CHAPTER X

CALCULUS OF VARIATIONS

10.1. SCOPE OF THE CHAPTER

This chapter is devoted to a brief study of the calculus of variations that deals with determining functions rendering a functional defined as an integral over a function space stationary. However, in contrast to the classical approach, we will be employing exterior analysis throughout the discussion. We shall start with a relatively simple case in Sec. 10.2 by assuming that a functional is specified by an integral of a function over a region of a smooth manifold. This function that is frequently called the Lagrangian function, will be supposed to depend on independent variables, a certain number of dependent variables and their first order partial derivatives. We then examine the conditions under which the functional so formed becomes stationary, that is, its first order variation vanishes. We show in Sec. 10.3 that a function that renders such a functional stationary must satisfy the Euler-Lagrange equations which arise from the vanishing of the Lie derivative of an appropriately prescribed form with respect to an isovector field of the contact ideal. We discuss in Sec. 10.4 the variational symmetries that leave the functional invariant, the Noetherian vector fields that conduce to such symmetries and conservation laws generated by these symmetries. Finally, Sec. 10.5 is concerned with the case in which the Lagrangian function involves partial derivatives of dependent variables up to any finite order. This is, however, handled by resorting to previous findings.

10.2. STATIONARY FUNCTIONALS

Let us consider an n-dimensional connected region B_n of an n-dimensional smooth manifold. We denote the piecewise smooth boundary of this region by ∂B_n. The local coordinates of the manifold are given by x^i, $1 \leq i \leq n$. Suppose that sufficiently differentiable functions $u^\alpha : B_n \to \mathbb{R}$ defined on B_n constitute the set $\mathcal{B} = \{u^\alpha(x^i),\ 1 \leq \alpha \leq N\}$. Let $L(\mathbf{x}, \mathbf{u})$ be a real-valued function depending on functions u^α and their partial

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derivatives of various orders. The mapping $A : \mathcal{B} \to \mathbb{R}$ defined by

$$A(\mathbf{u}) = \int_{B_n} L(\mathbf{x}, \mathbf{u})\, dx^1 dx^2 \cdots dx^n$$

is called a ***functional*** or an ***action integral*** whereas the function L is named as a ***Lagrangian function*** [French mathematician Joseph-Louis Lagrange (1736-1813)]. The source of these terms is the terminology widely used in the analytical mechanics. The ***calculus of variations*** copes with the task of determining the functions **u** that extremise the action integral, namely, either maximise or minimise it. We here deal with this problem as an application of the exterior analysis. We first start with a simpler functional that helps us to comprehend much better our treatment. The general case will be considered later. Thus, we select the Lagrangian function L in the form

$$L(\mathbf{x}, \mathbf{u}) = L(x^i, u^\alpha, v_i^\alpha),\ \ v_i^\alpha = u_{,i}^\alpha = \frac{\partial u^\alpha}{\partial x^i}.$$

We again denote the $(n + N + nN)$-dimensional smooth contact manifold whose coordinate cover is given by $\{x^i, u^\alpha, v_i^\alpha\}$ by $\mathcal{C}_1$ and consider the closed contact ideal $\mathcal{I}_1 = \overline{\mathcal{I}}(\sigma^\alpha, d\sigma^\alpha)$ of the exterior algebra $\Lambda(\mathcal{C}_1)$. We know that the contact forms are given by

$$\sigma^\alpha = du^\alpha - v_i^\alpha dx^i \in \Lambda^1(\mathcal{C}_1)$$

We characterise the set of all *regular mappings* from the region B_n into the manifold $\mathcal{C}_1$ as follows

$$\mathcal{R}(B_n) = \{\phi : B_n \to \mathcal{C}_1 : \phi^*\mathcal{I}_1 = 0,\ \phi^*\mu \neq 0\} \tag{10.2.1}$$

where $\mu = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$ is the volume form. Thus, we must have $\phi^*\sigma^\alpha = 0$ with $1 \leq \alpha \leq N$. According to this definition, regular mappings may be taken as *mappings depending on the same independent variables x^i annihilating the ideal $\mathcal{I}_1$*. Hence, if $\phi \in \mathcal{R}(B_n)$, then it is represented by the relations

$$\phi^* x^i = x^i,\ \ \phi^* u^\alpha = \phi^\alpha(\mathbf{x}),\ \ \phi^* v_i^\alpha = \phi_{,i}^\alpha(\mathbf{x}) \tag{10.2.2}$$

Therefore, the action integral corresponding to a mapping $\phi \in \mathcal{R}(B_n)$ can be expressed as

$$A(\phi) = \int_{B_n} \phi^*(L\mu), \tag{10.2.3}$$

namely, it becomes the integral of the form $L\mu = L(x^i, u^\alpha, v_i^\alpha)\mu \in \Lambda^n(\mathcal{C}_1)$

on B_n. If we take into account two regular mappings $\phi, \psi \in \mathcal{R}(B_n)$, then the difference of corresponding action integrals are obviously given by

$$A(\phi) - A(\psi) = \int_{B_n} \big[\phi^*(L\mu) - \psi^*(L\mu)\big].$$

Let us note that although the mappings ϕ and ψ are prescribed by different functions $\phi^\alpha(x^i)$ and $\psi^\alpha(x^i)$, their arguments $\{x^i\}$ remain the same. To make the comparison of two action integrals more productive, let us embed the mappings ϕ and ψ into a 1-parameter family of regular mappings in such a way that $\phi = \psi$ for the value $s = 0$ of the parameter. In order to determine such a family systematically, we make use of an isovector field of the contact ideal. If a vector field

$$U = X^i \frac{\partial}{\partial x^i} + U^\alpha \frac{\partial}{\partial u^\alpha} + V_i^\alpha \frac{\partial}{\partial v_i^\alpha}$$

is an isovector field of the ideal $\mathcal{I}_1$, we then know from Sec. 9.3 that its components are given by

$$X^i = X^i(\mathbf{x}, \mathbf{u}), \quad U^\alpha = U^\alpha(\mathbf{x}, \mathbf{u}), \quad V_i^\alpha = D_i^{(0)}(U^\alpha - v_j^\alpha X^j)$$

for $N > 1$. The flow generated by this isovector leaves the ideal $\mathcal{I}_1$ invariant. Here, we have employed the notation

$$D_i^{(0)} = \frac{\partial}{\partial x^i} + v_i^\alpha \frac{\partial}{\partial u^\alpha}$$

that has already been introduced on p. 506. However, since we do not want to transform the independent variables x^i, we have to take $X^i = 0$. Such an isovector field will be called a ***vertical isovector field***. Hence, vertical isovector fields of the ideal $\mathcal{I}_1$ are characterised by

$$V = U^\alpha(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u^\alpha} + D_i^{(0)} U^\alpha(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial v_i^\alpha}.$$

When $N = 1$, the condition $X^i = 0$ requires that $\partial F / \partial v_i = 0$. We thus get $U = F = F(\mathbf{x}, u)$ and the structure of vertical isovectors does not change except we have to take, of course, $\alpha = 1$. Consequently, the flow arisen from a vertical isovector field is obtained as the solution of the set of ordinary differential equations

$$\frac{d\overline{x}^i}{ds} = 0, \frac{d\overline{u}^\alpha}{ds} = U^\alpha(\overline{x}^j, \overline{u}^\beta), \frac{d\overline{v}_i^\alpha}{ds} = \overline{D}_i^{(0)} U^\alpha(\overline{\mathbf{x}}, \overline{\mathbf{u}}) = \frac{\partial U^\alpha}{\partial \overline{x}^i} + \overline{v}_i^\beta \frac{\partial U^\alpha}{\partial \overline{u}^\beta}$$

under the initial conditions $\overline{x}^i(0) = x^i, \overline{u}^\alpha(0) = u^\alpha, \overline{v}_i^\alpha(0) = v_i^\alpha$. It is clear that $\overline{x}^i(s) = x^i$. As is well-known, this flow is specified by the mapping $\psi_V(s) = e^{sV} : \mathcal{C}_1 \to \mathcal{C}_1$. As we had mentioned before, the MacLaurin series of this mapping for very small values of the parameter s about $s = 0$ may be expressed as

$$\overline{x}^i(s) = x^i, \ \overline{u}^\alpha(s) = u^\alpha + U^\alpha(x^j, u^\beta)s + o(s),$$
$$\overline{v}_i^\alpha(s) = v_i^\alpha + D_i^{(0)}U^\alpha(x^j, u^\beta)s + o(s)$$

Let us now define the mapping $\phi_V(s) = \psi_V(s) \circ \phi = e^{sV} \circ \phi$ where ϕ is a regular mapping. We obviously get $\phi_V(0) = \phi$ and since

$$\phi_V(s)^*\omega = \phi^*(e^{s\pounds_V}\omega) = 0$$

for all $\omega \in \mathcal{I}_1$, the 1-parameter mappings $\phi_V(s)$ are also regular. For a regular mapping $\phi(\mathbf{x})$ and small values of the parameter s, the mapping $\phi_V(s)$ can be represented as follows

$$\overline{x}^i(s) = x^i, \ \overline{\phi}^\alpha(s) = \phi^\alpha + U^\alpha(x^j, \phi^\beta)s + o(s),$$
$$\overline{\phi}_{,i}^\alpha(s) = \phi_{,i}^\alpha + \Big(\frac{\partial}{\partial x^i} + \phi_{,i}^\beta\frac{\partial}{\partial \phi^\beta}\Big)U^\alpha(x^j, \phi^\gamma)s + o(s).$$

Let us now define a ***variation operator*** δ_V such that the ***variation*** $\delta_V\phi$ means that

$$\delta_V\phi^\alpha = U^\alpha(x^j, \phi^\beta) = \phi^*\mathbf{i}_V(du^\alpha),$$
$$\delta_V\phi_{,i}^\alpha = D_i^{(0)}U^\alpha(x^j, \phi^\beta) = \phi^*\mathbf{i}_V(dv_i^\alpha).$$

It is immediately seen that $\delta_V\phi_{,i}^\alpha = D_i^{(0)}\delta_V\phi^\alpha$. Therefore, the operators of variation and partial differentiation commute. We can thus write

$$\overline{\phi}^\alpha(s) = \phi^\alpha + (\delta_V\phi^\alpha)s + o(s), \ \overline{\phi}_{,i}^\alpha(s) = \phi_{,i}^\alpha + (D_i^{(0)}\delta_V\phi^\alpha)s + o(s).$$

*The action functional $A : \mathcal{R}(B_n) \to \mathbb{R}$ becomes stationary at a regular mapping ϕ if and only if the **variation of the functional** $A(\phi)$ satisfies the condition*

$$\delta_V A(\phi) = \lim_{s\to 0}\frac{A\big(\phi_V(s)\big) - A(\phi)}{s} = 0 \qquad (10.2.4)$$

for every vertical isovector field V verifying the boundary conditions $\delta_V\phi^\alpha|_{\partial B_n} = 0$. In fact, under this circumstances, the functional $A\big(\phi_V(s)\big)$ that may be expressible in the neighbourhood of the mapping ϕ for small values of the parameter s as

$$A\big(\phi_V(s)\big) = \int_{B_n} \phi^*\big[e^{s\pounds_V}(L\mu)\big] = A(\phi) + \delta_V A(\phi)\, s + o(s),$$

does not experience a change of first order and it is reduced to the form

$$A\big(\phi_V(s)\big) = A(\phi) + o(s).$$

Let $n \geq 1$. If $o(s) \simeq s^{2n}$ then ϕ corresponds to an extremum point while if $o(s) \simeq s^{2n+1}$ then to an inflection point. The restriction $\delta_V \phi^\alpha|_{\partial B_n} = 0$ on the boundary means that admissible functions ϕ^α take on specified values on ∂B_n and all varied functions $\phi^\alpha + (\delta_V \phi^\alpha)s$ assume the same values on the boundary.

10.3. EULER-LAGRANGE EQUATIONS

Let us consider the action functional

$$A(\phi) = \int_{B_n} \phi^*(L\mu)$$

where $L \in \Lambda^0(\mathcal{C}_1)$ corresponding to a regular mapping ϕ. We would like to evaluate the expression (10.2.4) when V is a vertical isovector field of the contact ideal. We easily obtain

$$\begin{aligned}\delta_V A(\phi) &= \lim_{s\to 0} \frac{1}{s} \int_{B_n} \big[\phi_V(s)^*(L\mu) - \phi^*(L\mu)\big] \\ &= \lim_{s\to 0} \int_{B_n} \frac{\phi^*(e^{s\pounds_V} - 1)(L\mu)}{s}\end{aligned}$$

from which we immediately arrive at the result

$$\delta_V A(\phi) = \int_{B_n} \phi^* \pounds_V (L\mu).$$

We shall now try to convert this expression into another one on which we will be able to work more efficiently although it would creates exactly the same effect. We define the ***generalised Cartan n-form*** $J \in \Lambda^n(\mathcal{C}_1)$ [it had been essentially proposed by Cartan as a 1-form within a limited scope. *See* Cartan (1922)] by the relation

$$J = \frac{\partial L}{\partial v_i^\alpha}\, \sigma^\alpha \wedge \mu_i = \frac{\partial L}{\partial v_i^\alpha}(du^\alpha \wedge \mu_i - v_i^\alpha \mu). \qquad (10.3.1)$$

It is clear that $J \in \mathcal{I}_1$ so that we automatically find $\phi^* J = 0$. On the other hand, since V is an isovector of the contact ideal, we have $£_V J \in \mathcal{I}_1$ and we thus obtain $\phi^* £_V J = 0$. Therefore, we are able to introduce the relations

$$\begin{aligned} A(\phi) &= \int_{B_n} \phi^*(L\mu + J), \qquad (10.3.2) \\ \delta_V A(\phi) &= \int_{B_n} \phi^* £_V (L\mu + J). \end{aligned}$$

Furthermore, according to the Cartan magic formula employed in evaluating the Lie derivatives, we write

$$£_V(L\mu + J) = \mathbf{i}_V\big(d(L\mu + J)\big) + d\big(\mathbf{i}_V(L\mu + J)\big).$$

