets of a smooth manifold M. We assume that U, V and $U \cap V$ are de Rham manifolds. Then $U \cup V$ is also a de Rham manifold.*

Let us consider the following Mayer-Vietoris sequences associated with de Rham cohomology and singular cohomology that are exact in view of Theorem 8.8.1:

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
H^{k-1}(U) \oplus H^{k-1}(V) & \xrightarrow{\simeq} & \mathcal{H}^{k-1}(U) \oplus \mathcal{H}^{k-1}(V) \\
\downarrow & & \downarrow \\
H^{k-1}(U \cap V) & \xrightarrow{\simeq} & \mathcal{H}^{k-1}(U \cap V) \\
\downarrow & & \downarrow \\
H^{k}(U \cup V) & \xrightarrow{\mathfrak{I}_k} & \mathcal{H}^{k}(U \cup V) \\
\downarrow & & \downarrow \\
H^{k}(U) \oplus H^{k}(V) & \xrightarrow{\simeq} & \mathcal{H}^{k}(U) \oplus \mathcal{H}^{k}(V) \\
\downarrow & & \downarrow \\
H^{k}(U \cap V) & \xrightarrow{\simeq} & \mathcal{H}^{k}(U \cap V) \\
\downarrow & & \downarrow \\
H^{k+1}(U \cup V) & \xrightarrow{\mathfrak{I}_{k+1}} & \mathcal{H}^{k+1}(U \cup V) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

Our assumption dictates that the two homomorphisms before the homomorphism $\mathfrak{I}_k : H^k(U \cup V) \to \mathcal{H}^k(U \cup V)$ and the other two after that are isomorphisms denoted by the symbol $\simeq$. Then Theorem 1.2.2 (the five lemma) states that $\mathfrak{I}_k$ must be an isomorphism. Hence $U \cup V$ is a de Rham manifold. □

Finally, we have to prove the following lemma:

Lemma 8.8.5. *Let M be a smooth m-dimensional second countable manifold. Assume that $P(U)$ denotes a property associated with an open subset U of M satisfying the four conditions given below*:

(i). *$P(\emptyset)$ is true.*

(ii). *$P(U)$ is true for any U diffeomorphic to a convex open subset of $\mathbb{R}^m$.*

(iii). *If $P(U), P(V)$ and $P(U \cap V)$ are true, then $P(U \cup V)$ is also true.*

(vi). *If $\{U_i : i \in \mathbb{N}\}$ is a sequence of pairwise disjoint open subsets and $P(U_i)$ is true for each $i \in \mathbb{N}$, then $P\big(\bigcup_{i=1}^{n} U_i\big)$ is also true.*

In that case $P(M)$ will also be true. This property satisfies also the above conditions for all convex open subsets of $\mathbb{R}^m$.

Since M is second countable, then it is expressed as a countable union of open sets. Every open set is covered by open sets of some charts. Hence, every open set $U \subseteq M$ is diffeomorphic to an open set of $\mathbb{R}^m$. Therefore, to prove the lemma it suffices to show that the property is true for an open set in $\mathbb{R}^m$. Since $\mathbb{R}^m$ is second countable [*see p.* 70] an open set is expressible as a countable union of open balls B. We know that open balls in $\mathbb{R}^m$ are convex sets [*see p.* 328]. Thus, $P(\mathrm{B})$ is true on open balls. Moreover, it is straightforward to see that intersection of two open balls is also convex. Hence, (iii) implies that countable unions of open balls are convex. Thus, open sets in $\mathbb{R}^m$ are convex and in view of (ii) $P(U)$ is true. Let us now suppose that $P(U)$ and $P(V)$ are true. Since $U \cap V$ is an open set, it is diffeomorphic to a convex open set in $\mathbb{R}^m$ so $P(U \cap V)$ is true. By (iii), we conclude that $P(U \cup V)$ is also true. Therefore the property must be true for a countable union of open sets. Consequently $P(M)$ is true. □

We can now easily prove the de Rham theorem.

Theorem 8.8.2. *Let M be a locally compact, second countable and oriented smooth manifold. The homomorphism $\overset{\circ}{\mathcal{F}}_k : H^k(M) \to \mathcal{H}_k(M)$ is an isomorphism.*

In order to prove this theorem, we have to show that such a manifold is a de Rham manifold. Let us define a property P associated with an open subset of M as being a de Rham manifold. The condition (i) in Lemma 8.8.5 is met due to the fact that $H^k(\emptyset) = 0$ because there are no k-forms on the empty set and $\mathcal{H}_k(\emptyset) = 0$ since there are no homomorphisms from $\emptyset$ to $\mathbb{R}$. Lemmas 8.8.2 and 8.8.4 indicate that P satisfies the conditions (ii) and

(iii) while Lemma 8.8.3 implies that the condition (iv) is also satisfied. Thus M becomes a de Rham manifold. □

Since $\mathcal{H}_k(M)$ and $H_k(M)^*$ are isomorphic, then the homomorphism $H^k(M) \to H_k(M)^*$ is also an isomorphism for such manifolds. Of course, the foregoing results will naturally be valid for compact, second countable manifolds.

An interested reader is suggested to consult to Hodge (1952) and de Rham (1955) for a more detailed proof of the de Rham theorem. For a sheaf-theoretic treatment that is probably the most direct and elegant way to show this theorem we refer to Singer and Thorpe (1967) or Warner (1971). However, to investigate the theory sheaves transcends the intended level of this work.

When $\overset{\circ}{\mathcal{F}}_k$ is an isomorphism and $H^k(M)$ *is finite-dimensional*, then the dual space $H_k(M)^*$ and, consequently, the vector space $H_k(M)$ are of finite and the same dimension. Hence, if $b_k(M) < \infty$, we find

$$b_k(M) = b\big(H^k(M)\big) = b\big(H_k(M)^*\big) = b\big(H_k(M)\big).$$

Isomorphism implies that $\overset{\circ}{\mathcal{F}}_k$ is a bijective, namely, injective and surjective mapping. *Injectiveness requires that if* $\overset{\circ}{\mathcal{F}}_k([\omega]) = \mathbf{0} \in H_k(M)^*$, *then we get* $[\omega] = [\mathbf{0}]$. The ***period*** of a closed form ω, consequently, of the equivalence class produced by this form over a cycle c_k is defined by

$$\pi(c_k) = \int_{c_k} \omega.$$

Therefore, vanishing of the functional $\overset{\circ}{\mathcal{F}}_k([\omega])$ means that all periods of the equivalence class $[\omega]$ are zero. The equality $[\omega] = [\mathbf{0}]$ implies that the form ω is exact. On the other hand, the Stokes theorem indicates that if a form ω is exact, then all of its periods vanish on every cycle c_k. Hence, it follows from de Rham's theorem that *a closed k-form ω is exact if and only if all of its periods are zero*. *That the mapping $\overset{\circ}{\mathcal{F}}_k$ is surjective amounts to say that every linear functional on the vector space $H_k(M)$ is generated through a closed k-form*. Since such a linear functional is prescribed by its value on every cycle, de Rham's theorem leads to the following conclusion: *when we assign a number $\pi(c_k) \in \mathbb{R}$ to every cycle $c_k \in \overset{\circ}{C}_k(M)$, there exists a closed k-form ω_0 admitting these numbers as its periods, namely, verifying the relation $\pi(c_k) = \int_{c_k}\omega_0$ for every k-cycle c_k, if only these numbers satisfy the conditions $\pi\Big(\sum_i a_i c_k^{(i)}\Big) = \sum_i a_i \pi\big(c_k^{(i)}\big), a_i \in \mathbb{R}$ and if $c_k \in B_k(M)$, then one must have $\pi(c_k) = 0$.*

8.9. HARMONIC FORMS. THEORY OF HODGE-DE RHAM

Let $(M,\boldsymbol{\mathcal{G}})$ be an m-dimensional complete Riemannian manifold. A form $\omega\in\Lambda^k(M)$ is given by $\omega=\dfrac{1}{k!}\,\omega_{i_1\cdots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k}$. We know that the *Hodge dual* of this form is the form $*\omega=\dfrac{1}{k!}\,\omega^{i_1\cdots i_k}\mu_{i_k\cdots i_1}\in\Lambda^{m-k}(M)$ [*see* (5.9.20)]. Contravariant components in this expression are related to the covariant components by $\omega^{i_1\cdots i_k}=g^{i_1j_1}\cdots g^{i_kj_k}\omega_{j_1\cdots j_k}$. If we take two forms $\omega,\sigma\in\Lambda^k(M)$ into consideration, we have already known that the identity (5.9.27) enables us to write

$$\begin{aligned}\omega\wedge*\sigma=\sigma\wedge*\omega&=\frac{1}{k!}\,\omega_{i_1\cdots i_k}\sigma^{i_1\cdots i_k}\mu\\&=\frac{1}{k!}\,\omega^{i_1\cdots i_k}\sigma_{i_1\cdots i_k}\mu\in\Lambda^m(M)\end{aligned}\tag{8.9.1}$$

where μ is the volume form given by (5.9.13) or (5.9.14). Thus, for every form $\omega\in\Lambda^k(M)$ we can write

$$\omega\wedge*\omega=\frac{1}{k!}\,\omega_{i_1\cdots i_k}\omega^{i_1\cdots i_k}\mu.$$

Because the Riemannian manifold is complete, we may assume that the metric tensor is positive definite so that we must have

$$\omega_{i_1\cdots i_k}\omega^{i_1\cdots i_k}=g_{i_1j_1}\cdots g_{i_kj_k}\omega^{i_1\cdots i_k}\omega^{j_1\cdots j_k}>0\tag{8.9.2}$$

if $\omega\neq 0$. Hence, an ***inner product*** on $\Lambda^k(M)$, that is a vector space on real numbers $\mathbb{R}$, may be defined as follows

$$(\omega,\sigma)_k=(\sigma,\omega)_k=\int_M\omega\wedge*\sigma=\frac{1}{k!}\int_M\omega_{i_1\cdots i_k}\sigma^{i_1\cdots i_k}\mu\in\mathbb{R}\tag{8.9.3}$$