If we utilise the Stokes theorem by taking notice of the commutation rule $\phi^* d = d\phi^*$ we arrive at the result

$$\begin{aligned} \delta_V A(\phi) &= \int_{B_n} \big[\phi^*\{\mathbf{i}_V\big(d(L\mu + J)\big)\} + d\{\phi^*\big(\mathbf{i}_V(L\mu + J)\big)\}\big] \\ &= \int_{B_n} \phi^*\{\mathbf{i}_V\big(d(L\mu + J)\big)\} + \int_{\partial B_n} \phi^*\big(\mathbf{i}_V(L\mu + J)\big). \end{aligned}$$

But because $X^i = 0$, we get $\mathbf{i}_V \mu = 0$ and $\mathbf{i}_V \mu_i = 0$ so that we obtain

$$\begin{aligned} \phi^*\big(\mathbf{i}_V(L\mu + J)\big)\big|_{\partial B_n} &= \phi^*\Big(\frac{\partial L}{\partial v_i^\alpha} U^\alpha \mu_i\Big)\Big|_{\partial B_n} \\ &= \phi^*\Big(\frac{\partial L}{\partial v_i^\alpha} \mu_i\Big)\, \delta_V \phi^\alpha\Big|_{\partial B_n} = 0 \end{aligned}$$

since $\delta_V \phi^\alpha$ vanishes on the boundary. We thus conclude that

$$\delta_V A(\phi) = \int_{B_n} \phi^*\{\mathbf{i}_V\big(d(L\mu + J)\big)\}. \qquad (10.3.3)$$

By making use of the relation $d\sigma^\alpha = -\, dv_i^\alpha \wedge dx^i$, we can now introduce the *Cartan* $(n+1)$*-form* C as follows

$$\begin{aligned} C = d(L\mu + J) = dL \wedge \mu + dJ &= \frac{\partial L}{\partial u^\alpha}\, du^\alpha \wedge \mu + \frac{\partial L}{\partial v_i^\alpha}\, dv_i^\alpha \wedge \mu \\ &\quad + d\Big(\frac{\partial L}{\partial v_i^\alpha}\Big) \wedge \sigma^\alpha \wedge \mu_i - \frac{\partial L}{\partial v_i^\alpha}\, dv_i^\alpha \wedge \mu \\ &= \sigma^\alpha \wedge E_\alpha \end{aligned}$$

where the n-forms

$$E_\alpha = \frac{\partial L}{\partial u^\alpha}\,\mu - d\Big(\frac{\partial L}{\partial v_i^\alpha}\Big) \wedge \mu_i \in \Lambda^n(K_1) \tag{10.3.4}$$

will henceforth be called as ***Euler-Lagrange forms***.

Theorem 10.3.1. *A regular mapping $\phi : B_n \to \mathcal{C}_1$ described by the relations* (10.2.2) *renders the action functional A stationary if only if the equations*

$$\phi^* E_\alpha = 0$$

are satisfied in all interior points of the region B_n.

Since $X^i = 0$, we can write

$$\mathbf{i}_V(C) = U^\alpha E_\alpha - \sigma^\alpha \wedge \mathbf{i}_V(E_\alpha)$$

Hence, because of the relations $\phi^*\sigma^\alpha = 0$, we find from (10.3.3) that

$$\delta_V A(\phi) = \int_{B_n} \phi^* U^\alpha\, \phi^* E_\alpha = \int_{B_n} \delta_V \phi^\alpha\, \phi^* E_\alpha.$$

If $\phi^* E_\alpha = 0$, we naturally get $\delta_V A(\phi) = 0$ for all vertical isovectors. Conversely, let us assume that the condition $\delta_V A(\phi) = 0$ holds *for all vertical isovectors*. Then, it can easily be shown that the equations

$$\begin{aligned}\phi^* E_\alpha &= \Big[\frac{\partial L}{\partial \phi^\alpha} - \frac{\partial}{\partial x^i}\Big(\frac{\partial L}{\partial \phi^\alpha_{,i}}\Big)\Big]\mu \\ &= \mathcal{E}_\alpha(L)\mu = 0\end{aligned}$$

are to be satisfied at every interior point of B_n due to the fact that the integrand above is a continuous function (this is known as the fundamental lemma of the calculus of variations). Consequently, the action functional becomes stationary for functions $u^\alpha = \phi^\alpha(\mathbf{x})$ where $\mathbf{x} \in \overset{\circ}{B}_n$ if and only if they are determined as solutions of the following N second order quasilinear partial differential equations called ***Euler-Lagrange equations***

$$\begin{aligned}\mathcal{E}_\alpha(L) &= \frac{\partial L}{\partial u^\alpha} - \frac{\partial}{\partial x^i}\Big(\frac{\partial L}{\partial u^\alpha_{,i}}\Big) = \frac{\partial L}{\partial u^\alpha} - \mathcal{D}_i\Big(\frac{\partial L}{\partial u^\alpha_{,i}}\Big) \\ &= \frac{\partial L}{\partial u^\alpha} - \frac{\partial^2 L}{\partial u^\alpha_{,i}\partial x^i} - \frac{\partial^2 L}{\partial u^\alpha_{,i}\partial u^\beta}\frac{\partial u^\beta}{\partial x^i} - \frac{\partial^2 L}{\partial u^\alpha_{,i}\partial u^\beta_{,j}}\frac{\partial^2 u^\beta}{\partial x^i \partial x^j} = 0\end{aligned} \tag{10.3.5}$$

and satisfy the prescribed conditions on the boundary ∂B_n. Here, the operator $\mathcal{D}_i$ is defined as $\mathcal{D}_i = \phi^* D_i^{(1)}$ where

$$D_i^{(1)} = \frac{\partial}{\partial x^i} + v_i^\alpha \frac{\partial}{\partial u^\alpha} + v_{ij}^\alpha \frac{\partial}{\partial v_j^\alpha}. \qquad \square$$

Instead of a vertical isovector V, let us now consider any isovector $U = X^i \partial/\partial x^i + V$ of the contact ideal $\mathcal{I}_1$. Let us suppose that a regular mapping $\phi : B_n \to \mathcal{C}_1$ is satisfying the equations $\phi^* E_\alpha = 0$. Since we have now $X^i \neq 0$, we find that

$$\mathbf{i}_U(L\mu + J) = \Big[\Big(L\delta_j^i - v_j^\alpha \frac{\partial L}{\partial v_i^\alpha}\Big)X^j + \frac{\partial L}{\partial v_i^\alpha}U^\alpha\Big]\mu_i - X^j \frac{\partial L}{\partial v_i^\alpha}\sigma^\alpha \wedge \mu_{ji}.$$

Therefore, $\phi^*\big(\mathbf{i}_U(L\mu + J)\big)$ is no longer zero on ∂B_n. In this case, the variation of the action functional A must be calculated as follows

$$\delta_U A(\phi) = \int_{\partial B_n} \phi^*\big(\mathbf{i}_U(L\mu + J)\big).$$

Some properties of Euler-Lagrange forms can be readily observed:

(i). *Euler-Lagrange forms are balance n-forms.*

In fact, if we write

$$\Sigma_\alpha^i(x^j, u^\beta, v_j^\beta) = -\frac{\partial L}{\partial v_i^\alpha},$$
$$\Sigma_\alpha(x^j, u^\beta, v_j^\beta) = \frac{\partial L}{\partial u^\alpha}$$

we obtain the balance forms

$$E_\alpha = d\Sigma_\alpha^i \wedge \mu_i + \Sigma_\alpha \mu \in \Lambda^n(\mathcal{C}_1)$$

and Euler-Lagrange equations becomes expressible in the form of balance equations

$$\frac{\partial \Sigma_\alpha^i}{\partial x^i} + \Sigma_\alpha = 0.$$

However, the converse statement is generally not true, that is, every balance forms cannot be represented as the Euler-Lagrange forms.

(ii). *The balance n-forms $\omega_\alpha = d\Sigma_\alpha^i \wedge \mu_i + \Sigma_\alpha \mu$ are to be the Euler-Lagrange forms corresponding to a variational principle if and only if the $(n+1)$-form $\Omega \wedge \mu$ is closed. The 1-form Ω is defined by*

$$\Omega = -\Sigma_\alpha^i\, dv_i^\alpha + \Sigma_\alpha\, du^\alpha \in \Lambda^1(\mathcal{C}_1). \tag{10.3.6}$$

If the forms ω_α are Euler-Lagrange forms we can write

$$\Omega \wedge \mu = \Big(\frac{\partial L}{\partial v_i^\alpha}\,dv_i^\alpha + \frac{\partial L}{\partial u^\alpha}\,du^\alpha\Big) \wedge \mu = dL \wedge \mu = d(L\mu)$$

because $dx^i \wedge \mu = 0$. Thus the form $\Omega \wedge \mu$ is closed. Conversely, let us assume that the form $\Omega \wedge \mu$ is closed, namely, $d(\Omega \wedge \mu) = 0$. Since we are occupied in the Euclidean manifold, the Poincaré lemma will be valid. Thus, there exists a form $\lambda \in \Lambda^n(\mathcal{C}_1)$ such that $\Omega \wedge \mu = d\lambda$. μ is a simple n-form. As a result, the form $\Omega \wedge \mu$ and thus $d\lambda$ become simple $(n+1)$-forms. Let us write $\lambda = L\mu + \Lambda$ where $L \in \Lambda^0(\mathcal{C}_1)$. The form $\Lambda \in \Lambda^n(\mathcal{C}_1)$ cannot contain the form μ. In this case, we have to write $d\lambda = dL \wedge \mu + d\Lambda = \Omega \wedge \mu$. But, we get $d\Lambda = 0$ because μ is not present in the form Λ. We thus obtain $\Omega \wedge \mu = dL \wedge \mu$ and

$$\Sigma_\alpha^i = -\frac{\partial L}{\partial v_i^\alpha}, \quad \Sigma_\alpha = \frac{\partial L}{\partial u^\alpha}.$$

Hence, the forms ω_α are Euler-Lagrange forms.

Example 10.3.1. Let us consider one dimensional non-linear diffusion equation

$$\frac{\partial^2 u}{\partial x^2} = \kappa(u)\frac{\partial u}{\partial t}, \ \ \kappa(u) \neq 0.$$

On denoting $x^1 = x$, $x^2 = t$, $v_1 = u_x$, $v_2 = u_t$, $\mu = dx \wedge dt$, $\mu_1 = dt$ and $\mu_2 = -dx$, the balance form producing this equation can be written as

$$\omega = dv_1 \wedge dt + dK \wedge dx = dv_1 \wedge \mu_1 - dK \wedge \mu_2$$

where we defined $\kappa(u) = K'(u)$. We thus have $\Sigma^1 = v_1$, $\Sigma^2 = -K(u)$, $\Sigma = 0$ and find

$$\Omega = -v_1 dv_1 + K(u)\,dv_2 = -\frac{1}{2}\,d(v_1^2) + K(u)\,dv_2.$$

Therefore, we obtain $d(\Omega \wedge \mu) = d\Omega \wedge \mu = \kappa(u)\,du \wedge dv_2 \wedge dx \wedge dt \neq 0$ which implies that this balance form cannot be derived from a variational principle. ■

We shall now try to show that every balance forms that cannot be generated by a variational principle can be transformed into Euler-Lagrange forms by suitably enlarging that system. We first assume that an arbitrary system of balance forms

$$\omega_\alpha = d\Sigma_\alpha^i \wedge \mu_i + \Sigma_\alpha \mu$$

is given on the contact manifold $\mathcal{C}_1$ whose coordinate cover is $\{x^i, u^\alpha, v_i^\alpha\}$.

In the first step, let us extend the $(n+N+nN)$-dimensional manifold $\mathcal{C}_1$ to an $(n+2N+2nN)$-dimensional manifold $\overline{\mathcal{C}}_1$ by introducing the *adjoint variables* w^α and w_i^α so that its coordinate cover is $\{x^i, u^\alpha, v_i^\alpha, w^\alpha, w_i^\alpha\}$. We then define, respectively, the additional contact and balance forms depending also on adjoint variables

$$\overline{\sigma}^\alpha = dw^\alpha - w_i^\alpha dx^i, \quad \overline{\omega}_\alpha = dW_\alpha^i \wedge \mu_i + W_\alpha \mu, \quad W_\alpha^i, W_\alpha \in \Lambda^0(\overline{\mathcal{C}}_1)$$

where the functions W_α^i and W_α are undetermined as yet. We can thus easily prove the following theorem.

Theorem 10.3.2. *The enlarged balance system* $(\omega_\alpha, \overline{\omega}_\alpha)$ *admits a variational principle if the additional functions* W_α^i *and* W_α *are chosen as follows*

$$\begin{aligned} W_\alpha^i &= -\frac{\partial}{\partial v_i^\alpha}\left(-\Sigma_\beta^j\, w_j^\beta + \Sigma_\beta\, w^\beta\right), \\ W_\alpha &= \frac{\partial}{\partial u^\alpha}\left(-\Sigma_\beta^i\, w_i^\beta + \Sigma_\beta\, w^\beta\right). \end{aligned}$$

Let us introduce a function $L \in \Lambda^0(\overline{\mathcal{C}}_1)$ by the relation

$$L = -\Sigma_\alpha^i\, w_i^\alpha + \Sigma_\alpha\, w^\alpha. \tag{10.3.7}$$

Obviously L depends linearly on the adjoint variables. This definition leads directly to the results

$$\Sigma_\alpha^i = -\frac{\partial L}{\partial w_i^\alpha}, \quad \Sigma_\alpha = \frac{\partial L}{\partial w^\alpha}, \quad W_\alpha^i = -\frac{\partial L}{\partial v_i^\alpha}, \quad W_\alpha = \frac{\partial L}{\partial u^\alpha}.$$

Hence, the form Ω defined above and associated with the enlarged system becomes

$$\begin{aligned} \Omega &= -\Sigma_\alpha^i\, dw_i^\alpha - W_\alpha^i\, dv_i^\alpha + \Sigma_\alpha\, dw^\alpha + W_\alpha\, du^\alpha \\ &= \frac{\partial L}{\partial w_i^\alpha}\, dw_i^\alpha + \frac{\partial L}{\partial v_i^\alpha}\, dv_i^\alpha + \frac{\partial L}{\partial w^\alpha}\, dw^\alpha + \frac{\partial L}{\partial u^\alpha}\, du^\alpha \\ &= dL - \frac{\partial L}{\partial x^i}\, dx^i. \end{aligned}$$

in view of the definitions given above. It then follows that

$$\Omega \wedge \mu = dL \wedge \mu = d(L\mu)$$

which means that the form $\Omega \wedge \mu$ is now closed. Consequently, the balance forms $(\omega_\alpha, \overline{\omega}_\alpha)$ are Euler-Lagrange forms corresponding to a variational principle that can be written as

$$
\begin{aligned}
E_\alpha &= \frac{\partial L}{\partial w^\alpha}\,\mu - d\Big(\frac{\partial L}{\partial w_i^\alpha}\Big)\wedge\mu_i = d\Sigma_\alpha^i\wedge\mu_i + \Sigma_\alpha\mu,\\
\overline{E}_\alpha &= \frac{\partial L}{\partial u^\alpha}\,\mu - d\Big(\frac{\partial L}{\partial v_i^\alpha}\Big)\wedge\mu_i = dW_\alpha^i\wedge\mu_i + W_\alpha\mu\\
&= d\Big(\frac{\partial \Sigma_\beta^j}{\partial v_i^\alpha}\,w_j^\beta - \frac{\partial \Sigma_\beta}{\partial v_i^\alpha}w^\beta\Big)\wedge\mu_i + \Big(\frac{\partial \Sigma_\beta}{\partial u^\alpha}w^\beta - \frac{\partial \Sigma_\beta^i}{\partial u^\alpha}w_i^\beta\Big)\wedge\mu
\end{aligned}
$$

Therefore, if $\phi : B_n \to \overline{\mathcal{C}}_1$ is a regular mapping,, that is, if it satisfies the conditions $\phi^*\sigma^\alpha = 0$, $\phi^*\overline{\sigma}^\alpha = 0$, $\phi^*\mu \neq 0$, then the functions $u^\alpha = \phi^\alpha(\mathbf{x})$, $w^\alpha = \psi^\alpha(\mathbf{x})$ that render the action functional

$$
A(\phi) = \int_{B_n}\phi^*\big[L(x^i,u^\alpha,v_i^\alpha,w^\alpha,w_i^\alpha)\mu\big]
$$

stationary are found as solutions of differential equations

$$
\frac{\partial L}{\partial w^\alpha} - \frac{\partial}{\partial x^i}\Big(\frac{\partial L}{\partial w_{,i}^\alpha}\Big) = 0,\quad \frac{\partial L}{\partial u^\alpha} - \frac{\partial}{\partial x^i}\Big(\frac{\partial L}{\partial u_{,i}^\alpha}\Big) = 0 \qquad (10.3.8)
$$

corresponding to $\phi^*E_\alpha = 0$ and $\phi^*\overline{E}_\alpha = 0$, respectively. The Lagrangian function L is given by (10.3.7). Generally non-linear partial differential equations $(10.3.8)_1$ depend only on x^i, u^α and $u_{,i}^\alpha$. They are exactly in the form of balance equations and determine only the functions u^α whereas $(10.3.8)_2$ yield the *adjoint equations*

$$
\begin{aligned}
&\frac{\partial \Sigma_\beta^j}{\partial u_{,i}^\alpha}\frac{\partial^2 w^\beta}{\partial x^i\partial x^j} + \Big[\mathcal{D}_i\Big(\frac{\partial \Sigma_\beta^j}{\partial u_{,i}^\alpha}\Big) - \frac{\partial \Sigma_\beta^j}{\partial u^\alpha} - \frac{\partial \Sigma_\beta}{\partial u_{,i}^\alpha}\,\delta_i^j\Big]\frac{\partial w^\beta}{\partial x^j} \qquad (10.3.9)\\
&\qquad\qquad + \Big[\frac{\partial \Sigma_\beta}{\partial u^\alpha} - \mathcal{D}_i\Big(\frac{\partial \Sigma_\beta}{\partial u_{,i}^\alpha}\Big)\Big]w^\beta = 0
\end{aligned}
$$

where the operator $\mathcal{D}_i$ was introduced on p. 663. The equations (10.3.9) are linear second order partial differential equations to determine the functions w^α since their coefficients are now, in principle, known functions in terms of $u^\alpha(\mathbf{x})$. □

Example 10.3.2. Let us reconsider the field equations taken into account in Example 10.3.1 on the enlarged manifold $\overline{\mathcal{C}}_1$ this time. With the Lagrangian function

$$
L = -v_1w_1 + K(u)w_2
$$

we then obtain

$$W^1 = -\frac{\partial L}{\partial v_1} = w_1,\ W^2 = -\frac{\partial L}{\partial v_2} = 0,\ \ W = \frac{\partial L}{\partial u} = \kappa(u)w_2.$$

Hence, the adjoint balance form becomes

$$\overline{E} = dw_1 \wedge \mu_1 + \kappa(u)\, w_2\, \mu = dw_1 \wedge dt + \kappa(u)\, w_2\, dx \wedge dt$$

and Euler-Lagrange equations corresponding to the enlarged system are found as

$$\frac{\partial^2 u}{\partial x^2} - \kappa(u)\frac{\partial u}{\partial t} = 0,\ \ \frac{\partial^2 w}{\partial x^2} + \kappa(u)\frac{\partial w}{\partial t} = 0.$$

If we determine the function u from the first equation, then we see that the adjoint equation turns out to be linear in terms of w. Solutions of these equations stationarises the functional

$$A(u,w) = \int_{B_2}\Big(-\frac{\partial u}{\partial x}\frac{\partial w}{\partial x} + K(u)\frac{\partial w}{\partial t}\Big)dxdt. \qquad \blacksquare$$

We shall now try to obtain another relation associated with a regular mapping $\phi \in \mathcal{R}(B_n)$ satisfying the Euler-Lagrange equations. Let us introduce the tensor

$$T^i_j(L) = v^\alpha_j \frac{\partial L}{\partial v^\alpha_i} - L\delta^i_j \tag{10.3.10}$$

that depends linearly on the Lagrangian function L. Inspired by the analytical mechanics, we shall also call (10.3.10) as the ***energy-momentum tensor***. *For any regular mapping ϕ, the following identity holds*:

$$\phi^*(v^\alpha_i E_\alpha + dT^j_i \wedge \mu_j) = -\phi^*\frac{\partial L}{\partial x^i}\mu. \tag{10.3.11}$$

Indeed, routine calculations yield

$$\begin{aligned}v^\alpha_i E_\alpha + dT^j_i \wedge \mu_j &= v^\alpha_i \frac{\partial L}{\partial u^\alpha}\mu - v^\alpha_i d\Big(\frac{\partial L}{\partial v^\alpha_j}\Big) \wedge \mu_j + \frac{\partial L}{\partial v^\alpha_j} dv^\alpha_i \wedge \mu_j \\ &\quad + v^\alpha_i d\Big(\frac{\partial L}{\partial v^\alpha_j}\Big) \wedge \mu_j - dL \wedge \mu_i \\ &= \Big(v^\alpha_i \frac{\partial L}{\partial u^\alpha} - \frac{\partial L}{\partial x^i}\Big)\mu - \frac{\partial L}{\partial u^\alpha}\, du^\alpha \wedge \mu_i + \frac{\partial L}{\partial v^\alpha_j}(dv^\alpha_i \wedge \mu_j - dv^\alpha_j \wedge \mu_i).\end{aligned}$$

We thus obtain

$$\phi^*(v_i^\alpha E_\alpha + dT_i^j \wedge \mu_j) = \Big(u_{,i}^\alpha \phi^* \frac{\partial L}{\partial u^\alpha} - \phi^* \frac{\partial L}{\partial x^i}\Big)\mu - \phi^* \frac{\partial L}{\partial u^\alpha} u_{,i}^\alpha \mu$$
$$+ \phi^* \frac{\partial L}{\partial v_j^\alpha}(u_{,ij} - u_{,ji})\mu = \, - \phi^* \frac{\partial L}{\partial x^i}\mu.$$

If the mapping ϕ is making the action functional stationary, then we ought to get $\phi^* E_\alpha = 0$. Therefore, the foregoing relations lead in that case to

$$\phi^* dT_i^j \wedge \mu_j = d\phi^* T_i^j \wedge \mu_j = \frac{\partial(\phi^* T_i^j)}{\partial x^j}\mu = \, - \phi^* \frac{\partial L}{\partial x^i}\mu$$

where partial derivatives with respect to the variables x^j are actually designating total derivatives with respect to those variables. We finally arrive at the relations

$$\frac{\partial(\phi^* T_i^j)}{\partial x^j} = \mathcal{D}_j(\phi^* T_i^j) = \frac{\partial T_i^j}{\partial x^j} + u_{,j}^\alpha \frac{\partial T_i^j}{\partial u^\alpha} + u_{,jk}^\alpha \frac{\partial T_i^j}{\partial u_{,k}^\alpha} = \, - \phi^* \frac{\partial L}{\partial x^i}.$$

If the function L does not depend explicitly on the variables x^i, then we get $\partial L / \partial x^i = 0$ so that we reach to the result

$$\frac{\partial(\phi^* T_i^j)}{\partial x^j} = 0.$$

We thus produce n conservation laws [*see* (8.7.3)] associated with this variational principle. However, if the function L is explicitly dependent on the variables x^i, then we plainly conclude that the conserved quantities are $\phi^*(T_i^j + L\delta_i^j)$.

Example 10.3.3. We choose the Lagrangian function as

$$L = \frac{1}{2}\sum_{i=1}^{n} v_i^2$$

where $v_i = \partial u / \partial x^i$ with $1 \leq i \leq n$. Since we have

$$\frac{\partial L}{\partial x^i} = 0, \;\; \frac{\partial L}{\partial u} = 0, \;\; \frac{\partial L}{\partial v_i} = v_i,$$

the corresponding Euler-Lagrange equation turns out to be none other than the familiar Laplace equation

$$- u_{,ii} = \, - \nabla^2 u = 0.$$

The energy-momentum tensor

$$T_{ij} = v_i v_j - \tfrac{1}{2} v_k v_k \delta_{ij}$$

associated with this case is obviously symmetric and the conservation law takes the form

$$\frac{\partial}{\partial x^j}(u_{,i}u_{,j} - \tfrac{1}{2}u_{,k}u_{,k}\delta_{ij}) = 0.$$

Hence, if a function u satisfies the Laplace equation inside a region B_n, then the relation

$$\int_{\partial B_n}(u_{,i}u_{,j}n_j - \tfrac{1}{2}u_{,k}u_{,k}n_i)dS = 0$$

must hold on the boundary ∂B_n of that region. This relation may also be represented in the form

$$\int_{\partial B_n}\Big(\frac{\partial u}{\partial n}\boldsymbol{\nabla} u - \frac{1}{2}|\boldsymbol{\nabla} u|^2\mathbf{n}\Big)ds = 0$$

where $\partial u/\partial n$ denotes the normal derivative $u_{,j}n_j$. ■

10.4. NOETHERIAN VECTOR FIELDS

Let us consider a group of transformations acting on the graph space G replacing the variables x^i, u^α by the new variables $\overline{x}^i, \overline{u}^\alpha$. If this group leaves the action integral

$$A(\mathbf{u}) = \int_{B_n} L\Big(x^i, u^\alpha, \frac{\partial u^\alpha}{\partial x^i}\Big)dx^1dx^2\cdots dx^n$$

invariant, that is, if it enforces the satisfaction of the numerical equality

$$\int_{B_n} L\Big(x^i, u^\alpha, \frac{\partial u^\alpha}{\partial x^i}\Big)dx^1\cdots dx^n = \int_{\overline{B}_n} L\Big(\overline{x}^i, \overline{u}^\alpha, \frac{\partial \overline{u}^\alpha}{\partial \overline{x}^i}\Big)d\overline{x}^1\cdots d\overline{x}^n,$$

then it is called a ***variational symmetry group***. That a variational symmetry anticipating the transformations of independent variables as well gives rise to a conservation law is established by the Noether theorem in its most illustrative form [German mathematician Emmy Amalie Noether (1882-1935)]. This theorem has proven to be a principal agent in revealing important results in many physical areas. In this section, we shall try to discuss the Noether theorem and some of its generalisations by an approach based on exterior analysis.

Let $U \in T(\mathcal{C}_1)$ be an isovector field of the contact ideal $\mathcal{I}_1$ and $\phi : B_n \to \mathcal{C}_1$ be a regular mapping satisfying the condition (10.2.1). We consider the action functional

$$A(\phi) = \int_{B_n} \phi^*(L\mu + J)$$

where the form J is given by (10.3.1). An *isovector field* U of the contact ideal $\mathcal{I}_1$ is a ***Noetherian vector field*** if and only if the condition

$$\pounds_U(L\mu + J) = 0 \tag{10.4.1}$$

is fulfilled. We denote the set of all Noetherian vector fields associated with a given Lagrangian function $L \in \Lambda^0(\mathcal{C}_1)$ by $\mathcal{N}(L, \mathcal{C}_1)$. It is quite clear that *the form* $L\mu + J \in \Lambda^n(\mathcal{C}_1)$ *will be invariant with respect to a Noetherian vector field*.

An *isovector field* U of the contact ideal $\mathcal{I}_1$ is called a ***Noetherian vector field of the first kind*** if and only if it satisfies the condition

$$\pounds_U(L\mu + J) \in \mathcal{I}_1 \tag{10.4.2}$$

We denote the set of all Noetherian vector fields of the first kind associated with a given Lagrangian function L by $\mathcal{N}_1(L, \mathcal{C}_1)$.

An *isovector field* U of the contact ideal $\mathcal{I}_1$ is called a ***Noetherian vector field of the second kind*** if and only if it satisfies the condition

$$\pounds_U(L\mu + J) - d\Omega(U) \in \mathcal{I}_1 \tag{10.4.3}$$

for a form $\Omega(U) \in \Lambda^{n-1}(\mathcal{C}_1)$ depending on the vector U. We denote the set of all Noetherian vector fields of the second kind associated with a given Lagrangian function L by $\mathcal{N}_2(L, \mathcal{C}_1)$.