due to the property (8.9.1). It is easily verified that (8.9.3) obeys all rules imposed on an inner product [*see p.* 68]. *When M is a compact manifold, the integral* (8.9.3) *will always exist*. We immediately recognise that the mapping $(\,\cdot\,,\,\cdot\,)_k:\Lambda^k(M)\times\Lambda^k(M)\to\mathbb{R}$ so defined is a *symmetric bilinear functional* on real numbers. Because of (8.9.2) we get $(\omega,\omega)_k\geq 0$ which becomes zero if and only if $\omega=0$. The non-negative number

$$\|\omega\|_k=\sqrt{(\omega,\omega)_k}\geq 0\tag{8.9.4}$$

may now be called the ***norm*** of a form $\omega\in\Lambda^k(M)$. Since $\Lambda^k(M)$ equipped with (8.9.3) becomes an ***inner product space***, the well known ***Schwarz***

inequality must be satisfied:

$$(\omega, \sigma)_k \leq \|\omega\|_k \|\sigma\|_k. \tag{8.9.5}$$

This inner product on $\Lambda^k(M)$ can easily be extended to an inner product on the graded algebra $\Lambda(M) = \overset{m}{\underset{k=0}{\oplus}} \Lambda^k(M)$. We take the forms $\omega \in \Lambda^k(M)$ and $\sigma \in \Lambda^l(M)$ into account so that we get $\omega \wedge *\sigma \in \Lambda^{m+k-l}(M)$. This form vanishes identically if $k > l$. Its degree is less than m if $k < l$, hence its integral over M cannot be defined. If we adopt the convention that such an integral also vanishes, we can then define the inner product of two arbitrary k-form ω and l-form σ in $\Lambda(M)$ in the following manner

$$(\omega, \sigma) = \begin{cases} (\omega, \sigma)_k & \textit{if} \;\; k = l, \\ \quad = 0 & \textit{if} \;\; k > l, \\ \quad = 0 & \textit{if} \;\; k < l. \end{cases}$$

This definition amounts to admit that the vector spaces $\Lambda^k(M)$ and $\Lambda^l(M)$ are *orthogonal* with respect to this inner product whenever $k \neq l$. Henceforth, by adopting this definition we shall not designate the inner product as dependent on the index k. In view of the definition (8.9.3), we obtain the following relation for forms $\omega, \sigma \in \Lambda^k(M)$

$$\begin{aligned} (*\omega, *\sigma) &= \int_M *\omega \wedge **\sigma = (-1)^{k(m-k)} \int_M *\omega \wedge \sigma \\ &= (-1)^{2k(m-k)} \int_M \sigma \wedge *\omega = \int_M \sigma \wedge *\omega \\ &= \int_M \omega \wedge *\sigma = (\omega, \sigma). \end{aligned}$$

This means that the Hodge star operator $* : \Lambda^k(M) \to \Lambda^{m-k}(M)$, which is obviously a linear operator, preserves the inner product. It is well known that a linear operator between two inner product spaces that preserves the inner product is called a ***unitary*** or ***conformal operator***. Consequently, *the Hodge star operator* $*$ *is a unitary or conformal operator on the exterior algebra with respect to the inner product so defined*.

Let us now consider the forms $\omega \in \Lambda^{k-1}(M)$ and $\sigma \in \Lambda^k(M)$ and evaluate the exterior derivative of the form $\omega \wedge *\sigma \in \Lambda^{m-1}(M)$ to obtain:

$$d(\omega \wedge *\sigma) = d\omega \wedge *\sigma + (-1)^{k-1}\omega \wedge d(*\sigma) \in \Lambda^m(M).$$

On making use of the relation $*\delta = (-1)^k d*$ between the operators of co-differential δ and the exterior derivative d [*see p.* 283] we arrive at the expression

$$d(\omega \wedge *\sigma) = d\omega \wedge *\sigma - \omega \wedge *\delta\sigma.$$

Then the Stokes theorem yields

$$\int_M (d\omega \wedge *\sigma - \omega \wedge *\delta\sigma) = \int_M d(\omega \wedge *\sigma) = \int_{\partial M} \omega \wedge *\sigma$$

so that we get

$$\int_M d\omega \wedge *\sigma = \int_M \omega \wedge *\delta\sigma + \int_{\partial M} \omega \wedge *\sigma.$$

On recalling the definition of the inner product, we thus conclude that

$$(d\omega, \sigma) = (\omega, \delta\sigma) + \int_{\partial M} \omega \wedge *\sigma.$$

If M is a manifold without boundary ($\partial M = \emptyset$), we necessarily have to write $\int_{\partial M} \omega \wedge *\sigma = 0$ to obtain

$$(d\omega, \sigma) = (\omega, \delta\sigma). \tag{8.9.6}$$

According to the foregoing relation, we are led to the following conclusion: *let $\Lambda(M)$ be the graded exterior algebra on a compact manifold without boundary. The operators on the exterior algebra $d : \Lambda^k(M) \to \Lambda^{k+1}(M)$ and $\delta : \Lambda^k(M) \to \Lambda^{k-1}(M)$ are* ***adjoint operators*** *on $\Lambda(M)$.* This result will also be valid for all forms with compact support on a manifold with boundary. Because such forms will necessarily vanish on the boundary of the manifold.

The Laplace-de Rham operator $\delta d + d\delta = \Delta : \Lambda^k(M) \to \Lambda^k(M)$ was defined by (5.9.31). When $\omega, \sigma \in \Lambda^k(M)$, (8.9.6) together with the symmetry of the inner product leads to the result

$$\begin{aligned}(\Delta\omega, \sigma) &= (\delta d\omega, \sigma) + (d\delta\omega, \sigma) = (d\omega, d\sigma) + (\delta\omega, \delta\sigma) \\ &= (\omega, \delta d\sigma) + (\omega, d\delta\sigma) = \big(\omega, (\delta d + d\delta)\sigma\big) \\ &= (\omega, \Delta\sigma).\end{aligned}$$

Hence, with respect to this inner product the operator Δ on $\Lambda(M)$ becomes a ***self-adjoint operator*** if M is a manifold without boundary. It follows from the above relation that we obtain

$$(\Delta\omega, \omega) = (d\omega, d\omega) + (\delta\omega, \delta\omega) = \|d\omega\|^2 + \|\delta\omega\|^2 \geq 0 \tag{8.9.7}$$

for all $\omega \in \Lambda(M)$. Since $(\varDelta\omega, \omega) \geq 0$ for all $\omega \in \Lambda(M)$ we may describe $\varDelta$ as a ***positive definite*** or ***elliptic operator***.

A form $\omega \in \Lambda(M)$ is called a ***harmonic form*** if $\varDelta\omega = 0$.

Theorem 8.9.1. *Let M be a compact and oriented Riemannian manifold without boundary. We consider a form $\omega \in \Lambda^k(M)$. The form ω is harmonic if and only if $d\omega = 0$ and $\delta\omega = 0$.*

If $d\omega = 0$ and $\delta\omega = 0$, then the definition leads to $\varDelta\omega = 0$. Conversely let us assume that $\varDelta\omega = 0$. Then it follows from (8.9.7) that $0 = (0, \omega) = \|d\omega\|^2 + \|\delta\omega\|^2$. Hence, we get $\|d\omega\| = \|\delta\omega\| = 0$ from which we deduce that $d\omega = 0$ and $\delta\omega = 0$. □

Consequently, *all harmonic forms are also closed on manifolds to which the above theorem might be applied*. We had previously mentioned that all harmonic forms $\omega \in \Lambda^k(M)$ holding the condition $\varDelta\omega = 0$ constitute the following subspace

$$\mathrm{H}^k(M) = \{\omega \in \Lambda^k(M) : \varDelta\omega = 0\} = \mathcal{N}(\varDelta)$$

on real numbers.

We had seen that $\varDelta f = \delta df = \nabla^2 f$ when $f \in \Lambda^0(M)$. The operator ∇^2 was defined by (5.9.33). Therefore, the solution of the Laplace equation $\nabla^2 f = 0$ on a compact manifold without boundary must satisfy the condition $df = 0$. Hence, *the harmonic function f can only be a constant number*. This result is a sort of generalisation of the well known ***Liouville theorem*** [French mathematician Joseph Liouville (1809-1882)].

Probably the most important theorem concerning harmonic forms has been demonstrated by Hodge. Because the proof of this theorem is quite difficult and requires a rather good knowledge of functional analysis and properties of elliptic operators, we shall not be able to present its proof here in its full generality.

Theorem 8.9.2. (The Hodge Decomposition Theorem). *Let M be a compact and oriented Riemannian manifold without boundary. For each form $\omega \in \Lambda^k(M)$, there exist forms $\alpha \in \Lambda^{k-1}(M)$, $\beta \in \Lambda^{k+1}(M)$ and $\gamma \in \mathrm{H}^k(M)$ so that one can express ω as*

$$\omega = d\alpha + \delta\beta + \gamma \tag{8.9.8}$$

and this representation is unique. We can thus write symbolically

$$\Lambda^k(M) = d\Lambda^{k-1}(M) \oplus \delta\Lambda^{k+1}(M) \oplus \mathrm{H}^k(M).$$

All these subspaces are mutually orthogonal.

We can easily show the orthogonality of subspaces. Since $\delta^2 = 0$, we find that

$$(d\alpha, \delta\beta) = (\alpha, \delta^2\beta) = 0.$$

On the other hand, if $\Delta\gamma = 0$, then we have $d\gamma = 0, \delta\gamma = 0$ so that we are led to the results

$$(d\alpha, \gamma) = (\alpha, \delta\gamma) = 0, \quad (\delta\beta, \gamma) = (\beta, d\gamma) = 0.$$

The difficult part of the theorem is to show the existence of the forms α, β, γ satisfying the relation (8.9.8). As to this part, the interested readers may be referred to Warner (1971, Ch. 6). In order to prove the uniqueness, let us suppose that there are two representations of this form:

$$\omega = d\alpha_1 + \delta\beta_1 + \gamma_1 = d\alpha_2 + \delta\beta_2 + \gamma_2.$$