Theorem 10.4.1. *The sets* $\mathcal{N}(L, \mathcal{C}_1)$, $\mathcal{N}_1(L, \mathcal{C}_1)$ *and* $\mathcal{N}_2(L, \mathcal{C}_1)$ *constitute Lie subalgebras on* $\mathbb{R}$ *of the Lie algebra of isovectors of the contact ideal* $\mathcal{I}_1$. *These Lie algebras satisfy the set inclusion relations below*

$$\mathcal{N}(L, \mathcal{C}_1) \subset \mathcal{N}_1(L, \mathcal{C}_1) \subset \mathcal{N}_2(L, \mathcal{C}_1).$$

If $U_1, U_2 \in \mathcal{N}_2(L, \mathcal{C}_1)$, *then the following rules for* $\Omega(U)$ *must be valid*

$$\begin{gathered} d\Omega(a_1U_1 + a_2U_2) - d\big[a_1\Omega(U_1) + a_2\Omega(U_2)\big] \in \mathcal{I}_1, \\ d\Omega([U_1, U_2]) - d\big[\pounds_{U_1}\Omega(U_2) - \pounds_{U_2}\Omega(U_1)\big] \in \mathcal{I}_1 \end{gathered} \tag{10.4.4}$$

where $a_1, a_2 \in \mathbb{R}$.

The set inclusion relations are immediately observable from the definitions. Indeed, by taking $d\Omega = 0$ in $\mathcal{N}_2(L, \mathcal{C}_1)$ we obtain $\mathcal{N}_1(L, \mathcal{C}_1)$ from which we get, in turn, $\mathcal{N}(L,\mathcal{C}_1)$ by simply considering only the zero n-form in $\mathcal{I}_1$. Because of the inclusion relations, it suffices to show that $\mathcal{N}_2(L,\mathcal{C}_1)$ is a Lie subalgebra.

Let us assume that $U_1, U_2 \in \mathcal{N}_2(L,\mathcal{C}_1)$ and $a_1, a_2 \in \mathbb{R}$. Since we can write

$$£_{a_1U_1+a_2U_2}(L\mu + J) = (a_1 £_{U_1} + a_2 £_{U_2})(L\mu + J)$$

we obtain

$$£_{a_1U_1+a_2U_2}(L\mu + J) - a_1 d\Omega(U_1) - a_2 d\Omega(U_2) \in \mathcal{I}_1.$$

According to $(10.4.4)_1$, we find that $a_1U_1 + a_2U_2 \in \mathcal{N}_2(L,\mathcal{C}_1)$. Therefore, $\mathcal{N}_2(L,\mathcal{C}_1)$ becomes a linear vector space on $\mathbb{R}$. On the other hand, in view of (5.11.12), we are able to write

$$£_{[U_1,U_2]}(L\mu + J) = (£_{U_1}£_{U_2} - £_{U_2}£_{U_1})(L\mu + J).$$

Since the operators $£_U$ and d commute, we get

$$£_{[U_1,U_2]}(L\mu + J) - d\big[£_{U_1}\Omega(U_2) - £_{U_2}\Omega(U_1)\big] \in \mathcal{I}_1.$$

If we note $(10.4.4)_2$, we are led to $[U_1,U_2] \in \mathcal{N}_2(L,\mathcal{C}_1)$. This proves that, the vector space $\mathcal{N}_2(L,\mathcal{C}_1)$ is a Lie algebra. In a similar fashion, we can easily show that vector spaces $\mathcal{N}_1(L,\mathcal{C}_1)$ and $\mathcal{N}(L,\mathcal{C}_1)$ are also Lie algebras. □

Let U be an isovector field of the contact ideal $\mathcal{I}_1$. The value of the action functional under the group of transformation, or under the flow, generated by this vector on the manifold $\mathcal{C}_1$ may be computed through the relation

$$A\big(\phi_U(s)\big) = \int_{B_n} \phi^*\big[e^{s£_U}(L\mu + J)\big].$$

We represent the isovector field by

$$U = X^i \frac{\partial}{\partial x^i} + U^\alpha \frac{\partial}{\partial u^\alpha} + V_i^\alpha \frac{\partial}{\partial v_i^\alpha}.$$

We know that the isovector components are given by

$$\begin{aligned} &X^i = X^i(\mathbf{x},\mathbf{u}), \quad U^\alpha = U^\alpha(\mathbf{x},\mathbf{u}),\\ &V_i^\alpha = D_i^{(0)}(U^\alpha - v_j^\alpha X^j), \ \ D_i^{(0)} = \frac{\partial}{\partial x^i} + v_i^\alpha \frac{\partial}{\partial u^\alpha} \end{aligned}$$

when $N > 1$. By making use of (10.3.2) we can write

$$A\big(\phi_U(s)\big) - A(\phi) = \int_{B_n} \phi^* \pounds_U (L\mu + J) + o(s).$$

whence follows the *condition of infinitesimal invariance* as

$$\phi^* \pounds_U (L\mu + J) = \phi^* \pounds_U (L\mu) + \phi^* \pounds_U J = 0$$

Since U is an isovector of the contact ideal, we have $\pounds_U J \in \mathcal{I}_1$. We thus obtain $\phi^* \pounds_U (L\mu + J) = \phi^* \pounds_U (L\mu)$. On the other hand, the relation

$$\pounds_U (L\mu) = \mathbf{i}_U (dL \wedge \mu) + d\, \mathbf{i}_U (L\mu) = U(L)\mu - X^i dL \wedge \mu_i + d(LX^i \mu_i)$$

yields

$$\pounds_U (L\mu) = U(L)\mu + L dX^i \wedge \mu_i = \big[U(L) + LD_i X^i\big]\mu + L \frac{\partial X^i}{\partial u^\alpha}\, \sigma^\alpha \wedge \mu_i.$$

Consequently, the *criterion for infinitesimal invariance* takes the form

$$\phi^* \pounds_U (L\mu) = \phi^* \big[U(L) + LD_i X^i\big]\mu = 0 \tag{10.4.5}$$

where the function $\phi^* U(L)$ is obviously defined as

$$\phi^* U(L) = \phi^* \Big[X^i \frac{\partial L}{\partial x^i} + U^\alpha \frac{\partial L}{\partial u^\alpha} + V_i^\alpha \frac{\partial L}{\partial v_i^\alpha}\Big].$$

However, as is clearly emphasised in the following theorem, the global invariance will also be realised if this criterion occurs.

Let us now attempt to reveal the interrelation between Noetherian vector fields and the invariance of the action functional and conservation laws.

Theorem 10.4.2 (The Noether Theorem). *Let $U \in \mathcal{N}(L, \mathcal{C}_1)$ be a Noetherian vector field. For every regular mapping $\phi \in \mathcal{R}(B_n)$, we recover the global invariance*

$$A\big(\phi_U(s)\big) = A(\phi).$$

Furthermore, if the regular mapping ϕ satisfies the Euler-Lagrange equations, then the identity

$$d\phi^* \big(\mathbf{i}_U (L\mu + J)\big) = 0$$

is also fulfilled. This identity may be as well written in the form

$$d\phi^* \mathcal{J} = 0$$

where the ***current form*** *$\mathcal{J} \in \Lambda^{n-1}(\mathcal{C}_1)$ is defined by*

$$\mathcal{J} = \Big(U^\alpha \frac{\partial L}{\partial v_i^\alpha} - X^j T_j^i\Big)\mu_i$$

Therefore, one has $\phi^*\mathcal{J} = J^i\mu_i$. *The functions* J^i *are given by*

$$J^i = \phi^*\Big[U^\alpha \frac{\partial L}{\partial v_i^\alpha} - X^j T_j^i\Big] = \phi^*\Big[(U^\alpha - X^j v_j^\alpha)\frac{\partial L}{\partial v_i^\alpha} + LX^i\Big] \qquad (10.4.6)$$

Since $U \in \mathcal{N}(L, \mathcal{C}_1)$, the relation $\pounds_U(L\mu + J) = 0$ is satisfied so that we obtain

$$\begin{aligned} A\big(\phi_U(s)\big) - A(\phi) &= \int_{B_n} \phi^*\big[\big(e^{s\pounds_U} - I\big)(L\mu + J)\big] \\ &= \int_{B_n} \phi^*\Big(s\pounds_U + \frac{s^2}{2!}\pounds_U^2 + \cdots\Big)(L\mu + J) = 0 \end{aligned}$$

implying that $A\big(\phi_U(s)\big) = A(\phi)$ for every mapping $\phi \in \mathcal{R}(B_n)$. Hence, the integral curves of a Noetherian vector field generate transformation groups eliciting variational symmetries. On the other hand, $\pounds_U(L\mu + J) = 0$ can be expressed explicitly as

$$\mathbf{i}_U\big(d(L\mu + J)\big) + d\big(\mathbf{i}_U(L\mu + J)\big) = 0$$

so that we can write $d\big(\mathbf{i}_U(L\mu + J)\big) = -\,\mathbf{i}_U\big(d(L\mu + J)\big) = -\,\mathbf{i}_U C$. We thus get

$$d\big(\mathbf{i}_U(L\mu + J)\big) = -\,(U^\alpha - v_i^\alpha X^i)E_\alpha + \sigma^\alpha \wedge \mathbf{i}_U(E_\alpha).$$

Let us now assume that $\phi^* E_\alpha = 0$. Since we also have $\phi^*\sigma^\alpha = 0$, the above expression results in

$$\phi^* d\big(\mathbf{i}_U(L\mu + J)\big) = d\phi^*\big(\mathbf{i}_U(L\mu + J)\big) = 0.$$

If we recall the definition (10.3.10), we understand that we can write

$$\mathbf{i}_U(L\mu + J) = \mathcal{J} - X^j \frac{\partial L}{\partial v_i^\alpha}\,\sigma^\alpha \wedge \mu_{ji} \qquad (10.4.7)$$

and we obtain $\phi^*\big(\mathbf{i}_U(L\mu + J)\big) = \phi^*\mathcal{J}$ whence we deduce that $d\phi^*\mathcal{J} = 0$. This relation gives then rise to the integral conservation law below

$$\int_{B_n} d\phi^*\mathcal{J} = \int_{\partial B_n} \phi^*\mathcal{J} = 0.$$

At interior points of the region B_n, the equation $d\phi^*\mathcal{J} = 0$ produces the conservation law in the differential form

$$\frac{\partial J^i}{\partial x^i} = \frac{\partial}{\partial x^i}\phi^*\Big[(U^\alpha - X^j u^\alpha_{,j})\frac{\partial L}{\partial u^\alpha_{,i}} + LX^i\Big] = 0. \qquad \square$$

Theorem 10.4.3. *Let* $U \in \mathcal{N}_1(L, \mathcal{C}_1)$. *For every regular mapping* $\phi \in \mathcal{R}(B_n)$, *the global invariance*

$$A\big(\phi_U(s)\big) = A(\phi)$$

is valid. Furthermore, if the regular mapping ϕ *satisfies Euler-Lagrange equations, then the following identity*

$$d\phi^*\mathcal{J} = 0$$

is fulfilled.

If $U \in \mathcal{N}_1(L, \mathcal{C}_1)$, then the relation $\pounds_U(L\mu + J) \in \mathcal{I}_1$ must be satisfied. Since $\phi^*\mathcal{I}_1 = 0$, we evidently obtain exactly the same result as in Theorem 10.4.2. However, although the results of those two theorems seem to be remarkably alike, we have to point out that they possess actually rather different structures because variational symmetry groups are generated by significantly different Noetherian vector fields in Theorems 10.4.2 and 10.4.3. $\square$

Theorem 10.4.4. *Let* $U \in \mathcal{N}_2(L, \mathcal{C}_1)$. *For every regular mapping* $\phi \in \mathcal{R}(B_n)$ *the relation*

$$A\big(\phi_U(s)\big) = A(\phi) + \int_{\partial B_n} \phi^* \int_0^s e^{t\pounds_U}\Omega(U)\, dt$$

is valid. Furthermore, if the regular mapping ϕ *satisfies Euler-Lagrange equations, then the following identity*

$$d\phi^*(\mathcal{J} - \Omega) = 0$$

is fulfilled.

If $U \in \mathcal{N}_2(L, \mathcal{C}_1)$, we have the relation $\pounds_U(L\mu + J) - d\Omega(U) = \alpha$ where $\alpha \in \mathcal{I}_1$. On the other hand, utilising the commutation rule $d \circ \pounds_U = \pounds_U \circ d$ we come up with the expression

$$\begin{aligned}\big(e^{s\pounds_U} - I\big)(L\mu + J) &= \int_0^s \frac{d}{dt}\big[e^{t\pounds_U}(L\mu + J)\big]\, dt \\ &= \int_0^s e^{t\pounds_U}\pounds_U(L\mu + J)\, dt\end{aligned}$$

$$= \int_0^s e^{t\pounds_U} d\Omega(U)\, dt + \int_0^s e^{t\pounds_U} \alpha\, dt$$
$$= d\int_0^s e^{t\pounds_U} \Omega(U)\, dt + \int_0^s e^{t\pounds_U} \alpha\, dt.$$

In the last line, we have made use of the plainly observable rule $e^{t\pounds_U} \circ d = d \circ e^{t\pounds_U}$. Because $\phi^*\alpha = 0$, we thus conclude that

$$\begin{aligned} A\big(\phi_U(s)\big) &= A(\phi) + \int_{B_n} \phi^*\big[\big(e^{s\pounds_U} - I\big)(L\mu + J)\big] \\ &= A(\phi) + \int_{B_n} d\,\phi^* \int_0^s e^{t\pounds_U} \Omega(U)\, dt \\ &= A(\phi) + \int_{\partial B_n} \phi^* \int_0^s e^{t\pounds_U} \Omega(U)\, dt. \end{aligned}$$