Hence, if we denote $\alpha = \alpha_1 - \alpha_2, \beta = \beta_1 - \beta_2, \gamma = \gamma_1 - \gamma_2$, we realise that the condition below should be satisfied:

$$d\alpha + \delta\beta + \gamma = 0.$$

On evaluating the exterior derivative of that expression, we get $d\delta\beta = 0$ from which we deduce that $0 = (d\delta\beta, \beta) = (\delta\beta, \delta\beta) = \|\delta\beta\|^2$ and $\delta\beta = 0$. So the foregoing equality is reduced to $d\alpha + \gamma = 0$. The co-differential of this last expression yields $\delta d\alpha = 0$. Thus, we can obtain at once $0 = (\delta d\alpha, \alpha) = (d\alpha, d\alpha) = \|d\alpha\|^2$ and $d\alpha = 0$ implying that $\gamma = 0$. Hence, we find that $d\alpha_1 = d\alpha_2, \delta\beta_1 = \delta\beta_2, \gamma_1 = \gamma_2$. □

Theorem 8.9.3. *Let M be a compact and oriented Riemannian manifold without boundary. The solution of the equation $\Delta\omega = \sigma$ where the form $\sigma \in \Lambda^k(M)$ is prescribed does exist if and only if the form σ is orthogonal to the vector space $\mathrm{H}^k(M)$, in other words, if $(\sigma, \lambda) = 0$ for all $\lambda \in \mathrm{H}^k(M)$.*

Let us first assume that $\Delta\omega = \sigma$ and $\lambda \in \mathrm{H}^k(M)$. We then obtain

$$(\sigma, \lambda) = (\Delta\omega, \lambda) = (\omega, \Delta\lambda) = (\omega, 0) = 0.$$

Conversely, we suppose that the form σ satisfies the condition $(\sigma, \lambda) = 0$ for all $\lambda \in \mathrm{H}^k(M)$. On utilising the Hodge decomposition, we may write $\sigma = d\alpha + \delta\beta + \gamma$ so that the above condition yields

$$\begin{aligned} 0 &= (\sigma, \gamma) = (d\alpha, \gamma) + (\delta\beta, \gamma) + (\gamma, \gamma) \\ &= (\alpha, \delta\gamma) + (\beta, d\gamma) + (\gamma, \gamma) \\ &= (\gamma, \gamma) = \|\gamma\|^2 \end{aligned}$$

and we find that $\gamma = 0$. Hence, the form σ can only take the shape $\sigma = d\alpha + \delta\beta$. Let us now write $\omega = \omega_1 + \omega_2$ and try to determine the solutions of the equations $\Delta\omega_1 = d\alpha$ and $\Delta\omega_2 = \delta\beta$ separately. If we employ the representation $\alpha = d\alpha_1 + \delta\beta_1 + \gamma_1$, we get $d\alpha = d\delta\beta_1$. If we in turn write $\beta_1 = d\alpha_2 + \delta\beta_2 + \gamma_2$ and note that $d^2 = 0$, we obtain

$$d\alpha = d\delta\beta_1 = d\delta\, d\alpha_2 = (d\delta + \delta d)d\alpha_2 = \Delta(d\alpha_2).$$

Therefore, the equation $\Delta\omega_1 = d\alpha$ admits a solution in the form $\omega_1 = d\alpha_2$. Similarly, we take $\beta = d\alpha_3 + \delta\beta_3 + \gamma_3$ to obtain $\delta\beta = \delta d\alpha_3$ by noting that $\delta^2 = 0$. By using the representation $\alpha_3 = d\alpha_4 + \delta\beta_4 + \gamma_4$, we find that

$$\delta\beta = \delta d\alpha_3 = \delta d\delta\beta_4 = (\delta d + d\delta)\delta\beta_4 = \Delta(\delta\beta_4).$$

Therefore, we conclude that the equation $\Delta\omega_2 = \delta\beta$ admits a solution in the form $\omega_2 = \delta\beta_4$. Ultimately, we find that the equation $\Delta\omega = \sigma$ possesses a solution in the form $\omega = d\alpha_2 + \delta\beta_4$. □

We have above touched upon the fact that harmonic forms are closed. Hence, there exists a linear operator $\mathcal{I} : \mathrm{H}^k(M) \to \mathcal{C}^k(M)$ embedding $\mathrm{H}^k(M)$ into $\mathcal{C}^k(M)$. Let $\pi : \mathcal{C}^k(M) \to H^k(M)$ be the linear canonical mapping. We are thus led to the conclusion that there exists a linear transformation $\psi = \pi \circ \mathcal{I} : \mathrm{H}^k(M) \to H^k(M)$ between the vector space of harmonic k. forms and the relevant cohomology group.