Employing this time, the relation (10.4.7) we can reach to the expression

$$\begin{aligned} \pounds_U(L\mu + J) &= \mathbf{i}_U(\sigma^\alpha)E_\alpha - \sigma^\alpha \wedge \mathbf{i}_U(E_\alpha) + d\big(\mathbf{i}_U(L\mu + J)\big) \\ &= \mathbf{i}_U(\sigma^\alpha)E_\alpha + d\mathcal{J} - X^j \frac{\partial L}{\partial v_i^\alpha}\, d\sigma^\alpha \wedge \mu_{ji} \\ &\quad + \sigma^\alpha \wedge \Big[d\Big(X^j \frac{\partial L}{\partial v_i^\alpha}\Big) \wedge \mu_{ji} - \mathbf{i}_U(E_\alpha)\Big] \\ &= d\Omega + \alpha \end{aligned}$$

so that we can write $d(\mathcal{J} - \Omega) + \mathbf{i}_U(\sigma^\alpha)E_\alpha \in \mathcal{I}_1$. This relation helps us to deduce the result

$$\phi^* d(\mathcal{J} - \Omega) = d\phi^*(\mathcal{J} - \Omega) = 0$$

whenever $\phi^* E_\alpha = 0$. Then the conservation law in integral form leads to the boundary integral

$$\int_{\partial B_n} \phi^*(\mathcal{J} - \Omega) = 0$$

when we employ the Stokes theorem. If we particularly choose $\Omega = \Omega^i \mu_i$, then we obtain the conservation law

$$\frac{\partial \phi^*(J^i - \Omega^i)}{\partial x^i} = 0$$

at interior points of B_n. □

Example 10.4.1. As an application of the Noether theorem, we will obtain once more the conservation laws given by the relations (8.7.7) of the equations of motion of an hyperelastic medium in Sec. 8.7 by means of variational symmetries. We employ the notation in Example 8.7.4 and denote the isovector field by

$$U = \Phi_K \frac{\partial}{\partial X_K} + \Psi \frac{\partial}{\partial t} + \Omega_k \frac{\partial}{\partial x_k} + U_{kK} \frac{\partial}{\partial F_{kK}} + U_k \frac{\partial}{\partial v_k}$$

where $F_{kK} = \partial x_k / \partial X_K$, $v_k = \partial x_k / \partial t$. Introducing the Lagrangian function $L = \Sigma(F_{kK}) - \frac{1}{2}\rho_0 v_k v_k = \Sigma(\mathbf{F}) - \frac{1}{2}\rho_0 |\mathbf{v}|^2$, we may define the action functional by

$$A(\phi) = \int_{B_4} \phi^*(L\mu)$$

where $\mu = dX_1 \wedge dX_2 \wedge dX_3 \wedge dt$ is the volume form. Then the Euler-Lagrange equations (10.3.5) give the equations of motion (8.7.4)

$$-\frac{\partial}{\partial X_K}\Big(\frac{\partial \Sigma}{\partial F_{kK}}\Big) + \rho_0 \frac{\partial v_k}{\partial t} = 0$$

in the absence of body forces. The components U_{kK} and U_k of the isovector field U can be written as

$$U_{kK} = \frac{\partial \mathcal{F}_k}{\partial X_K} + F_{lK} \frac{\partial \mathcal{F}_k}{\partial x_l}, \;\; U_k = \frac{\partial \mathcal{F}_k}{\partial t} + v_l \frac{\partial \mathcal{F}_k}{\partial x_l}$$

where $\mathcal{F}_k = \Omega_k - F_{kL}\Phi_L - v_k \Psi$. The isovector field U satisfies the criterion for infinitesimal invariance (10.4.5) provided that

$$\phi^*\Big[U_{kK} \frac{\partial \Sigma}{\partial F_{kK}} - \rho_0 U_k v_k + L\Big(\frac{\partial \Phi_K}{\partial X_K} + F_{kK} \frac{\partial \Phi_K}{\partial x_k} + \frac{\partial \Psi}{\partial t} + v_k \frac{\partial \Psi}{\partial x_k}\Big)\Big] = 0.$$

Keeping in mind this relation, we then obtain from (10.4.6)

$$\begin{aligned} J_K =& \frac{\partial \Sigma}{\partial F_{kK}} \Omega_k + \Big(L\delta_{KL} - F_{kL} \frac{\partial \Sigma}{\partial F_{kK}}\Big)\Phi_L - v_k \frac{\partial \Sigma}{\partial F_{kK}} \Psi \\ J_4 =& -\rho_0 v_k \Omega_k + \rho_0 F_{kL} v_k \Phi_L + (L + \rho_0 v_k v_k)\Psi \end{aligned}$$

and the conservation law takes the form

$$\frac{\partial J_K}{\partial X_K} + \frac{\partial J_4}{\partial t} = 0.$$

To find the general expressions satisfying the criterion of infinitesimal invariance for an arbitrary stress potential Σ is next to impossible. However, we can designate some conservation laws just by inspection corresponding to certain isovector fields satisfying the criterion of infinitesimal invariance trivially.

(i). We take $\Phi_K = 0, \Psi = 0, \Omega_k = a_k =$ *constant*. (10.4.5) is clearly satisfied. Since then

$$J_K = \frac{\partial \Sigma}{\partial F_{kK}} a_k, \quad J_4 = -\rho_0 v_k a_k$$

the conservation law is found to be

$$a_k \Big[\frac{\partial}{\partial X_K} \Big(\frac{\partial \Sigma}{\partial F_{kK}} \Big) - \rho_0 \frac{\partial v_k}{\partial t} \Big] = 0.$$

Because the coefficients a_k are arbitrary, we thus get

$$\frac{\partial T_{Kk}}{\partial X_K} - \rho_0 \frac{\partial v_k}{\partial t} = 0$$

[*see* $(8.7.7)_2$].

(ii). We take $\Phi_K = 0, \Psi = -A =$ *constant*, $\Omega_k = 0$. (10.4.5) is satisfied automatically. We then get

$$J_K = \frac{\partial \Sigma}{\partial F_{kK}} v_k A, \quad J_4 = -(L + \rho_0 v_k v_k) A = -\Big(\Sigma + \frac{1}{2} \rho_0 v_k v_k \Big) A$$

so that the conservation law becomes

$$\frac{\partial}{\partial X_K} (T_{Kk} v_k) - \frac{\partial}{\partial t} \Big(\Sigma + \frac{1}{2} \rho_0 v_k v_k \Big) = 0$$

[*see* $(8.7.7)_1$].

(iii). We take $\Phi_K = A_K =$ *constant*, $\Psi = 0, \Omega_k = 0$. (10.4.5) is satisfied identically and we have

$$J_K = \Big(L \delta_{KL} - F_{kL} \frac{\partial \Sigma}{\partial F_{kK}} \Big) A_L, \quad J_4 = \rho_0 F_{kL} v_k A_L.$$

Hence, the corresponding conservation laws are

$$\frac{\partial}{\partial X_K} \Big[\Big(\Sigma - \frac{1}{2} \rho_0 v_k v_k \Big) \delta_{KL} - T_{Kk} x_{k,L} \Big] + \rho_0 \frac{\partial}{\partial t} (x_{k,L} v_k) = 0$$

[*see.* $(8.7.7)_4$].

(iv). We take $\Phi_K = 0, \Psi = 0, \Omega_k = e_{klm} a_l x_m$ where $a_l =$ *constant.* In this case we obtain

$$U_{kK} = e_{klm} a_l F_{mK}, \quad U_k = e_{klm} a_l v_m$$

and (10.4.5) cannot satisfied unless

$$a_l \Big(e_{klm} F_{mK} \frac{\partial \Sigma}{\partial F_{kK}} - e_{klm} \rho_0 v_m v_k \Big) = a_l e_{klm} F_{mK} \frac{\partial \Sigma}{\partial F_{kK}} = 0.$$

Due to $(8.7.5)_2$, $F_{mK} T_{Kk}$ is symmetric in indices k and m. It is immediate then to observe that the above condition is satisfied and the current components

$$J_K = e_{klm} \frac{\partial \Sigma}{\partial F_{kK}} x_m a_l, \quad J_4 = \, - \, e_{klm} \rho_0 v_k x_m a_l$$

yield the conservation laws

$$\frac{\partial}{\partial X_K} (e_{klm} x_m T_{Kk}) - \rho_0 \frac{\partial}{\partial t} (e_{klm} x_m v_k) = 0$$

[*see* $(8.7.7)_3$]. It is seen that the above introduced invariances of the action functional characterising the motion of an hyperelastic material give rise to the following conservation laws: *invariance under spatial translations leads to the balance of linear momentum, invariance under time translations leads to the conservation of energy, invariance under spatial rotations leads to the balance of angular momentum and invariance under material translations leads to the J-integral of fracture mechanics.* ■

Since variational symmetries leave action functionals invariant, the transformations of the extremals that are solutions of Euler-Lagrange equations under these groups yield solutions of the same equations. Therefore, a variational symmetry group of an action functional becomes likewise a symmetry group of Euler-Lagrange equations arisen from that functional. However, the converse statement is generally not true. It is possible to observe that although Euler-Lagrange equations may possess some symmetry groups, these groups may not correspond to a variational symmetry.

10.5. VARIATIONAL PROBLEM FOR A GENERAL ACTION FUNCTIONAL

As in Sec. 10.2, we consider a functional A associated with functions $u^\alpha(\mathbf{x})$, where $\alpha = 1, \dots, N$, defined on a region $B_n \subseteq \mathbb{R}$. However, this time we shall assume that the Lagrangian function L generating this functional is also dependent on the partial derivatives of functions u^α up to and including the order m. Hence, such a functional is expressible as

$$A(\mathbf{u}) = \int_{B_n} L(x^i, u^\alpha, u^\alpha_{,i_1}, u^\alpha_{,i_1 i_2}, \dots, u^\alpha_{,i_1 i_2 \cdots i_m})\, dx^1 dx^2 \cdots dx^n.$$

As in (9.2.4), we introduce auxiliary variables $u^\alpha_{,i_1\cdots i_r} = v^\alpha_{i_1\cdots i_r}, 0 \le r \le m$, $1 \le i_1, \cdots, i_r \le n$ to construct the contact manifold $\mathcal{C}_m$ whose coordinate cover is now given by $\{x^i, v^\alpha_{i_1 i_2\cdots i_r} : 0 \le r \le m\}$. The closed contact ideal of this manifold is

$$\mathcal{I}_m = \overline{\mathcal{I}}(\sigma^\alpha, \sigma^\alpha_{i_1}, \sigma^\alpha_{i_1 i_2}, \dots \sigma^\alpha_{i_1 i_2 \cdots i_{m-1}}; d\sigma^\alpha_{i_1 i_2 \cdots i_{m-1}}).$$

As we have done before, we define

$$\sigma^\alpha_{i_1 i_2 \cdots i_r} = dv^\alpha_{i_1 i_2 \cdots i_r} - v^\alpha_{i_1 i_2 \cdots i_r i} dx^i, 0 \le r \le m-1.$$

We denote the set all *regular mappings* from B_n into the manifold $\mathcal{C}_m$ by

$$\mathcal{R}(B_n) = \{\phi : B_n \to \mathcal{C}_m : \phi^* \mathcal{I}_m = 0,\ \phi^* \mu \neq 0\}.$$

Consequently, we must have $\phi^* \sigma^\alpha_{i_1 i_2 \cdots i_r} = 0, 1 \le \alpha \le N, 0 \le r \le m-1$. The condition $\phi^* d\sigma^\alpha_{i_1 \cdots i_{m-1}} = 0$ is satisfied automatically due to the commutation rule $\phi^* d = d\phi^*$. We know that we can write

$$d\sigma^\alpha_{i_1 i_2 \cdots i_{m-1}} \wedge \mu_{i_m} = -\, dv^\alpha_{i_1 \cdots i_{m-1} i_m} \wedge \mu.$$

We thus obviously get $\phi^* dv^\alpha_{i_1 \cdots i_m} = 0$. According to this definitions the regular mappings are *functions depending on the same independent variables and annihilating the ideal* $\mathcal{I}_m$. Hence, if $\phi \in \mathcal{R}(B_n)$, then it must be represented by the relations

$$x^i = x^i, \quad v^\alpha_{i_1 \cdots i_r} = \phi^\alpha_{,i_1 \cdots i_r}(\mathbf{x}), \quad 0 \le r \le m. \tag{10.5.1}$$

Therefore, the Lagrangian function

$$L(x^i, u^\alpha, v^\alpha_{i_1}, v^\alpha_{i_1 i_2}, \dots, v^\alpha_{i_1 i_2 \cdots i_m}) \in \Lambda^0(\mathcal{C}_m)$$

is dependent only on the coordinates of the manifold $\mathcal{C}_m$. In order to be able to utilise the exterior analysis that involves only first order derivatives, we shall not take into account the relations between arguments of L as types of derivatives when it was pulled back by ϕ. We, therefore, write $\phi^* L$ as follows

$$\phi^* L = L\big(\mathbf{x}, u^\alpha(\mathbf{x}), v^\alpha_{i_1}(\mathbf{x}), v^\alpha_{i_1 i_2}(\mathbf{x}), \dots, v^\alpha_{i_1 i_2 \cdots i_m}(\mathbf{x})\big).$$

But, with the aim of revealing the existing interrelations between these variables, we shall resort to the method of *Lagrange multipliers* introduced by Lagrange to provide an effective strategy to determine extrema of functions subject to constraints. To further simplify the analysis we now define a form $\mathcal{L} \in \Lambda^n(\mathcal{C}_m)$ by the expression below

$$\begin{aligned}\mathcal{L} &= L\mu + \sum_{r=0}^{m-1} \lambda_\alpha^{i_1\cdots i_r i}(\mathbf{x})\, \sigma^\alpha_{i_1 i_2\cdots i_r} \wedge \mu_i + \sum_{r=0}^{m-1} \frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i}}\, \sigma^\alpha_{i_1 i_2\cdots i_r} \wedge \mu_i \\ &= \Big[L - \sum_{r=0}^{m-1}\Big(\lambda_\alpha^{i_1\cdots i_r i}(\mathbf{x}) + \frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i}}\Big) v^\alpha_{i_1 i_2\cdots i_r i}\Big]\mu \\ &\quad + \sum_{r=0}^{m-1}\Big(\lambda_\alpha^{i_1\cdots i_r i}(\mathbf{x}) + \frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i}}\Big) dv^\alpha_{i_1 i_2\cdots i_r} \wedge \mu_i \end{aligned} \tag{10.5.2}$$

where the Lagrange multipliers $\lambda_\alpha^{i_1\cdots i_r i}(\mathbf{x}), 0 \le r \le m-1$ are presently arbitrary functions. Since (10.5.2) implies that $\mathcal{L} - L\mu \in \mathcal{I}_m$, it is clear that $\phi^*\mathcal{L} = \phi^*(L\mu)$. Hence, there will be no harm in writing the action functional in the form

$$A(\phi) = \int_{B_n} \phi^*\mathcal{L}. \tag{10.5.3}$$

In order to construct the regular mappings, we again make use of the isovector fields of the contact ideal. If $N > 1$, we know that such a vector is expressible as

$$U = X^i(\mathbf{x},\mathbf{u})\frac{\partial}{\partial x^i} + U^\alpha(\mathbf{x},\mathbf{u})\frac{\partial}{\partial u^\alpha} + \sum_{r=1}^{m} V^\alpha_{i_1\cdots i_r}\frac{\partial}{\partial v^\alpha_{i_1\cdots i_r}} \tag{10.5.4}$$