Theorem 8.9.4. *Let M be a compact and oriented Riemannian manifold without boundary. The vector spaces $\mathrm{H}^k(M)$ and $H^k(M)$ are isomorphic.*

In order to prove this theorem, we have to show that the linear operator ψ introduced above is bijective. Let us first assume that $\omega \in \mathrm{H}^k(M)$ and $[\omega] = \psi(\omega) = [\mathbf{0}] \in H^k(M)$. This means that ω is an exact form and one writes $\omega = d\sigma$. But, we get $(\omega, d\sigma) = (\delta\omega, \sigma) = 0$ since $\delta\omega = 0$. We thus arrive at the relation $(\omega, \omega) = \|\omega\|^2 = 0$ implying that $\omega = 0$. This amounts to say that ψ is injective. We now consider an arbitrary cohomology class $[\omega] \in H^k(M)$. ω is a representative of this equivalence class. Due to the Hodge decomposition theorem, we can write $\omega = d\alpha + \delta\beta + \gamma$. Since $d\omega = 0$, we find that $d\delta\beta = 0$. It then follows just as above that $\delta\beta = 0$ which implies that *a closed form ω is represented as $\omega = d\alpha + \gamma$*. This of course gives $[\omega] = [\gamma]$ so that one is able to write $[\omega] = \psi(\gamma)$ where $\gamma \in \mathrm{H}^k(M)$. Consequently, we see that ψ is surjective. As a result, ψ is identified as an isomorphism. Hence, the vector spaces $\mathrm{H}^k(M)$ and $H^k(M)$ are isomorphic. Accordingly, we can say that *every cohomology class has a harmonic representative* in manifolds complying with the assumptions of the theorem. □

According to a property that we shall again not be able prove here, the null space of the operator Δ, or more generally of a linear elliptic operator, is finite-dimensional if M is a compact Riemannian manifold [interested readers may be referred to Warner (1971)]. Therefore, on such a manifold the vector space $\mathrm{H}^k(M)$ is finite-dimensional. Since isomorphic spaces must have the same dimension we can now state that *dimensions of cohomology groups, that is, Betti numbers on a compact and oriented Riemannian manifold without boundary are all finite*.

We have seen while proving the above theorem that any closed form $\omega \in \mathcal{C}^k(M)$ on a compact and oriented Riemannian manifold without boundary is expressible as $\omega = d\alpha + \gamma$ where $\alpha \in \Lambda^{k-1}(M), \gamma \in \mathrm{H}^k(M)$. If c_k is a k-cycle, then we can write

$$\begin{aligned}\pi(c_k) &= \int_{c_k} \omega = \int_{c_k} d\alpha + \int_{c_k} \gamma \\ &= \int_{\partial c_k} \alpha + \int_{c_k} \gamma = \int_{c_k} \gamma.\end{aligned}$$

This is tantamount to say that *there exists a unique harmonic k-form γ possessing the same periods as a closed k-form ω on such kind of manifolds.*

8.10. POINCARE DUALITY

Let M be an m-dimensional compact, oriented, smooth Riemannian manifold without boundary. We shall now introduce a bilinear functional $P : H^k(M) \times H^{m-k}(M) \to \mathbb{R}$ through the following relation

$$P([\omega], [\sigma]) = \int_M \omega \wedge \sigma, \tag{8.10.1}$$

where $[\omega] \in H^k(M)$ and $[\sigma] \in H^{m-k}(M)$, and the forms $\omega \in \mathcal{C}^k(M)$ and $\sigma \in \mathcal{C}^{m-k}(M)$ are arbitrary representatives of these cohomology classes. In order that the functional (8.10.1) known as the ***Poincaré form*** proves to be meaningful, it must be independent of the selection of the representatives of equivalence classes. This property can be shown quite easily. We consider the forms $\alpha \in \Lambda^{k-1}(M)$, $\beta \in \Lambda^{m-k-1}(M)$. If we note that $d\omega = 0$ and $d\sigma = 0$, the Stokes theorem results in the expression

$$\begin{aligned}\int_M (\omega + d\alpha) \wedge (\sigma + d\beta) &= \int_M (\omega \wedge \sigma + d\alpha \wedge \sigma + \omega \wedge d\beta + d\alpha \wedge d\beta) \\ &= \int_M \omega \wedge \sigma + \int_M d(\alpha \wedge \sigma) + (-1)^k \int_M d(\omega \wedge \beta) + \int_M d(\alpha \wedge d\beta) \\ &= \int_M \omega \wedge \sigma + \int_{\partial M} \alpha \wedge \sigma + (-1)^k \int_{\partial M} \omega \wedge \beta + \int_{\partial M} \alpha \wedge d\beta = \int_M \omega \wedge \sigma.\end{aligned}$$

Boundary integrals vanish because we have assumed that $\partial M = \emptyset$. Hence, we can write $P([\omega],[\sigma]) = P(\omega,\sigma)$ for arbitrary representatives. We shall now demonstrate that this bilinear form is non-degenerate. To this end, it would suffice to determine an equivalence class $[\sigma] \neq [\mathbf{0}]$ such that $P([\omega],[\sigma]) \neq 0$ whenever $[\omega] \neq [\mathbf{0}]$. Let $[\omega] \in H^k(M)$ be a non-zero cohomology class. We choose the form $\omega \in \mathrm{H}^k(M)$ as the harmonic representative of that cohomology class. We thus write $\Delta\omega = 0$. The form ω cannot be identically zero since $[\omega] \neq [\mathbf{0}]$. Let us next consider the Hodge dual $*\omega \in \Lambda^{m-k}(M)$ of the form ω. We had obtained the relation $*\Delta = \Delta *$ on p. 284. We thus find $\Delta * \omega = * \Delta\omega = 0$ implying that $*\omega \in \mathrm{H}^{m-k}(M)$. This form can be chosen as the harmonic representative of the cohomology class $[*\omega] \in H^{m-k}(M)$. Next, we take $[\sigma] = [*\omega]$. Hence, we conclude that

$$P(\omega, *\omega) = \int_M \omega \wedge *\omega = (\omega,\omega) = \|\omega\|^2 \neq 0$$

On the other hand, this relation signifies that $P(\omega,*\omega) = 0$ if and only if $\omega = 0$. Hence, the bilinear form P is non-degenerate. Let us fix a class $[\omega]$ in (8.10.1). Then a linear functional $L([\omega])$ on the vector space $H^{m-k}(M)$ can be introduced by the relation

$$L([\omega])\,([\sigma]) = P([\omega],[\sigma]).$$

Thus the bilinear form $P([\omega],[\sigma])$ induces a linear transformation

$$L : H^k(M) \to \left[H^{m-k}(M)\right]^*. \tag{8.10.2}$$

We can realise right away that the non-degeneracy of the bilinear form P secures that the linear operator L is injective. On the other hand, we know that the dimensions of $H^k(M)$ and $H^{m-k}(M)$, consequently, that of the dual $\left[H^{m-k}(M)\right]^*$ is finite. In this case, L becomes an isomorphism so that the spaces $H^k(M)$ and $\left[H^{m-k}(M)\right]^*$ are isomorphic. Because a finite-dimensional vector space and its dual are isomorphic, we thus infer that the spaces $H^k(M)$ and $H^{m-k}(M)$ are isomorphic. This property is called the ***Poincaré duality***. Therefore, we can regard these two spaces as the same as far as their algebraic properties are concerned. Hence, the Betti numbers of compact, oriented Riemannian manifolds without boundary must satisfy the relation

$$b_k(M) = b\big(H^k(M)\big) = b\big(H^{m-k}(M)\big) = b_{m-k}(M). \tag{8.10.3}$$

If the dimension of the manifold is an odd number, then its Euler-Poincaré characteristic becomes

$$\chi(M) = \sum_{k=0}^{m}(-1)^k b_k(M) = 0.$$

In fact, we find in this case $(-1)^{m-k} b_{m-k}(M) = (-1)^{k+1} b_k(M)$ and the corresponding terms cancel each other in the above sum.

According to the Poincaré duality, the vector spaces $H^m(M)$, $H^0(M)$ are isomorphic in a compact, oriented Riemannian manifold without boundary. We know that $H^0(M) = \mathbb{R}$ when M is *connected*. Thus, in this sort of manifolds the cohomology group $H^m(M)$ is isomorphic to $\mathbb{R}$. Furthermore, it is possible to show that $H^{m-1}(M) = 0$ if the manifold M is *simply connected*, that is, if every closed curve on M can be contracted smoothly to a point inside the curve. Indeed, due to the Poincaré duality, the vector spaces $H^{m-1}(M)$ and $H^1(M)$ are isomorphic. In local coordinates, let us write a form $\omega \in \Lambda^1(M)$ as $\omega = \omega_i dx^i$ where $i = 1, \ldots, m$. If ω is closed, then the condition

$$d\omega = \omega_{i,j} dx^j \wedge dx^i = 0 \quad \text{or} \quad \omega_{[i,j]} = 0$$

must be satisfied. If M is simply connected, then it is well known that the general solution of the following system of partial differential equations

$$\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} = 0$$

is provided as follows

$$\omega_i = \frac{\partial f}{\partial x^i}$$

where $f \in \Lambda^0(M)$. Thereby, we get $\omega = f_{,i}\, dx^i = df$. Thus, on such kind of manifolds every closed 1-form is exact. Therefore, we find $H^1(M) = 0$, and consequently, $H^{m-1}(M) = 0$.

VIII. EXERCISES

8.1. Show that a k-dimensional submanifold of a manifold M is orientable if one can find a k-form that vanishes nowhere on this submanifold.

8.2. Show that the Cartesian product of orientable manifolds is also an orientable manifold.

8.3. Show that the Cartesian product of non-orientable manifolds is also a non-orientable manifold.

8.4. Show that the Klein bottle is non-orientable.

8.5. Show that the Lie groups $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are orientable.

8.6. Show that if a form $\omega \in \Lambda^k(M)$ satisfies the condition $\int_{\sigma_k} \omega = 0$ on *every* singular k-simplex, then one has $\omega = 0$.

8.7. Show that if forms $\omega_1, \omega_2 \in \Lambda^k(M)$ satisfy the condition $\int_{\sigma_k} \omega_1 = \int_{\sigma_k} \omega_2$ on *every* singular k-simplex σ_k, then one has $\omega_1 = \omega_2$.

8.8. Show that a form $\omega \in \Lambda^k(M)$ turns out to be closed if it satisfies the condition $\int_{\partial\sigma_{k+1}} \omega = 0$ on *every* singular $(k+1)$-simplex.

8.9. Show that the volume form on the hyperbolic plane H^2 (*see* Exercise **7.9**) is given by

$$\mu = \frac{r}{\sqrt{1+r^2}}\, dr \wedge d\theta = \sinh s\, ds \wedge d\theta.$$

Find the volume of the subregion of H^2 satisfying the condition $1 \leq x_0 \leq 2$.