[*see p.* 506]. The functions $V^\alpha_{i_1\cdots i_r}$ can be calculated through the recurrence relations

$$V^\alpha_{i_1\cdots i_r i} = D_i^{(r)}(V^\alpha_{i_1\cdots i_r} - v^\alpha_{i_1\cdots i_r j}X^j). \tag{10.5.5}$$

The operator $D_i^{(r)}$ is defined by the relation (9.3.18) and accounts for the *total derivative* of any function $f\big(\mathbf{x}, u^\alpha(\mathbf{x}), v^\alpha_{i_1}(\mathbf{x}), \dots, v^\alpha_{i_1\cdots i_r}(\mathbf{x})\big)$ with respect to the variable x^i. Under the mapping ϕ, we get

$$\phi^* D_i^{(r)} f = \frac{\partial \phi^* f}{\partial x^i}.$$

Regular mappings can now be selected as flows generated by isovector fields of the contact manifold. However, since we do not wish to change the independent variables, we have to take $X^i = 0$. As we have mentioned before, this property allows us to convey all findings established in this situation directly to the case $N = 1$. We thus consider the ***vertical isovector fields*** in the form

$$V = U^\alpha(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial u^\alpha} + \sum_{r=1}^{m} V^\alpha_{i_1\cdots i_r}\frac{\partial}{\partial v^\alpha_{i_1\cdots i_r}}. \tag{10.5.6}$$

where we have of course the recurrence relations $V^\alpha_{i_1\cdots i_r i} = D^{(r)}_i V^\alpha_{i_1\cdots i_r}$. The flow created by a vertical isovector field will be obtained, as usual, by integrating the set of ordinary differential equations

$$\frac{d\overline{x}^i}{ds} = 0,$$
$$\frac{d\overline{v}^\alpha_{i_1\cdots i_r}}{ds} = V^\alpha_{i_1\cdots i_r}(\overline{x}^j, \overline{u}^\beta, \overline{v}^\alpha_{i_1}, \ldots, \overline{v}^\alpha_{i_1\cdots i_r}), \;\; 0 \le r \le m$$

under the initial conditions $\overline{x}^i(0) = x^i, \overline{v}^\alpha_{i_1\cdots i_r}(0) = v^\alpha_{i_1\cdots i_r}, 0 \le r \le m$. It is obvious that $\overline{x}^i(s) = x^i$. We know that this flow is represented by the mapping $\psi_V(s) = e^{sV} : \mathcal{C}_m \to \mathcal{C}_m$. Expanding this exponential mapping into a Maclaurin series about $s = 0$ and retain only up to the first order terms, we can write

$$\overline{x}^i(s) = x^i, \;\; \overline{u}^\alpha(s) = u^\alpha + U^\alpha(x^j, u^\beta)s + o(s),$$
$$\overline{v}^\alpha_{i_1\cdots i_r}(s) = v^\alpha_{i_1\cdots i_r} + V^\alpha_{i_1\cdots i_r}s + o(s), \qquad 1 \le r \le m$$

for small values of the parameter s. Let us now define the mapping $\phi_V(s) = \psi_V(s) \circ \phi = e^{sV} \circ \phi$ for a regular mapping ϕ. We have $\phi_V(0) = \phi$ and

$$\phi_V(s)^*\omega = \phi^*(e^{s\pounds_V}\omega) = 0$$

for every form $\omega \in \mathcal{I}_m$. We introduce the variation operator δ_V as follows

$$\delta_V\phi^\alpha = \phi^* U^\alpha; \quad \delta_V\phi^\alpha_{,i_1\cdots i_r} = \phi^* V^\alpha_{i_1\cdots i_r}, \;\; 1 \le r \le m\,.$$

The action functional $A : \mathcal{R}(B_n) \to \mathbb{R}$ becomes stationary at a regular mapping ϕ if and only if the variation of the functional $A(\phi)$ vanishes, that is, if the relation

$$\delta_V A(\phi) = \lim_{s\to 0}\frac{A\big(\phi_V(s)\big) - A(\phi)}{s} = 0$$

is satisfied for any vertical isovector field V of the contact ideal satisfying the boundary conditions $\delta_V\phi^\alpha_{,i_1\cdots i_r}\big|_{\partial B_n} = 0$, $0 \le r \le m-1$. These boundary conditions mean that the values of the functions $\phi^\alpha_{,i_1\cdots i_r}$, $0 \le r \le m-1$ on the boundary ∂B_n are prescribed and for any isovector *field V all varied functions $\phi^\alpha_{,i_1\cdots i_r}+\delta_V\phi^\alpha_{,i_1\cdots i_r}s, 0\le r\le m-1$ take on the same values on the boundary.*

With the aim of determining the regular mapping ϕ that makes the functional $A(\phi)$ stationary, we have to discuss again the satisfaction of the condition

$$\delta_V A(\phi)=\int_{B_n}\phi^*\pounds_V\mathcal{L}=0. \tag{10.5.7}$$

Since V is an isovector field, we must have $\pounds_V(\mathcal{L}-L\mu)\in\mathcal{I}_m$ so that we get $\phi^*\pounds_V\mathcal{L}=\phi^*\pounds_V(L\mu)$. If we substitute into the expression (10.5.7) the Lie derivative $\pounds_V\mathcal{L}=\mathbf{i}_V(d\mathcal{L})+d\mathbf{i}_V(\mathcal{L})$ and recall the commutation rule $d\circ\phi^*=\phi^*\circ d$, then the application of the Stokes theorem casts (10.5.7) into the form

$$\delta_V A(\phi)=\int_{B_n}\phi^*\mathbf{i}_V(d\mathcal{L})+\int_{\partial B_n}\phi^*\mathbf{i}_V(\mathcal{L})=0.$$

On the other hand, because of the relation

$$\mathbf{i}_V(\mathcal{L})=\sum_{r=0}^{m-1}\Big(\lambda^{i_1\cdots i_r i}_\alpha(\mathbf{x})+\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i}}\Big)V^\alpha_{i_1\cdots i_r}\mu_i=\sum_{r=0}^{m-1}\Lambda^{i_1\cdots i_r i}_\alpha V^\alpha_{i_1\cdots i_r}\mu_i$$

we obtain

$$\phi^*\mathbf{i}_V(\mathcal{L})|_{\partial B_n}=\sum_{r=0}^{m-1}\big(\phi^*\Lambda^{i_1\cdots i_r i}_\alpha\big)\delta_V v^\alpha_{i_1\cdots i_r}\mu_i\Big|_{\partial B_n}=0$$

where we have obviously defined

$$\Lambda^{i_1\cdots i_r i}_\alpha=\lambda^{i_1\cdots i_r i}_\alpha(\mathbf{x})+\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i}}.$$

Next, we shall evaluate the form $d\mathcal{L}$. After some almost trivial manipulations, we end up with the expression

$$\begin{aligned}d\mathcal{L}=&\Big(\frac{\partial L}{\partial u^\alpha}-\frac{\partial\lambda^i_\alpha}{\partial x^i}\Big)\sigma^\alpha\wedge\mu-\sum_{r=1}^{m-1}\Big(\frac{\partial\lambda^{i_1\cdots i_r i}_\alpha}{\partial x^i}+\lambda^{i_1\cdots i_r}_\alpha\Big)\sigma^\alpha_{i_1\cdots i_r}\wedge\mu\\&+\sum_{r=0}^{m-1}d\Big(\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i}}\Big)\wedge\sigma^\alpha_{i_1\cdots i_r}\wedge\mu_i-\lambda^{i_1\cdots i_m}_\alpha dv^\alpha_{i_1\cdots i_m}\wedge\mu.\end{aligned}$$

But we can write

$$\sum_{r=0}^{m-1} d\Big(\frac{\partial L}{\partial v^{\alpha}_{i_1\cdots i_r i}}\Big) \wedge \sigma^{\alpha}_{i_1\cdots i_r} \wedge \mu_i = -\sum_{r=0}^{m-1} \frac{\partial}{\partial x^i}\Big(\frac{\partial L}{\partial v^{\alpha}_{i_1\cdots i_r i}}\Big)\sigma^{\alpha}_{i_1\cdots i_r} \wedge \mu$$
$$+\sum_{s=0}^{m}\sum_{r=0}^{m-1} \frac{\partial^2 L}{\partial v^{\alpha}_{i_1\cdots i_r i}\partial v^{\beta}_{j_1\cdots j_s}}\, dv^{\beta}_{j_1\cdots j_s} \wedge \sigma^{\alpha}_{i_1\cdots i_r} \wedge \mu_i.$$

Thus, introducing this expression into $d\mathcal{L}$ and arranging the terms appropriately we eventually obtain

$$d\mathcal{L} = \Big[\frac{\partial L}{\partial u^{\alpha}} - \frac{\partial \lambda^i_{\alpha}}{\partial x^i} - \frac{\partial}{\partial x^i}\Big(\frac{\partial L}{\partial v^{\alpha}_i}\Big)\Big]\sigma^{\alpha} \wedge \mu - \sum_{r=1}^{m-1}\Big[\frac{\partial \lambda^{i_1\cdots i_r i}_{\alpha}}{\partial x^i} + \lambda^{i_1\cdots i_r}_{\alpha}$$
$$+ \frac{\partial}{\partial x^i}\Big(\frac{\partial L}{\partial v^{\alpha}_{i_1\cdots i_r i}}\Big)\Big]\sigma^{\alpha}_{i_1\cdots i_r} \wedge \mu - \lambda^{i_1\cdots i_m}_{\alpha} dv^{\alpha}_{i_1\cdots i_m} \wedge \mu$$
$$+ \sum_{s=0}^{m-1}\sum_{r=0}^{m-1} \frac{\partial^2 L}{\partial v^{\alpha}_{i_1\cdots i_r i}\partial v^{\beta}_{j_1\cdots j_s}}\, \sigma^{\beta}_{j_1\cdots j_s} \wedge \sigma^{\alpha}_{i_1\cdots i_r} \wedge \mu_i$$
$$+ \sum_{r=0}^{m-1} \frac{\partial^2 L}{\partial v^{\alpha}_{i_1\cdots i_r i}\partial v^{\beta}_{j_1\cdots j_m}}\, dv^{\beta}_{j_1\cdots j_m} \wedge \sigma^{\alpha}_{i_1\cdots i_r} \wedge \mu_i.$$

From the above expression we deduce that

$$\mathbf{i}_V(d\mathcal{L}) = U^{\alpha}\Big[\frac{\partial L}{\partial u^{\alpha}} - \frac{\partial \lambda^i_{\alpha}}{\partial x^i} - \frac{\partial}{\partial x^i}\Big(\frac{\partial L}{\partial v^{\alpha}_i}\Big)\Big]\mu$$
$$-\sum_{r=1}^{m-1}\Big[\frac{\partial \lambda^{i_1\cdots i_r i}_{\alpha}}{\partial x^i} + \lambda^{i_1\cdots i_r}_{\alpha}$$
$$+ \frac{\partial}{\partial x^i}\Big(\frac{\partial L}{\partial v^{\alpha}_{i_1\cdots i_r i}}\Big)\Big] V^{\alpha}_{i_1\cdots i_r}\mu - \lambda^{i_1\cdots i_m}_{\alpha} V^{\alpha}_{i_1\cdots i_m}\mu$$
$$+ \sum_{s=0}^{m-1}\sum_{r=0}^{m-1} \frac{\partial^2 L}{\partial v^{\alpha}_{i_1\cdots i_r i}\partial v^{\beta}_{j_1\cdots j_s}}\,(V^{\beta}_{j_1\cdots j_s}\sigma^{\alpha}_{i_1\cdots i_r} - V^{\alpha}_{i_1\cdots i_r}\sigma^{\beta}_{j_1\cdots j_s}) \wedge \mu_i$$
$$+ \sum_{r=0}^{m-1} \frac{\partial^2 L}{\partial v^{\alpha}_{i_1\cdots i_r i}\partial v^{\beta}_{j_1\cdots j_m}}\,(V^{\beta}_{j_1\cdots j_m}\sigma^{\alpha}_{i_1\cdots i_r} - V^{\alpha}_{i_1\cdots i_r} dv^{\beta}_{j_1\cdots j_m}) \wedge \mu_i.$$

However, since $\phi^*\mathcal{I}_m = 0$ we finally obtain

$$\int_{B_n} \phi^*\mathbf{i}_V(d\mathcal{L}) = \int_{B_n}\Big[\phi^* F_{\alpha}\delta_V u^{\alpha} - \sum_{r=1}^{m-1}\phi^* F^{i_1\cdots i_r}_{\alpha}\delta_V v^{\alpha}_{i_1\cdots i_r} - \lambda^{i_1\cdots i_m}_{\alpha}\delta_V v^{\alpha}_{i_1\cdots i_m}\Big]\mu$$

where we have defined

$$F_\alpha = \frac{\partial L}{\partial u^\alpha} - \frac{\partial \lambda^i_\alpha}{\partial x^i} - \frac{\partial}{\partial x^i}\Big(\frac{\partial L}{\partial v^\alpha_i}\Big),$$
$$F^{i_1\cdots i_r}_\alpha = \frac{\partial \lambda^{i_1\cdots i_r i}_\alpha}{\partial x^i} + \lambda^{i_1\cdots i_r}_\alpha + \frac{\partial}{\partial x^i}\Big(\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i}}\Big), \quad 1 \le r \le m-1.$$

Since we have assumed that all variations $\delta_V v^\alpha_{i_1\cdots i_r}, 0 \le r \le m$ are independent, the condition

$$\int_{B_n} \phi^* \mathbf{i}_V(d\mathcal{L}) = 0$$

is satisfied if and only if the inverse recurrence relations

$$\phi^*\mathcal{F}_\alpha = \phi^*\Big[\frac{\partial L}{\partial u^\alpha} - \frac{\partial \lambda^i_\alpha}{\partial x^i} - \frac{\partial}{\partial x^i}\Big(\frac{\partial L}{\partial v^\alpha_i}\Big)\Big] = 0, \tag{10.5.8}$$
$$\phi^*\mathcal{F}^{i_1\cdots i_r}_\alpha = \phi^*\Big[\frac{\partial \lambda^{i_1\cdots i_r i}_\alpha}{\partial x^i} + \lambda^{i_1\cdots i_r}_\alpha + \frac{\partial}{\partial x^i}\Big(\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i}}\Big)\Big] = 0, 1 \le r \le m-1$$
$$\lambda^{i_1\cdots i_m}_\alpha = 0$$

are verified. We shall try to find the solution of these recurrence relations by discarding the operator ϕ^* for notational simplicity. Therefore, $\partial/\partial x^i$ will denote the total derivative with respect to that variable and we should, in fact, employ the substitution $v^\alpha_{i_1\cdots i_r} = u^\alpha_{,i_1\cdots i_r}$. Noting that $\lambda^{i_1\cdots i_m}_\alpha = 0$, we extract from $(10.5.8)_2$, respectively,

$$\lambda^{i_1\cdots i_{m-1}}_\alpha = -\frac{\partial}{\partial x^{i_m}}\Big(\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_{m-1} i_m}}\Big)$$
$$\lambda^{i_1\cdots i_{m-2}}_\alpha = -\frac{\partial \lambda^{i_1\cdots i_{m-2} i_{m-1}}_\alpha}{\partial x^{i_{m-1}}} - \frac{\partial}{\partial x^{i_{m-1}}}\Big(\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_{m-2} i_{m-1}}}\Big)$$
$$= -\frac{\partial}{\partial x^{i_{m-1}}}\Big(\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_{m-2} i_{m-1}}}\Big) + \frac{\partial^2}{\partial x^{i_{m-1}}\partial x^{i_m}}\Big(\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_{m-2} i_{m-1} i_m}}\Big).$$

. . . .

We shall now show by mathematical induction that

$$\lambda^{i_1\cdots i_r}_\alpha = \sum_{s=1}^{m-r} (-1)^s \frac{\partial^s}{\partial x^{i_{r+1}}\cdots\partial x^{i_{r+s}}}\Big(\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i_{r+1}\cdots i_{r+s}}}\Big). \tag{10.5.9}$$

Indeed, let us assume that

$$\lambda_\alpha^{i_1\cdots i_r i_{r+1}} = \sum_{s=1}^{m-r-1}(-1)^s \frac{\partial^s}{\partial x^{i_{r+2}}\cdots\partial x^{i_{r+s+1}}}\Big(\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i_{r+1} i_{r+2}\cdots i_{r+s+1}}}\Big).$$

Because of the relation

$$\begin{aligned}
&-\frac{\partial \lambda_\alpha^{i_1\cdots i_r i_{r+1}}}{\partial x^{i_{r+1}}}\\
&\qquad = \sum_{s=1}^{m-r-1}(-1)^{s+1} \frac{\partial^{s+1}}{\partial x^{i_{r+1}}\partial x^{i_{r+2}}\cdots\partial x^{i_{r+s+1}}}\Big(\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i_{r+1} i_{r+2}\cdots i_{r+s+1}}}\Big)\\
&\qquad\qquad = \sum_{t=2}^{m-r}(-1)^t \frac{\partial^t}{\partial x^{i_{r+1}}\cdots\partial x^{i_{r+t}}}\Big(\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i_{r+1}\cdots i_{r+t}}}\Big),
\end{aligned}$$

we conclude that

$$\begin{aligned}
\lambda_\alpha^{i_1\cdots i_r} &= -\frac{\partial \lambda_\alpha^{i_1\cdots i_r i_{r+1}}}{\partial x^{i_{r+1}}} - \frac{\partial}{\partial x^{i_{r+1}}}\Big(\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i_{r+1}}}\Big)\\
&= \sum_{s=1}^{m-r}(-1)^s \frac{\partial^s}{\partial x^{i_{r+1}}\cdots\partial x^{i_{r+s}}}\Big(\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i_{r+1}\cdots i_{r+s}}}\Big), 1\le r\le m-1.
\end{aligned}$$

It is clear that the terms corresponding to the case $s > m-r$ must be supposed to be nil. Hence, we get

$$\begin{aligned}
\lambda_\alpha^{i_1} &= \sum_{s=1}^{m-1}(-1)^s \frac{\partial^s}{\partial x^{i_2}\cdots\partial x^{i_{1+s}}}\Big(\frac{\partial L}{\partial v^\alpha_{i_1 i_2\cdots i_{1+s}}}\Big)\\
&= \sum_{s=2}^{m}(-1)^{s-1} \frac{\partial^{s-1}}{\partial x^{i_2}\cdots\partial x^{i_s}}\Big(\frac{\partial L}{\partial v^\alpha_{i_1 i_2\cdots i_s}}\Big)
\end{aligned}$$

and the equations $(10.5.8)_1$ take finally the form below in which Lagrange multipliers are eliminated

$$\begin{aligned}
&\frac{\partial L}{\partial u^\alpha} - \frac{\partial \lambda_\alpha^{i_1}}{\partial x^{i_1}} - \frac{\partial}{\partial x^{i_1}}\Big(\frac{\partial L}{\partial v^\alpha_{i_1}}\Big) =\\
&\qquad \frac{\partial L}{\partial u^\alpha} - \frac{\partial}{\partial x^{i_1}}\Big(\frac{\partial L}{\partial v^\alpha_{i_1}}\Big) + \sum_{t=2}^{m}(-1)^t \frac{\partial^t}{\partial x^{i_1}\cdots\partial x^{i_t}}\Big(\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_t}}\Big)\\
&\qquad\qquad = \frac{\partial L}{\partial u^\alpha} + \sum_{s=1}^{m}(-1)^s \frac{\partial^s}{\partial x^{i_1}\cdots\partial x^{i_s}}\Big(\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_s}}\Big) = 0.
\end{aligned}$$

Consequently, the functions $u^\alpha = \phi^\alpha(\mathbf{x})$ that render the action functional stationary must satisfy the *Euler-Lagrange equations*

$$\mathcal{E}_\alpha(L) = \sum_{s=0}^{m}(-1)^s \frac{\partial^s}{\partial x^{i_1}\cdots\partial x^{i_s}}\Big(\frac{\partial L}{\partial u^\alpha_{,i_1\cdots i_s}}\Big) = 0 \qquad (10.5.10)$$

where $\alpha = 1,\ldots,N$. *We have to take heed that the operators $\partial/\partial x^i$ here represent total derivatives with respect to the independent variables x^i*. For instance, if $m=2$, then the Lagrangian function is $L = L(x^i, u^\alpha, u^\alpha_{,i}, u^\alpha_{,ij})$ and the Euler-Lagrange equations turn out to be the following fourth order quasilinear partial differential equations

$$\begin{aligned}
&\frac{\partial L}{\partial u^\alpha} - \frac{\partial}{\partial x^i}\Big(\frac{\partial L}{\partial u^\alpha_{,i}}\Big) + \frac{\partial^2}{\partial x^i\partial x^j}\Big(\frac{\partial L}{\partial u^\alpha_{,ij}}\Big) = \frac{\partial L}{\partial u^\alpha} - \frac{\partial^2 L}{\partial x^i\partial u^\alpha_{,i}} + \frac{\partial^3 L}{\partial x^i\partial x^j\partial u^\alpha_{,ij}} \\
&\quad + \Big(2\frac{\partial^3 L}{\partial x^j\partial u^\beta\partial u^\alpha_{,ij}} - \frac{\partial^2 L}{\partial u^\beta\partial u^\alpha_{,i}}\Big)\frac{\partial u^\beta}{\partial x^i} + \Big(\frac{\partial^2 L}{\partial u^\beta\partial u^\alpha_{,ij}} + 2\frac{\partial^3 L}{\partial x^k\partial u^\beta_{,j}\partial u^\alpha_{,ik}} \\
&\quad - \frac{\partial^2 L}{\partial u^\alpha_{,i}\partial u^\beta_{,j}}\Big)\frac{\partial^2 u^\beta}{\partial x^i\partial x^j} + \Big(\frac{\partial^2 L}{\partial u^\beta_{,k}\partial u^\alpha_{,ij}} + 2\frac{\partial^3 L}{\partial x^l\partial u^\alpha_{,il}\partial u^\beta_{,jk}} \\
&\quad - \frac{\partial^2 L}{\partial u^\alpha_{,i}\partial u^\beta_{,jk}}\Big)\frac{\partial^3 u^\beta}{\partial x^i\partial x^j\partial x^k} + \frac{\partial^3 L}{\partial u^\beta\partial u^\gamma\partial u^\alpha_{,ij}}\frac{\partial u^\gamma}{\partial x^i}\frac{\partial u^\beta}{\partial x^j} \\
&\quad + 2\frac{\partial^3 L}{\partial u^\beta\partial u^\gamma_{,k}\partial u^\alpha_{,ij}}\frac{\partial u^\beta}{\partial x^j}\frac{\partial^2 u^\gamma}{\partial x^i\partial x^k} + 2\frac{\partial^3 L}{\partial u^\beta\partial u^\alpha_{,ij}\partial u^\gamma_{,kl}}\frac{\partial u^\beta}{\partial x^j}\frac{\partial^2 u^\gamma}{\partial x^i\partial x^k\partial x^l} \\
&\quad + \frac{\partial^3 L}{\partial u^\beta_{,k}\partial u^\gamma_{,l}\partial u^\alpha_{,ij}}\frac{\partial^2 u^\beta}{\partial x^j\partial x^k}\frac{\partial^2 u^\gamma}{\partial x^i\partial x^l} + 2\frac{\partial^3 L}{\partial u^\beta_{,k}\partial u^\alpha_{,ij}\partial u^\gamma_{,lm}}\frac{\partial^2 u^\beta}{\partial x^j\partial x^k}\frac{\partial^3 u^\gamma}{\partial x^i\partial x^l\partial x^m} \\
&\quad + \frac{\partial^3 L}{\partial u^\alpha_{,ij}\partial u^\beta_{,kl}\partial u^\gamma_{,mn}}\frac{\partial^3 u^\beta}{\partial x^j\partial x^k\partial x^l}\frac{\partial^3 u^\gamma}{\partial x^i\partial x^m\partial x^n} \\
&\quad + \frac{\partial^2 L}{\partial u^\alpha_{,ij}\partial u^\beta_{,kl}}\frac{\partial^4 u^\beta}{\partial x^i\partial x^j\partial x^k\partial x^l} = 0.
\end{aligned}$$

As a simple example, let us consider the Lagrangian function

$$L = \frac{1}{2}(u_{xx} + u_{yy})^2$$

where $u = u(x,y)$. Since

$$\frac{\partial L}{\partial u_{xx}} = \frac{\partial L}{\partial u_{yy}} = u_{xx} + u_{yy}$$

The Euler-Lagrange equation yields the *biharmonic equation*

$$(u_{xx}+u_{yy})_{xx}+(u_{xx}+u_{yy})_{yy}=u_{xxxx}+2u_{xxyy}+u_{yyyy}=\nabla^4u=0.$$

As another example, let us take $L=u_{xx}u_{yy}-u_{xy}^2$. Since

$$\frac{\partial L}{\partial u_{xx}}=u_{yy},\ \frac{\partial L}{\partial u_{yy}}=u_{xx},\ \frac{\partial L}{\partial u_{xy}}=\ -2u_{xy}$$

the Euler-Lagrange equation is satisfied identically:

$$\mathcal{E}(L)=u_{yyxx}+u_{xxyy}-2u_{xyxy}\equiv 0.$$

In other words, the action functional becomes stationary for every function u obeying the boundary conditions. This Lagrangian function can also be written in the shape

$$L=(u_xu_{yy})_x-(u_xu_{xy})_y$$

so that it is in the form of a divergence. This is, in fact, a general feature of Lagrangian functions rendered by every functions obeying the boundary conditions. In effect, if we can express a Lagrangian functions as

$$L=\frac{\partial P^i}{\partial x^i}$$

where the functions P^i depend on x^j, u^α and derivatives of various orders of u^α, then the action integral may be transformed to an integral over the boundary through the Stokes theorem as follows

$$\int_{B_n}L\,dV=\int_{\partial B_n}P^in_i\,dS.$$

Hence, it becomes dependent only on data describing the behaviours of functions u^α on the boundary. Since we have assumed that the variations on the boundary vanish, this amounts to say that the integral on the left hand side above is not affected by variations inside the region. In consequence, the Euler-Lagrange equations are satisfied identically by all functions meeting the conditions on the boundary. *It can be proven that a divergence form for the Lagrangian function provides a necessary and sufficient condition for the existence of this property*.