8.10. U is an m-dimensional compact submanifold with boundary of an m-dimensional Riemannian manifold. Show that for a vector field $V \in T(M)$ one is able to write

$$\int_U \operatorname{div} V \mu = \int_{\partial U} \mathbf{i}_V \mu.$$

8.11. We consider the simplex $s_2 = (P_0, P_1, P_2)$ in $\mathbb{R}^2$ where $P_0 = (0,0)$, $P_1 = (1,0)$, $P_2 = (0,2)$. Evaluate the integral of the 1-form

$$\omega = (x^2 + 7y)\, dx + (y \sin y^2 - x)\, dy$$

on the cycle ∂s_2.

8.12. Show that the form $\omega = (2x + y \cos xy)\, dx + x \cos xy\, dy \in \Lambda^1(\mathbb{R}^2)$ is exact. Find the integral of this form on the cycle defined in Exercise. **8.11**.

8.13. Show that a form $\omega \in \Lambda^2(\mathbb{S}^2)$ is exact if only the following condition is met

$$\int_{\mathbb{S}^2} \omega = 0.$$

8.14. We consider the form $\omega = x^1 dx^2 \wedge dx^3 \in \Lambda^2(\mathbb{R}^3)$ where $(x^1, x^2, x^3) \in \mathbb{R}^3$. Show that

$$\int_{\mathbb{S}^2} \omega|_{\mathbb{S}^2} = \frac{4\pi}{3} R^3$$

where R is the radius of the sphere $\mathbb{S}^2$. Since the form $\omega|_{\mathbb{S}^2} \in \Lambda^2(\mathbb{S}^2)$ is clearly closed, this result indicates the fact that that every closed 2-form on $\mathbb{S}^2$ is not necessarily exact.

8.15. Show that every closed 1-form on $\mathbb{S}^2$ is exact..

8.16. Show that the restriction $\omega|_{\mathbb{S}^{n-1}} \in \Lambda^{n-1}(\mathbb{S}^{n-1})$ to the sphere $\mathbb{S}^{n-1}$ of the form $\omega = \epsilon_{ii_1\cdots i_{n-1}} x^i dx^{i_1} \wedge \cdots \wedge dx^{i_{n-1}} \in \Lambda^{n-1}(\mathbb{R}^n)$ is a closed form that is not exact and it does vanish nowhere on $\mathbb{S}^{n-1}$.

8.17. Let us consider the manifold $M = \mathbb{R}^n - \{\mathbf{0}\}$ and the form

$$\omega = \frac{x^1 dx^1 + x^2 dx^2 + \cdots + x^n dx^n}{\big((x^1)^2 + (x^2)^2 + \cdots + (x^n)^2\big)^{n/2}} \in \Lambda^1(M).$$

Determine the form $*\omega$ and show that it is closed. Evaluate the integral

$$\int_{\mathbb{S}^{n-1}} *\omega.$$

Is the form $*\omega$ exact?

8.18. Let us consider a form $\omega \in \Lambda^{m-1}(M)$ on an m-dimensional compact and orientable manifold M without boundary ($\partial M = \emptyset$). Show that there exists a point $p \in M$ such that $d\omega(p) = 0$.

8.19. Let G be a compact and oriented Lie group. We define the mapping $\iota(g) = g^{-1}$ for every $g \in G$. Show that we can write the following relation for every continuous function f on G

$$\int_G f = \int_G f \circ \iota.$$

8.20. Let us consider the functions $f, g \in \Lambda^0(\mathbb{R}^n)$ and a finite region $D \subset \mathbb{R}^n$. By employing the Stokes theorem, derive the Green formula given below

$$\int_{\partial D} \big(f*(dg) - g*(df)\big) = -\int_D (f\Delta g - g\Delta f)\mu.$$

8.21. Let us consider the manifold $M = \mathbb{R}^5$ with a coordinate cover (x, t, θ, u, v) and the forms

$$\omega^1 = du \wedge dt + dv \wedge dx \in \Lambda^2(\mathbb{R}^5), \quad \omega^2 = d\theta - u\,dx - v\,dt \in \Lambda^1(\mathbb{R}^5)$$

Let $\mathcal{I}$ be the ideal generated by these forms. Assume that the 2-dimensional solution submanifold of the ideal $\mathcal{I}$ is prescribed by the mapping $\theta = \theta(x, t)$, $u = u(x, t)$ and $v = v(x, t)$. Show that the *wave equation* $\theta_{xx} - \theta_{tt} = 0$ is satisfied on the solution submanifold. Determine the conservation laws of this equation.

8.22. Show that a form $\omega \in \Lambda^1(M)$ is exact if it satisfies the condition $\int_{\mathcal{C}} \omega = 0$ for *every* closed curve $\mathcal{C} \subset M$.

8.23. Show that a connected manifold M is simply connected if and only if one gets $H^1(M) = 0$.

8.24. Determine the de Rham cohomology of the annular region depicted by the condition $1 < \sqrt{x_1^2 + x_2^2} < 2$ in $\mathbb{R}^2$.