The result (10.5.10) can also be obtained from the equations (10.3.5) as a conditional variation problem. The function $L(x^i,u^\alpha,u^\alpha_{,i_1},\dots,u^\alpha_{,i_1\cdots i_m})$ can be written in the form $L(x^i,u^\alpha,v^\alpha_{i_1},\dots,v^\alpha_{i_1\cdots i_m})$ together with the additional conditions $v^\alpha_{i_1\cdots i_ri}=v^\alpha_{i_1\cdots i_r,i}$. In this case, the arguments of L are interrelated. In order to replace these relations in the action functional, we again use the method of Lagrange multipliers and define the function

$$\mathfrak{L} = L + \sum_{r=0}^{m-1} \lambda_\alpha^{i_1\cdots i_r i}(\mathbf{x})(v^\alpha_{i_1\cdots i_r i} - v^\alpha_{i_1\cdots i_r, i}) \tag{10.5.11}$$

where $\lambda_\alpha^{i_1\cdots i_r i}(\mathbf{x})$ are Lagrange multipliers. Therefore, the action functional becomes stationary if the following equations are to be satisfied

$$\frac{\partial \mathfrak{L}}{\partial v^\alpha_{i_1\cdots i_r}} - \frac{\partial}{\partial x^i}\Big(\frac{\partial \mathfrak{L}}{\partial v^\alpha_{i_1\cdots i_r, i}}\Big) = 0,\ \ 0 \le r \le m.$$

It then follows from (10.5.11) that

$$\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r}} + \lambda_\alpha^{i_1\cdots i_r} + \frac{\partial \lambda_\alpha^{i_1\cdots i_r i}}{\partial x^i} = 0,\ \ 0 < r \le m-1. \tag{10.5.12}$$

We obviously find

$$\lambda_\alpha^{i_1\cdots i_m} = -\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_m}}$$

for $r = m$ and

$$\frac{\partial L}{\partial u^\alpha} + \frac{\partial \lambda_\alpha^i}{\partial x^i} = 0$$

for $r = 0$. Utilising these results, one can readily show that the solution of the recurrence relation (10.5.12) is still given by the equations (10.5.10).

The Euler-Lagrange equations (10.5.10) can also be written as balance equations

$$\frac{\partial \Sigma^i_\alpha}{\partial x^i} + \Sigma_\alpha = 0$$

if we introduce the definitions

$$\Sigma^i_\alpha = \sum_{r=1}^{m}(-1)^r \frac{\partial^{r-1}}{\partial x^{i_1}\cdots\partial x^{i_{r-1}}}\Big(\frac{\partial L}{\partial u^\alpha_{,ii_1\cdots i_{r-1}}}\Big),\ \ \Sigma_\alpha = \frac{\partial L}{\partial u^\alpha}$$

However, balance equations are evidently not expressible in general in the form of the Euler-Lagrange equations.

The variational symmetries that pave the way for the Noether theorem were dealt with in Sec. 10.4. We shall now try to generalise those results for the functional given by (10.5.3). Let us consider an isovector field U of the contact ideal given by (10.5.4) together with (10.5.5). We know that we have in the first order

$$A\big(\phi_U(s)\big) = \int_{B_n} \phi^*\left(e^{s£_U}\mathcal{L}\right) \\ = A(\phi) + \delta_U A(\phi)\, s + o(s)$$

so that we can write

$$\delta_U A(\phi) = \int_{B_n} \phi^* £_U \mathcal{L}.$$

Since the contact ideal is invariant under the isovector field U, the criterion of infinitesimal invariance is again recovered from the condition $\phi^* £_U \mathcal{L} = \phi^* £_U(L\mu) = 0$ giving

$$\phi^* £_U(L\mu) = \phi^*\big[U(L) + LD_i X^i\big]\mu = 0$$

as in (10.4.5). However, this time we have

$$\phi^* U(L) = \phi^*\Big[X^i \frac{\partial L}{\partial x^i} + U^\alpha \frac{\partial L}{\partial u^\alpha} + \sum_{r=1}^{m} V^\alpha_{i_1\cdots i_r} \frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r}}\Big].$$

We shall now attempt to transform the expression $\phi^* £_U \mathcal{L} = 0$ into another form in case the regular mapping ϕ satisfies the Euler-Lagrange equations (10.5.10) or, in other words, the relations (10.5.8) leading to those equations. Because of the relation $\phi^* £_U \mathcal{L} = 0$, we obtain owing to the Cartan formula

$$\phi^* d(\mathbf{i}_U \mathcal{L}) = d\phi^*(\mathbf{i}_U \mathcal{L}) = -\,\phi^* \mathbf{i}_U(d\mathcal{L}). \tag{10.5.13}$$

Let us now try to calculate this expression explicitly. On making use of the formula for $d\mathcal{L}$ given on p. 684, we first get

$$\begin{aligned}\mathbf{i}_U(d\mathcal{L}) = (U^\alpha - v^\alpha_j X^j)\Big[\Big(\frac{\partial L}{\partial u^\alpha} - \frac{\partial \lambda^i_\alpha}{\partial x^i}\Big)\mu - d\Big(\frac{\partial L}{\partial v^\alpha_i}\Big) \wedge \mu_i\Big] \\ - \sum_{r=1}^{m-1}(V^\alpha_{i_1\cdots i_r} - v^\alpha_{i_1\cdots i_r j}X^j)\Big[\Big(\frac{\partial \lambda^{i_1\cdots i_r i}_\alpha}{\partial x^i} + \lambda^{i_1\cdots i_r}_\alpha\Big)\mu \\ + d\Big(\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i}}\Big) \wedge \mu_i\Big] \\ - \lambda^{i_1\cdots i_m}_\alpha V^\alpha_{i_1\cdots i_m}\mu + \Big\{ - X^j\Big(\frac{\partial L}{\partial u^\alpha} - \frac{\partial \lambda^i_\alpha}{\partial x^i}\Big)\sigma^\alpha \wedge \mu_j\end{aligned}$$

$$+\sum_{r=1}^{m-1} X^j\Big(\frac{\partial\lambda_\alpha^{i_1\cdots i_r i}}{\partial x^i}+\lambda_\alpha^{i_1\cdots i_r}\Big)\sigma^\alpha_{i_1\cdots i_r}\wedge\mu_j$$

$$+\sum_{r=1}^{m-1} U\Big(\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i}}\Big)\sigma^\alpha_{i_1\cdots i_r}\wedge\mu_i$$

$$+\sum_{r=1}^{m-1} X^j\, d\Big(\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i}}\Big)\wedge\sigma^\alpha_{i_1\cdots i_r}\wedge\mu_{ji}+X^i\lambda_\alpha^{i_1\cdots i_m}dv^\alpha_{i_1\cdots i_m}\wedge\mu_i\Big\}$$

where the terms within braces belong to the contact ideal $\mathcal{I}_m$. On the other hand, owing to the relations

$$\phi^* d\Big(\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i}}\Big)=\frac{\partial}{\partial x^j}\,\phi^*\Big(\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i}}\Big)dx^j$$

and $\phi^*\mathcal{I}_m=0$, we conclude that

$$\phi^*\mathbf{i}_U(d\mathcal{L})=\phi^*(U^\alpha-v^\alpha_j X^j)\,\phi^*\Big[\frac{\partial L}{\partial u^\alpha}-\frac{\partial\lambda^i_\alpha}{\partial x^i}-\frac{\partial}{\partial x^i}\Big(\frac{\partial L}{\partial v^\alpha_i}\Big)\Big]\mu$$

$$-\sum_{r=1}^{m-1}\phi^*(V^\alpha_{i_1\cdots i_r}-v^\alpha_{i_1\cdots i_r j}X^j)\,\phi^*\Big[\frac{\partial\lambda_\alpha^{i_1\cdots i_r i}}{\partial x^i}+\lambda_\alpha^{i_1\cdots i_r}+\frac{\partial}{\partial x^i}\Big(\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i}}\Big)\Big]\mu$$

$$-\lambda_\alpha^{i_1\cdots i_m}\,\phi^* V^\alpha_{i_1\cdots i_m}\mu.$$

If the mapping ϕ is a solution of the Euler-Lagrange equations the right hand side in the foregoing expression vanishes in view of (10.5.8) so that we obtain

$$\phi^*\mathbf{i}_U(d\mathcal{L})=0.$$

Thus, it follows from (10.5.13) that

$$d\phi^*(\mathbf{i}_U\mathcal{L})=d\phi^*\mathcal{J}=0$$

where the form $\mathcal{J}\in\Lambda^{n-1}(\mathcal{C}_m)$ is defined by

$$\mathcal{J}=\mathbf{i}_U\mathcal{L}=LX^i\mu_i$$

$$+\sum_{r=0}^{m-1}\Big(\lambda_\alpha^{i_1\cdots i_r i}+\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i}}\Big)(V^\alpha_{i_1\cdots i_r}-v^\alpha_{i_1\cdots i_r j}X^j)\mu_i$$

$$-\sum_{r=0}^{m-1}\Big(\lambda_\alpha^{i_1\cdots i_r i}+\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i}}\Big)X^j\sigma^\alpha_{i_1\cdots i_r}\wedge\mu_{ji}$$

Hence, we can again write $\phi^*\mathcal{J} = J^i \mu_i$ where the functions J^i are determined as follows

$$
\begin{aligned}
J^i &= \phi^*\Big[LX^i + \sum_{r=0}^{m-1}\Big(\lambda_\alpha^{i_1\cdots i_r i} + \frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i}}\Big)(V^\alpha_{i_1\cdots i_r} - v^\alpha_{i_1\cdots i_r j}X^j)\Big] \\
&= \phi^*\Big[LX^i + \sum_{r=0}^{m-1}(V^\alpha_{i_1\cdots i_r} - v^\alpha_{i_1\cdots i_r j}X^j)\Big\{\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i}} \qquad (10.5.14) \\
&\qquad + \sum_{s=2}^{m-r}(-1)^{s-1}\frac{\partial^{s-1}}{\partial x^{i_{r+1}}\cdots\partial x^{i_{r+s-1}}}\Big(\frac{\partial L}{\partial v^\alpha_{i_1\cdots i_r i_{r+1}\cdots i_{r+s-1} i}}\Big)\Big\}\Big]
\end{aligned}
$$

Accordingly, the relation $d\phi^*\mathcal{J} = 0$ leads again to the conservation law

$$\frac{\partial J^i}{\partial x^i} = 0.$$

It is immediate to observe that these expressions are also valid in the case $N = 1$. But, it is evident that we have then to employ isovector components generated by a function $F(x^i, u, v_i)$.

For $m = 1$, we obtain from (10.5.14)

$$
\begin{aligned}
J^i &= \phi^*\Big[LX^i + \frac{\partial L}{\partial v_i^\alpha}(U^\alpha - v_j^\alpha X^j)\Big], \\
\phi^* v_i^\alpha &= u^\alpha_{,i}.
\end{aligned}
$$

This result is exactly the same as the expression (10.4.6) previously obtained. When we choose $m = 2$, we find that

$$
\begin{aligned}
J^i = \phi^*\Big[LX^i &+ \Big\{\frac{\partial L}{\partial v_i^\alpha} - \frac{\partial}{\partial x^k}\Big(\frac{\partial L}{\partial v^\alpha_{ik}}\Big)\Big\}(U^\alpha - v_j^\alpha X^j) \\
&+ \frac{\partial L}{\partial v^\alpha_{ik}}(V_k^\alpha - v^\alpha_{jk}X^j)\Big]
\end{aligned}
$$

where we have $\phi^* v_i^\alpha = u^\alpha_{,i}$, $\phi^* v^\alpha_{ij} = u^\alpha_{,ij}$ and the functions V_i^α are expressible as

$$V_i^\alpha = \frac{\partial U^\alpha}{\partial x^i} + v_i^\beta \frac{\partial U^\alpha}{\partial u^\beta} - v_j^\alpha \frac{\partial X^j}{\partial x^i} - v_j^\alpha v_i^\beta \frac{\partial X^j}{\partial u^\beta}$$

so that we can write

$$\phi^* V_i^\alpha = \frac{\partial \phi^* U^\alpha}{\partial x^i} - u^\alpha_{,j}\frac{\partial \phi^* X^j}{\partial x^i}$$

X. EXERCISES

10.1. Show that the balance equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = c(u)\frac{\partial u}{\partial t}, \quad c(u) \neq 0$$

modelling the propagation of one-dimensional waves in a dissipative medium cannot be derived from a variational principle. Determine the enlarged system that gives rise to a variational principle and find this principle.

10.2. The following non-linear ordinary second order equation

$$u'' + \frac{k}{x}\,u' - \lambda\, x^\alpha u^\beta = 0$$

is known as the *Emden-Fowler equation* [Swiss astrophysicist and meteorologist Jacob Robert Emden (1862-1940) and English physicist and astronomer Sir Ralph Howard Fowler (1889-1944)]. k, λ, α and β are given real constants.

(a). Show that this equation is derivable from a variational principle (*Hint*: multiply the differential equation by x^k).

(b). Show that this variational principle has a variational symmetry in the form of scaling transformation if the condition $\alpha = [(\beta-1)k - \beta - 3]/2$ is satisfied.

(c). Utilising that symmetry, show that the Emden-Fowler equation has a solution in the form

$$u(x) = \Big(- \frac{(1-k)^2}{4\lambda}\Big)^{1/(\beta-1)} x^{(1-k)/2}.$$

For instance, when we take $k=2,\ \alpha=0,\ \beta=5$ and $\lambda=-1$ we get $u = 1/\sqrt{2x}$.

10.3. Determine the variational symmetry groups of the variational principle generating the Laplace equation $\Delta u = u_{,ii} = 0$ in $\mathbb{R}^n$ and find the corresponding conservation laws.

10.4. Determine the variational symmetries of a variational principle generating the following differential equations

$$\Delta u^1 + \lambda u^2 = 0, \quad \Delta u^2 + \lambda u^1 = 0$$

in $\mathbb{R}^n$ where λ is a constant. Find the corresponding conservation laws.

10.5. The *Navier equations* governing the motion of a homogeneous and isotropic linearly elastic three-dimensional body are given by

$$(c_1^2 - c_2^2)\frac{\partial^2 u_j}{\partial x_i \partial x_j} + c_2^2 \frac{\partial^2 u_i}{\partial x_j \partial x_j} = \frac{\partial^2 u_i}{\partial t^2}$$

where u_i with $i = 1, 2, 3$ denote the Cartesian components of the displacement vector field. The constants c_1 and c_2 are the velocities of propagation of the longitudinal and transversal waves in the medium, respectively. Find a variational principle generating these equations. Determine the variational symmetries and corresponding conservation laws.

10.6. Find the variational principle generating the equation $\Delta^2 u + \lambda u = 0$ in $\mathbb{R}^n$. Determine the variational symmetries and corresponding conservation laws.

10.7. Show that the *BBM* (*Benjamin-Bona-Mahony*) *equation*

$$u_t + u_x + uu_x - u_{xxt} = 0$$

that is encountered in the shallow water theory in hydrodynamics is derivable from a variational principle after the substitution $u = v_x$. Find the variational symmetries and corresponding conservation laws.